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New zero-density estimates for the Beurling zeta function

by

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Abstract. In two previous papers the second author proved some Carlson type density theorems for zeroes in the critical strip for Beurling zeta functions satisfying Axiom A of Knopfmacher. In the first of these invoking two additonal conditions were needed, while in the second an explicit, fully general result was obtained. Subsequently, Frederik Broucke and Gregory Debruyne obtained, via a different method, a general Carlson type density theorem with an even better exponent, and recently Frederik Broucke improved this further, getting $N(\sigma,T) \leq T^{a(1-\sigma)}$ with any $a > \frac{4}{1-\theta}$. Broucke employed a new mean value estimate of the Beurling zeta function, without using the method of Halász and Montgomery.

Here we elaborate a new approach of the first author, using the classical zero detecting sums coupled with a kernel function technique and Halász' method, but otherwise arguing in an elementary way avoiding e.g. mean value estimates for Dirichlet polynomials. We make essential use of the additional assumptions that the Beurling system of integers consists of natural numbers, and that the system satisfies the Ramanujan condition. This way we give a new variant of the Carlson type density estimate with strength similar to Turán's 1954 result for the Riemann zeta function, coming close even to the Density Hypothesis for σ close to 1.

1. Zero density results for the Riemann zeta function. In the classical case of natural numbers $\mathbb{N}$ and primes $\mathbb{P}$, it has been known since Riemann's time that the distribution of the zeroes of the (analytic continuation of) the Riemann zeta function ζ is decisive in questions of prime distribution. As all the nontrivial zeroes are in the critical strip $0 < \Re s < 1$ and their number up to height T is asymptotically $N(T) \sim \frac{T}{2\pi} \log T$, the estimations of the number $N(\sigma, T)$ of zeroes in the halfplane $\Re s > \sigma$ and up to height T by quantities of smaller order could point towards validity of the Riemann Hypothesis. We are discussing here the so-called *density estimates*,

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first for the Riemann zeta function itself, and then, more generally, we obtain new density estimates for Beurling number systems as well.

Bohr and Landau [2] proved in 1914 that $N(\sigma, T) = o(N(T))$ if $\sigma > 1/2$. This was the first estimate of the kind that we call density estimates.

A few years later Carlson [7] proved

$$N(\sigma, T) \ll_\varepsilon T^{A(\sigma)(1-\sigma)+\varepsilon} \quad \text{for any } \varepsilon > 0 \text{ and } \sigma \geq 1/2, \tag{1}$$

where he obtained

$$A(\sigma) \leq 4\sigma. \tag{2}$$

Number-theoretic consequences in connection with the Landau problem on the difference of consecutive primes made it of top interest to improve these bounds as much as possible. Landau conjectured that there is a prime between any two squares [15]. In particular, it was shown that an almost exact positive answer to Landau's question would follow from the so-called *Density Hypothesis* (DH) which asserts that

$$N(\sigma, T) \ll T^{2(1-\sigma)} \log^C T \quad \text{with some } C > 0 \text{ for all } \sigma \geq 1/2, \tag{3}$$

or, in a slightly weaker form, using the notation (1),

$$A(\sigma) \leq 2 \quad \text{for all } \sigma \geq 1/2 \quad (\Longleftrightarrow N(\sigma, T) \ll_\varepsilon T^{2(1-\sigma)+\varepsilon}). \tag{4}$$

Clearly, the Riemann Hypothesis contains the Density Hypothesis. However, Ingham [12] was able to demonstrate DH from another, weaker assumption, the so-called *Lindelöf Hypothesis* (LH), which reads

$$\mu\left(\tfrac{1}{2}\right) = 0, \text{ where } \mu(\alpha) := \inf\{\mu : |\zeta(\sigma + it)| \leq T^\mu \text{ for } \sigma \geq \alpha,\ 1 < |t| \leq T\}. \tag{5}$$

In 1954 Turán [23] used his celebrated power-sum method [22] (see also [24]) to give a different proof of Ingham's result that LH implies DH, and he almost achieved DH in the vicinity of the boundary line $\sigma = 1$. Let us put

$$s = \sigma + it, \qquad \eta := 1 - \sigma, \qquad B(\eta) := \frac{1}{\eta} A(1 - \eta). \tag{6}$$

Then Turán's result [23] was, with a small positive constant c_1,

$$N(1 - \eta, T) \ll T^{2\eta + \eta^{1.14}} \log^6 T \quad \text{for } \eta < c_1, \tag{7}$$

or with our new notation

$$B(\eta) \leq 2 + \eta^{0.14} \quad \text{for } \eta < c_1. \tag{8}$$

The next breakthrough improvement, yielding in particular DH for a fixed strip near the 1-line, was obtained by Halász and Turán [11]. They combined Turán's celebrated power sum theory with the best known Korobov–Vinogradov estimates on the growth of the Riemann zeta function and Halász' pioneering idea of [10], which will also play a decisive role in the main argument of the present work.

A special feature of our approach to the general case of Beurling systems is that we invoke our better knowledge of the behavior of the Riemann zeta function when addressing questions of zero density for certain Beurling systems. In particular, we will employ the best known growth bound, due to Bourgain [3],

$$|\zeta(1/2+it)| \ll_\varepsilon (|t|+1)^{\frac{13}{84}+\varepsilon} \qquad (\forall \varepsilon > 0). \tag{9}$$

2. Beurling systems and density theorems for Beurling zeta functions. Beurling systems were introduced by Beurling [1], and were subsequently put into a more abstract – and hence more general – framework (see e.g. [14]). Beurling systems can thus be viewed in two alternative ways, one being that they are just an arbitrary semigroup $\mathcal{G}$ of elements g, admitting an (essentially, i.e. up to ordering) unique factorization into prime elements $p \in \mathcal{P} \subset \mathcal{G}$. In this "arithmetical semigroup" point of view the elements can be algebraic numbers, ideals of integer rings of algebraic number fields, finite Abelian groups, and many more systems which admit a suitable Krull–Schmidt type decomposition leading to a multiplicative structure. In this abstract setting a key role is played by the multiplicative norming of elements, i.e. there is a norm $|\cdot| : \mathcal{G} \to [1,\infty)$ which is multiplicative ($|gh| = |g|\,|h|$) and its image is locally finite. That maps, in a unique way, the abstract structure to $\mathbb{R}_+$, so that one can view it as included in the real number setting where the primes $\mathcal{P}$ are just an arbitrary sequence of reals, nondecreasing and tending to infinity, and freely generating the respective system $\mathcal{N}$ of Beurling integers. Although we will refer back to this first interpretation here and there, the main setting of the paper is the latter real number setting.

Let $\mathcal{P}$ be a sequence of Beurling primes and $\mathcal{N}$ the semigroup generated by them, that is, the system of Beurling integers. Together, these form a Beurling number system $\mathcal{B} := (\mathcal{P}, \mathcal{N})$. Due to its Euler product form, the Beurling zeta function does not vanish in its halfplane of convergence:

$$\zeta_{\mathcal{B}}(s) := \sum_{n\in\mathcal{N}} \frac{1}{n^s} = \prod_{p\in\mathcal{P}} \frac{1}{1-1/p^s} \neq 0. \tag{10}$$

For a general overview of the analytic theory of Beurling number systems we refer to [9].

Beurling systems may or may not have an analytic continuation over the boundary line of convergence of their Dirichlet series expansion. However, if they do, then there are basically two possibilities: either the analytic continuation is governed by a nice asymptotic formula with power-type error for the number of integers $\mathcal{N}(x) := \#\{n \le x : n \in \mathcal{N}\}$, or the Beurling zeta function must show extremely irregular behavior with large values and no reasonable (polynomial) bounds. The latter possibility was pointed out by Frederik Broucke; for an explanation see e.g. [6, Remark 4.1]. In turn, if $\zeta_{\mathcal{B}}$ does not admit polynomial bounds in any halfplane larger than the halfplane of convergence, then there is no hope to control its behavior and to derive finer results on zero distribution and hence prime distribution.

Therefore, to talk about analytic continuation of $\zeta_{\mathcal{B}}$ and to have a chance to come up with a successful analysis using it even in the critical strip, we assume that the integers are "well-behaved". That is, in this work we assume the so-called *Axiom A* (in its normalized form to $\delta = 1$) of Knopfmacher (see pages 73–79 of his fundamental book [14]).

Definition 1 (Axiom A). We say that $\mathcal{N}$ (or, loosely speaking, $\zeta_{\mathcal{B}}$) satisfies *Axiom A* – more precisely, Axiom $A(A, \kappa, \theta)$ with suitable constants $A, \kappa > 0$ and $0 \leq \theta < 1$ – if the remainder term $\mathcal{R}(x) := \mathcal{N}(x) - \kappa x$ satisfies

$$|\mathcal{R}(x)| \leq Ax^{\theta} \quad (x \geq 1). \tag{11}$$

Historically, the first result which could be considered a weak form of a density estimate was worked out by Kahane [13]. However, it was part of an indirect proof and he made a very strong number-theoretical assumption. Under that restrictive assumption he could prove, however, that on a given line $\Re s = a$ in the critical strip, with $a > (1+\theta)/2$, the number of $\zeta_{\mathcal{B}}$-zeroes is $O(T)$.

The first zero density result for general Beurling zeta functions was obtained in [18] under two extra assumptions on the Beurling system $\mathcal{B}$. One was that the Beurling number system remains within the realm of natural numbers – a strong, but useful assumption, emphasized also by Knopfmacher [14, pp. 57–58]. We will term this *the integrality condition*, meaning that $|\cdot| : \mathcal{G} \to \mathbb{N}$, that is, the norm $|g|$ of any $g \in \mathcal{G}$ is a natural number. Alternatively, we may state it as $\mathcal{N} \subset \mathbb{N}$.

Further, in [18] an averaged form of the Ramanujan condition was assumed in proving the density bound

$$B_{\mathcal{B}}(\eta) \leq \frac{6 - 2\theta}{1 - \theta}. \tag{12}$$

In the present work we will also need a form of the Ramanujan condition, but given that our main reference [16] used it in its sharper, pointwise version, we settle with this version here, too. For its formulation, let us introduce the arithmetical function $G(\nu) := \#\{n \in \mathcal{N} : |n| = \nu\}$, the number of Beurling integers having a given norm (value). With this notation the Beurling zeta function can be written as

$$\zeta_{\mathcal{B}}(s) = \sum_{n \in \mathcal{N}} \frac{1}{n^s} = \sum_{\nu=1}^{\infty} \frac{G(\nu)}{\nu^s}. \tag{13}$$

Then the Beurling system, or, loosely speaking, $\zeta_{\mathcal{B}}$ satisfies the *Ramanujan condition* if $\log G(\nu) = o(\log \nu)$, that is, if for any $\delta > 0$ we have $G(\nu) \leq \nu^{\delta}$ for $\nu > \nu_0(\delta)$.

The two extra conditions (integrality and the averaged Ramanujan condition) of the result (12) were removed in [21] at the expense of a worse exponent. Almost simultaneously and independently, however, Frederik Broucke and Gregory Debruyne [5] succeeded in proving an improved bound also without relying on these additional assumptions. Moreover, they also showed that assuming only Axiom A, there are Beurling zeta functions $\zeta_{\mathcal{B}}$ with $B_{\mathcal{B}}(\eta) > c_1$ for some sufficiently small positive constant c_1. Finally, very recently Frederik Broucke [4] further improved the exponent in a general density theorem for Beurling zeta functions satisfying Axiom A, obtaining

$$B_{\mathcal{B}}(\eta) \leq \frac{4}{1 + 2\eta - \theta}. \tag{14}$$

To end this introduction let us point out that the new development of having zero density theorems for zeta functions of well-behaved systems of Beurling integers and primes opened up the way to achieve many strong number-theoretical advances in Beurling's theory. These advances are likely to be totally impossible to reach without such tools. The interested reader may consult [19, 20, 5, 6].

3. The aim of the paper. Our present goal is to demonstrate that returning to the extra assumptions of integrality and the Ramanujan condition, a new result, significantly stronger than (14), can be obtained. For example, if $\theta = 0$, then we can almost reach DH for small values of η, similarly to the 1954 result of Turán for the classical case; see Corollary 1 below.

It is of interest that while Turán used his power sum method to show (8), here we need only Halász' idea [10] to show a general form of (16), somewhat weaker if $\theta > 0$. Namely, we will prove

THEOREM 1. *Under Axiom A, the integrality condition, and the Ramanujan condition for a Beurling system $\mathcal{B}$, we have, as long as $0 < \eta < 1 - \theta$, the estimates*

$$B_{\mathcal{B}}(\eta) = \begin{cases} \frac{2}{1-\theta-\eta} & \textit{if } 0 < \eta < \min\left(\frac{4}{29}, \frac{8+13\theta}{71}\right), \\ \frac{26/21}{1-4\eta} & \textit{if } \frac{8+13\theta}{71} \leq \eta < \frac{4}{29} \ \left(\textit{empty for } \theta \geq \frac{4}{29}\right), \\ \frac{2}{1-\theta-\eta} & \textit{if } \frac{4}{29} \leq \eta < \theta \ \left(\textit{empty for } \theta < \frac{4}{29}\right), \\ \frac{2}{1-2\eta} & \textit{if } \max\left(\theta, \frac{4}{29}\right) \leq \eta. \end{cases} \tag{15}$$

We did not directly insert into the definition of the various ranges for η the generally required condition that $\eta < 1 - \theta$. Note that taking into account that restriction, or even already in view of the conditions mentioned, the actual ranges for the various estimates may well be empty for specific values of the parameter θ; only the very first range – written out in full as $0 < \eta < \min\left(1-\theta, \frac{4}{29}, \frac{8+13\theta}{71}\right)$ – must be nonempty for all values of $\theta \in [0, 1)$.

Specializing to $\theta = 0$ and to small η only, we obtain

Corollary 1. *Let $\mathcal{B}$ be a Beurling number system satisfying the integrality condition and also Axiom A with $\theta = 0$. Then for the Beurling zeta function $\zeta_{\mathcal{B}}$ of this system we have a zero density bound with*

$$B_{\mathcal{B}}(\eta) = \frac{2}{1-\eta} \; (= 2 + O(\eta)) \quad \textit{for } \eta \le \frac{8}{71}. \tag{16}$$

4. The main tool. Our main tool is a slightly strengthened variant of a general zero density type theorem proved recently by the first named author [16]. This deals with general (but ordinary, i.e. with integer powers in the denominator) Dirichlet series.

Assuming $f_1 \neq 0$, we consider the (Dirichlet inverse to each other) arithmetical functions f_n and g_n, both satisfying the Ramanujan condition, and the corresponding reciprocal pair of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{f_n}{n^s}, \qquad M(s) = \frac{1}{f(s)} = \sum_{n=1}^{\infty} \frac{g_n}{n^s}, \tag{17}$$

which by the Ramanujan condition must be analytic for $\sigma > 1$, and satisfy $M(s)f(s) = 1$ there.

Remark 1. If f_n is completely multiplicative as a function of n then $f_1 = 1$ and $g(n) = \mu(n) f_n$.

Further we suppose that with some fixed constant $\alpha_f < 1$, the function $f(s)$ can be continued analytically to the halfplane $\sigma > \alpha_f$, save a simple pole at $s = 1$ with residue f_0. For any such f we define the "generalized Lindelöf function" as follows:

(18)
$$\mu_f(\sigma_0) := \inf\{\mu : |f(\sigma+it)| \le T^{\mu} \text{ for } \sigma \ge \sigma_0, 1 \le |t| \le T\} < \infty \text{ for } \sigma_0 > \alpha_f.$$

This is clearly Lindelöf's classical μ-function if $f(s)$ is chosen to be $\zeta(s)$. For a general Dirichlet series, finiteness of $\mu_f(\sigma)$ is a standing assumption for us, which is satisfied in our application to Beurling zeta functions in Section 5.2. In the following technical definition the function $\lambda_f^{(0)}$ will depend on μ_f, and in particular $\lambda_\zeta^{(0)}$ on μ_ζ. Let

$$\begin{aligned} \lambda_f^{(0)}(\eta) &:= \inf_{a>0,\,(a+1)\eta<1-\alpha_f} \frac{\mu_f(1-(a+1)\eta)}{a\eta}, \\ \lambda_\zeta^{(0)}(\eta) &:= \inf_{b>0} \frac{\mu_\zeta(1-(b+1)\eta)}{b\eta}. \end{aligned} \tag{19}$$

Note that we do not assume anything about the size of the parameter b in the above definition for $\lambda_\zeta^{(0)}(\eta)$; this is an essential change compared to the original version in [16] of the main auxiliary theorem that we want to use.

In the following we do not need *the exact* values of these Lindelöf-type functions (which is rather fortunate, given that not even the precise value of μ_ζ is known), but we will be satisfied with any function $\lambda_f(\eta) \geq \lambda_f^{(0)}(\eta)$, in particular with $\lambda_\zeta(\eta) \geq \lambda_\zeta^{(0)}(\eta)$. Even if precise values are not known, strong estimates on the growth of the Riemann zeta function ζ will furnish us reasonably strong bounds $\lambda_\zeta(\eta)$.

As is usual, we denote the number of zeroes (counted with multiplicities) as

$$N_f(1-\eta, T) := \#\{\rho : f(\rho) = 0,\, \Re\rho \in [1-\eta, 1],\, |\Im\rho| \leq T\}. \tag{20}$$

The general density estimate which we apply in this work reads as follows.

LEMMA 1. *Let f and g be reciprocal Dirichlet series satisfying the above assumptions and the Ramanujan condition* $\log|f_n| = o(\log n)$ *and* $\log|g_n| = o(\log n)$.

Let $0 < \eta < 1 - \alpha_f$ be arbitrary, and let $\lambda_f(\eta)$ and $\lambda_\zeta(\eta)$ be some estimator functions satisfying $\lambda_f(\eta) \geq \lambda_f^{(0)}(\eta)$ and $\lambda_\zeta(\eta) \geq \lambda_\zeta^{(0)}(\eta)$, where $\lambda_f^{(0)}(\eta)$, $\lambda_\zeta^{(0)}(\eta)$ are defined according to (18) *and* (19). *Denote*

$$B_f(\eta) := \max(2\lambda_f(\eta), 4\lambda_\zeta(2\eta)). \tag{21}$$

If the condition

$$\lambda_f(\eta) \geq \lambda_\zeta(2\eta) \tag{22}$$

holds true, then we have the zero density estimate

$$N_f(1-\eta, T) \ll_{\eta,\varepsilon} T^{B_f(\eta)\eta+\varepsilon}. \tag{23}$$

Proof. This is essentially a part of Theorem 2 in [16], listed under formula (2.9) there.

However, here we changed the definition of $\lambda_\zeta^{(0)}(\eta)$ somewhat, allowing the b-parameter, originally in [16] restricted to satisfy some upper bound, to become unbounded.

It is clear from the original proof that

(i) we can allow $\lambda_f(\eta) = \lambda_\zeta(2\eta)$, too (no need for strict inequality, as in the original formulation);
(ii) in the definition of the original $\lambda_f(\eta)$ function (which corresponds to $\lambda_f^{(0)}(\eta)$ here), we can allow any upper estimation, as we formally described above in detail;

(iii) when defining $\lambda_\zeta^{(0)}(\eta)$, actually there is no need for any upper bound for b, as the original proof works without change even for larger values of b, given the analytic continuation (and known bounds of the growth) of the Riemann zeta function on the whole complex plane.

For completeness we briefly sketch (a slightly simplified form of) the proof of Theorem 2 of [16] with an emphasis on the main ideas of the proof.

The classical zero density theorems start from the seemingly contradictory facts that

$$f(s)M(s)=1 \quad \text{for } \sigma>1, \tag{24}$$

whereas at a zero ρ of $f(s)$ we have

$$f(\rho)=0. \tag{25}$$

This, however, might actually occur only for $\sigma\le 1$ – as shown in, e.g., [9], and for the Beurling zeta functions, even only for $\sigma<1-c/\log t$.

So if we try to approximate $M(s)$ by

$$M_X(s):=\sum_{n\le X} g_n n^{-s} \tag{26}$$

with

$$X=T^\varepsilon \tag{27}$$

then $M_X(s)$ will be an entire function, and (24), (25) turn into

$$f(s)M_X(s)=1+\sum_{n\ge X} a_n n^{-s}, \quad a_n:=\sum_{d|n,\, n/d\le X} f(d)g(n/d)\ll n^{o(1)}, \tag{28}$$

$$f(\rho)M_X(\rho)=0. \tag{29}$$

In order to approximate $f(s)$ with a finite Dirichlet polynomial we can introduce another parameter Y and a kernel $Y^s\exp(s^2/\mathcal{L})$ (as in [16]) where $\mathcal{L}:=\log Y$ or the more usual $Y^s\Gamma(s)$ where a good (and simple) choice for Y will be

$$Y=T^{\lambda_f(\eta)+\Delta} \quad \text{with a small } \Delta>0. \tag{30}$$

We will choose a maximal set of K separated zeroes of $f(s)$ with

$$\rho_j=1-\eta_j+i\gamma_j,\quad \eta_j\le\eta,\quad \gamma_j\in[T/2,T],\quad |\gamma_j-\gamma_\nu|\ge\mathcal{L}^2\ (j\ne\nu). \tag{31}$$

Using a well-known Mellin transform we obtain $I_j:=I(\rho_j)$ with

(32)

$$\begin{aligned} I(\rho)&:=e^{-1/Y}+\sum_{n>X} a_n n^{-\rho}e^{-n/Y}=\frac{1}{2\pi i}\int_{(2)} f(\rho+s)M_X(\rho+s)Y^s\Gamma(s)\,ds \\ &=o(1)+\frac{1}{2\pi i}\int_{(-a'\eta)} f(\rho+s)M_X(\rho+s)Y^s\Gamma(s)\,ds, \end{aligned}$$

where $o(1)$ replaces the residue at the pole $s = 1 - \rho$ which is negligible due to $\Gamma(1-\rho) \ll e^{-T/2}$ and where $a' = a$ is the place where $\lambda_f(\eta)$ assumes its minimum value (or, if only an infimum exists, then a value which is very near to the infimum). The choice of Y ensures that the integral on the RHS of (32) is $o(1)$.

Due to the exponential decay of the sum on the LHS of (32), by a dyadic division of the interval $[X, Y\mathcal{L}]$ we obtain a $U \in [X, Y\mathcal{L}]$ such that

$$\sum_{j=1}^{K} \Big| \sum_{n \in I(U)} a_n^* n^{-\rho_j} \Big| \gg \frac{K}{\mathcal{L}}, \quad |a_n^*| \le a_n, \quad I(U) \subset [U, 2U]. \tag{33}$$

We will raise the Dirichlet polynomial on the LHS of (33) to a minimal power h with $U^h \ge Z := T^{\lambda_\zeta(2\eta)+\Delta}$. The choice of Z is motivated by the argument which follows later and the condition that the partial sums of $\zeta(2s)$ (not $f(s)$) should be of size $O(T^{-\varepsilon})$ in any interval of type $I(N)$ with $N \ge Z$. Using Perron's formula we can show (see [16, Lemma 1]) in fact, for $\sigma \ge 1/2 + \varepsilon$,

$$S := \sum_{n \in I(N)} n^{-\sigma - it} \ll \frac{\mathcal{L}N^{1-\sigma}}{|t|} + O(T^{-\varepsilon/2}) \quad \text{for } 1 \le |t| \le T,\ N \ge Z. \tag{34}$$

Raising (33) to the hth power and applying the Hölder inequality we obtain, with some b_n,

$$\sum_{j=1}^{K} \Big| \sum_{n \in I(M)} b_n n^{-\rho_j} \Big| \gg \frac{K}{\mathcal{L}^h}, \quad |b_n| \le \tau_h(n) n^{o(1)}, \quad M \in [U^h, (2U)^h], \tag{35}$$

with the generalized divisor function $\tau_h(n) \ll n^{o(1)}$, where $h \ll \lambda_f(2\eta)/\varepsilon \ll 1/\varepsilon$.

Let us define φ_j with $|\varphi_j| = 1$ $(j = 1, \ldots, K)$ by

$$\Big| \sum_{n \in I(M)} b_n n^{-\rho_j} \Big| = \varphi_j \sum_{n \in I(M)} b_n n^{-\rho_j}. \tag{36}$$

The crucial idea of Halász is to square the LHS of (35), interchange the summations over j and n and use the Cauchy–Schwarz inequality for the sum when n runs through $I(M)$ with

$$b_n n^{-\rho_j} = b_n n^{-1/2} n^{-1/2 + \eta_j - i\gamma_j} = d_n e_n(j) \quad (n \in I(M)). \tag{37}$$

Proceeding in such a way it turns out that in the resulting estimate the exact choice of b_n will be irrelevant; it is sufficient to have $b_n = n^{o(1)}$ and we need an estimate only for the partial sums of $\zeta(2s)$ (cf. (34)), independently of the particular choice of the functions $f(s)$.

Separating the diagonal terms ($j=\nu$), in case $M \ge Z$ we obtain, from (33)–(37),

(38)

$$\begin{aligned} K^2\mathcal{L}^{-2h} &\ll \Big(\sum_{j=1}^{K}\sum_{n\in I(M)} b_n n^{-\rho_j}\Big)^2 = \Big(\sum_{n\in I(M)} d_n \sum_{j=1}^{K}\varphi_j e_n^{(j)}\Big)^2 \\ &\ll \Big(\sum_{n\in I(M)}\frac{|b_n|^2}{n}\Big)\Big(\sum_{j=1}^{K}\sum_{\nu=1}^{K}\varphi_j\overline{\varphi_\nu}\sum_{n\in I(M)}\frac{1}{n^{1-\eta_j-\eta_\nu+i(\gamma_j-\gamma_\nu)}}\Big) \\ &\ll T^{o(1)}\Big(K(K-1)T^{-\varepsilon/2}+\mathcal{L}M^{2\eta}\sum_{j=1}^{K}\sum_{\nu=1,\,\nu\ne j}^{K}\frac{1}{|\gamma_j-\gamma_\nu|}+KM^{2\eta}\Big) \\ &\ll T^{o(1)}(K^2T^{-\varepsilon/2}+KM^{2\eta}+KM^{2\eta}). \end{aligned}$$

Since the first term on the RHS is much smaller than the LHS, we obtain

$$K \ll \mathcal{L}^{2h}T^{o(1)}M^{2\eta} \ll T^{o(1)}M^{2\eta}. \tag{39}$$

If $U^2 \ge Z$ we can choose $h=2$, otherwise we have an integer power h with $M \asymp U^h \in [Z, Z^{3/2}]$. So since $U \le Y$, we deduce from (39) that

$$\begin{aligned} K &\ll T^{o(1)}(\max(Y^2, Z^{3/2}))^{2\eta} \\ &\ll T^{o(1)}\max(T^{(2\lambda_f(\eta)+2\Delta)2\eta}, T^{(3\lambda_\zeta(2\eta)+3\Delta)\eta}), \end{aligned} \tag{40}$$

which proves Lemma 1 since Δ can be arbitrarily small. ∎

5. The proof of Theorem 1

5.1. Computing estimates for the estimator $\lambda_\zeta(2\eta)$ of the Riemann zeta function. We start by finding a sufficiently good estimator function λ_ζ (which we will later use at the value of 2η, not at η). Note that here we deal with the Riemann zeta function, so that there is no upper bound on the admissible values of η; however, since for the Beurling zeta function we have the natural bound $0<\eta<1-\theta\le 1$, we do not extend over $0<\eta<1$ (or, if considering values at 2η, we restrict equivalently to $0<\eta<1/2$).

We work with two different choices of the appropriate parameter b, depending on the size of η. More precisely, for $\eta \le 4/29$ we will use Bourgain's estimate (9) together with the choice $b := \frac{1}{4\eta}-1 \Leftrightarrow (b+1)2\eta = 1/2$, which leads to $\mu_\zeta(1-(b+1)2\eta)) = \mu_\zeta(1/2) \le 13/84$ and hence to the estimator function

$$\lambda_\zeta(2\eta) = \frac{13/84}{1/2-2\eta} = \frac{13/42}{1-4\eta}, \tag{41}$$

obviously satisfying $\lambda_\zeta(2\eta) \ge \lambda_\zeta^{(0)}(2\eta)$ for this range of values of η.

If $4/29 \leq \eta \leq 1/4$, then we will use the basic estimate $\mu_\zeta(0) = 1/2$. Choosing $b := \frac{1}{2\eta} - 1 \Leftrightarrow (b+1)2\eta = 1$ in this case, we find from (19) that we can take

$$\lambda_\zeta(2\eta) = \frac{1/2}{1-2\eta}. \tag{42}$$

If $\eta > 1/4$ then $\lambda_\zeta(2\eta) > 1$, hence by (21), $B_{\mathcal{B}}(\eta) \geq 4\lambda_\zeta(2\eta) > 1/\eta$, which is weaker than the trivial estimate 1 – see Section 6.

5.2. Estimator for the Beurling zeta function $\zeta_{\mathcal{B}}$. For the Beurling zeta function of our Beurling system $\mathcal{B}$, we cannot take values with $\Re s \leq \theta$, so that we must restrict to $\eta < 1-\theta$, and in the construction of $\lambda^{(0)}_{\zeta_{\mathcal{B}}}(\eta)$ and $\lambda_{\zeta_{\mathcal{B}}}(\eta)$ even to $1-(a+1)\eta > \theta$, i.e. $(a+1)\eta < 1-\theta$.

We will extend $(a+1)\eta$ close to this limit. A reference to [21, Lemma 5] (or [17, Lemma 2.5] with full proof) furnishes

$$\mu_{\zeta_{\mathcal{B}}}(\theta+\varepsilon) \leq \frac{1-\theta+\varepsilon}{1-\theta}, \tag{43}$$

which in turn allows one to set $a := \frac{1-\theta-\varepsilon}{\eta} - 1 \Leftrightarrow (a+1)\eta = 1-\theta-\varepsilon$ and derive with this value the estimator functions

$$\lambda^{(\varepsilon)}_{\zeta_{\mathcal{B}}}(\eta) = \frac{1-\theta+\varepsilon}{(1-\theta)(1-\theta-\eta-\varepsilon)}.$$

As $\lambda^{(\varepsilon)}_{\zeta_{\mathcal{B}}}(\eta) \geq \lambda^{(0)}_{\zeta_{\mathcal{B}}}(\eta)$ for all $\varepsilon > 0$, we can in fact let $\varepsilon \to 0$ and obtain the new estimator function

$$\lambda_{\zeta_{\mathcal{B}}}(\eta) = \frac{1}{1-\theta-\eta}. \tag{44}$$

Unlike the estimator functions for the Riemann zeta function, where we distinguished two different cases with two different estimates (41) and (42), the estimate (44) is "universal" (the same formula) for all admissible values of $0 < \eta < 1-\theta$.

5.3. Verification of the conditions of the key lemma. Next we check the conditions of the key lemma in order to apply it for admissible values of η. We always assume that $\eta < 1-\theta$ without explicit mention.

For the small values $0 < \eta \leq 4/29$, an easy calculation furnishes that $\lambda_{\zeta_{\mathcal{B}}}(\eta) = \frac{1}{1-\theta-\eta} \geq \frac{1}{1-\eta} \geq \frac{13/42}{1-4\eta} = \lambda_\zeta(2\eta)$, the condition in (22). Let now $4/29 \leq \eta \leq 1/4$. Then we find $\lambda_{\zeta_{\mathcal{B}}}(\eta) = \frac{1}{1-\theta-\eta} \geq \frac{1}{1-\eta} \geq \frac{1}{2-4\eta} = \lambda_\zeta(2\eta)$, that is, condition (22) of Lemma 1 is again satisfied.

All in all, with the estimator functions worked out above for the Riemann zeta ζ and the Beurling zeta $\zeta_{\mathcal{B}}$, the last condition (22) of Lemma 1, i.e. the inequality $\lambda_{\zeta_{\mathcal{B}}}(\eta) \geq \lambda_\zeta(2\eta)$, holds true for the entire domain $0 < \eta < 1-\theta$.

5.4. End of the proof of Theorem 1. For all $0 < \eta < 1-\theta$, we can apply Lemma 1, which yields

$$B_{\mathcal{B}}(\eta) = \max(2\lambda_{\zeta_{\mathcal{B}}}(\eta), 4\lambda_{\zeta}(2\eta)) \tag{45}$$
$$= \begin{cases} \max\left(\frac{2}{1-\theta-\eta}, \frac{26/21}{1-4\eta}\right) & \text{if } \eta \le \min(4/29, 1-\theta), \\ \max\left(\frac{2}{1-\theta-\eta}, \frac{2}{1-2\eta}\right) & \text{if } 4/29 \le \eta \le \min(1/4, 1-\theta). \end{cases}$$

For $\eta \le 4/29$, the first term is maximal precisely when $\eta \le (8+13\theta)/71$. For $4/29 \le \eta \le 1/4$, the first term gives the maximum if and only if $\eta \le \theta$. Winding up these partial case calculations furnishes (15). ∎

6. Conclusion. In contrast to zeta functions with a functional equation, there is no asymptotic formula for the number of zeroes of a Beurling zeta function in the critical strip or in a halfplane strictly in the critical strip. This is not just a weakness of methods for a proof; Beurling systems with only finitely many zeroes (or no zeroes at all) do exist [19, Theorem 7.4] (though one may recall that this phenomenon is known only for $\theta \ge 1/2$ and strips $\Re s \ge \sigma > \theta$). Nevertheless, an upper estimate of the same order of $T \log T$ as in the Riemann zeta case is known: see [8, Theorem 2] or [17, Lemma 3.5]. Therefore, a density bound exceeding $T^{1+\varepsilon}$ does not provide anything new. In light of this the part for $\eta \ge \frac{1-\theta}{3}$ of the above results becomes worthless: they can be substituted by the better $B^{\star}_{\mathcal{B}} = 1/\eta$ (equivalent to $N(\sigma, T) \ll T^{1+\varepsilon}$ for all $\varepsilon > 0$). Also, one may note that our bounds above become inferior to the recent results of Broucke [4, (14)] for larger values of η. For example, assuming $\theta > 1/5$, his estimate is stronger than ours for $\eta > \frac{1-\theta}{4}$. However, for $\eta < \frac{1-\theta}{4}$, our results provide sharper estimates, even though only under the two assumptions on integrality and the Ramanujan condition.

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