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Precompactness notions in Kaplansky–Hilbert modules and extensions with discrete spectrum

by

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Abstract. This paper is a continuation of our work on the functional-analytic core of the classical Furstenberg–Zimmer theory. We introduce and study (in the framework of lattice-ordered spaces) the notions of total order-boundedness and uniform total order-boundedness. Either one generalizes the concept of ordinary precompactness known from metric space theory. These new notions are then used to define and characterize "compact extensions" of general measure-preserving systems (with no restrictions on the underlying probability spaces or on the acting groups). In particular, it is (re)proved that compact extensions and extensions with discrete spectrum are one and the same thing. Finally, we show that under natural hypotheses a subset of a Kaplansky–Banach module is totally order-bounded if and only if it is cyclically compact (in the sense of Kusraev).

1. Introduction. The Furstenberg–Zimmer structure theorem is one of the great milestones in structural ergodic theory. It goes back to the seminal works of Zimmer [Zim76a, Zim76b] and Furstenberg [Fur77]. The heart of the matter is a diligent examination of *extensions* of measure-preserving systems and a fundamental *dichotomy*: either an extension is "weakly mixing" or it has an in some sense "well-structured" intermediate extension. Zimmer showed that "well-structured" could mean "with relative discrete spectrum" or, equivalently, "isometric". Furstenberg recovered Zimmer's results, and added "compact" in a later paper [FK78]; see also [Fur81].

Both these authors relied on technical assumptions on the underlying probability space (a standard Lebesgue space) and on the acting group (Zimmer: standard Borel group; Furstenberg: $\mathbb{Z}^d$). After the strong popularization of structural ergodic theory in the aftermath of the Fields medals for Gowers (1998) and Tao (2004), there was a growing interest in freeing the result

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from the above mentioned restrictions and to gain an understanding of it in terms of more abstract structures.

Relatively recently, this has been achieved by several different (but related) approaches. Our own one, in a joint work with N. Edeko, makes use of *Kaplansky–Hilbert modules* (the theory which had to a large extent to be developed for this purpose); cf. [EHK24]. This approach differs from others in that the main dichotomy for extensions of measure-preserving systems is reduced to an abstract, purely functional-analytic statement about so-called *KH-dynamical systems*. In particular, the Hilbert space structure of the surrounding L^2-space is not employed.

However, the dichotomy proved in [EHK24] involves only the notion of "discrete spectrum" for the structured part of the extension. And it remained open how to define "compact" extensions in this abstract setting and to prove that compact extensions are the same as extensions with discrete spectrum. It is the purpose of this paper (announced at the end of [EHK24]) to fill this gap.

The paper is organized as follows. In Section 2 we first recall the notion of a lattice-normed space (but refer to Appendix A for all the other related notions like order-convergence, Stone algebra or Kaplansky–Hilbert module). Then we define *totally order-bounded* and *uniformly totally order-bounded* subsets of such spaces as generalizations of ordinary precompactness. Moreover, we establish a couple of relevant technical properties (Lemma 2.4). The main result is that a bounded subset of a finite-rank KH-module is uniformly totally order-bounded (Proposition 2.5). Finally, following Tao's definition in [Tao09] an alternative compactness notion involving *zonotopes* is considered and equivalence with uniform total order-boundedness is proved.

In Section 3 we recall from [EHK24] the relevant notions from the theory of (abstract) measure-preserving systems and their extensions, in particular extensions with discrete spectrum. Next, we introduce *compact extensions* (employing the abstract precompactness notions from Section 2). The main results are then Theorem 3.6 and its Corollary 3.7, in which compact extensions are characterized in several ways, and in particular as extensions with discrete spectrum. The proof of this characterization, however, does not solely rely on results from abstract KH-dynamical systems theory, but involves the surrounding Hilbert space structure in a crucial way. Whether this can be avoided remains open, see Remark 3.8.

In the final Section 4, we relate the precompactness notions from Section 2 to the notion of *cyclical compactness* appearing in the work of Kusraev [Kus00] and mentioned at several places in [EHK24]. As a main result we find that under natural assumptions a mix-complete subset of a lattice-normed space is totally order-bounded if and only if it is relatively cyclically compact (Proposition 4.5).

Let us make some remarks concerning originality. As mentioned above, the equality of compact extensions and extensions of discrete spectrum (in Corollary 3.7) is classical under the usual assumptions in standard ergodic theory. In the framework of "abstract" measure-preserving systems—and hence without restrictions on the underlying probability space or on the acting group—it has been proved by Jamneshan in [Jam23, Thm. 4.1]. Moreover, also the connections with notions from conditional set theory and Boolean-valued analysis have been pointed out at several occasions in Jamneshan's work, see [Jam23, Remarks 3.5, 4.3, 4.5, 4.7, and 6.4].

Our contribution in this article regards less the results themselves but rather the theoretical framework to obtain them. It emphasizes the purely functional-analytic essence of (the largest part of) the results and is another step towards explaining how the plethora of concepts/results for extensions appearing in the ergodic-theoretic literature can be stated/proved in purely functional-analytic terms. Where in the usual approach to extensions of measure-preserving systems, arguments are motivated by *analogies* with classical concepts or results from Banach or Hilbert space theory, we show that there is a purely functional-analytic theory that renders the results actual *generalizations*.

We also hope that our work (together with [EHK24] and the upcoming article [HK] on Furstenberg's main theorem) will make the powerful Furstenberg–Zimmer theory more accessible, in particular for readers with a functional-analytic background.

2. Total order-boundedness. In this section we generalize in several ways the notion of total boundedness (= precompactness) for subsets of normed spaces to subsets of lattice-normed spaces. To this aim, we fix (once and for all) a commutative unital C^*-algebra $\mathbb{A}$ and consider it as a Banach lattice in the canonical way.

A *lattice-normed space* over $\mathbb{A}$ is a vector space E together with a mapping $|\cdot|\colon E \to \mathbb{A}_+$ with the following properties:

$$|x| = 0 \Leftrightarrow x = 0, \quad |\lambda x| = |\lambda|\,|x|, \quad |x+y| \le |x| + |y| \quad (x, y \in E,\ \lambda \in \mathbb{C}).$$

The *closed ball* and the *open ball* in this norm with center $x \in E$ and radius $t \in \mathbb{A}_+$ are the sets

$$\mathrm{B}_E[x;t] := \{y \in E \mid |x-y| \le t\} \quad \text{and} \quad \mathrm{B}_E(x;t) := \{y \in E \mid |x-y| < t\},$$

respectively.

Each lattice-normed space carries a natural norm given by $\|x\|_E := \big\||x|\big\|_{\mathbb{A}}$ for $x \in E$. For $r \in \mathbb{R}_{\ge 0}$ one has $|x| \le r\mathbf{1}$ iff $\|x\|_E \le r$, and hence

$$\mathrm{B}_E[x; r\mathbf{1}] = \{y \in E \mid |y-x| \le r\} = \{y \in E \mid \|y-x\|_E \le r\} = \mathrm{B}_E[x;r],$$

where the last set is the closed ball with respect to the norm.

A lattice-normed space over $\mathbb{A} = \mathbb{C}$ is nothing other than a normed space, and in this case $\|\cdot\|_E$ coincides with the original norm. So, a lattice-normed space is an analogue of a normed space, but with elements of $\mathbb{A}_+$ taking the role of the possible values for the "norm". If E is even a lattice-normed *module*, then elements of $\mathbb{A}$ take the role of the scalars in all respects, and the analogy is even more striking.

The elementary theory of lattice-normed spaces is strongly analogous to the theory of normed spaces, but with the usual notions of convergence, closedness, continuity and completeness suitably modified (namely to *order-convergence*, *order-closedness*, *order-continuity*, and *order-completeness*). It reduces to the classical theory in the case $\mathbb{A} = \mathbb{C}$, and is only slightly more difficult in the general case. However, as we certainly cannot speak of common knowledge here, we have collected basic definitions and some elementary statements in Appendix A. For proofs and further information we refer to [EHK24, Sec. 1]. The presentation there centers around *Kaplansky–Hilbert modules*, which are instances of lattice-normed spaces most relevant for us.

Let us now turn to the main topic of this section.

DEFINITION 2.1. Let E be a lattice-normed space over a unital commutative C^*-algebra $\mathbb{A}$. A subset $M \subseteq E$ is *totally order-bounded*, or *order-precompact*, if there is a net $(u_\alpha)_\alpha$ in $\mathbb{A}_+$ decreasing to 0 and such that for every α there is a finite set $F \subseteq E$ with

$$\inf_{y \in F} |x - y| \leq u_\alpha \quad \text{for every } x \in M.$$

And M is *uniformly totally order-bounded* if for every $\varepsilon \in \mathbb{R}_{>0}$ there is a finite set $F \subseteq E$ such that

$$\inf_{y \in F} |x - y| \leq \varepsilon \mathbf{1} \quad \text{for every } x \in M.$$

REMARK 2.2. (1) Uniform total boundedness is related to the notion of relative uniform convergence in vector lattices [Kus00, Sec. 1.3.4]. It generalizes in the most straightforward way ordinary precompactness in a normed space to subsets of lattice-normed spaces, namely by replacing the ordinary norm by the lattice-norm.

In the context of extensions X|Y of probability spaces, uniformly totally order-bounded subsets of $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ are sometimes called *conditionally precompact.* See also Remark 3.4 below.

(2) Observe the validity of the implications

$$\begin{aligned} M \text{ uniformly totally order-bounded} &\implies M \text{ totally order-bounded} \\ &\implies M \text{ (order-)bounded.} \end{aligned}$$

The first implication is trivial; the second one follows from

$$|x| \leq \inf_{y\in F} |x-y| + \sup_{y\in F} |y| \quad \text{for all } x \in E \text{ and all finite } F \subseteq E.$$

Evidently, if $\mathbb{A} = \mathbb{C}\mathbf{1}$, then "totally order-bounded = uniformly order-bounded = precompact".

From now on we suppose that $\mathbb{A}$ *is order-complete as a lattice-ordered space over itself*. That is, $\mathbb{A}$ is a *Stone algebra* (see Appendix A). Then we may rephrase the definitions above in terms of (order-)convergence. To this end, consider the set

$$\mathcal{P}_{\mathrm{fin}}(E) = \{F \subseteq E \mid F \text{ finite}\}$$

to be upwards directed by inclusion. To a given bounded subset $M \subseteq E$ we associate the decreasing net

$$F \mapsto \sup_{x\in M} \inf_{y\in F} |x-y| \quad (F \in \mathcal{P}_{\mathrm{fin}}(E)).$$

Then M is totally order-bounded if and only if

$$\inf_{F\in\mathcal{P}_{\mathrm{fin}}(E)} \sup_{x\in M} \inf_{y\in F} |x-y| = 0,$$

which means that the said net order-converges/decreases to zero in $\mathbb{A}_+$. Similarly, M is uniformly totally order-bounded if and only if

$$\inf_{F\in\mathcal{P}_{\mathrm{fin}}(E)} \Big\| \sup_{x\in M} \inf_{y\in F} |x-y| \Big\|_{\mathbb{A}} = 0,$$

i.e., if the net $F \mapsto \sup_{x\in M} \inf_{y\in F} |x-y|$ norm-converges to zero in $\mathbb{A}$.

As mentioned, every uniformly totally order-bounded subset is totally order-bounded. The following example shows that the converse is false, even for subsets of Kaplansky–Hilbert modules (KH-modules).

Example 2.3. Let H be an infinite-dimensional Hilbert space with norm $\|\cdot\|_H$. The space $E := \ell^\infty(\mathbb{N}; H)$ of bounded H-valued sequences is a KH-module over $\mathbb{A} := \ell^\infty = \ell^\infty(\mathbb{N};\mathbb{C})$, the space of all bounded scalar sequences. The lattice-valued norm is

$$|f| := (n \mapsto \|f(n)\|_H).$$

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system in H. We claim that the set

$$M := \{\mathbf{1}_{\{k\}} \otimes e_j \mid k, j \in \mathbb{N},\ 1 \leq j \leq k\}$$

is totally order-bounded, but not uniformly totally order-bounded in E. (We write $f \otimes e$ for the function $(n \mapsto f(n)e)\colon \mathbb{N} \to H$ whenever $f \in \ell^\infty$ and $e \in H$.)

To prove the first claim, we take $n \in \mathbb{N}$ and let

$$F_n := \{0\} \cup \{\mathbf{1} \otimes e_l \mid 1 \leq l \leq n\}.$$

Fix $1 \le j \le k$. Then, since $g = 0 \in F_n$,

$$\inf_{g \in F_n} |\mathbf{1}_{\{k\}} \otimes e_j - g| \le \mathbf{1}_{\{k\}}.$$

If, in addition, $j \le n$, then

$$\inf_{g \in F_n} |\mathbf{1}_{\{k\}} \otimes e_j - g| \le |\mathbf{1}_{\{k\}} \otimes e_j - \mathbf{1} \otimes e_j| = \mathbf{1}_{\{k\}^c}.$$

This shows that

$$\inf_{g \in F_n} |\mathbf{1}_{\{k\}} \otimes e_j - g| = 0 \quad \text{on } \{1, \ldots, n\}.$$

It follows that

$$\sup_{f \in M} \inf_{g \in F_n} |f - g| \le \sqrt{2}\, \mathbf{1}_{\{1,\ldots,n\}^c} \searrow 0 \quad (n \to \infty),$$

proving the (first) claim.

For the proof of the second, observe that if $F = \{g_1, \ldots, g_d\} \subseteq E$ is a finite set and $n > d$, then the set

$$\bigcup_{j=1}^{d} \mathrm{B}_H\big(g_j(n); \tfrac{1}{2}\sqrt{2}\,\big)$$

cannot contain all the basis vectors $e_1, \ldots, e_n$. Hence, there is $1 \le i \le n$ with

$$\min_{j \le d} \|e_i - g_j(n)\|_H \ge \tfrac{1}{2}\sqrt{2}.$$

This leads to

$$\min_{j \le d} |\mathbf{1}_{\{n\}} \otimes e_i - g_j| \ge \tfrac{1}{2}\sqrt{2}\, \mathbf{1}_{\{n\}},$$

violating the uniform total order-boundedness of M.

In the following lemma we collect some elementary properties of (uniformly) totally order-bounded sets.

LEMMA 2.4. *Let E, E_1, E_2 be lattice-ordered spaces over a Stone algebra $\mathbb{A}$.*

(a) *Each finite subset of E is uniformly totally order-bounded. If $M \subseteq E$ is (uniformly) totally order-bounded, then so is each subset of M.*
(b) *If $M, N \subseteq E$ are (uniformly) totally order-bounded, then so are $M + N$ and $M \cup N$.*
(c) *Suppose that $m \colon E \times E_1 \to E_2$ is a bilinear mapping with $|m(x, y)| \le |x|\,|y|$ for all $x \in E$, $y \in E_1$. If $M \subseteq E$ and $N \subseteq E_1$ are (uniformly) totally order-bounded, then so is $m(M, N) \subseteq E_2$.*
(d) *Let $M \subseteq E$ and suppose that $N \subseteq \mathbb{A}_+$ satisfies $\inf N = 0$ ($\inf_{t \in N} \|t\|_{\mathbb{A}} = 0$) and for each $t \in N$ there is a (uniformly) totally order-bounded set $M_t \subseteq E$ with*

$$M \subseteq M_t + \mathrm{B}_E[0; t].$$

Then M is (uniformly) totally order-bounded.

(e) *If $M \subseteq E$ is (totally) order-bounded, then so is its order-closure* $\mathrm{ocl}(M)$.

(f) *Suppose that $T\colon E \to E_1$ is linear and order-bounded, i.e. there is $c \in \mathbb{R}_{>0}$ with $|Tx| \le c|x|$ for all $x \in E$. Then if $M \subseteq E$ is (uniformly) totally order-bounded, so is $T(M) \subseteq E_1$.*

(g) *If $M \subseteq E$ is bounded and $V \subseteq E$ is arbitrary, then*

$$\inf\Big\{\sup_{x\in M}\inf_{y\in F}|x-y| \;\Big|\; F \subseteq V \text{ finite}\Big\} = \inf\Big\{\sup_{x\in M}\inf_{y\in F}|x-y| \;\Big|\; F \subseteq \mathrm{ocl}(V) \text{ finite}\Big\}.$$

In particular, if M is totally order-bounded and V is order-dense in E, then

$$\inf_{F\in\mathcal{P}_{\mathrm{fin}}(V)}\sup_{x\in M}\inf_{y\in F}|x-y| = 0.$$

(h) *Suppose that E is a lattice-normed module, $r \in \mathbb{R}_{>0}$ and $M \subseteq \mathrm{B}_E[0;r]$. Then for each finite set $F \subseteq E$ there is a finite set $F' \subseteq \mathrm{B}_E[0;2r]$ with $\#F' \le \#F$ and*

$$\inf_{y'\in F'}|x-y| \le \inf_{y\in F}|x-y|.$$

In particular, one may replace general finite subsets F of E by finite subsets of $\mathrm{B}_E[0;2r]$ in the definition of (uniformly) totally order-bounded sets.

Proof. (a) is trivial.

(b) The proof of the statement about $M \cup N$ follows from the fact that in $\mathbb{A}$ from $u_\alpha \searrow 0$ and $v_\alpha \searrow 0$ it follows that $u_\alpha \vee v_\beta \searrow 0$. (This is obviously true if $u_\alpha, v_\alpha \in \mathbb{R}\mathbf{1}$, and this covers the "uniform" case of the statement. But it is even true in every vector lattice: if $h \le u_\alpha \vee v_\alpha$ for each α, then $h \le u_\beta \vee v_\alpha \le u_\beta + v_\alpha$, and hence $h - v_\alpha \le u_\beta$, for $\beta \ge \alpha$. This yields $h - v_\alpha \le 0$ for all α and hence $h \le 0$. See also [LZ71, Thm. 15.9].)

Let $G, H \subseteq E$ be finite. Then

$$|(a+b)-(z+w)| \le |a-z| + |b-w| \quad (a \in M,\, b \in N,\, z \in G,\, w \in H).$$

This implies

$$\inf_{y\in F}|(a+b)-y| \le \inf_{z\in G}|a-z| + \inf_{w\in H}|b-w| \quad (a \in M,\, b \in N)$$

whenever $F \subseteq E$ is finite with $F \supseteq G + H$. And this implies

$$\sup_{x\in M+N}\inf_{y\in F}|x-y| = \sup_{a\in M,\, b\in N}\inf_{y\in F}|(a+b)-y| \le \sup_{a\in M}\inf_{z\in G}|a-z| + \sup_{b\in N}\inf_{w\in H}|b-w|$$

for $G+H \subseteq F \subseteq E$ finite. This yields (b).

(c) Define $A := \sup_{a\in M}|a|$ and $B := \sup_{b\in N}|b|$. Let $G \subseteq E$ and $H \subseteq E_1$ be finite. Then

$$|m(a,b) - m(z,w)| = |m(a,b-w) - m(a-z,b-w) + m(a-z,b)| \\ \leq A|b-w| + |a-z|\,|b-w| + B|a-z|$$

for $a \in M$, $b \in N$, $z \in G$, $w \in H$. This implies

$$\inf_{y \in m(G,H)} |m(a,b) - y| = \inf_{z \in G,\, w \in H} |m(a,b) - m(z,w)| \\ \leq A \inf_{w \in H} |b-w| + \inf_{z \in G} |a-z| \inf_{w \in H} |b-w| + B \inf_{z \in G} |a-z|.$$

(Here we use the fact that $\inf_\alpha f g_\alpha = f \inf_\alpha g_\alpha$ in $\mathbb{A}$ whenever $f \in \mathbb{A}_+$.) From this we may pass to

$$\sup_{x \in m(M,N)} \inf_{y \in m(G,H)} |x-y| \\ \leq A \sup_{b \in N} \inf_{w \in H} |b-w| + \sup_{a \in M} \inf_{z \in G} |a-z| \sup_{b \in N} \inf_{w \in H} |b-w| + B \sup_{A \in M} \inf_{z \in G} |a-z|.$$

And this yields (c).

(d) Fix $t \in N$ and $F \subseteq E$ finite. For $x \in M$ let $x_t \in M_t$ with $|x - x_t| \leq t$. Then

$$|x-y| \leq t + |x_t - y| \quad (y \in F)$$

and hence

$$\inf_{y \in F} |x-y| \leq t + \sup_{z \in M_t} \inf_{y \in F} |z-y|.$$

It follows that

$$\sup_{x \in M} \inf_{y \in F} |x-y| \leq t + \sup_{z \in M_t} \inf_{y \in F} |z-y|,$$

and this implies (d).

(e) Let $f \in \mathbb{A}_+$ and $F \subseteq E$ be finite. Then the set of all $x \in E$ satisfying

$$\inf_{y \in F} |x-y| \leq f$$

is order-closed in E (simply because the mapping $x \mapsto \inf_{y \in F} |x-y|$ is order-continuous). Hence, if each $x \in M$ satisfies the inequality, then also each $x \in \operatorname{ocl}(M)$ does. This proves the claim.

(f) This follows from

$$\inf_{y \in F} |Tx - Ty| \leq c \inf_{y \in F} |x-y| \quad \text{for all } x \in M \text{ and all } F \subseteq E \text{ finite.}$$

(g) Fix $d \in \mathbb{N}$. For $y \in V^d$, $z \in \operatorname{ocl}(V)^d$ and $x \in M$ one has

$$\inf_{j=1,\ldots,d} |x - y_j| \leq |x - z_k| + \sum_{j=1}^{d} |y_j - z_j| \quad (k = 1, \ldots, d).$$

Taking first the infimum with respect to k and then the supremum with respect to $x \in M$ yields

$$\sup_{x\in M}\inf_{j=1,\dots,d}|x-y_j| \le \sup_{x\in M}\inf_{j=1,\dots,d}|x-z_j| + |z-y|_1$$

with $|\cdot|_1$ being the lattice-norm on E^d. By Lemma A.1 (and an induction argument), $\mathrm{ocl}(V)^d = \mathrm{ocl}(V^d)$, hence taking the infimum over all $y \in V^d$ we obtain (see also the inclusion (A.1))

$$\inf_{y\in V^d}\sup_{x\in M}\inf_{j=1,\dots,d}|x-y_j| \le \sup_{x\in M}\inf_{j=1,\dots,d}|x-z_j|.$$

And this implies

$$\inf_{y\in V^d}\sup_{x\in M}\inf_{j=1,\dots,d}|x-y_j| \le \inf_{z\in\mathrm{ocl}(V)^d}\sup_{x\in M}\inf_{j=1,\dots,d}|x-z_j|.$$

Finally, take the infimum with respect to $d \in \mathbb{N}$ to obtain the non-trivial inequality between the two quantities in the claim. The additional statement follows readily.

(h) Let $F \subseteq E$ be an arbitrary finite set. Fix $y \in F$, let $p := [\![\, |y| \le 2r\mathbf{1} \,]\!] := \mathbf{1}_{[|y|\le 2r]^\circ}$ and define $y' := py$. Then $|y'| \le 2r\mathbf{1}$ and $p^c 2|x| \le p^c 2r \le p^c|y|$, and hence

$$|x-y'| = p|x-y| + p^c|x| \le p|x-y| + p^c(|y|-|x|) \le |x-y| \qquad (x \in M).$$

It follows that for $x \in M$,

$$\inf_{y'\in F'}|x-y'| \le \inf_{y\in F}|x-y|,$$

where $F' = \{y' \mid y \in F\}$. ■

For Kaplansky–Hilbert modules of finite rank (see [EHK24, Sec. 2.5]) we obtain the following Heine–Borel-type theorem.

PROPOSITION 2.5. *Let $\mathbb{A}$ be a Stone algebra. Then $\mathrm{B}_{\mathbb{A}}[0;1]$ is uniformly totally order-bounded. More generally, let E be a KH-module of finite rank over $\mathbb{A}$. Then for $M \subseteq E$ the following assertions are equivalent:*

(i) *M is order-bounded.*
(ii) *M is totally order-bounded.*
(iii) *M is uniformly totally order-bounded.*

Proof. Since $\mathbb{A}$ is a rank-one KH-module over itself, the first assertion is a corollary of the second. However, we shall need the special case in the proof of the more general statement.

For $\varepsilon \in \mathbb{R}_{>0}$ find a finite set $F \subseteq \mathbb{C}$ with $\mathrm{B}_{\mathbb{C}}[0;1] \subseteq \bigcup_{z\in F}\mathrm{B}_{\mathbb{C}}[z;\varepsilon]$. Then

$$\inf_{z\in F}|f-z\mathbf{1}| \le \varepsilon\mathbf{1} \qquad \text{for each } f \in \mathrm{B}_{\mathbb{A}}[0;1].$$

Now let E be a KH-module over $\mathbb{A}$ and let $e_1,\dots,e_d$ be a suborthonormal basis for E [EHK24, Sec. 2.3]. Since $M \subseteq E$ is order-bounded, there is

$c \in \mathbb{R}_{\geq 0}$ with $|f| \leq c\mathbf{1}$ for all $f \in M$. Given $f \in M$ we may write

$$f = \sum_{j=1}^{d} \lambda_j e_j$$

for certain $\lambda_j \in |e_j|\mathbb{A}$. Then

$$c^2\mathbf{1} \geq |f|^2 = \sum_{j=1}^{d} |\lambda_j|^2$$

and hence $|\lambda_j| \leq c\mathbf{1}$ for each $j = 1, \ldots, d$. This shows that

$$M \subseteq \sum_{j=1}^{d} \mathrm{B}_{\mathbb{A}}[0; c] \cdot e_j.$$

Since $\mathrm{B}_{\mathbb{A}}[0; c] \subseteq \mathbb{A}$ is uniformly totally order-bounded (by what we have already shown), the claim follows from Lemma 2.4. ∎

In the measure-theoretic situation we have the following close connection between the two notions of total order-boundedness.

Proposition 2.6. *Let E be a KH-module over $\mathrm{L}^\infty(\mathrm{Y})$ for some probability space $\mathrm{Y} = (Y, \Sigma_{\mathrm{Y}}, \mu_{\mathrm{Y}})$. Then for a bounded subset $M \subseteq E$ the following assertions are equivalent:*

(i) *M is totally order-bounded.*
(ii) *For every $\varepsilon \in \mathbb{R}_{>0}$ there is a measurable set $A \in \Sigma_{\mathrm{Y}}$ with $\mu_{\mathrm{Y}}(A^c) \leq \varepsilon$ such that $\mathbf{1}_A M$ is uniformly totally order-bounded.*

Proof. Assume (i) and fix $\varepsilon \in \mathbb{R}_{>0}$. Note that in $\mathrm{L}^\infty(\mathrm{Y}; \mathbb{R})$ the infimum of a bounded subset must coincide with the infimum of some countable subset (see, e.g. [EFHN15, Thm. 7.6]). Hence, we find a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of E such that

$$0 = \inf_{F \in \mathcal{P}_{\mathrm{fin}}(E)} \sup_{x \in M} \inf_{y \in F} |x - y| = \inf_{n \in \mathbb{N}} \sup_{x \in M} \inf_{y \in F_n} |x - y|.$$

Without loss of generality we may suppose that $(F_n)_{n\in\mathbb{N}}$ is increasing, i.e., $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$. Then the sequence $(\sup_{x\in M} \inf_{y \in F_n} |x - y|)_{n\in\mathbb{N}}$ decreases and thus order-converges to zero. In particular, it converges to zero almost everywhere (see [EHK24, Lemma 7.5]). By Egorov's theorem we thus find $A \in \Sigma_{\mathrm{Y}}$ with $\mu_{\mathrm{Y}}(A^c) \leq \varepsilon$ such that the sequence

$$n \mapsto \mathbf{1}_A \sup_{x \in M} \inf_{y \in F_n} |x - y| = \sup_{x \in M} \inf_{y \in F_n} |\mathbf{1}_A x - \mathbf{1}_A y|$$

converges to zero in the norm of $\mathrm{L}^\infty(\mathrm{Y})$.

Conversely, assume (ii) holds and set $c := \sup\{\|x\| \mid x \in M\} \in [0, \infty)$. For every $n \in \mathbb{N}$ we then find $A_n \in \Sigma_{\mathrm{Y}}$ with $\mu_{\mathrm{Y}}(A_n^c) \leq \frac{1}{n}$ and a finite subset

$F_n \subseteq E$ such that

$$\sup_{x\in M}\inf_{y\in F_n} |\mathbf{1}_{A_n}x - y| \le \tfrac{1}{n}\mathbf{1}.$$

But then

$$\sup_{x\in M}\inf_{y\in F_n} |x - y| \le \tfrac{1}{n}\mathbf{1} + c\mathbf{1}_{A_n^c}$$

for every $n \in \mathbb{N}$. Since $\inf_{n\in\mathbb{N}}\big(\frac{1}{n}\mathbf{1} + c\mathbf{1}_{A_n^c}\big) = 0$, we obtain (i). ∎

We conclude this section by comparing the two notions of total order-boundedness with yet another similar property. For this we suppose E to be not just a lattice-ordered space over $\mathbb{A}$, but a lattice-ordered *module*, see [EHK24, Def. 1.3].

DEFINITION 2.7. Let E be a lattice-ordered module over the commutative unital C^*-algebra $\mathbb{A}$. For a finite set $F \subseteq E$ let

$$\mathrm{Z}_F := \Big\{ \sum_{y\in F} \lambda_y y \;\Big|\; \lambda \in \mathbb{A}^F,\ \sup_{y\in F} |\lambda_y| \le \mathbf{1} \Big\}$$

be the $\mathbb{A}$-*zonotope* in E determined by F. A subset M of E has *property* (CP) if for each $\varepsilon \in \mathbb{R}_{>0}$ there is a finite set $F \subseteq E$ such that

$$M \subseteq \mathrm{Z}_F + \mathrm{B}_E[0;\varepsilon].$$

In the context of extensions of measure-preserving systems, property (CP) was introduced by Tao in [Tao09, Def. 2.13.7]. Our terminology is reminiscent of "conditionally precompact", which is the term Tao uses. See also Remark 3.4 below.

PROPOSITION 2.8. *Let E be a lattice-normed module over a Stone algebra $\mathbb{A}$. Then a subset $M \subseteq E$ has property* (CP) *if and only if it is uniformly totally order-bounded.*

Proof. The first implication is rather trivial. If M is uniformly totally order-bounded and $\varepsilon \in \mathbb{R}_{>0}$, then there is a finite set $F \subseteq E$ with $\inf_{y\in F} |x-y| \le \varepsilon\mathbf{1}$ for each $x \in M$. Hence, given $x \in M$ one finds idempotents $p_y = p_y^2 \in \mathbb{A}$ with $\sum_{y\in F} p_Y = \mathbf{1}$ and $p_y|x - y| \le \varepsilon\mathbf{1}$ for each $y \in F$. It follows that $x \in \sum_{y\in F} p_y y + \mathrm{B}_E[0;\varepsilon] \subseteq \mathrm{Z}_F + \mathrm{B}_E[0;\varepsilon]$.

For the second implication note that

$$\mathrm{Z}_F = \sum_{y\in F} \mathrm{B}_{\mathbb{A}}[0;1]\cdot y$$

is uniformly totally order-bounded, by Lemma 2.4 and Proposition 2.5. Hence, if M has (CP), then M is uniformly totally order-bounded, again by Lemma 2.4. ∎

3. Compact extensions. In this section we apply the abstract terminology and results from Section 2 to the structure theory of measure-preserving dynamics. To wit, we characterize extensions with discrete spectrum as compact extensions. In the classical case of $\mathbb{Z}^d$-dynamics on separable probability spaces, this is due to Furstenberg [Fur81, Thm. 6.13]. The general case is due to Jamneshan [Jam23, Thm. 4.1].

We briefly describe the set-up, for details see [EHK24]. Let G be a group (any group, no topology). A *measure-preserving* G-system is a pair $(\mathrm{X}; T)$ consisting of a probability space X and a representation $T = (T_t)_{t\in G} \colon G \to \mathscr{L}(\mathrm{L}^1(\mathrm{X}))$ of G as *Markov embeddings* (= unital, integral-preserving linear lattice homomorphisms). An *extension* of two measure-preserving systems $(\mathrm{Y}; S), (\mathrm{X}; T)$ is a Markov embedding $J \colon \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ with $JS_t = T_t J$ for all $t \in G$. We write $J \colon (\mathrm{Y}; S) \to (\mathrm{X}; T)$ to denote the extension, or just $\mathrm{X}|\mathrm{Y}$, for the sake of simplicity.

Given an extension $\mathrm{X}|\mathrm{Y}$, the space $\mathrm{L}^2(\mathrm{X})$ becomes an $\mathrm{L}^\infty(\mathrm{Y})$-module in a natural way. (Basically, J identifies $\mathrm{L}^2(\mathrm{Y})$ with $\mathrm{L}^2(X, \mathcal{F}, \mu_{\mathrm{X}})$ for some sub-σ-algebra $\mathcal{F}$ of Σ_{X}.) Along with the extension comes the *conditional expectation operator* $\mathbb{E}_{\mathrm{Y}} \colon \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$, which on the level of L^2-spaces is nothing other than the adjoint $\mathbb{E}_{\mathrm{Y}} = J^*$ of J. It coincides with the classical conditional expectation under the identification of $\mathrm{L}^1(\mathrm{Y})$ with its range under J.

It turns out that the space

$$\mathrm{L}^2(\mathrm{X}|\mathrm{Y}) := \{f \in \mathrm{L}^2(\mathrm{X}) \mid \mathbb{E}_{\mathrm{Y}}|f|^2 \in \mathrm{L}^\infty(\mathrm{Y})\}$$

is a Kaplansky–Hilbert module over the Stone algebra $\mathbb{A} := \mathrm{L}^\infty(\mathrm{Y})$ with respect to the inner product

$$(f|g)_{\mathrm{Y}} := \mathbb{E}_{\mathrm{Y}}(f\overline{g})$$

(see [EHK24, Prop. 7.6]). The lattice-valued norm of an element $f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ is

$$|f|_{\mathrm{Y}} := \sqrt{(f|f)_{\mathrm{Y}}} = \sqrt{\mathbb{E}_{\mathrm{Y}}|f|^2}.$$

Note the difference between $|f|_{\mathrm{Y}}$ and $|f|$, the latter being the usual (pointwise) modulus of f.

Since $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ is G-invariant and $(T_t f|T_t g)_{\mathrm{Y}} = S_t(f|g)_{\mathrm{Y}}$, one arrives at a *KH-dynamical system* as defined in [EHK24, Sec. 5.1]; see also [EHK24, Sec. 7.2].

On $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ we have three natural notions of convergence: convergence with respect to the norm

$$\|f\|_{\mathrm{L}^2(\mathrm{X}|\mathrm{Y})} = \big\||f|_{\mathrm{Y}}\big\|_{\mathrm{L}^\infty(\mathrm{Y})},$$

convergence with respect to the L^2-topology as a subspace of $\mathrm{L}^2(\mathrm{X})$, and order-convergence (see Appendix A). The following result relates these notions.

LEMMA 3.1. *Let* $\mathrm{X}|\mathrm{Y}$ *be an extension of measure-preserving systems. Then the following assertions hold:*

(a) *Within* $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ *each norm-convergent net and each order-convergent net is* L^2*-convergent (to the same limit).*
(b) *If* $(f_n)_n$ *is a sequence in* $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ *which* L^2*-converges to* $f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$, *then there is a subsequence* $(h_n)_n$ *with the following property: for each* $\delta \in \mathbb{R}_{>0}$ *there is* $E \in \Sigma_{\mathrm{Y}}$ *with* $\mu_{\mathrm{Y}}(E^c) < \delta$ *and* $\mathbf{1}_E h_n \to \mathbf{1}_E f$ *in the norm of* $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$.
(c) *If* $M \subseteq \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ *is an* $\mathrm{L}^\infty(\mathrm{Y})$*-submodule, then in* $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ *its order-closure coincides with its* L^2*-closure. In particular,* M *is order-dense in* $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ *iff* M *is dense in* $\mathrm{L}^2(\mathrm{X})$.
(d) *If* $M \subseteq \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ *is (uniformly) totally order-bounded, then so are the sets*
$$|M| := \{\, |f| \mid f \in M\} \quad \textit{and} \quad M' := \{\overline{f} \mid f \in M\}.$$

Proof. For (a) see [EHK24, Lemma 7.5(ii)] and its proof, for (b) see the proof of [EHK24, Lemma 7.5(iii)]; (c) is a consequence of (b) as in [EHK24, Lemma 7.5.(iii)] and the fact that $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ is dense in $\mathrm{L}^2(\mathrm{X})$. For the proof of (d) we observe that for $f, g \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ one has
$$\big||f| - |g|\big|_{\mathrm{Y}} \le |f - g|_{\mathrm{Y}}$$
as a consequence of the reverse triangle inequality for the modulus. And this implies
$$\sup_{u \in |M|} \inf_{v \in |F|} |u - v|_{\mathrm{Y}} = \sup_{f \in M} \inf_{g \in F} \big||f| - |g|\big|_{\mathrm{Y}} \le \sup_{f \in M} \inf_{g \in F} |f - g|_{\mathrm{Y}}$$
for any finite $F \subseteq \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$. The proof for M' is similar. ∎

We proceed with recalling a definition from [EHK24](1).

DEFINITION 3.2. For an extension $\mathrm{X}|\mathrm{Y}$ of measure-preserving G-systems its (*relative*) *Kronecker subspace* is
$$\mathscr{E}(\mathrm{X}|\mathrm{Y}) := \mathrm{cl}_{\mathrm{L}^2} \bigcup \{\text{finitely generated and } G\text{-invariant } \mathrm{L}^\infty(\mathrm{Y})\text{-submodules of } \mathrm{L}^2(\mathrm{X})\}.$$
The extension $\mathrm{X}|\mathrm{Y}$ has *discrete spectrum* if $\mathscr{E}(\mathrm{X}|\mathrm{Y}) = \mathrm{L}^2(\mathrm{X})$.

In [EHK24], extensions with discrete spectrum were approached through the theory of KH-dynamical systems. In particular, it was proved there that
$$\mathscr{E}(\mathrm{X}|\mathrm{Y}) = \mathrm{cl}_{\mathrm{L}^2}\, \mathrm{FM}_T(\mathrm{X}|\mathrm{Y}),$$

(1) Actually, the definition in [EHK24] is slightly different, but proved to be equivalent in [EHK24, Prop. 8.5].

where

$$\mathrm{FM}_T(\mathrm{X}|\mathrm{Y}) := \bigcup\{G\text{-invariant finite-rank KH-submodules of } \mathrm{L}^2(\mathrm{X}|\mathrm{Y})\},$$

see [EHK24, Prop. 8.5].

Next, we define compact extensions.

DEFINITION 3.3. Let X|Y be an extension of measure-preserving G-systems $(\mathrm{X};T)$ and $(\mathrm{Y};S)$. An element $f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ is said to be *conditionally almost periodic* if its *orbit* $T_G f := \{T_t f \mid t \in G\}$ is uniformly totally order-bounded in $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$, i.e., if for each $\varepsilon \in \mathbb{R}_{>0}$ there is a finite set $F \subseteq \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ such that

$$\inf_{g\in F} |T_t f - g|_{\mathrm{Y}} \le \varepsilon \mathbf{1} \quad \text{for all } t \in G.$$

The extension X|Y is *compact* if the set

$$\mathrm{AP}_T(\mathrm{X}|\mathrm{Y}) := \{f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y}) \mid f \text{ is conditionally almost periodic}\}$$

is dense in $\mathrm{L}^2(\mathrm{X})$.

REMARK 3.4. Given an extension X|Y it is common to call a subset M of $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ *conditionally precompact* if it is uniformly totally order-bounded. By Proposition 2.8, this is consistent with Tao's definition of this term in [Tao09, Def. 2.13.7]. Hence, with this terminology a function f is conditionally almost periodic if its orbit is conditionally precompact. (And this is still consistent with [Tao09, Def. 2.13.7].)

The following result is a straightforward consequence of Lemmas 2.4(d) and 3.1.

PROPOSITION 3.5. *For an extension* X|Y *of measure-preserving G-systems the sets*

$$\mathrm{AP}_T(\mathrm{X}|\mathrm{Y}) \quad \textit{and} \quad \{f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y}) \mid T_G f \textit{ totally order-bounded}\}$$

are norm-closed G-invariant, conjugation- and modulus-invariant $\mathrm{L}^\infty(\mathrm{Y})$*-submodules of* $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$.

With these preparations at hand, we are now in a position to formulate and prove the main result of this section.

THEOREM 3.6. *Let* X|Y *be an extension of measure-preserving G-systems. Then the following assertions hold:*

(a) $\mathrm{FM}_T(\mathrm{X}|\mathrm{Y}) \subseteq \mathrm{AP}_T(\mathrm{X}|\mathrm{Y}) \subseteq \{f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y}) \mid T_G f \textit{ totally order-bounded}\}$.
(b) *The* L^2*-closures of all three sets in* (a) *coincide with* $\mathscr{E}(\mathrm{X}|\mathrm{Y})$.
(c) *For each* $f \in \mathrm{L}^2(\mathrm{X})$,

$$f \in \mathscr{E}(\mathrm{X}|\mathrm{Y}) \iff \forall\delta \in \mathbb{R}_{>0}\, \exists E \in \Sigma_{\mathrm{Y}} : \mu_{\mathrm{Y}}(E^c) \le \delta \wedge \mathbf{1}_E f \in \mathrm{AP}_T(\mathrm{X}|\mathrm{Y}).$$

Proof. (a) If M is a G-invariant, finite-rank KH-submodule of $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ and $f \in M$, then the orbit $T_G f$ is a bounded subset of M and hence uniformly totally order-bounded, by Proposition 2.5. Hence $\mathrm{FM}_T(\mathrm{X}|\mathrm{Y}) \subseteq \mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$. The second inclusion is trivial, as each uniformly totally order-bounded set is totally order-bounded.

(c) We shall prove the equivalence

$$f \in \mathrm{cl}_{\mathrm{L}^2} \mathrm{AP}_T(\mathrm{X}|\mathrm{Y}) \iff \forall \delta \in \mathbb{R}_{>0}\, \exists E \in \Sigma_{\mathrm{Y}} : \mu_{\mathrm{Y}}(E^c) \leq \delta \wedge \mathbf{1}_E f \in \mathrm{AP}_T(\mathrm{X}|\mathrm{Y}).$$

Then (c) follows as soon as we prove (b) (see below).

The implication "$\Leftarrow$" is straightforward. For the converse, fix a function $f \in \mathrm{cl}_{\mathrm{L}^2} \mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$ and some $\delta \in \mathbb{R}_{>0}$. In the first step we suppose in addition that $f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$. By assumption we find a sequence $(f_n)_n$ in $\mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$ with $f_n \to f$ in L^2. Since $f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$, by Lemma 3.1 we may pass to a subsequence $(h_n)_n$ such that there is a subset $E \in \Sigma_{\mathrm{Y}}$ with $\mu_{\mathrm{Y}}(E^c) < \delta$ and $\mathbf{1}_E h_n \to \mathbf{1}_E f$ in the norm of $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$. Since $\mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$ is a norm-closed submodule of $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$, it follows that $\mathbf{1}_E f \in \mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$ as required.

Finally, for general $f \in \mathrm{cl}_{\mathrm{L}^2} \mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$ we can find $B \in \Sigma_{\mathrm{Y}}$ such that $\mathbf{1}_B f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y})$ and $\mu_{\mathrm{Y}}(B^c) < \delta/2$. (For example, let $B := [\mathbb{E}_{\mathrm{Y}}|f|^2 < N]$ for $N > 0$ large.) Since $\mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$ is an $\mathrm{L}^\infty(\mathrm{Y})$-module, we see that $\mathbf{1}_B f \in \mathrm{cl}_{\mathrm{L}^2} \mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$, and we can apply what we have already proved (with δ replaced by $\delta/2$). This yields a set $C \in \Sigma_{\mathrm{Y}}$ with $\mu_{\mathrm{Y}}(C) < \delta/2$ such that $\mathbf{1}_{B\cap C} f = \mathbf{1}_C(\mathbf{1}_B f) \in \mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$. Taking $E := B \cap C$ then yields what is desired.

(b) By the inclusions in (a) it suffices to establish the implication

$$f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y}),\ T_G f \text{ totally order-bounded} \implies f \in \mathscr{E}(\mathrm{X}|\mathrm{Y}).$$

By [EHK24, Thm. 6.9] we may write

$$\mathrm{L}^2(\mathrm{X}|\mathrm{Y}) = \mathrm{L}^2(\mathrm{X}|\mathrm{Y})_{\mathrm{ds}} \oplus \mathrm{L}^2(\mathrm{X}|\mathrm{Y})_{\mathrm{wm}}$$

as an orthogonal sum of KH-modules, where

$$\mathrm{L}^2(\mathrm{X}|\mathrm{Y})_{\mathrm{ds}} = \mathrm{ocl}\big(\mathrm{FM}_T(\mathrm{X}|\mathrm{Y})\big) \subseteq \mathscr{E}(\mathrm{X}|\mathrm{Y}).$$

Let Q be the orthogonal projection onto the KH-submodule $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})_{\mathrm{wm}}$. Then $QT_t = T_t Q$ for each $t \in G$, and hence $T_G Q f = Q T_G f$ is totally order-bounded whenever $T_G f$ is, by Lemma 2.4(f). Therefore, it remains to show

$$f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y})_{\mathrm{wm}},\ T_G f \text{ totally order-bounded} \implies f = 0.$$

In order to do this, pick a function f satisfying the stated hypotheses. In addition, we may and do suppose that $|f|_{\mathrm{Y}} \leq 1$.

Fix $\varepsilon \in \mathbb{R}_{>0}$. Since $\mathrm{L}^\infty(\mathrm{X})$ is order-dense in $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$, by Lemma 2.4(g) the net

$$F \mapsto \sup_{t\in G} \inf_{g\in F} |T_t f - g|_{\mathrm{Y}}$$

decreases to 0, where F ranges over the finite subsets of $\mathrm{L}^\infty(\mathrm{X})$. Since the $\mathrm{L}^1(\mathrm{Y})$-norm is order-continuous [EFHN15, Sec. 7.2], there is a finite set $F \subseteq \mathrm{L}^\infty(\mathrm{X})$ with

$$\int_{\mathrm{Y}} \sup_{t\in G} \inf_{g\in F} |T_t f - g|_{\mathrm{Y}} \leq \varepsilon.$$

Now for $t \in G$ and $h \in F$ we certainly have

$$|T_t f|_{\mathrm{Y}}^2 \leq |(T_t f|T_t f - h)_{\mathrm{Y}}| + |(T_t f|h)_{\mathrm{Y}}| \leq |T_t f - h|_{\mathrm{Y}} + \sum_{g\in F} |(T_t f|g)_{\mathrm{Y}}|,$$

and taking the infimum over h we arrive at

$$|T_t f|_{\mathrm{Y}}^2 \leq \inf_{g\in F} |T_t f - g|_{\mathrm{Y}} + \sum_{g\in F} |(T_t f|g)_{\mathrm{Y}}| \leq \sup_{s\in G} \inf_{g\in F} |T_s f - g|_{\mathrm{Y}} + \sum_{g\in F} |(T_t f|g)_{\mathrm{Y}}|.$$

Integrating over Y we find, with $n := \#F$ being the cardinality of F, that

$$\begin{aligned} \|f\|_2^2 = \|T_t f\|_2^2 = \int_{\mathrm{Y}} |T_t f|_{\mathrm{Y}}^2 &\leq \varepsilon + \sum_{g\in F} \|(T_t f|g)_{\mathrm{Y}}\|_1 \\ &\leq \varepsilon + n^{1/2} \Big(\sum_{g\in F} \|(T_t f|g)_{\mathrm{Y}}\|_2^2 \Big)^{1/2} \end{aligned}$$

for each $t \in G$. Hence, it suffices to prove

$$\inf_{t\in G} \sum_{g\in F} \|(T_t f|g)_{\mathrm{Y}}\|_2^2 = 0.$$

But this is true by [EHK24, Prop. 8.8], since $F \subseteq \mathrm{L}^\infty(\mathrm{X})$. ∎

As a corollary we obtain the announced characterization of extensions with discrete spectrum.

Corollary 3.7. *Let* X|Y *be an extension of measure-preserving systems. Then the following assertions are equivalent:*

(i) *The extension* X|Y *has discrete spectrum.*
(ii) *The extension* X|Y *is compact, i.e., the space* $\mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$ *is dense in* $\mathrm{L}^2(\mathrm{X})$.
(iii) *The set* $\{f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y}) \mid T_G f \text{ totally order-bounded}\}$ *is dense in* $\mathrm{L}^2(\mathrm{X})$.
(iv) *For every* $f \in \mathrm{L}^2(\mathrm{X})$ *and every* $\delta \in \mathbb{R}_{>0}$ *there exists a measurable set* $E \in \Sigma_{\mathrm{Y}}$ *with* $\mu_{\mathrm{Y}}(E^c) \leq \delta$ *and such that* $\mathbf{1}_E f \in \mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$.

Remark 3.8. By Lemma 3.1 we can reformulate assertion (ii) in Corollary 3.7 as

(ii)′ *The set* $\mathrm{AP}_T(\mathrm{X}|\mathrm{Y})$ *is order-dense in* $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$.

And (iii) can be reformulated as

(iii)′ *The set* $\{f \in \mathrm{L}^2(\mathrm{X}|\mathrm{Y}) \mid T_G f \text{ totally order-bounded}\}$ *is order-dense in* $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$.

Since also (i) can be reformulated entirely in terms of KH-modules, the equivalence of (i), (ii), and (iii) could actually be formulated for arbitrary KH-dynamical systems. However, whereas the implications (i)⇒(ii)⇒(iii) are still valid in this more general situation, we do not know whether this is also true for the implication (ii)⇒(i). Our proof of Corollary 3.7 hinges on the Birkhoff–Alaoglu theorem, which refers to the global L^2-dynamics, a feature not present in the abstract framework of KH-dynamical systems.

4. Relative cyclical compactness. In this section we shall relate the notions of total order-boundedness and *relative cyclical compactness* as defined in [Kus00]. For this we boldly assume the reader to be familiar with the relevant notions.

As before, let E be a lattice-normed space over the Stone algebra $\mathbb{A}$. We let $\mathbb{B}$ be the complete Boolean algebra of idempotents on $\mathbb{A}$ and freely identify $\mathbb{A}$ with $\mathrm{C}(\Omega)$ and $\mathbb{B}$ with $\mathrm{Clop}(\Omega)$, where Ω is the (extremally disconnected) Gelfand space of $\mathbb{A}$.

There is a canonical $\mathbb{B}$-set structure (2) on E defined by

$$[\![\, x \neq y \,]\!] := \mathrm{supp}(|x-y|), \quad [\![\, x = y \,]\!] := [\![\, x \neq y \,]\!]^c \quad (x, y \in E).$$

Here we employ the notion of the *support* of an element of $\mathbb{A}$ as defined in [EHK24, Sect. 2.1]. It was mentioned there that under the identification $\mathbb{A} = \mathrm{C}(\Omega)$ one has $[\![\, x = y \,]\!] = \mathbf{1} - \mathrm{supp}(|x-y|) = \mathbf{1}_{[|x-y|=0]^\circ}$.

In this terminology, Kusraev's axioms for a $\mathbb{B}$-set read $(x, y, z \in E)$:

(1) $[\![\, x = y \,]\!] = [\![\, y = x \,]\!]$,
(2) $[\![\, x = y \,]\!] = \mathbf{1} \Leftrightarrow x = y$,
(3) $[\![\, x = y \,]\!]\,[\![\, y = z \,]\!] \leq [\![\, x = z \,]\!]$.

Observe that one can take $E = \mathbb{A}$ here.

Suppose F, G are $\mathbb{B}$-sets. Then a mapping $f \colon F \to G$ is a $\mathbb{B}$-*set map* (a "non-expanding" map in Kusraev's terminology) if

$$[\![\, x = y \,]\!] \leq [\![\, f(x) = f(y) \,]\!] \quad (x, y \in F).$$

It is easy to see that for $z \in E$ the mapping

$$E \to \mathbb{A}, \qquad x \mapsto |x - z|,$$

is a $\mathbb{B}$-set map.

A family $(p_\alpha)_\alpha$ in $\mathbb{B}$ is a *partition of unity* if

$$p_\alpha \wedge p_\beta = 0 \quad \text{whenever } \alpha \neq \beta \text{ and } \bigvee_\alpha p_\alpha = \mathbf{1}.$$

(2) For the definition of a $\mathbb{B}$-set see [Kus00, A.12]. Note, however, that Kusraev uses the symbol "$[\![\, x = y \,]\!]$" for something different than we do.

Note that we allow $p_\alpha = 0$ in this definition. We recall the following (well-known) result.

Lemma 4.1 (Exhaustion principle/disjointification). *Let $(t_\alpha)_\alpha$ be any family in $\mathbb{B}$ with $\bigvee_\alpha t_\alpha = \mathbf{1}$. Then there is a partition of unity $(p_\alpha)_\alpha$ with $p_\alpha \le t_\alpha$ for each α.*

Proof. Let $E := \{e \in \mathbb{B} \mid \exists \alpha : e \le t_\alpha\}$ and let $S \subseteq E$ be a maximal antichain, i.e. a maximal subset of pairwise disjoint elements. (This exists by a standard application of Zorn's lemma.) By maximality, each upper bound of S is also an upper bound of all the t_α, hence $\bigvee S = \mathbf{1}$.

For each $s \in S$ let $\alpha(s)$ be an index with $s \le t_{\alpha(s)}$. (This is a direct application of the Axiom of Choice.) Now define

$$p_\alpha := \bigvee \{s \mid \alpha(s) = \alpha\}$$

for each index α (with $\bigvee \emptyset = 0$ as usual). Then it is routine to verify that $(p_\alpha)_\alpha$ has the required properties.

Alternative proof (*sketch*). By the well-ordering principle we may suppose that the index set is well-ordered. Define

$$q_\alpha := \bigvee \{t_\beta \mid \beta < \alpha\} \quad \text{and} \quad p_\alpha := t_\alpha \wedge q_\alpha^c$$

for each α. Then, clearly, the p_α are pairwise disjoint. Using the defining property of a well-ordering, one shows that

$$t_\alpha \le \bigvee_{\beta \le \alpha} p_\alpha$$

is true for all α, and hence $\bigvee_\alpha p_\alpha = \mathbf{1}$ as desired.

Third proof (*sketch*). Let

$$\mathcal{M} := \{(s_\alpha) \mid \forall \alpha : \mathbb{B} \ni s_\alpha \le t_\alpha \text{ and } \forall \alpha, \beta : \alpha \ne \beta \Rightarrow s_\alpha \wedge s_\beta = 0\}.$$

The set $\mathcal{M}$ is partially ordered (componentwise). If $(p_\alpha)_\alpha$ is a maximal element of $\mathcal{M}$ (which exists by Zorn's lemma), then it satisfies the requirements. ∎

Remark 4.2. The first step in the (first) proof of Lemma 4.1 is corollary (1) of Kusraev's *exhaustion principle*, see [Kus00, 1.1.6]. Although not stated explicitly, the lemma is applied a couple of times in [Kus00] (e.g. in the proof of Theorem 5.3.6). And the remaining arguments are given in the proof of Theorem 8.1.8.

The second proof is the transfinite version of the usual "disjointification" procedure known from elementary measure theory courses. One can find it, basically, in Sikorski's book [Sik69, Thm. 2.20.2].

Let $(p_\alpha)_\alpha$ be a partition of unity in $\mathbb{B}$, and $(x_\alpha)_\alpha$ a family (over the same index set) of elements in E. An element $x \in E$ is called the *mixing* of $(x_\alpha)_\alpha$

over $(p_\alpha)_\alpha$ if

$$p_\alpha \le [\![x = x_\alpha]\!] \quad \text{for all } \alpha.$$

The mixing element x is uniquely determined by this condition and one writes $x = \sum_\alpha p_\alpha x_\alpha$.

Mixings are preserved under $\mathbb{B}$-set maps. In the particular case of the $\mathbb{B}$-set map $x \mapsto |x - z|$ from E to $\mathbb{A}$, this means

$$x = \sum_\alpha p_\alpha x_\alpha \implies |z - x| = \sum_\alpha p_\alpha |x_\alpha - z|.$$

Of course, a mixing need not exist. A subset M of E is called *mix-complete* if every mixing of elements of M (i.e., every mixing of any family in M over any partition of unity of $\mathbb{B}$) exists in M. And it is called $\mathbb{B}$-*cyclic* (by Kusraev [Kus00, 7.3.3]) or *boundedly mix-complete* (by us) if every mixing of any *bounded* family in M exists in M. If mixings exist in M just for all *finite* partitions of unity, the set M is called *finitely mix-complete*.

Note that in the expression $x = \sum_\alpha p_\alpha x_\alpha$ the sum is to be understood in a purely formal way, just expressing that x is the mixing of the bounded family $(x_\alpha)_\alpha$ over the partition of unity $(p_\alpha)_\alpha$. However, if E is a lattice-normed *module*, one may interpret the sum as the net of partial sums

$$F \mapsto \sum_{\alpha \in F} p_\alpha x_\alpha \quad (F \subseteq \Lambda \text{ finite}),$$

and the identity $x = \sum_\alpha p_\alpha x_\alpha$ as stating that this net is order-convergent to x. In particular, each lattice-normed module is finitely mix-complete, and each Kaplansky–Banach module is boundedly mix-complete (= $\mathbb{B}$-cyclic).

The following characterization links all these notions.

THEOREM 4.3. *For a lattice-normed space E over a Stone algebra $\mathbb{A}$ the following assertions are equivalent:*

(i) *E is finitely mix-complete and order-complete.*
(ii) *E is boundedly mix-complete (= $\mathbb{B}$-cyclic) and norm-complete.*
(iii) *E is a Kaplansky-Banach module over $\mathbb{A}$ (for some (unique) multiplication $\mathbb{A} \times E \to E$).*
(iv) *E is a Banach–Kantorovich space (= decomposable, order-complete, lattice-normed space* [Kus00, 2.2.1]*).*
(v) *E is disjointly decomposable and order-complete.*

Proof. The equivalence of (i) and (ii) is basically [Kus00, Thm. 2.2.3]; the implications (i),(ii)⇒(iii) follow from [Kus00, 2.1.8]; (iii)⇒(iv) holds since each Kaplansky–Banach module is decomposable (in the sense of [Kus00, 2.1.1]); (iv)⇒(v) is trivial.

We prove the implication (v)⇒(i). Suppose that $p \in \mathbb{B}$ and $x_1, x_2 \in E$. For each $j = 1, 2$ we can write $|x_j| = p|x_j| + p^c|x_j|$. Disjoint decomposability yields elements $u_j, v_j \in E$ with $x_j = u_j + v_j$ and $|u_j| = p|x_j|$, $|v_j| = p^c|x_j|$. Define $x := u_1 + v_2$. Then

$$[\![\, x = x_1 \,]\!] \geq p \quad \text{and} \quad [\![\, x = x_2 \,]\!] \geq p^c$$

and hence $x = px_1 + p^c x_2$ in the sense of mixings. ∎

Let again E be a lattice-normed space over $\mathbb{A}$. The *mix-closure* of $M \subseteq E$ in E is

$$\mathrm{mix}(M) = \Big\{ x \in E \;\Big|\; \exists (p_m)_{m\in M} \text{ partition of unity} : x = \sum_{m\in M} p_m m \Big\}.$$

And M is *mix-closed* in E if $\mathrm{mix}(M) = M$. It is straightforward that $\mathrm{mix}(\mathrm{mix}(M)) = \mathrm{mix}(M)$, and hence $\mathrm{mix}(M)$ is mix-closed whatever M is.

We are now approaching the main result of this section. Formulating it requires the notion of a (*relatively*) *cyclically compact* subset, see [Kus00, 8.5.1]. However, we shall not work with the original definition, but with the following equivalent characterization provided by [Kus00, Thm. 8.5.2].

LEMMA 4.4. *Let E be a Kaplansky–Banach module. A mix-complete subset $M \subseteq E$ is relatively cyclically compact in E iff it has the following property: For each $\varepsilon \in \mathbb{R}_{>0}$ there is a countable partition of unity $(q_n)_n$ in $\mathbb{B}$ and a sequence $(F_n)_n$ of finite subsets of E such that for each $x \in M$ there is $z_n \in \mathrm{mix}(F_n)$ with $q_n|x - z_n| \leq \varepsilon\mathbf{1}$.*

Proof. See [Kus00, Thm. 8.5.2] and its proof. The original formulation in [Kus00] requires the finite sets F_n to be subsets of M itself. However, the proof literally yields only $F_n \subseteq E$ and not $F_n \subseteq M$. ∎

PROPOSITION 4.5. *Let E be a Kaplansky–Banach module over a Stone algebra $\mathbb{A}$. Then for $M \subseteq E$ the following assertions are equivalent:*

(i) *M is totally order-bounded.*
(ii) *M is bounded and $\mathrm{mix}(M)$ is relatively cyclically compact.*

Proof. (i)⇒(ii): Since M is totally order-bounded, it is bounded. For each $F \subseteq E$ the mapping $x \mapsto \inf_{y\in F} |x - y|$ is a $\mathbb{B}$-set map and hence respects mixings. Therefore,

$$\sup_{x\in M} \inf_{y\in F} |x - y| = \sup_{z\in \mathrm{mix}(M)} \inf_{y\in F} |z - y|$$

for each finite $F \subseteq E$. It follows that $\mathrm{mix}(M)$ is totally order-bounded. In particular, as E is boundedly mix-complete and $\mathrm{mix}(M)$ is mix-closed in E, $\mathrm{mix}(M)$ is mix-complete. So we may suppose without loss of generality that M is mix-complete.

To show that M is relatively cyclically compact, we apply Lemma 4.4. By hypothesis,

$$\inf_{F\in \mathcal{P}_{\mathrm{fin}}(E)} \sup_{x\in M} \inf_{y\in F} |x - y| = 0.$$

As M is bounded, we may apply Lemma 2.4(h) and find $r \in \mathbb{R}_{>0}$ such that

$$\inf_{F \in \mathcal{P}_{\text{fin}}(E_r)} \sup_{x \in M} \inf_{y \in F} |x - y| = 0,$$

where $E_r := \mathrm{B}_E[0; 2r]$, for short.

Now fix $\varepsilon \in \mathbb{R}_{>0}$. For each $F \in \mathcal{P}_{\text{fin}}(E_r)$ let $h_F := \sup_{x \in M} \inf_{y \in F} |x - y| \in \mathbb{A}_+$. Define $\tilde{p}_F := [h_F \le \varepsilon]^\circ \in \mathbb{B}$. Then $h_F \ge \varepsilon$ on $\tilde{p}_F^c$, and so $\bigvee_F \tilde{p}_F = \mathbf{1}$. By disjointification (Lemma 4.1) there is a partition of unity $(p_F)_{F \in \mathcal{P}_{\text{fin}}(E_r)}$ such that $p_F \le \tilde{p}_F$ for each F, i.e.,

$$p_F \sup_{x \in M} \inf_{y \in F} |x - y| \le \varepsilon \mathbf{1} \quad (F \subseteq E_r \text{ finite}).$$

Let $q_n := \bigvee \{p_F \mid F \subseteq E_r,\ \#F = n\}$ for $n \in \mathbb{N}$. Then $(q_n)_n$ is a partition of unity in $\mathbb{B}$. For fixed $n \in \mathbb{N}$ and $F \subseteq E_r$ with $\#F = n$ we write $F = \{y_1^F, \dots, y_n^F\}$. Define the (bounded!) mixings

$$y_j^n := \sum_{\#F = n} p_F y_j^F + \sum_{\#G \ne n} p_G 0 \quad (j = 1, \dots, n).$$

Now, let $F_n := \{y_1^n, \dots, y_n^n\}$. In this case, if $F \subseteq E_r$ with $\#F = n$, then one has, for $x \in M$,

$$p_F \inf_{y \in F_n} |x - y| = p_F \inf_{j=1,\dots,n} |x - y_j^n| = p_F \inf_{j=1,\dots,n} |x - y_j^F| = p_F \inf_{y \in F} |x - y| \le \varepsilon \mathbf{1}.$$

This yields $q_n \inf_{y \in F_n} |x - y| \le \varepsilon \mathbf{1}$. But then it is easy to find $z_n \in \operatorname{mix}(F_n)$ with $q_n |x - z| \le \varepsilon \mathbf{1}$.

(ii)$\Rightarrow$(i): It suffices to show that $\operatorname{mix}(M)$ is totally order-bounded. Since M is bounded, so is $\operatorname{mix}(M)$. As in the proof of the implication (i)$\Rightarrow$(ii), we conclude that $\operatorname{mix}(M)$ is mix-complete. Hence as above we may suppose without loss of generality that M is mix-complete, and apply Lemma 4.4.

Fix $\varepsilon \in \mathbb{R}_{>0}$ and pick $(q_n)_n$ and $(F_n)_n$ as in the lemma. Now fix $n \in \mathbb{N}$ and write $F_n = \{y_1, \dots, y_d\}$. For each given $x \in M$ we then find a partition of unity $(p_j)_{j=1}^d$ such that the mixing $z_n := \sum_{j=1}^d p_j y_j$ satisfies $q_n |x - z_n| \le \varepsilon \mathbf{1}$. The mapping $z \mapsto |x - z|$ is a $\mathbb{B}$-set map, hence it follows that

$$q_n \inf_{y \in F_n} |x - y| \le \sum_{j=1}^d p_j q_n |x - y_j| = q_n |x - z_n| \le \varepsilon \mathbf{1}.$$

This implies, since M is bounded, that $q_n \sup_{x \in M} \inf_{y \in F_n} |x - y| \le \varepsilon \mathbf{1}$ and hence

$$q_n \inf_{F \in \mathcal{P}_{\text{fin}}(E)} \sup_{x \in M} \inf_{y \in F} |x - y| \le \varepsilon \mathbf{1} \quad \text{for each } n \in \mathbb{N}.$$

Since $\bigvee_n q_n = \mathbf{1}$, we obtain

$$\inf_{F \in \mathcal{P}_{\text{fin}}(E)} \sup_{x \in M} \inf_{y \in F_n} |x - y| \le \varepsilon \mathbf{1},$$

and this implies (i) as $\varepsilon > 0$ was arbitrary. ∎

Appendix A. Background. In this appendix we collect some basic notions from the theory of lattice-normed spaces; see [EHK24, Sec. 1.2], in particular [EHK24, Def. 1.5 and Rem. 1.6(1)].

Let $\mathbb{A}$ be a commutative unital C^*-algebra. A *lattice-normed space* over $\mathbb{A}$ is a $\mathbb{C}$-vector space E together with a mapping $|\cdot|\colon E \to \mathbb{A}_+$ with the following properties:

$$|x| = 0 \Leftrightarrow x = 0, \quad |\lambda x| = |\lambda|\,|x|, \quad |x+y| \le |x| + |y| \quad (x, y \in E,\ \lambda \in \mathbb{C}).$$

Each lattice-normed space carries a natural norm given by $\|x\|_E := \big\||x|\big\|_{\mathbb{A}}$ for $x \in E$.

The algebra $\mathbb{A}$ is a lattice-normed space over itself, with $|\cdot|$ being the usual modulus mapping. The induced norm is the natural one.

A lattice-normed space over $\mathbb{A} = \mathbb{C}\mathbf{1}$ is just an ordinary normed space.

A *lattice-normed module* is a lattice-normed space together with a bilinear mapping

$$\mathbb{A} \times E \to E, \quad (\lambda, x) \mapsto \lambda x,$$

that turns E into an $\mathbb{A}$-module such that

$$|\lambda f| = |\lambda|\,|f| \quad (\lambda \in \mathbb{A},\ f \in E).$$

A *pre-Hilbert lattice-normed module* is a lattice-normed module E together with an $\mathbb{A}$-sesquilinear mapping

$$E \times E \to \mathbb{A}, \quad (x, y) \mapsto (x|y),$$

satisfying $(x|x) = |x|^2$ for all $x \in E$.

A subset $M \subseteq E$ is *order-bounded* if there is $f \in \mathbb{A}_+$ such that $|x| \le f$ for all $x \in M$. Clearly, M is order-bounded iff it is norm-bounded.

A net $(u_i)_i$ in $\mathbb{A}_+$ *decreases to* 0 (symbolically: $u_i \searrow 0$) if $(u_i)_i$ is decreasing with $\inf_i u_i = 0$.

A net $(x_\alpha)_\alpha$ is *order-convergent* to $x \in E$ if there is a net $(u_i)_i$ in $\mathbb{A}_+$ decreasing to 0 and with the property that

$$\forall i\, \exists \alpha_i\, \forall \alpha \ge \alpha_i : |x_\alpha - x| \le u_i.$$

In this case, x is the *order-limit* of $(x_\alpha)_\alpha$ and as such is uniquely determined.

If a net $(x_\alpha)_\alpha$ converges to some $x \in E$, then it order-converges to x.

A mapping $f\colon E \to F$ between lattice-normed spaces E and F is *order-continuous* if it preserves order-convergence, i.e., whenever some (equivalently, some bounded) net $(x_\alpha)_\alpha$ order-converges to some x in E, then Tx_α order-converges to Tx.

The vector space operations and the modulus mapping $|\cdot|\colon E \to \mathbb{A}$ are order-continuous.

An order-continuous linear mapping is norm-continuous, and hence maps bounded sets to bounded sets.

A subset $M \subseteq E$ is *order-closed* if it contains the order-limit of each (equivalently, each bounded) net in M which is order-convergent (in E). The *order-closure* $\mathrm{ocl}(M)$ is the smallest order-closed subset of E containing M, i.e.,

$$\mathrm{ocl}(M) = \bigcap \{N \subseteq E \mid M \subseteq N,\ N \text{ order-closed}\}.$$

One can show that

$$\mathrm{ocl}(M) \subseteq \Big\{x \in E \;\Big|\; \inf_{z\in M} |x - z| = 0\Big\} \tag{A.1}$$

as the right-hand side set is order-closed; see [EHK24, proof of Lemma 1.13].

Lemma A.1. *Let E, F be lattice-normed spaces over $\mathbb{A}$. Then $E \times F$ is a lattice-normed space over $\mathbb{A}$ with respect to the lattice-norm*

$$|(x, y)|_1 := |x| + |y| \quad (x \in E,\ y \in F).$$

Furthermore, for all subsets $M \subseteq E$ and $N \subseteq F$,

$$\mathrm{ocl}(M \times N) = \mathrm{ocl}(M) \times \mathrm{ocl}(N).$$

Proof. It is clear that $E \times F$ is lattice-normed over $\mathbb{A}$ with respect to $|\cdot|_1$. A net $((x_\alpha, y_\alpha))_\alpha$ in $E \times F$ order-converges to $(x, y) \in E \times F$ iff $x_\alpha \to x$ and $y_\alpha \to y$ in order. Hence $\mathrm{ocl}(M) \times \mathrm{ocl}(N)$ is order-closed. If $A \subseteq E \times F$ is order-closed with $M \times N \subseteq A$, then for each $x \in M$ the set $\{y \in F \mid (x, y) \in A\}$ is order-closed and contains N, and hence contains $\mathrm{ocl}(N)$. This yields $M \times \mathrm{ocl}(N) \subseteq Q$. By symmetry, $\mathrm{ocl}(M) \times \mathrm{ocl}(N) \subseteq A$ and this concludes the proof. ∎

A net $(x_\alpha)_\alpha$ is *order-Cauchy* if the net $(|x_\alpha - x_\beta|)_{(\alpha,\beta)}$ order-converges to 0. A lattice-normed space E is *order-complete* if each (equivalently, each bounded) order-Cauchy net in E is order-convergent.

Each order-Cauchy net ist norm-Cauchy, and each order-complete space is norm-complete.

A commutative unital C^*-algebra $\mathbb{A}$ is a *Stone algebra* if it is order-complete (as a lattice-normed space over itself). Equivalently, $\mathbb{A}$ is Dedekind-complete as a Banach lattice; see [EHK24, Sec. 1.3].

By Gelfand's theorem, we may identify $\mathbb{A} \cong \mathrm{C}(\Omega)$, where Ω is a compact Hausdorff space. Then $\mathbb{A}$ is a Stone algebra iff Ω is extremally disconnected.

If E is a lattice-normed space over a Stone algebra $\mathbb{A}$, then a bounded net $(x_\alpha)_\alpha$ in E order-converges to some $x \in E$ iff there is a net $(u_\alpha)_\alpha$ (same index set!) in $\mathbb{A}_+$ decreasing to 0 with $|x - x_\alpha| \le u_\alpha$ for all α; see also [EHK24, Lemma 1.10].

A *Kaplansky–Hilbert module* is an order-complete pre-Hilbert lattice-normed module over a Stone algebra; see [EHK24, Def. 2.1].

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References

[EHK24] N. Edeko, M. Haase, and H. Kreidler, *A decomposition theorem for unitary group representations on Kaplansky–Hilbert modules and the Furstenberg–Zimmer structure theorem*, Anal. Math. 50 (2024), 377–454.

[EFHN15] T. Eisner, B. Farkas, M. Haase, and R. Nagel, *Operator Theoretic Aspects of Ergodic Theory*, Springer, 2015.

[Fur77] H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. Anal. Math. 31 (1977), 204–256.

[Fur81] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, 1981.

[FK78] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for commuting transformations*, J. Anal. Math. 34 (1978), 275–291.

[HK] M. Haase and H. Kreidler, *Tensor products of Kaplansky–Hilbert modules and discrete spectrum in conditional products*, work in progress.

[Jam23] A. Jamneshan, *An uncountable Furstenberg–Zimmer structure theory*, Ergodic Theory Dynam. Systems 43 (2023), 2404–2436.

[Kus00] A. G. Kusraev, *Dominated Operators*, Kluwer, 2000.

[LZ71] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*. North-Holland, 1971.

[Sik69] R. Sikorski, *Boolean Algebras*, 3rd ed., Springer, 1969.

[Tao09] T. Tao, *Poincaré's Legacies, Part I*, Amer. Math. Soc., 2009.

[Zim76a] R. J. Zimmer, *Extensions of ergodic group actions*, Illinois J. Math. 20 (1976), 373–409.

[Zim76b] R. J. Zimmer, *Ergodic actions with generalized discrete spectrum*, Illinois J. Math. 20 (1976), 555–588.

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