COLLOQUIUM MATHEMATICUM

VOL. 179 2025 NO. 2

BOUNDEDNESS OF THE GENERALIZED RIESZ POTENTIAL IN CENTRAL MORREY–ORLICZ SPACES

BY

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Abstract. We investigate the boundedness of the generalized Riesz potential in central Morrey–Orlicz spaces. We also present necessary and sufficient conditions for the generalized Riesz potential to be finite under certain assumptions on the kernel.

1. Generalized Riesz potential.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function. The *generalized Riesz potential* $I_\rho f$ is defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \rho(|x-y|) f(y)\,dy, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $\rho\colon (0,\infty) \to (0,\infty)$ is a Lebesgue measurable function. When $\rho(t) = t^{\alpha-n}$ for some $0 < \alpha < n$, the operator I_ρ coincides with the classical Riesz potential I_α.

The study of generalized fractional integral operators dates back to the 1950s and 1960s, with early contributions by [9], [10, pp. 146–172 (pp. 144–172 in the English version)], [25] and [26]. The problem of determining the finiteness of the Riesz potential naturally arises within the framework of potential theory.

In 1966, N. S. Landkof established a necessary and sufficient condition for the almost everywhere finiteness of the Riesz potential I_α, presented in his fundamental monograph [12, pp. 84–85] (see also pp. 61–62 in the English version). An alternative proof of an equivalent result on the finiteness of the Riesz potential was later provided by M. Essén [1, Lemma 8.20] and independently by Y. Mizuta [17, Chapter 2, Theorem 1.1].

In this paper, we extend these findings by deriving a necessary and sufficient condition for the almost everywhere finiteness of the generalized Riesz potential I_ρ under specific assumptions on the kernel ρ.

2020 *Mathematics Subject Classification*: Primary 46E30; Secondary 42B20, 42B35.
Key words and phrases: generalized Riesz potential, fractional integrals, Orlicz functions, central Morrey–Orlicz spaces.
Received 16 May 2025; revised 31 August 2025.
Published online 4 November 2025 in Open Access (under CC-BY license).

DOI: 10.4064/cm9633-9-2025

Furthermore, we investigate the boundedness of I_ρ in central Morrey–Orlicz spaces. The boundedness of generalized Riesz potentials in Orlicz spaces was examined by E. Nakai [18], while their behaviour in Morrey spaces was studied by Eridani et al. [6, 7]. We also refer to E. Pustylnik's article [21], where the generalized Riesz potential was analysed in rearrangement-invariant spaces.

The boundedness of I_ρ in Morrey–Orlicz spaces was investigated by E. Nakai [19, 20], with further developments by Kawasumi et al. [8]. More recently, V. I. Burenkov and M. A. Senouci [2, 3] examined the generalized Riesz potential in central Morrey spaces. We also refer to [24, Vol. II], where Y. Sawano et al. investigated boundedness of the generalized Riesz potential in generalized Morrey and Morrey–Orlicz spaces.

Building on our prior work in [4], which focused on the classical Riesz potential in central Morrey–Orlicz spaces, this paper extends those results to the generalized Riesz potential in the same setting.

2. Convergence of the generalized Riesz potential. First, we prove a necessary and sufficient condition for finiteness almost everywhere in $\mathbb{R}^n$ of the generalized Riesz potential under certain assumptions on the kernel ρ. For any Radon measure μ on $\mathbb{R}^n$ we define the generalized Riesz potential of μ as

$$I_\rho \mu(x) = \int_{\mathbb{R}^n} \rho(|x-y|)\, d\mu(y).$$

In order to derive further results, we require additional assumptions on the function ρ. We say that $\rho \in D$ if for some $c_1, c_2 > 0$,

$$c_1 \rho(t) \le \rho(s) \le c_2 \rho(t) \quad \text{for all } s, t > 0 \text{ with } t/2 \le s \le 2t.$$

Note that if $\rho_1, \rho_2 \in D$, then the pointwise product $\rho_1 \cdot \rho_2$ is in D, and also $\rho \in D$ where

$$\rho(t) = \begin{cases} \rho_1(t), & 0 < t \le 1, \\ \rho_2(t), & 1 < t < \infty. \end{cases}$$

LEMMA 2.1 ([3, Remark 3.5]). *If $\rho \in D$, then for any $a > 0$ there exist constants $C_{1a}, C_{2a} > 0$ such that*

$$C_{1a} \rho(t) \le \rho(at) \le C_{2a} \rho(t)$$

for any $t > 0$.

For $n \in \mathbb{N}$, we say that $\rho \in D_n$ if $\rho \in D$ and

$$\int_0^r \rho(t) t^{n-1}\, dt < \infty \quad \text{for all } r > 0. \tag{2.1}$$

EXAMPLE 2.2 ([3, Example 2]). For $0 < \alpha < n$ and $\beta_1, \beta_2 \in \mathbb{R}$ let $\rho(t) = t^{\alpha-n}\varphi_{\beta_1,\beta_2}(t)$ with

$$\varphi_{\beta_1,\beta_2}(t) = \begin{cases} (1+|\ln t|)^{\beta_1}, & 0 < t \le 1, \\ (1+|\ln t|)^{\beta_2}, & t \ge 1. \end{cases}$$

Then $\rho \in D_n$.

We say that a function $\rho\colon (0,\infty) \to (0,\infty)$ is *almost-decreasing* if there exists a positive constant $A > 0$ such that $\rho(t) \le A\rho(s)$ for all $0 < s < t$.

EXAMPLE 2.3 ([20, p. 209]). For $\beta > 0$ let

$$k(t) = \begin{cases} (\ln 1/t)^{-\beta-1} & \text{for small } t > 0, \\ (\ln t)^{\beta-1} & \text{for large } t > 0. \end{cases}$$

Then $k \in D$. For $0 < \alpha < n$ the function $\rho(t) = t^{\alpha-n}k(t)$ is in D_n since

$$\int_0^r \rho(t)t^{n-1}\,dt = \int_0^r t^{\alpha-1}k(t)\,dt \le r^\alpha \int_0^r \frac{k(t)}{t}\,dt < \infty.$$

Moreover, if $\alpha < n-1$, $n \ge 2$, then ρ is almost-decreasing. In fact, since $k(t)/t$ is almost-decreasing, for $s < t$ we have

$$\rho(t) = t^{\alpha-n+1}\frac{k(t)}{t} \le s^{\alpha-n+1}A\frac{k(s)}{s} = A\rho(s).$$

EXAMPLE 2.4. The assumption $\rho \in D$ is stronger than $\rho \in \Delta_2$ even for decreasing functions (obviously, a decreasing or almost-decreasing function satisfies the Δ_2-condition). For example, $\rho(t) = e^{-t}$ is decreasing but for $s = t/2$ we have

$$\frac{\rho(t/2)}{\rho(t)} = e^{t/2} \to \infty \qquad \text{as } t \to \infty$$

and the estimate from below does not hold.

The following theorem is an extension of the result proved by N. S. Landkof in [12, pp. 84–85 (see also pp. 61–62 in the English version)] for the classical Riesz potential I_α. Everywhere below, $B(x,r)$ denotes the open ball with center at $x \in \mathbb{R}^n$ and radius $r > 0$, that is, $B(x,r) = \{y \in \mathbb{R}^n\colon |x-y| < r\}$, and v_n denotes the Lebesgue measure of a unit ball in $\mathbb{R}^n$, that is, $v_n = |B(0,1)| = \pi^{n/2}/\Gamma(n/2+1)$.

THEOREM 2.5. *Let $\rho \in D_n$ be almost-decreasing. Then $I_\rho\mu < \infty$ a.e. if and only if*

$$\int_{|y|>1} \rho(|y|)\,d\mu(y) < \infty. \tag{2.2}$$

Proof. Let first $I_\rho\mu(x) < \infty$ a.e. and suppose that (2.2) is not true. Then for any $r > 0$ we have

$$\int\limits_{|y|>r} \rho(|y|)\, d\mu(y) = \infty.$$

Indeed, this is obvious for $0 < r \le 1$. If $r > 1$ then

$$\int\limits_{|y|>r} \rho(|y|)\, d\mu(y) = \int\limits_{|y|>1} \rho(|y|)\, d\mu(y) - \int\limits_{1\le|y|<r} \rho(|y|)\, d\mu(y) = \infty,$$

since

$$\int\limits_{1\le|y|<r} \rho(|y|)\, d\mu(y) \le A\rho(1)\mu(B(0,r)) < \infty,$$

where $\rho(|y|) \le A\rho(1)$ because ρ is almost-decreasing on $(0,\infty)$.

Then using the fact that $\rho \in D$ is almost-decreasing on $(0,\infty)$ we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^n} \rho(|x-y|)\, d\mu(y) &\ge \int\limits_{|y|>|x|+1} \rho(|x-y|)\, d\mu(y)\\ &= \int\limits_{|y|>|x|+1} \rho\bigg(\frac{|x-y|}{|y|}|y|\bigg)\, d\mu(y)\\ &\ge \frac{1}{A}\int\limits_{|y|>|x|+1} \rho(2|y|)\, d\mu(y)\\ &\ge \frac{c_1}{A}\int\limits_{|y|>|x|+1} \rho(|y|)\, d\mu(y) = \infty,\end{aligned}$$

since for $|y| > |x| + 1$ we have

$$\frac{|x-y|}{|y|} \le \frac{|x|+|y|}{|y|} = 1 + \frac{|x|}{|y|} \le 1 + \frac{|x|}{|x|+1} \le 2,$$

and we arrive at a contradiction.

Next, we will show that (2.2) implies finiteness of $I_\rho\mu$ a.e. First note that for any $r > 0$, $|x| < r$, constant $c_3 = c_3(r) > \max\{1, 1/r\}$ and any $|y| > c_3 r$ we have

$$\begin{aligned}\frac{|y|}{|x-y|} &\le \frac{|x-y|+|x|}{|x-y|} = 1 + \frac{|x|}{|x-y|} \le 1 + \frac{|x|}{|y|-|x|}\\ &\le 1 + \frac{r}{c_3 r - r} = \frac{c_3}{c_3-1},\end{aligned}$$

and so

$$\begin{aligned}\int\limits_{|x|<r} \rho(|x-y|)\, dx &= \int\limits_{|x|<r} \rho\bigg(\frac{|x-y|}{|y|}|y|\bigg)\, dx \le A\int\limits_{|x|<r} \rho\bigg(\frac{c_3-1}{c_3}|y|\bigg)\, dx\\ &\le AC_{2a} v_n r^n \rho(|y|),\end{aligned}$$

where we have used the fact that ρ is almost-decreasing on $(0,\infty)$ and satisfies the statement of Lemma 2.1 with $a = \frac{c_3-1}{c_3}$.

Thus, by Fubini's theorem we get

$$\begin{aligned}\int\limits_{|x|<r} I_\rho\mu(x)\,dx &= \int\limits_{|y|\le c_3 r}\int\limits_{|x|<r} \rho(|x-y|)\,dx\,d\mu(y)\\ &\quad+\int\limits_{|y|>c_3 r}\int\limits_{|x|<r} \rho(|x-y|)\,dx\,d\mu(y)\\ &\le \int\limits_{|y|\le c_3 r}\int\limits_{|x-y|<(c_3+1)r} \rho(|x-y|)\,dx\,d\mu(y)\\ &\quad+AC_{2a}v_n r^n\int\limits_{|y|>c_3 r}\rho(|y|)\,d\mu(y).\end{aligned}$$

Since ρ satisfies (2.1) it follows by (2.2) that

$$\begin{aligned}\int\limits_{|x|<r} I_\rho\mu(x)\,dx &\le v_{n-1}\int\limits_{|y|\le c_3 r}\Big(\int\limits_0^{(c_3+1)r}\rho(t)t^{n-1}\,dt\Big)\,d\mu(y)\\ &\quad+AC_{2a}v_n r^n\int\limits_{|y|>c_3 r}\rho(|y|)\,d\mu(y)\\ &\le v_{n-1}\mu(B(0,c_3 r))\int\limits_0^{(c_3+1)r}\rho(t)t^{n-1}\,dt\\ &\quad+AC_{2a}v_n r^n\int\limits_{|y|>c_3 r}\rho(|y|)\,d\mu(y)<\infty.\end{aligned}$$

Since $\int_{|x|<r} I_\rho\mu(x)\,dx<\infty$ for any $r>0$, we see that $I_\rho\mu<\infty$ a.e. ∎

In 1979, T. Kurokawa and Y. Mizuta [11, Lemma 2.1] provided an equivalent formulation of the result stated in Theorem 2.5 for the classical Riesz potential I_α. An earlier discussion of this result appears in Y. Mizuta's 1977 paper [16], where it was mentioned in Remark 1. Although no proof was provided, both papers reference Landkof's original result [12, pp. 61–62].

In 1996, Y. Mizuta presented an independent proof of this result in his book [17, Chapter 2, Theorem 1.1]. Interestingly, this later work makes no reference to Landkof's result. Similarly, a more recent book, [24, Vol. I, Lemma 180], presents a necessary and sufficient condition for $I_\alpha f(x)$ to converge absolutely, but also fails to cite Landkof's original result.

Below we present a generalization of Theorem 1.1 from [17, Chapter 2] to the generalized Riesz potential. In order to prove Theorem 2.7 we need the following lemma:

LEMMA 2.6. *For any $x \in \mathbb{R}^n$, $r > 0$ and $y \in \mathbb{R}^n \setminus B(x,r)$ there exists $M = M(|x|, r) > 0$ such that*

$$\frac{1}{M}(1+|y|) \leq |x-y| \leq M(1+|y|), \tag{2.3}$$

where $M = \max\left\{2 + \frac{1+|x|}{r}, 1+|x|\right\}$.

Proof. First note that for any $x, y \in \mathbb{R}^n$ we have

$$|x-y| \leq |x| + |y| \leq \max\{1, |x|\}(1+|y|) \leq (1+|x|)(1+|y|).$$

Let now $E_k = \{y \in \mathbb{R}^n \colon 2^k r \leq |x-y| < 2^{k+1} r\}$ for $k \in \mathbb{N} \cup \{0\}$. Then

$$\{y \in \mathbb{R}^n \colon |x-y| \geq r\} = \bigcup_{k=0}^{\infty} E_k$$

and for any $y \in E_k$ we get

$$|x-y| \geq \frac{2^k r(1+|y|)}{1+|y|} \geq \frac{2^k r}{1+|y-x|+|x|}(1+|y|) > \frac{2^k r}{1+|x|+2^{k+1}r}(1+|y|).$$

Since the function $f(t) = \frac{t}{1+|x|+2t}$ is increasing for $t \geq r$, it follows that

$$|x-y| \geq \frac{r}{1+|x|+2r}(1+|y|)$$

and we arrive at (2.3) with $M = \max\left\{2 + \frac{1+|x|}{r}, 1+|x|\right\}$. ∎

THEOREM 2.7. *Let $\rho \in D_n$ be almost-decreasing. For any Radon measure μ on $\mathbb{R}^n$ the following statements are equivalent:*

(R1) *$I_\rho \mu < \infty$ a.e.;*
(R2) *$\int_{\mathbb{R}^n} \rho(1+|y|)\, d\mu(y) < \infty$;*
(R3) *$\int_{\mathbb{R}^n \setminus B(x,r)} \rho(|x-y|)\, d\mu(y) < \infty$ for any $x \in \mathbb{R}^n$ and any $r > 0$;*
(R4) *$\int_{\mathbb{R}^n \setminus B(x,r)} \rho(|x-y|)\, d\mu(y) < \infty$ for some $x \in \mathbb{R}^n$ and some $r > 0$.*

Proof. We follow the ideas of [17, Chapter 2, Theorem 1.1]. By Lemma 2.6, we have $1+|y| \leq M|x-y|$ for any $|x-y| \geq r$ with $r > 0$. Applying Lemma 2.1 with $a = M$, we find that

$$\begin{aligned}\int_{\mathbb{R}^n} \rho(1+|y|)\, d\mu(y) &\geq \frac{1}{A} \int_{\mathbb{R}^n \setminus B(x,r)} \rho(M|x-y|)\, d\mu(y) \\ &\geq \frac{C_{1a}}{A} \int_{\mathbb{R}^n \setminus B(x,r)} \rho(|x-y|)\, d\mu(y).\end{aligned}$$

Thus, (R2) implies (R3) and thereafter (R4).

Next, for some $x_0 \in \mathbb{R}^n$ and $r_0 > 0$, using (2.3) and applying Lemma 2.1 with $a = 1/M$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \rho(1+|y|)\,d\mu(y) &= \int_{B(x_0,r_0)} \rho(1+|y|)\,d\mu(y) + \int_{\mathbb{R}^n\setminus B(x_0,r_0)} \rho(1+|y|)\,d\mu(y)\\ &\le A\rho(1)\mu(B(x_0,r_0)) + A\int_{|y-x_0|\ge r_0} \rho\Big(\frac{1}{M}|x_0-y|\Big)\,d\mu(y)\\ &\le A\rho(1)\mu(B(x_0,r_0)) + AC_{2a}\int_{|y-x_0|\ge r_0} \rho(|x_0-y|)\,d\mu(y),\end{aligned}$$

which shows that (R4) implies (R2). Thus, (R2), (R3) and (R4) are equivalent.

Note that (R1) implies (R4). Finally, we will show that (R1) follows from (R3). Indeed, by Fubini's theorem we obtain

$$\begin{aligned}\int_{B(0,N)} \Big(\int_{B(x,r)} \rho(|x-y|)\,d\mu(y)\Big)\,dx &= \int_{B(0,N+r)} \Big(\int_{B(y,r)} \rho(|x-y|)\,dx\Big)\,d\mu(y)\\ &= v_{n-1}\int_{B(0,N+r)} \Big(\int_0^r \rho(t)t^{n-1}\,dt\Big)\,d\mu(y).\end{aligned}$$

Since ρ satisfies (2.1) it follows that $\int_{B(0,N)}(\int_{B(x,r)} \rho(|x-y|)\,d\mu(y))\,dx < \infty$ for any $N > 0$ and therefore (R3) implies (R1). Thus, all four conditions of the theorem are equivalent. ∎

3. Central Morrey–Orlicz spaces. In this paper we consider the generalized Riesz potential in central Morrey–Orlicz spaces. These spaces are a generalization of Orlicz spaces and central Morrey spaces (on $\mathbb{R}^n$). Central Morrey–Orlicz spaces appeared for the first time in [14, 15] where the authors investigated the boundedness of the Hardy–Littlewood maximal operator and the Calderón–Zygmund operator on those spaces. Below we recall the definition and some properties of these spaces.

A function $\Phi\colon [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is an increasing continuous and convex function with $\Phi(0) = 0$. Everywhere in this paper we will consider generalized Riesz potentials in central Morrey–Orlicz spaces defined by Orlicz functions.

For any Orlicz function Φ, the number $\lambda \in \mathbb{R}$ and an open ball $B_r = \{x \in \mathbb{R}^n\colon |x| < r\}$, $r > 0$, we define the *central Morrey–Orlicz space* $M^{\Phi,\lambda}(0)$ to consist of all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M^{\Phi,\lambda}(0)} = \sup_{r>0} \|f\|_{\Phi,\lambda,B_r} < \infty,$$

where

$$\|f\|_{\Phi,\lambda,B_r} = \inf\left\{\varepsilon > 0\colon \frac{1}{|B_r|^\lambda}\int_{B_r}\Phi\left(\frac{|f(x)|}{\varepsilon}\right)dx \le 1\right\}.$$

The properties of these spaces can be found in [4, 5]. If $\Phi(u) = u^p$, $1 \le p < \infty$, and $\lambda \in \mathbb{R}$, then we get the *classical central Morrey space* $M^{\Phi,\lambda}(0) = M^{p,\lambda}(0)$, and if $\lambda = 0$ then $M^{\Phi,0}(0) = L^\Phi(\mathbb{R}^n)$ is a classical Orlicz space.

To each Orlicz function Φ one can associate another convex function Φ^*, the *complementary function* to Φ, which is defined by

$$\Phi^*(v) = \sup_{u>0}\,[uv - \Phi(u)] \quad \text{for } v \ge 0.$$

Observe that $\Phi^*(v)$ can be 0 on an interval $[0,a]$ for some $a = a_\Phi \ge 0$. Additionally, it can attain the value $\Phi^*(v) = \infty$ for $v \ge b$, where $0 < b = b_\Phi < \infty$. Moreover, $\Phi^*(v)$ is an increasing convex function for $a_\Phi < v < b_\Phi$ (cf. [13, pp. 51–52]).

LEMMA 3.1. *Let Φ be an Orlicz function, Φ^* its complementary function, $0 \le \lambda \le 1$ and $r > 0$. Then the following hold:*

(i) $$\int_{B_r}|f(x)g(x)|\,dx \le 2\,|B_r|^\lambda\,\|f\|_{\Phi,\lambda,B_r}\|g\|_{\Phi^*,\lambda,B_r};$$

(ii) $$\|\chi_{B(x_0,r_0)}\|_{\Phi^*,\lambda,B_r} \le \frac{|B_r \cap B(x_0,r_0)|}{|B_r|^\lambda}\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r \cap B(x_0,r_0)|}\right),$$

where $B_r \cap B(x_0,r_0) \ne \emptyset$ for $x_0 \in \mathbb{R}^n$ and $r_0 > 0$; in particular,

$$\|\chi_{B_r}\|_{\Phi^*,\lambda,B_r} \le \frac{\Phi^{-1}(|B_r|^{\lambda-1})}{|B_r|^{\lambda-1}};$$

(iii) $$\|\chi_{B_t}\|_{\Phi,\lambda,B_r} = \frac{1}{\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r\cap B_t|}\right)} \quad \text{and} \quad \|\chi_{B_t}\|_{M^{\Phi,\lambda}(0)} = \frac{1}{\Phi^{-1}(|B_t|^{\lambda-1})}$$

for any $t > 0$.

The proof of this lemma can be found in [5, Lemma 1].

4. Boundedness of the generalized Riesz potential in central Morrey–Orlicz spaces. In this section we prove the boundedness of the generalized Riesz potential I_ρ in central Morrey–Orlicz spaces. This is a generalization of our previous results from [4, 5] on the boundedness of the classical Riesz potential in these spaces. Below we will use estimates from [14] for the Hardy–Littlewood maximal operator, defined for $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ by

$$Mf(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy.$$

We say that $\Phi^* \in \Delta_2$ if $0 < \Phi^*(u) < \infty$ and there exists a constant $K_2 > 1$ such that $\Phi^*(2u) \le K_2 \Phi^*(u)$ for all $u > 0$. For an Orlicz function Φ and $0 \le \lambda \le 1$, the operator M is bounded on $M^{\Phi,\lambda}(0)$ provided $\Phi^* \in \Delta_2$, that is, there exists a constant $C_0 > 1$ such that

$$\|Mf\|_{M^{\Phi,\lambda}(0)} \le C_0 \|f\|_{M^{\Phi,\lambda}(0)} \quad \text{for all } f \in M^{\Phi,\lambda}(0) \tag{4.1}$$

(see [14, Theorem 6(i)]).

For an almost-decreasing function $\rho \in D_n$ let us denote

$$\widehat{\rho}(r) = \int_0^r \rho(t^{1/n})\, dt. \tag{4.2}$$

Note that the requirement (2.1) on ρ is equivalent to $\widehat{\rho}(r) < \infty$ for all $r > 0$, that is,

$$\widehat{\rho}(r) = \int_0^r \rho(t^{1/n})\, dt = n \int_0^{r^{1/n}} \rho(t) t^{n-1}\, dt < \infty \quad \text{for all } r > 0.$$

We will need the following Hedberg-type pointwise estimate.

LEMMA 4.1. *If $f\colon \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable and $\rho \in D_n$ is almost-decreasing, then for all $x \in \mathbb{R}^n$ and $r > 0$,*

$$\int_{|y-x|\le r} \rho(|x-y|)|f(y)|\, dy \le C_H \widehat{\rho}(|B_r|) Mf(x), \tag{4.3}$$

where $\widehat{\rho}(r)$ is defined in (4.2), $C_H = \frac{2^n A^2 C_{2a}}{n \ln 2}$ and C_{2a} is the constant in Lemma 2.1 with $a = \frac{1}{2} v_n^{-1/n}$.

Proof. The proof is similar to the proof of [4, Lemma 2]. Since ρ is almost-decreasing on $(0, \infty)$ it follows that for any $x \in \mathbb{R}^n$ and $r > 0$,

$$\begin{aligned} \int_{|y-x|\le r} \rho(|x-y|)|f(y)|\, dy &= \sum_{k=0}^{\infty} \int_{r2^{-k-1}<|y-x|\le r2^{-k}} \rho(|x-y|)|f(y)|\, dy \\ &\le \frac{A}{n \ln 2} Mf(x) \sum_{k=0}^{\infty} |B_{r2^{-k}}| \rho(r2^{-k-1}) \int_{|B_{r2^{-k-1}}|}^{|B_{r2^{-k}}|} \frac{dt}{t}. \end{aligned}$$

By Lemma 2.1 with $a = \frac{1}{2} v_n^{-1/n}$ we have

$$\rho(r2^{-k-1}) = \rho\big(\tfrac{1}{2} v_n^{-1/n} |B_{r2^{-k}}|^{1/n}\big) \le C_{2a} \rho(|B_{r2^{-k}}|^{1/n}).$$

Thus,

$$\int_{|y-x|\le r} \rho(|x-y|)|f(y)|\,dy \le \frac{2^n A^2 C_{2a}}{n\ln 2} Mf(x) \sum_{k=0}^{\infty} \int_{|B_{r2^{-k-1}}|}^{|B_{r2^{-k}}|} \rho(t^{1/n})\,dt$$
$$= C_H \widehat{\rho}(|B_r|) Mf(x),$$

where $C_H = \frac{2^n A^2 C_{2a}}{n \ln 2}$. ■

THEOREM 4.2. *Let $\rho \in D_n$ be almost-decreasing. Let also Φ, Ψ be Orlicz functions, $0 < \lambda$, $\mu < 1$, $\lambda \ne \mu$ or $\lambda = 0$ and $0 \le \mu < 1$. Assume that there exist constants $C_1, C_2 \ge 1$ such that*

$$\int_u^{\infty} \widehat{\rho}(t) \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \le C_1 \Psi^{-1}(u^{\mu-1}) \quad \textit{for all } u > 0 \tag{4.4}$$

and

$$\Phi^{-1}\left(\frac{r^\lambda}{u}\right)\widehat{\rho}(u) + \int_u^r \widehat{\rho}(t)\Phi^{-1}\left(\frac{r^\lambda}{t}\right)\frac{dt}{t} \le C_2 \Psi^{-1}\left(\frac{r^\mu}{u}\right) \quad \textit{for all } r > u > 0, \tag{4.5}$$

where $\widehat{\rho}$ is defined by (4.2). If $\Phi^* \in \Delta_2$, then I_ρ is bounded from $M^{\Phi,\lambda}(0)$ to $M^{\Psi,\mu}(0)$, that is, there exists a constant $C_3 \ge 1$ such that $\|I_\rho f\|_{M^{\Psi,\mu}(0)} \le C_3 \|f\|_{M^{\Phi,\lambda}(0)}$ for all $f \in M^{\Phi,\lambda}(0)$.

Proof. We follow the proof of Theorem 1 in [4]. For any $x \in B_r$ and $f \in M^{\Phi,\lambda}(0)$ we consider two disjoint subsets

$$B_r^1 = \left\{ x \in B_r : \Phi\left(\frac{Mf(x)}{C_0\|f\|_{M^{\Phi,\lambda}(0)}}\right) \le |B_r|^{\lambda-1} \right\},$$
$$B_r^2 = \left\{ x \in B_r : \Phi\left(\frac{Mf(x)}{C_0\|f\|_{M^{\Phi,\lambda}(0)}}\right) > |B_r|^{\lambda-1} \right\}.$$

We have

$$|I_\rho f(x)| \le \int_{|y|\le 2r} \rho(|x-y|)|f(y)|\,dy + \int_{|y|>2r} \rho(|x-y|)|f(y)|\,dy$$
$$= I_1 f(x) + I_2 f(x).$$

Let first $x \in B_r^1$ and $|y| \le 2r$. Then $|x-y| \le |x| + |y| \le 3r$ and by Hedberg's pointwise estimate proved in Lemma 4.1 we have

$$I_1 f(x) = \int_{|y|\le 2r} \rho(|x-y|)|f(y)|\,dy \le \int_{|x-y|\le 3r} \rho(|x-y|)|f(y)|\,dy$$
$$\le C_H \widehat{\rho}(|B_{3r}|) Mf(x).$$

Since $x \in B_r^1$ it follows that $Mf(x) \le C_0 \|f\|_{M^{\Phi,\lambda}(0)} \Phi^{-1}(|B_r|^{\lambda-1})$, and we obtain

$$\begin{aligned} I_1 f(x) &\le C_H C_0 \|f\|_{M^{\Phi,\lambda}(0)} \Phi^{-1}(|B_r|^{\lambda-1}) \int_0^{|B_{3r}|} \rho(t^{1/n})\, dt \\ &\le 3^n C_H C_0 C_{23} \|f\|_{M^{\Phi,\lambda}(0)} \Phi^{-1}(|B_r|^{\lambda-1}) \int_0^{|B_r|} \rho(t^{1/n})\, dt \\ &= C_4 \|f\|_{M^{\Phi,\lambda}(0)} \Phi^{-1}(|B_r|^{\lambda-1}) \widehat{\rho}(|B_r|), \end{aligned}$$

where $C_{23} = C_{2a}$ with $a = 3$ and $C_4 = 3^n C_H C_0 C_{23}$.

Since $\widehat{\rho}$ is increasing on $(0, \infty)$ and Φ^{-1} is concave on $(0, \infty)$ it follows that

$$\begin{aligned} \int_u^\infty \widehat{\rho}(t) \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \ge \int_u^{2u} \widehat{\rho}(t) \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} &\ge \Phi^{-1}((2u)^{\lambda-1}) \int_u^{2u} \widehat{\rho}(t) \frac{dt}{t} \\ &\ge 2^{\lambda-1} \ln 2\ \widehat{\rho}(u) \Phi^{-1}(u^{\lambda-1}). \end{aligned}$$

Thus, by (4.4) we get

$$\begin{aligned} I_1 f(x) &\le \frac{C_4}{2^{\lambda-1}\ln 2} \|f\|_{M^{\Phi,\lambda}(0)} \int_{|B_r|}^\infty \widehat{\rho}(t) \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \\ &\le \frac{C_1 C_4}{2^{\lambda-1}\ln 2} \|f\|_{M^{\Phi,\lambda}(0)} \Psi^{-1}(|B_r|^{\mu-1}) \\ &\le \frac{2^{n(1-\mu)} C_1 C_4}{2^{\lambda-1}\ln 2} \|f\|_{M^{\Phi,\lambda}(0)} \Psi^{-1}(|B_{2r}|^{\mu-1}). \end{aligned}$$

Let now $x \in B_r^1$ and $|y| > 2r$. Since $|x| < r < |y|/2$, $|x - y| \ge |y| - |x| > |y|/2$ and ρ is almost-decreasing on $(0, \infty)$ it follows that $\rho(|x - y|) \le A\rho(|y|/2) \le c_2 A\rho(|y|)$. Thus, we obtain

$$\begin{aligned} I_2 f(x) &= \int_{|y|>2r} \rho(|x-y|)|f(y)|\, dy \le c_2 A \int_{|y|>2r} \rho(|y|)|f(y)|\, dy \\ &= c_2 A \sum_{k=0}^\infty \int_{r2^{k+1}<|y|\le r2^{k+2}} \rho(|y|)|f(y)|\, dy \\ &< c_2 A^2 \sum_{k=0}^\infty \int_{|y|\le r2^{k+2}} \rho(r2^{k+1})|f(y)|\, dy, \end{aligned}$$

where in the last inequality we have used the fact that ρ is almost-decreasing on $(0, \infty)$. By Lemma 3.1 we get

$$I_2 f(x) \leq \frac{2c_2 A^2 \|f\|_{M^{\Phi,\lambda}(0)}}{n \ln 2} \sum_{k=0}^{\infty} \rho(r2^{k+1}) |B_{r2^{k+2}}| \Phi^{-1}(|B_{r2^{k+2}}|^{\lambda-1}) \int_{|B_{r2^{k+1}}|}^{|B_{r2^{k+2}}|} \frac{dt}{t}$$

$$\leq \frac{2^{n+1} c_2^2 A^2 \|f\|_{M^{\Phi,\lambda}(0)}}{n \ln 2} \sum_{k=0}^{\infty} \rho(r2^{k+2}) \int_{|B_{r2^{k+1}}|}^{|B_{r2^{k+2}}|} \Phi^{-1}(t^{\lambda-1})\, dt.$$

By Lemma 2.1 we have

$$\rho(r2^{k+2}) = \rho(v_n^{-1/n} |B_{r2^{k+2}}|^{1/n}) \leq C_{2a} \rho(|B_{r2^{k+2}}|^{1/n}),$$

where $a = v_n^{-1/n}$. Thus, we obtain

$$I_2 f(x) \leq \frac{2^{n+1} c_2^2 A^3 C_{2a} \|f\|_{M^{\Phi,\lambda}(0)}}{n \ln 2} \sum_{k=0}^{\infty} \int_{|B_{r2^{k+1}}|}^{|B_{r2^{k+2}}|} \rho(t^{1/n}) \Phi^{-1}(t^{\lambda-1})\, dt$$

$$= C_5 \|f\|_{M^{\Phi,\lambda}(0)} \int_{|B_{2r}|}^{\infty} \rho(t^{1/n}) \Phi^{-1}(t^{\lambda-1})\, dt,$$

where $C_5 = \frac{2^{n+1} c_2^2 A^3 C_{2a}}{n \ln 2}$. Since $\rho(t^{1/n}) \leq A \widehat{\rho}(t)/t$ it follows by (4.4) that

$$I_2 f(x) \leq A C_5 \|f\|_{M^{\Phi,\lambda}(0)} \int_{|B_{2r}|}^{\infty} \widehat{\rho}(t) \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t}$$

$$\leq A C_1 C_5 \|f\|_{M^{\Phi,\lambda}(0)} \Psi^{-1}(|B_{2r}|^{\mu-1}).$$

Thus, for $x \in B_r^1$ we get

$$|I_\rho f(x)| \leq I_1 f(x) + I_2 f(x) \leq 2C_6 \|f\|_{M^{\Phi,\lambda}(0)} \Psi^{-1}(|B_{2r}|^{\mu-1}),$$

where $C_6 = C_1 \max\left\{\frac{2^{n(1-\mu)}}{2^{\lambda-1} \ln 2} C_4, AC_5\right\}$. Since $2^{n(\mu-1)} < 1$ it follows that

$$\int_{B_r^1} \Psi\left(\frac{|I_\rho f(x)|}{2C_6 \|f\|_{M^{\Phi,\lambda}(0)}}\right) dx \leq |B_r^1|\, |B_{2r}|^{\mu-1} \leq 2^{n(\mu-1)} |B_r|^\mu < |B_r|^\mu.$$

Let now $x \in B_r^2$. We write $I_\rho f(x)$ as follows:

$$|I_\rho f(x)| \leq \int_{|x-y|\leq\delta} \rho(|x-y|) |f(y)|\, dy + \int_{|x-y|>\delta} \rho(|x-y|) |f(y)|\, dy$$

$$=: I_3 f(x) + I_4 f(x),$$

where δ is defined in the same way as in [4]:

$$\Phi\left(\frac{Mf(x)}{C_0 \|f\|_{M^{\Phi,\lambda}(0)}}\right) = \frac{|B_r|^\lambda}{|B_\delta|}. \tag{4.6}$$

We note that in this case $|B_\delta| < |B_r|$. By Hedberg's pointwise estimate from

Lemma 4.1 and assumption (4.5) we get

$$I_3f(x) \le C_H\ \widehat{\rho}(|B_\delta|)Mf(x) \le C_HC_2\frac{\Psi^{-1}\big(\frac{|B_r|^\mu}{|B_\delta|}\big)}{\Phi^{-1}\big(\frac{|B_r|^\lambda}{|B_\delta|}\big)}Mf(x)$$
$$= C_HC_0C_2\|f\|_{M^{\Phi,\lambda}(0)}\Psi^{-1}\bigg(\frac{|B_r|^\mu}{|B_\delta|}\bigg).$$

To estimate $I_4f(x)$ we follow the idea from [4, Theorem 1]. By Hedberg's method we obtain

$$I_4f(x) = \int\limits_{|x-y|>\delta}\rho(|x-y|)|f(y)|\,dy = \sum_{k=0}^{\infty}\int\limits_{2^k\delta<|x-y|\le 2^{k+1}\delta}\rho(|x-y|)|f(y)|\,dy$$
$$\le A\sum_{k=0}^{\infty}\rho(2^k\delta)\int\limits_{B_{|x|+2^{k+1}\delta}}|f(y)|\chi_{B(x,2^{k+1}\delta)}(y)\,dy,$$

where in the last inequality we have used the fact that ρ is almost-decreasing on $(0,\infty)$. Applying Lemma 3.1 we obtain

$$I_4f(x) \le \frac{2A\|f\|_{M^{\Phi,\lambda}(0)}}{n\ln 2}\sum_{k=0}^{\infty}\rho(2^k\delta)|B(x,2^{k+1}\delta)|\Phi^{-1}\bigg(\frac{|B_{|x|+2^{k+1}\delta}|^\lambda}{|B(x,2^{k+1}\delta)|}\bigg)\int\limits_{|B(x,2^k\delta)|}^{|B(x,2^{k+1}\delta)|}\frac{dt}{t}.$$

By Lemma 2.1 with $a=\frac{1}{2}v_n^{-1/n}$ we have

$$\rho(2^k\delta) = \rho\big(\tfrac{1}{2}v_n^{-1/n}|B_{2^{k+1}\delta}|^{1/n}\big) \le C_{2a}\rho(|B_{2^{k+1}\delta}|^{1/n}).$$

Since ρ is almost-decreasing on $(0,\infty)$ and Φ^{-1} is concave, following the calculations in the proof of Theorem 1 in [4] we obtain

$$I_4f(x) \le \frac{2^{n+1}A^2C_{2a}\|f\|_{M^{\Phi,\lambda}(0)}}{n\ln 2}\sum_{k=0}^{\infty}\int\limits_{|B(x,2^k\delta)|}^{|B(x,2^{k+1}\delta)|}\rho(t^{1/n})\Phi^{-1}\bigg(\frac{4^{\lambda n}(\max\{|B_r|,t\})^\lambda}{t}\bigg)\,dt$$
$$\le \frac{2^{n+1}4^{\lambda n}A^2C_{2a}}{n\ln 2}\|f\|_{M^{\Phi,\lambda}(0)}\int\limits_{|B_\delta|}^{\infty}\rho(t^{1/n})\Phi^{-1}\bigg(\frac{(\max\{|B_r|,t\})^\lambda}{t}\bigg)\,dt$$
$$\le C_7\|f\|_{M^{\Phi,\lambda}(0)}\bigg(\int\limits_{|B_\delta|}^{|B_r|}\rho(t^{1/n})\Phi^{-1}\bigg(\frac{|B_r|^\lambda}{t}\bigg)\,dt + \int\limits_{|B_r|}^{\infty}\rho(t^{1/n})\Phi^{-1}(t^{\lambda-1})\,dt\bigg),$$

where $C_7 = \frac{2^{n+1}4^{\lambda n}A^2C_{2a}}{n\ln 2}$. Since $\rho(t^{1/n}) \le A\frac{\widehat{\rho}(t)}{t}$ it follows by (4.4) and (4.5) that

$$I_4 f(x) \le C_7 A \|f\|_{M^{\Phi,\lambda}(0)} \bigg(\int_{|B_\delta|}^{|B_r|} \widehat{\rho}(t) \Phi^{-1}\Big(\frac{|B_r|^\lambda}{t}\Big) \frac{dt}{t} + \int_{|B_r|}^{\infty} \widehat{\rho}(t)\Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \bigg)$$
$$\le C_7 A \|f\|_{M^{\Phi,\lambda}(0)} \bigg(C_2 \Psi^{-1}\Big(\frac{|B_r|^\mu}{|B_\delta|}\Big) + C_1 \Psi^{-1}(|B_r|^{\mu-1}) \bigg)$$
$$\le (C_1 + C_2) C_7 A \|f\|_{M^{\Phi,\lambda}(0)} \Psi^{-1}\Big(\frac{|B_r|^\mu}{|B_\delta|}\Big).$$

Thus, for $x \in B_r^2$ we get

$$|I_\rho f(x)| \le C_8 \|f\|_{M^{\Phi,\lambda}(0)} \Psi^{-1}\Big(\frac{|B_r|^\mu}{|B_\delta|}\Big)$$

with $C_8 = C_H C_0 C_2 + (C_1 + C_2) C_7 A$. Then, by the same calculations as in [4, proof of Theorem 1], we arrive at

$$\int_{B_r^2} \Psi\bigg(\frac{|I_\rho f(x)|}{C_8 \|f\|_{M^{\Phi,\lambda}(0)}} \bigg) dx \le |B_r|^\mu$$

and

$$\int_{B_r} \Psi\bigg(\frac{|I_\rho f(x)|}{C_3 \|f\|_{M^{\Phi,\lambda}(0)}} \bigg) dx \le |B_r|^\mu,$$

where $C_3 = 2\max\{2C_6, C_8\}$. ∎

Below we present an example for Theorem 4.2.

EXAMPLE 4.3. Let $0 < \alpha < n$, $0 \le \lambda < 1$, $1 < p < \frac{n(1-\lambda)}{\alpha}$, $0 \le a \le \sqrt{1 - 1/p} - (1 - 1/p)$ and

$$\Phi^{-1}(u) = \begin{cases} u^{1/p} & \text{for } 0 \le u \le 1, \\ u^{1/p}(1 + \ln u)^{-a} & \text{for } u \ge 1, \end{cases} \qquad \Psi^{-1}(u) = u^{1/q},$$

with $1 < p < q < \infty$. Let also $0 < \beta \le \frac{n-\alpha}{2}$ and

$$\rho(t) = \begin{cases} t^{\alpha-n}(1 - \ln t^2)^{-\beta}, & 0 < t \le 1, \\ t^{\alpha-n}, & t \ge 1. \end{cases} \tag{4.7}$$

If $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$, then all conditions of Theorem 4.2 are satisfied, and the Riesz potential I_ρ is bounded from $M^{\Phi,\lambda}(0)$ to $M^{\Psi,\mu}(0)$.

Proof. First note that the condition $0 < \beta \le \frac{n-\alpha}{2}$ ensures that ρ is almost-decreasing on $(0, \infty)$. Moreover, in [3, Example 2] it was shown that $\rho \in D_n$. In our earlier papers [4, Example 1] and [5, Example 2] we have shown that conditions (4.4) and (4.5) of Theorem 4.2 hold with $\rho(t) = t^{\alpha-n}$, that is,

$$\int_u^\infty t^{\alpha/n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \le C_1 \Psi^{-1}(u^{\mu-1}) \quad \text{for all } u > 0$$

and

$$u^{\alpha/n}\varPhi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_u^r t^{\alpha/n}\varPhi^{-1}\left(\frac{r^{\lambda}}{t}\right)\frac{dt}{t}\le C_2\,\varPsi^{-1}\left(\frac{r^{\mu}}{u}\right)\quad\text{for all } r>u>0.$$

Since $\widehat{\rho}(t)\le (n/\alpha)t^{\alpha/n}$ for any $t>0$ it follows that conditions (4.4) and (4.5) of Theorem 4.2 hold with ρ defined in (4.7). ∎

The operator I_ρ considered in Example 4.3 plays an important role in the theory of elliptic partial differential equations. For example, if P is a second order positive elliptic operator acting between $L_2(\Omega)$ and $C(\Omega)$, where Ω is some bounded set in $\mathbb{R}^n$, and we consider an equation $Pu=f$, then for a positive, increasing and concave function φ on $(0,\infty)$ we have an integral representation

$$\varphi(P^{-1})f(x)=\int_\Omega G_\varphi(x,y)f(y)\,dy$$

with the kernel

$$0\le G_\varphi(x,y)\le C|x-y|^{2-n}\varphi'(|x-y|^2),\qquad n\ge 3,$$

which means that

$$\int_\Omega G_\varphi(x,y)f(y)\,dy\le C\int_\Omega |x-y|^{2-n}\varphi'(|x-y|^2)f(y)\,dy,$$

where on the right-hand side we get the generalized Riesz potential I_ρ with the kernel $\rho(t)=t^{2-n}\varphi'(t^2)$. It is known (see [23]) that for $\varphi(t)=t^\tau$, $\tau\in(0,1)$, the operator $P^{-\tau}$ is bounded from L_2 to C for any $\tau>n/4$. Moreover, for $\tau=n/4$ this boundedness does not hold.

This prompted the question of finding a function $\psi(t)=t^{n/4}\varphi_0(t)$, as small as possible, such that $\psi(P^{-1})\colon L_2\to C$, where $\varphi_0(t)$ is a positive, increasing and concave function on $(0,\infty)$. This question was considered by Pustylnik [22, Theorem 3.1] and [23, Theorem 4]. In particular, for $\beta=1-1/n+\varepsilon$ and $\alpha=1$, $\varepsilon>0$, he showed that the boundedness of I_ρ with ρ defined in Example 4.3 from L_n to C ensures the $L_2\to C$ boundedness of the operator $\psi(P^{-1})=P^{1/2-n/4}\varphi(P^{-1})$, where

$$\varphi(t)=t^{1/2}\varphi_0(t)=\begin{cases} t^{1/2}(1-\ln t)^{-(1-1/n+\varepsilon)}, & 0<t\le 1,\\ t^{1/2}, & t\ge 1,\end{cases}$$

and $\rho(t)\approx t^{2-n}\varphi'(t^2)\approx t^{-n}\varphi(t^2)$.

Funding. The first author was supported by the G. S. Magnuson Foundation of the Royal Swedish Academy of Sciences (grant no. MG2023-0073). The second author was partially supported by the Poznań University of Technology under grant number 0213/SBAD/0122.

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