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Addendum and corrigendum to the paper "Nonlinear twists and moments of L-functions"

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by

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Abstract. We correct and refine a conjecture from [Acta Arith. 214 (2024), 153–171] on the analytic behaviour of the self-dual twist associated with the functions $F(s)$ in the extended Selberg class. The corrected version relates the order of the poles to those of the Rankin–Selberg convolution $F \times \overline{F}(s)$ and to the structural invariants $d_F(\ell)$.

In [4] we formulated the following conjecture predicting the basic analytic properties of the self-dual twist, defined for $\sigma > 1$ as

$$F^2_{\rm self}(s) = \sum_{n=1}^{\infty} \frac{a_{F^2}(n)}{n^s} e(-\kappa_{F^2} n^{1/d_F}), \qquad \kappa_{F^2} = d_F q_F^{-1/d_F}. \tag{1}$$

Here we keep the notation introduced in that paper. In particular, F denotes a function of positive degree d_F from the extended Selberg class $\mathcal{S}^\sharp$, and q_F denotes its conductor.

CONJECTURE 1. *Let $F \in \mathcal{S}^\sharp$ be of integer degree $d_F \geq 1$, with internal shift $\theta_F = 0$ and satisfying the Ramanujan conjecture. Then the self-dual twist $F^2_{\rm self}$ has meromorphic continuation to $\mathbb{C}$ with poles at most at the points*

$$s_\ell = \frac{1}{2} + \frac{1}{2d_F} - \frac{\ell}{d_F}, \qquad \ell \geq 0 \text{ integer},$$

and polynomial growth on vertical strips. Moreover, the pole at s_0 has order $d_F^2 + 1$, while the other poles have order either $d_F^2 + 1$ or 0.

While we believe that this conjecture correctly predicts the *location* of the poles of $F^2_{\rm self}$, it is unlikely to be valid in full generality concerning their *multiplicities*. Conjecture 1 may hold in several cases, but certainly not always.

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The main result of [4] shows that the existence of a suitable holomorphic continuation of $F^2_{\rm self}$ implies strong estimates for the second moment of F on the critical line. That paper considered only upper bounds, but a finer description of the analytic nature of the self-dual twist along similar lines would imply that

$$\int_{-T}^{T} |F(1/2+it)|^2\,dt \sim c_0(F)\,T(\log T)^m$$

as $T\to\infty$, where m denotes the order of the pole of $F^2_{\rm self}$ at $s=s_0$. In particular, Conjecture 1 agrees with the widely conjectured asymptotic formula

$$\int_{-T}^{T} |\zeta(1/2+it)|^{2k}\,dt \sim c_0(F)\,T(\log T)^{k^2}$$

for $F(s)=\zeta(s)^k$. However, it would fail in other instances, for example, when F is the L-function associated with a Hecke eigenform of any level. In that case, the predicted exponent of $\log T$ would be 4, contradicting a well known result by Good [1].

Here we correct Conjecture 1 so that it may hold for a larger subclass of functions from $\mathcal{S}^\sharp$. For $F\in\mathcal{S}^\sharp$ with $d_F>0$ we denote by

$$F\times\overline{F}(s)=\sum_{n=1}^{\infty}\frac{|a_F(n)|^2}{n^s}\qquad(\sigma>1)$$

its Rankin–Selberg convolution. We assume that there exists $\delta_F>0$ such that $F\times\overline{F}(s)$ has analytic continuation to the half-plane $\sigma>1-\delta_F$, apart from a pole at $s=1$ of order $m_{F\times\overline{F}}\le d_F^2$. Moreover, for $F\in\mathcal{S}^\sharp$, let $d_F(\ell)$, $\ell=0,1,\dots$, denote the structural invariants introduced in [2] (see also [3]).

Conjecture 2. *Let F satisfy the above assumptions. Then the self-dual twist $F^2_{\rm self}$ has meromorphic continuation to $\mathbb{C}$ with poles of order $m_{F\times\overline{F}}+1$ at the points*

$$\hat{s}_\ell=\frac{d_F+1}{2d_F}-\frac{\ell}{d_F}-i\theta_F,\qquad \ell\ge 0\ \textit{integer},\ d_F(\ell)\ne 0,$$

and polynomial growth on vertical strips.

The heuristic argument underlying Conjecture 2 runs as follows. Substituting

$$a_{F^2}(n)=\sum_{m\mid n}a_F(m)\,a_F\Big(\frac{n}{m}\Big)$$

into the definition of $F^2_{\rm self}$ given in (1) and interchanging the order of summation, for $\sigma > 1$ we obtain

$$F^2_{\mathrm{self}}(s) = \sum_{m=1}^{\infty} \frac{a_F(m)}{m^s} F(s, \kappa_{F^2} m^{1/d_F}), \tag{2}$$

where, for a generic $\alpha > 0$, $F(s, \alpha)$ denotes the standard twist of $F(s)$. We refer to [2] for the definition of the standard twist and a detailed analysis of its analytic properties; in particular, for the notion of the spectrum $\mathrm{Spec}(F)$ of F and its relation to the polar structure of $F(s, \alpha)$.

It is readily seen that $\kappa_{F^2} m^{1/d_F} \in \mathrm{Spec}(F)$ for all $m \geq 1$ with $a_F(m) \neq 0$. Hence, by [2, Theorem 3], each of the standard twists involved has a simple pole at $s = \hat{s}_\ell$, $\ell \geq 0$, with $d_F(\ell) \neq 0$, and with residue

$$c(\ell) \frac{\overline{a_F(m)}}{m^{1-\hat{s}_\ell}}, \qquad c(\ell) = c(\ell, F) \neq 0.$$

It is therefore natural to expect that the poles of F^2_{self} occur precisely at these points and at no others. For s near $\hat{s}_\ell$, we have the local approximation

$$F(s, \kappa_{F^2} m^{1/d_F}) \approx c(\ell) \frac{\overline{a_F(m)}}{m^{1-\hat{s}_\ell}} \frac{1}{s - \hat{s}_\ell}. \tag{3}$$

Substituting this into (2) yields

$$F^2_{\mathrm{self}}(s) \approx c(\ell) \sum_{m=1}^{\infty} \frac{|a_F(m)|^2}{m^{s+1-\hat{s}_\ell}} \cdot \frac{1}{s - \hat{s}_\ell}. \tag{4}$$

Thanks to our assumption, the Dirichlet series in (4) has a pole of order $m_{F\times\overline{F}}$ at $s = \hat{s}_\ell$. Consequently, the entire right-hand side exhibits a pole of order $m_{F\times\overline{F}} + 1$, as asserted by the conjecture.

It would be desirable to make rigorous the above heuristic considerations. To achieve this, one would need substantially finer information on the analytic behaviour of the standard twist, which however is not available at present.

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