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The set-theoretic Kaufmann–Clote question

by

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Abstract. Let M be the set theory obtained from ZF by removing the collection scheme, restricting separation to Δ_0-formulae and adding an axiom asserting that every set is contained in a transitive set. Let Π_n-$\mathsf{Collection}$ denote the restriction of the collection scheme to Π_n-formulae. We prove that for $n \geq 1$, if $\mathcal{M}$ is a model of $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$ and $\mathcal{N}$ is a *topless* Σ_{n+1}-elementary end extension of $\mathcal{M}$ that satisfies Π_{n-1}-$\mathsf{Collection}$, then Π_{n+1}-$\mathsf{Collection}$ holds in $\mathcal{M}$. Here *topless* indicates that $\mathcal{N}$ contains an ordinal that is not in $\mathcal{M}$, but no least ordinal that is not in $\mathcal{M}$. This result is used to show that for $n \geq 1$, the minimum model of $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$ has no Σ_{n+1}-elementary end extension that satisfies Π_{n-1}-$\mathsf{Collection}$, providing a negative answer to the generalisation of a question posed by Kaufmann.

1. Introduction. Paris and Kirby [PK] reveal the connection between fragments of the arithmetic collection scheme and the existence of proper Σ_n-elementary end extensions of models of a weak subsystem of arithmetic. They show that if $\mathcal{M}$ is a countable model of $I\Delta_0$, then for all $n \in \omega$ with $n \geq 1$, $\mathcal{M}$ satisfies $B\Sigma_{n+1}$ if and only if $\mathcal{M}$ has a proper Σ_{n+1}-elementary end extension. Here $I\Delta_0$ is the subsystem of Peano Arithmetic obtained by restricting induction to bounded formulae, and $B\Sigma_n$ is obtained from $I\Delta_0$ by adding the arithmetic collection scheme for Σ_n-formulae.

Let M^- be the set theory obtained from *Kripke–Platek Set Theory* by weakening the foundation available to only *set foundation*, removing Δ_0-$\mathsf{Collection}$ and adding an axiom, *transitive containment*, asserting that every set is contained in a transitive set (see Section 2 below for details). In [Kau, Theorem 1], Kaufmann proves the following set-theoretic analogue of Paris and Kirby's aforementioned result:

THEOREM 1.1 (Kaufmann). *Let $n \in \omega$ with $n \geq 1$. Let $\mathcal{M} = \langle M, \in \rangle$ be a model of M^-. Consider the statements*

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(I) *there exists* $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ *such that* $\mathcal{M} \prec_{e,n+1} \mathcal{N}$ *and* $M \neq N$*;*
(II) $\mathcal{M} \models \Pi_n\text{-}\mathsf{Collection}$.

If $\mathcal{M} \models \mathsf{V} = \mathsf{L}$*, then* (I)$\Rightarrow$(II)*. And, if* M *is countable, then* (II)$\Rightarrow$(I)*.*

In [Kau], it is not assumed that $\mathcal{M}$ satisfies set foundation and transitive containment, and (I)$\Rightarrow$(II) is proved from the weaker assumption that $\mathcal{M}$ is *resolvable*. Theorem 1.1 above also reformulates [Kau, Theorem 1] by making use of the equivalence of collection for Π_n-formulae with collection for Σ_{n+1}-formulae.

A result due to Simpson (see [Kau, Remark 2]) shows that for countable limit ordinals α, $\langle L_\alpha, \in \rangle$ has a topless Σ_1-elementary end extension $\mathcal{N}$ that satisfies Kripke–Platek Set Theory plus $\mathsf{V} = \mathsf{L}$ if and only if $\langle L_\alpha, \in \rangle$ satisfies Π_1-$\mathsf{Collection}$. If $\alpha = \omega_1^{\mathsf{CK}}$, then $\langle L_\alpha, \in \rangle$ is a model of Kripke–Platek Set Theory in which there is a Σ_1-fromula that defines a partial map from ω onto the ordinals and thus does not satisfy Π_1-$\mathsf{Collection}$ [Cho, Example 1.10(b)]. Therefore, $\langle L_{\omega_1^{\mathsf{CK}}}, \in \rangle$ is a transitive model of Kripke–Platek Set Theory with no proper Σ_1-elementary end extension that satisfies Kripke–Platek Set Theory. Using a generalisation of Simpson's result, [M25, §3] shows that there are transitive models of Π_n-$\mathsf{Collection}$ that have no proper Σ_{n+1}-elementary end extensions that satisfy Π_n-$\mathsf{Collection}$ (indeed, these transitive models can even satisfy powerset and full separation). Kaufmann [Kau, p. 102] asks:

Question 1.2. *If* $\langle L_\alpha, \in \rangle$ *has a proper* Σ_2*-elementary extension, does it necessarily have one satisfying* Δ_0-$\mathsf{Collection}$*?*

As shown in Lemma 3.1, each instance of Δ_0-$\mathsf{Collection}$ is Π_3. Using Theorem 1.1, the restriction of Question 1.2 to countable ordinals, α, asks: Does every $\langle L_\alpha, \in \rangle$ that satisfies Π_1-$\mathsf{Collection}$ have a proper Σ_2-elementary elementary end extension that satisfies Δ_0-$\mathsf{Collection}$? Clote [Clo, p. 39] (see also [CK, Problem 33]) asks a generalised version of Kaufmann's Question in the context of arithmetic: For $n \in \omega$ with $n \geq 1$, does every countable model of $B\Sigma_{n+1}$ have a proper Σ_{n+1}-elementary end extension that satisfies $B\Sigma_n$? This suggests the following set-theoretic generalisation of Question 1.2:

Question 1.3 (Set-theoretic Kaufmann–Clote question). *Let* $n \in \omega$ *with* $n \geq 1$*. Does every* $\langle L_\alpha, \in \rangle$ *such that* α *is countable and* $\langle L_\alpha, \in \rangle$ *satisfies* Π_n-$\mathsf{Collection}$ *have a proper* Σ_{n+1}*-elementary end extension that satisfies* Π_{n-1}-$\mathsf{Collection}$*?*

Let M be the set theory obtain by adding the powerset axiom to M^-. This article provides a negative answer to Question 1.3 (and in particular a negative answer to Question 1.2) by showing that if $n \in \omega$ with $n \geq 1$ and α is the least ordinal such that $\langle L_\alpha, \in \rangle$ satisfies $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$, then $\langle L_\alpha, \in \rangle$ has no proper Σ_{n+1}-elementary end extension that satisfies Π_{n-1}-$\mathsf{Collection}$.

This result is obtained from the following strong variant of the aforementioned result of Simpson:

MAIN THEOREM 1.4. *Let $n \in \omega$ with $n \geq 1$. Let $\mathcal{M}$ be a model of* $\mathsf{M} + \Pi_n$-$\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. *If $\mathcal{N}$ is a topless Σ_{n+1}-elementary end extension of $\mathcal{M}$ that satisfies* Π_{n-1}-$\mathsf{Collection}$, *then $\mathcal{M}$ satisfies* Π_{n+1}-$\mathsf{Collection}$.

In contrast, recent work of Sun [Sun] shows that if $n \in \omega$ with $n \geq 1$, then every countable model of $B\Sigma_{n+1}$ has a proper Σ_{n+1}-elementary end extension that satisfies $B\Sigma_n$, providing a positive answer to Clote's question [Clo, p. 39].

Section 3 analyses the theory that Σ_{n+1}-elementarity transfers to a Σ_{n+1}-elementary end extension of a structure that satisfies $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$, showing that additional collection holding in the end extension allows us to transfer more foundation from the structure being extended. Section 4 proves the base case ($n = 1$) of the Main Theorem, and Section 5 proves the Main Theorem for $n \geq 2$.

2. Background. Throughout this paper, $\mathcal{L}$ will denote the language of set theory, first-order logic with equality ($=$) and a binary relation symbol $\in$. As usual, Δ_0 ($= \Sigma_0 = \Pi_0$), Σ_1, Π_1, ... denote the Lévy classes $\mathcal{L}$-formulae. Takahashi [Tak] introduces the class $\Delta_0^{\mathcal{P}}$ extending Δ_0 that consists of all $\mathcal{L}$-formulae whose quantifiers are bounded either by the membership relation ($\in$) or the subset relation ($\subseteq$). Let T be a theory in a language that contains $\mathcal{L}$ and let Γ be a collection of $\mathcal{L}$-formulae. We use Γ^T to denote the collection of $\mathcal{L}$-formulae that T proves are equivalent to a formula in Γ. We use Δ_n^T to denote the collection of formulae that T proves are equivalent to both a Σ_n-formula and a Π_n-formula.

We use $\mathsf{Separation}$ and $\mathsf{Collection}$ to denote the usual set-theoretic separation and collection schemes, respectively, that are used to axiomatise ZF. $\mathsf{Foundation}$ denotes the theorem scheme of ZF asserting that every nonempty definable (with parameters) class has an $\in$-minimal element. If Γ is a collection of formulae, then we use Γ-$\mathsf{Separation}$, Γ-$\mathsf{Collection}$ and Γ-$\mathsf{Foundation}$ to denote the restrictions of the schemes $\mathsf{Separation}$, $\mathsf{Collection}$ and $\mathsf{Foundation}$, respectively, to formulae in Γ. We use $\mathsf{Set\text{-}Foundation}$ to denote the single axiom asserting that every set has an $\in$-minimal element.

Let M^- be the $\mathcal{L}$-theory with axioms:

(Extensionality) $\forall x \forall y (x = y \Leftrightarrow \forall z (z \in x \Leftrightarrow z \in y))$;

(Pair) $\forall x \forall y \exists z (x \in z \wedge y \in z \wedge (\forall w \in z)(w = x \vee w = y))$;

(Union) $\forall x \exists y ((\forall z \in x)(\forall w \in z)(w \in y) \wedge (\forall w \in y)(\exists z \in x)(w \in z))$;

(Emptyset) $\exists x (\forall y \in x)(y \neq y)$;

(Infinity) $\exists x(\emptyset \in x \wedge (\forall y \in x)(\exists z \in x)$
$((\forall w \in z)(w = y \vee w \in y) \wedge (\forall w \in y)(w \in z) \wedge y \in z));$

(TCo) $\forall x \exists y((\forall z \in x)(z \in y) \wedge (\forall z \in y)(\forall w \in z)(w \in y));$

(Set-Foundation) $\forall x((\exists w \in x)(w = w) \Rightarrow (\exists y \in x)(\forall z \in y)(z \notin x));$

(Δ_0-Separation) for all Δ_0-formulae $\phi(x, \vec{z})$,

$$\forall \vec{z} \forall w \exists y((\forall x \in y)(x \in w \wedge \phi(x, \vec{z})) \wedge (\forall x \in w)(\phi(x, \vec{z}) \Rightarrow x \in y)).$$

This makes it clear that M^- is axiomatised by a collection of Π_2-sentences. The theory M is obtained from M^- by adding the Π_3-sentence

(Powerset) $\forall x \exists y \forall z(z \in y \Leftrightarrow z \subseteq x)$.

Kripke–Platek Set Theory (KP) is obtained from M^- by removing Infinity and adding Δ_0-Collection and Π_1-Foundation. The theory KPI is obtained from KP by adding Infinity. This differs from [Bar] and [Fri] where Kripke–Platek Set Theory is defined to include Foundation instead of the more restrictive Π_1-Foundation. It should also be noted that TCo is not usually included as an axiom of either KP or KPI. However, as is shown in [Mat01, Proposition 1.20], TCo is a consequence of the other axioms of KP making its addition here superfluous.

The presence of TCo and Set-Foundation in M^- means that for any collection Γ of formulae, Γ-Separation implies Γ-Foundation. The axiom TCo also facilitates the equivalence between Foundation and set induction. In the theory M^-, Γ-Foundation is equivalent to set induction for all formulae in $\neg\Gamma = \{\neg\phi \mid \phi \in \Gamma\}$. It is well known (by generalising the proof of [Bar, Theorem I.4.4], for example) that, over the theory M^-, Π_n-Collection is equivalent to Σ_{n+1}-Collection. [FLW, Theorem 4.13] shows that the theory $\mathsf{M}^- + \Pi_n$-Collection proves the following separation scheme:

(Δ_{n+1}-Separation) for all Σ_{n+1}-formulae $\phi(x, \vec{z})$, and for all Π_{n+1}-formulae $\psi(x, \vec{z})$,

$$\forall \vec{z}(\forall v(\phi(v, \vec{z}) \Leftrightarrow \psi(v, \vec{z})) \Rightarrow \forall w \exists y \forall x(x \in y \Leftrightarrow x \in w \wedge \phi(x, \vec{z}))).$$

Another important consequence of Collection is that the levels of Lévy hierarchy are essentially closed under bounded quantification. In particular, [FLW, Proposition 2.4] notes that if T is an $\mathcal{L}$-theory that proves $\mathsf{M}^- + \Pi_n$-Collection, then both Σ_{n+1}^T and Π_{n+1}^T are closed under bounded quantification.

We will use $\mathcal{M}, \mathcal{N}, \ldots$ to denote structures. If $\mathcal{M}$ is an $\mathcal{L}$-structure, then we will use M to denote the underlying set of $\mathcal{M}$ and $\in^\mathcal{M}$ to denote the interpretation of $\in$ in $\mathcal{M}$, i.e. $\mathcal{M} = \langle M, \in^\mathcal{M} \rangle$. Let $\mathcal{M} = \langle M, \in^\mathcal{M} \rangle$ and $\mathcal{N} = \langle N, \in^\mathcal{N} \rangle$.

- We say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$, and, abusing notation, write $\mathcal{M} \subseteq \mathcal{N}$, if $M \subseteq N$ and $\in^\mathcal{N}$ agrees with $\in^\mathcal{M}$ on $M \times M$.

- We say that $\mathcal{M}$ is an (Σ_n-)*elementary substructure* of $\mathcal{N}$, and write $\mathcal{M} \prec \mathcal{N}$ ($\mathcal{M} \prec_n \mathcal{N}$, respectively) if $\mathcal{M} \subseteq \mathcal{N}$ and for all $\mathcal{L}$-formulae (Σ_n-formulae, respectively) $\phi(\vec{x})$, and for all $\vec{a} \in M$, $\mathcal{M} \models \phi(\vec{a})$ if and only if $\mathcal{N} \models \phi(\vec{a})$.

Suppose that $\mathcal{M}$ and $\mathcal{N}$ satisfy M^-.

- We say $\mathcal{N}$ is an *end extension* of $\mathcal{M}$, and write $\mathcal{M} \subseteq_e \mathcal{N}$, if $\mathcal{M} \subseteq \mathcal{N}$ and for all $y \in M$ and all $x \in N$, if $\mathcal{N} \models x \in y$, then $x \in M$.
- We say that $\mathcal{N}$ is a *topless end extension* of $\mathcal{M}$, and write $\mathcal{M} \subseteq_{\text{topless}} \mathcal{N}$, if $\mathcal{M} \subseteq_e \mathcal{N}$ and $\mathcal{N}$ contains an ordinal that is not in $\mathcal{M}$, but no least ordinal that is not in $\mathcal{M}$.
- The structure $\mathcal{N}$ is a (*topless*) *powerset-preserving end extension* of $\mathcal{M}$, written $\mathcal{M} \subseteq_e^{\mathcal{P}} \mathcal{N}$ ($\mathcal{M} \subseteq_{\text{topless}}^{\mathcal{P}} \mathcal{N}$, respectively), if $\mathcal{M} \subseteq_e \mathcal{N}$ ($\mathcal{M} \subseteq_{\text{topless}} \mathcal{N}$, respectively) and for $y \in M$ and for all $x \in N$, if $\mathcal{N} \models (x \subseteq y)$, then $x \in M$.
- We call $\mathcal{N}$ a (*topless*) Σ_n-*elementary end extension* of $\mathcal{M}$, and write $\mathcal{M} \prec_{e,n} \mathcal{N}$ ($\mathcal{M} \prec_{\text{topless},n} \mathcal{N}$, respectively), if $\mathcal{M} \subseteq_e \mathcal{N}$ ($\mathcal{M} \subseteq_{\text{topless}} \mathcal{N}$, respectively) and $\mathcal{M} \prec_n \mathcal{N}$.

Note that Δ_0-properties of points in an $\mathcal{L}$-structure $\mathcal{M}$ are preserved in every end extension, and $\Delta_0^{\mathcal{P}}$-properties of points in an $\mathcal{L}$-structure $\mathcal{M}$ are preserved in every powerset-preserving end extension.

Both KP (see [FLW, Lemma 4.1]) and M (see [Mat01, Proposition 3.10]) are capable of defining a ternary relation $\models(u, m, x)$ that expresses that the formula with Gödel code m and parameters given by x holds in the structure $\langle u, \in\rangle$. We write $\langle u, \in\rangle \models \phi(x_1, \dots, x_k)$ instead of $\models(u, m, x)$, where $m = \ulcorner\phi(v_1, \dots, v_k)\urcorner$ and $x = \langle x_1, \dots, x_k\rangle$. As noted in [FLW, Lemma 4.1], the formula $\models(u, m, x)$ is Δ_1^{KP} ([1]). It is a consequence of [Mat01, Proposition 3.10], which makes no use of the axiom TCo, that the theory M proves that for all u, the collection $\{\langle m, x\rangle \in \omega \times u^{<\omega} \mid \models(u, x, m)\}$ is a set. For $\Gamma = \Sigma_n$ or Π_n, the theory KP defines a formula $\mathsf{Sat}_\Gamma(m, x)$ asserting that m is the Gödel code of a Γ-formula $\phi(v_1, \dots, v_k)$, $x = \langle x_1, \dots, x_k\rangle$ and $\phi(x_1, \dots, x_k)$ holds. The formula $\mathsf{Sat}_{\Delta_0}(n, x)$ is Δ_1^{KP}, and if $\Gamma = \Sigma_n$ or Π_n and $n \geq 1$, then $\mathsf{Sat}_\Gamma(m, x)$ is Γ^{KP}. We use $X \prec_n \mathbb{V}$ to abbreviate the formula (with parameter ω)

$$(\forall m \in \omega)(\forall x \in X^{<\omega})(\mathsf{Sat}_{\Sigma_n}(m, x) \Rightarrow \langle X, \in\rangle \models \mathsf{Sat}_{\Sigma_n}(m, x)),$$

asserting that $\langle X, \in\rangle$ is a substructure of the universe that preserves Σ_n-properties of tuples in X. The formula $X \prec_n \mathbb{V}$ is Π_n^{KPI}.

([1]) In [Mat06], Mathias shows that the formula $\models(u, m, x)$ is Δ_1 in several systems that are strictly weaker than KP.

We use Ord to denote to class of ordinals. As noted in [FLW, §4], the theory KP is capable of defining Gödel's constructible universe:

$$L_0 = \emptyset, \qquad L_\alpha = \bigcup_{\beta \in \alpha} L_\beta \quad \text{if } \alpha \text{ is a limit ordinal,}$$

$$L_{\alpha+1} = L_\alpha \cup \mathsf{Def}(L_\alpha), \quad \text{and} \quad L = \bigcup_{\alpha \in \mathsf{Ord}} L_\alpha,$$

where $\mathsf{Def}(X)$ is the set of subsets of X that are the extension of a first-order formula (with parameter) according to the structure $\langle X, \in \rangle$. In particular, the function $\alpha \mapsto L_\alpha$ is provably total in KP, and Δ_1^{KP}. The following axiom asserts that every set is a member of the constructible universe and can be expressed as a Π_2-sentence in KP:

$$(\mathsf{V} = \mathsf{L})\ \forall x \exists \alpha (x \in L_\alpha).$$

DEFINITION 2.1. Let T be an $\mathcal{L}$-theory. A transitive set M is the *minimum model* of T if $\langle M, \in \rangle \models T$ and for all transitive N such that $\langle N, \in \rangle \models T$, $M \subseteq N$.

Gostanian [Gos] shows that many natural subsystems of ZF have minimal models. In particular, the following is a consequence of [Gos, Theorem 1.6(b)]:

THEOREM 2.2. *Let $n \in \omega$ with $n \geq 1$. The theory* $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$ *has a minimum model. Moreover, the minimum model of* $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$ *is of the form L_α, where $\alpha \in \omega_1$.*

In the presence of enough $\mathsf{Foundation}$, the theory $\mathsf{M} + \Pi_{n+1}\mathsf{Collection}$ is capable of building a transitive model of $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$. Theorem 2.3 is a weakening of [M19, Theorem 4.4] obtained by observing that Π_n-$\mathsf{Foundation}$ implies that induction on the natural numbers holds for all Σ_n-formulae.

THEOREM 2.3. *Let $n \in \omega$ with $n \geq 1$. The theory*

$$\mathsf{M} + \Pi_{n+1}\text{-}\mathsf{Collection} + \Pi_{n+2}\text{-}\mathsf{Foundation}$$

proves that there exists a transitive model of $\mathsf{M} + \mathsf{Separation} + \Pi_n$-$\mathsf{Collection}$.

Theorems 2.2 and 2.3 immediately imply that the minimum models of $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$ and $\mathsf{M} + \Pi_{n+1}$-$\mathsf{Collection}$ do not coincide.

COROLLARY 2.4. *Let $n \in \omega$ with $n \geq 1$. Let $\alpha \in \omega_1$ be such that L_α is the minimum model of* $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$. *Then $\langle L_\alpha, \in \rangle$ does not satisfy* Π_{n+1}-$\mathsf{Collection}$.

[M19, §4] shows that, for $n \geq 1$, $\mathsf{M} + \Pi_n$-$\mathsf{Collection} + \Pi_{n+1}$-$\mathsf{Foundation}$ proves the consistency of $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$. Combined with the fact that, for all $n \geq 1$, $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$ proves that $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$ holds in its own constructible universe, this yields:

THEOREM 2.5. *Let $n \in \omega$ with $n \geq 1$. The theory* $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$ *does not prove* $\Pi_{n+1}\text{-}\mathsf{Foundation}$.

The set theory MOST, studied in [Mat01], is obtained from M by adding $\Delta_0\text{-}\mathsf{Collection}$, $\Sigma_1\text{-}\mathsf{Separation}$ and the Axiom of Choice (AC). In particular, MOST is a subtheory of $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. A key feature of MOST, and the source of its name, is the fact that it proves *Mostowski's Collapsing Lemma*: Every well-founded extensional relation is isomorphic to a unique transitive set by a unique isomorphism known as the *transitive collapse*. This, combined with the absoluteness of the levels of the constructible universe between transitive models and the first-order expressibility of being constructible, allows us to apply *Gödel's Condensation Lemma* in MOST. The following can be obtained as a consequence of [Mat01, Lemma 4.18] or by observing that the proof of [Dev, Theorem II.5.2] can be carried out in MOST.

LEMMA 2.6. *The theory* MOST *proves that for all limit ordinals δ, if $\langle X, \in \rangle \prec \langle L_\delta, \in \rangle$, then there exists $\alpha \leq \delta$ such that the transitive collapse of $\langle X, \in \rangle$ is isomorphic to $\langle L_\alpha, \in \rangle$.*

A key ingredient used in the results proved in Sections 4 and 5 is a generalisation of a result of Simpson (see [Kau, Remark 2]) that is proved in [M25, Theorem 3.1]:

THEOREM 2.7. *Let $n \in \omega$ with $n \geq 1$. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be such that* $\mathcal{M} \models \mathsf{KP} + \mathsf{V} = \mathsf{L}$. *Suppose that $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is such that $\mathcal{M} \prec_{\mathrm{topless}, n} \mathcal{N}$ and* $\mathcal{N} \models \mathsf{KP}$. *If* $\mathcal{N} \models \Pi_{n-1}\text{-}\mathsf{Collection}$ *or* $\mathcal{N} \models \Pi_{n+2}\text{-}\mathsf{Foundation}$, *then* $\mathcal{M} \models \Pi_n\text{-}\mathsf{Collection}$.

Theorem 2.7 is used in [M25, §3] to show that for $n \geq 1$, the minimum model of $\mathsf{M} + \mathsf{Separation} + \Pi_n\text{-}\mathsf{Collection}$ has no proper Σ_{n+1}-elementary end extension that satisfies either $\mathsf{KP} + \Pi_n\text{-}\mathsf{Collection}$ or $\mathsf{KP} + \Pi_{n+3}\text{-}\mathsf{Foundation}$.

3. The theory transferred to partially-elementary end extensions. In this section we show that for all $n \geq 2$, if $\mathcal{N}$ is a Σ_{n+1}-elementary end extension of $\mathcal{M}$ that satisfies $\mathsf{M}^- + \Pi_n\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$, then $\mathcal{N}$ satisfies $\mathsf{M}^- + \Pi_{n-2}\text{-}\mathsf{Collection} + \Sigma_{n-1}\text{-}\mathsf{Separation} + \mathsf{V} = \mathsf{L}$. Moreover, if in addition $\mathcal{M}$ satisfies $\mathsf{Powerset}$ and $\mathcal{N}$ satisfies $\Pi_{n-1}\text{-}\mathsf{Collection}$, then $\mathcal{N}$ satisfies $\Sigma_n\text{-}\mathsf{Separation}$. We also show that if $\mathcal{N}$ is a Σ_2-elementary end extension of $\mathcal{M}$ that satisfies $\mathsf{M}^- + \Pi_1\text{-}\mathsf{Collection}$ and $\mathcal{N}$ satisfies $\Delta_0\text{-}\mathsf{Collection}$, then $\mathcal{N}$ satisfies $\Sigma_1 \cup \Pi_1\text{-}\mathsf{Foundation}$.

LEMMA 3.1. *Let $n \in \omega$. The theory* $\mathsf{M}^- + \Pi_n\text{-}\mathsf{Collection}$ *is axiomatised by a collection of Π_{n+3}-sentences.*

Proof. We prove this by induction on n. We have already observed that M^- is axiomatised by a collection of Π_2-sentences. Every instance of Δ_0-$\mathsf{Collection}$ is of the form

$$\forall \vec{z} \forall w((\forall x \in w)\exists y \phi(x, y, \vec{z}) \Rightarrow \exists c(\forall x \in w)(\exists y \in c)\phi(x, y, \vec{z})),$$

where $\phi(x, y, \vec{z})$ is Δ_0. Each of these instances is equivalent to a Π_3-sentence. This shows that the lemma holds when $n = 0$. Now, let $k \geq 0$ and assume that Γ is a collection of Π_{k+3}-sentences that axiomatise $\mathsf{M}^- + \Pi_k$-$\mathsf{Collection}$. Let $\phi(x, y, \vec{z})$ be a Π_{k+1}-formula. The instance of Π_{k+1}-$\mathsf{Collection}$ for the formula $\phi(x, y, \vec{z})$ is

$$\forall \vec{z} \forall w((\forall x \in w)\exists y \phi(x, y, \vec{z}) \Rightarrow \exists c(\forall x \in w)(\exists y \in c)\phi(x, y, \vec{z})). \tag{3.1}$$

In the theory $\mathsf{M}^- + \Pi_k$-$\mathsf{Collection}$, (3.1) is equivalent to a Π_{k+4}-sentence. Therefore, there is a collection Λ of Π_{k+4}-sentences such that $\Gamma \cup \Lambda$ axiomatises $\mathsf{M}^- + \Pi_{k+1}$-$\mathsf{Collection}$. ∎

LEMMA 3.2. *Let $n \in \omega$. For each instance σ of Σ_{n+1}-$\mathsf{Separation}$, there exists a Π_{n+3}-sentence γ such that $\mathsf{M}^- + \Pi_n$-$\mathsf{Collection} \vdash (\sigma \Leftrightarrow \gamma)$.*

Proof. Let σ be an instance of Σ_{n+1}-$\mathsf{Separation}$. Therefore, σ is

$$\forall \vec{z} \forall w \exists y((\forall x \in w)(\phi(x, \vec{z}) \Rightarrow x \in y) \land (\forall x \in y)(x \in w \land \phi(x, \vec{z}))),$$

where $\phi(x, \vec{z})$ is a Σ_{n+1}-formula. In the theory $\mathsf{M}^- + \Pi_n$-$\mathsf{Collection}$, this sentence is equivalent to a Π_{n+3}-sentence. ∎

LEMMA 3.3. *Let $n \in \omega$. There exists a collection Γ of Π_{n+2}-sentences such that*

$$\mathsf{M}^- + \Pi_n\text{-}\mathsf{Collection} + \Gamma \vdash \Pi_{n+1} \cup \Sigma_{n+1}\text{-}\mathsf{Foundation}$$

and

$$\mathsf{M}^- + \Pi_n\text{-}\mathsf{Collection} + \Pi_{n+1} \cup \Sigma_{n+1}\text{-}\mathsf{Foundation} \vdash \Gamma.$$

Proof. $\Sigma_{n+1} \cup \Pi_{n+1}$-$\mathsf{Foundation}$ is the scheme: for all $\Sigma_{n+1} \cup \Pi_{n+1}$-formulae $\phi(x, \vec{z})$,

$$\forall \vec{z}\, (\exists x \phi(x, \vec{z}) \Rightarrow \exists y(\phi(y, \vec{z}) \land (\forall w \in y)(\neg \phi(w, \vec{z})))).$$

In the theory M^-, $\Sigma_{n+1} \cup \Pi_{n+1}$-$\mathsf{Foundation}$ is equivalent to the scheme: for all $\Sigma_{n+1} \cup \Pi_{n+1}$-formulae $\phi(x, \vec{z})$,

$$\forall \vec{z} \forall u((\exists x \in u)\phi(x, \vec{z}) \Rightarrow (\exists y \in u)(\phi(y, \vec{z}) \land (\forall w \in y)(\neg \phi(w, \vec{z})))). \tag{3.2}$$

In the theory $\mathsf{M}^- + \Pi_n$-$\mathsf{Collection}$, each instance (3.2) is equivalent to a Π_{n+2}-sentence. ∎

THEOREM 3.4. *Let $n \in \omega$ with $n \geq 2$. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{M} + \Pi_n$-$\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. If $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is such that $\mathcal{M} \prec_{e, n+1} \mathcal{N}$, then*

$$\mathcal{N} \models \mathsf{M} + \Pi_{n-2}\text{-}\mathsf{Collection} + \Sigma_{n-1}\text{-}\mathsf{Separation} + \mathsf{V} = \mathsf{L}.$$

Proof. Let $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be such that $\mathcal{M} \prec_{e,n+1} \mathcal{N}$. Since $n \geq 2$ and $\mathcal{M} \prec_{e,n+1} \mathcal{N}$, $\mathcal{N} \models \mathsf{M} + \mathsf{V} = \mathsf{L}$. So, by Lemma 3.1,

$$\mathcal{N} \models \mathsf{M} + \Pi_{n-2}\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}.$$

Therefore, Lemma 3.2 yields the conclusion. ∎

THEOREM 3.5. *Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be such that*

$$\mathcal{M} \models \mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}.$$

If $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is such that $\mathcal{M} \prec_{e,2} \mathcal{N}$ and $\mathcal{N} \models \Delta_0$-$\mathsf{Collection}$, then

$$\mathcal{N} \models \mathsf{KPI} + \Sigma_1\text{-}\mathsf{Foundation} + \mathsf{V} = \mathsf{L}.$$

Proof. Let $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be such that $\mathcal{M} \prec_{e,2} \mathcal{N}$ and $\mathcal{N} \models \Delta_0$-$\mathsf{Collection}$. Since $\mathcal{M} \prec_{e,2} \mathcal{N}$, $\mathcal{N} \models \mathsf{M}^- + \Delta_0\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. Therefore, Lemma 3.3 implies the conclusion. ∎

The presence of $\mathsf{Powerset}$ in M allows Σ_{n+1}-$\mathsf{Separation}$ to be derived from Π_n-$\mathsf{Collection}$ together with Π_n-$\mathsf{Foundation}$. The following is [M25, Theorem 5.9]:

THEOREM 3.6. *Let $n \in \omega$ with $n \geq 1$. The theory $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \Pi_{n+1}\text{-}\mathsf{Foundation}$ proves Σ_{n+1}-$\mathsf{Separation}$.*

This means that if we have enough partial-elementarity to transfer $\mathsf{Powerset}$ to the end extension and we know that the end extension satisfies enough collection, then more separation is also transferred to the end extension.

THEOREM 3.7. *Let $n \in \omega$ with $n \geq 2$. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. If $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is such that $\mathcal{M} \prec_{e,n+1} \mathcal{N}$ and $\mathcal{N} \models \Pi_{n-1}$-$\mathsf{Collection}$, then*

$$\mathcal{N} \models \mathsf{M} + \Pi_{n-1}\text{-}\mathsf{Collection} + \Sigma_n\text{-}\mathsf{Separation} + \mathsf{V} = \mathsf{L}.$$

Proof. Let $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ with $\mathcal{M} \prec_{e,n+1} \mathcal{N}$ and $\mathcal{N} \models \Pi_{n-1}$-$\mathsf{Collection}$. Since $n \geq 2$ and $\mathcal{M} \prec_{e,n+1} \mathcal{N}$, $\mathcal{N} \models \mathsf{M} + \Pi_{n-1}\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. So, by Lemma 3.3, $\mathcal{N} \models \mathsf{M} + \Pi_{n-1}\text{-}\mathsf{Collection} + \Pi_n\text{-}\mathsf{Foundation} + \mathsf{V} = \mathsf{L}$. Therefore, since $n \geq 2$, Theorem 3.6 yields the conclusion. ∎

4. Σ_2-elementary end extensions that satisfy Δ_0-Collection. In this section we show that if $\mathcal{M}$ is a model of $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$ that has a topless Σ_2-elementary end extension $\mathcal{N}$ that satisfies Δ_0-$\mathsf{Collection}$, then $\mathcal{M}$ must satisfy Π_2-$\mathsf{Collection}$. This result is used to provide a negative answer to Kaufmann's Question (Question 1.2) by showing that the minimum model of $\mathsf{M} + \Pi_1$-$\mathsf{Collection}$ has no proper Σ_2-elementary end extension that satisfies Δ_0-$\mathsf{Collection}$.

The following is a slight strengthening of [M25, Lemma 5.7] that is obtained by observing that the proof of [M25, Lemma 5.7] makes no use of the assumption that the end extension satisfies M:

LEMMA 4.1. *Let $\mathcal{M} = \langle M, \in \rangle$ and $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be such that $\mathcal{M} \models \mathsf{M}$. If $\mathcal{M} \prec_{e,1} \mathcal{N}$, then $\mathcal{M} \subseteq_e^{\mathcal{P}} \mathcal{N}$.*

Note that a proper Σ_2-elementary end extension of a model of $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection}$ that satisfies $\Delta_0\text{-}\mathsf{Collection}$ need not satisfy $\mathsf{Powerset}$ or the assertion that there is no greatest cardinal. Nevertheless, an overspill argument allows us to see that such an end extension must contain a new cardinal.

LEMMA 4.2. *Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. If $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is such that $\mathcal{M} \prec_{e,2} \mathcal{N}$, $M \neq N$ and $\mathcal{N} \models \Delta_0\text{-}\mathsf{Collection}$, then there exists $\kappa \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ such that*

$$\mathcal{N} \models (\kappa \textit{ is a cardinal}).$$

Proof. Let $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be such that $\mathcal{M} \prec_{e,2} \mathcal{N}$, $M \neq N$ and $\mathcal{N} \models \Delta_0\text{-}\mathsf{Collection}$. By Theorem 3.5,

$$\mathcal{N} \models \mathsf{KPI} + \Sigma_1\text{-}\mathsf{Foundation} + \mathsf{V} = \mathsf{L}.$$

By Lemma 4.1, $\mathcal{M} \subseteq_e^{\mathcal{P}} \mathcal{N}$. Using the fact that $M \neq N$, let $\beta \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$. Using $\Sigma_1\text{-}\mathsf{Foudation}$ in $\mathcal{N}$, let κ be the $\in$-least element of the class

$$A = \{\gamma \in \mathsf{Ord}^{\mathcal{N}} \mid \exists f (f : \gamma \to \beta \text{ is a bijection})\}.$$

Then $\mathcal{N} \models (\kappa \text{ is a cardinal})$. Now, suppose that $\kappa \in M$. But then, since $\mathcal{M} \subseteq_e^{\mathcal{P}} \mathcal{N}$, the well-ordering of κ with order-type β is also in M. Since $\mathcal{M} \models \mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$, this implies that $\beta \in M$, which is a contradiction. Therefore, $\kappa \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$. ∎

In the theory $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$, if $\kappa > \omega$ is cardinal, then $\langle L_\kappa, \in \rangle$ satisfies $\mathsf{KPI} + \mathsf{Separation}$, the universe is a powerset-preserving end extension of L_κ and the universe is a Σ_1-elementary end extension of L_κ. Here we verify that these properties of L_κ, where $\kappa > \omega$ is a cardinal, can be proved in the theory $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$.

LEMMA 4.3. *The theory $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$ proves that if $\kappa > \omega$ is a cardinal, $x \in L_\kappa$ and $y \subseteq x$, then $y \in L_\kappa$.*

Proof. Work in the theory $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. Let $\kappa > \omega$ be a cardinal. Let $x \in L_\kappa$ and let $y \subseteq x$. Let $\gamma \in \kappa$ be such that $x \in L_\gamma$. Note that $|L_\gamma| < \kappa$. Let λ be a limit ordinal such that $y, L_\gamma \in L_\lambda$. Taking a Skolem hull of $\langle L_\lambda, \in \rangle$, let $\langle X, \in \rangle \prec \langle L_\lambda, \in \rangle$ be such that $L_\gamma \cup \{y\} \subseteq X$ and $|X| < \kappa$. Taking the transitive collapse of $\langle X, \in \rangle$ and using Gödel's Condensation Lemma, we obtain $L_\beta \in L_\kappa$ such that $y \in L_\beta$. Therefore $y \in L_\kappa$. ∎

LEMMA 4.4. *The theory $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$ proves that if $\kappa > \omega$ is a cardinal, then $\langle L_\kappa, \in \rangle \models \mathsf{KPI} + \mathsf{Separation} + \mathsf{V} = \mathsf{L}$.*

Proof. Work in the theory $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. Let $\kappa > \omega$ be a cardinal. It follows from Lemma 4.3 and the fact that L_κ is a set that

$\langle L_\kappa, \in\rangle \models \mathsf{M}^- + \mathsf{Separation}$. To verify Δ_0-$\mathsf{Collection}$, let $\phi(x, y, \vec{z})$ be a Δ_0-formula. Let $\vec{a}, b \in L_\kappa$ be such that

$$\langle L_\kappa, \in\rangle \models (\forall x \in b)\exists y \phi(x, y, \vec{a}).$$

Let $\gamma \in \kappa$ be such that $\vec{a}, b \in L_\gamma$. Note that $|L_\gamma| < \kappa$. Let $\lambda > \kappa$ be a limit ordinal. Since $\phi(x, y, \vec{z})$ is Δ_0 and $L_\kappa \in L_\lambda$,

$$\langle L_\lambda, \in\rangle \models \exists c(\forall x \in b)(\exists y \in c)\phi(x, y, \vec{a}).$$

Taking the Skolem hull of $\langle L_\lambda, \in\rangle$, let $\langle X, \in\rangle \prec \langle L_\lambda, \in\rangle$ be such that $L_\gamma \subseteq X$ and $|X| < \kappa$. Taking the transitive collapse of $\langle X, \in\rangle$, we obtain $L_\beta \in L_\kappa$ and an isomorphism between $\langle X, \in\rangle$ and $\langle L_\beta, \in\rangle$ that is the identity on $\vec{a}, b$. Therefore,

$$\langle L_\beta, \in\rangle \models \exists c(\forall x \in b)(\exists y \in c)\phi(x, y, \vec{a}).$$

And so, since $\phi(x, y, \vec{a})$ is Δ_0,

$$\langle L_\kappa, \in\rangle \models \exists c(\forall x \in b)(\exists y \in c)\phi(x, y, \vec{a}).$$

This shows that $\langle L_\kappa, \in\rangle$ satisfies Δ_0-$\mathsf{Collection}$, and so

$$\langle L_\kappa, \in\rangle \models \mathsf{KPI} + \mathsf{Separation} + \mathsf{V} = \mathsf{L}. \blacksquare$$

LEMMA 4.5. *The theory* $\mathsf{M} + \Pi_1$-$\mathsf{Collection} + \mathsf{V} = \mathsf{L}$ *proves that if* $\kappa > \omega$ *is a cardinal, then for* $k \in \omega$ *and for all* $x \in L_\kappa$, *if* $\mathsf{Sat}_{\Sigma_1}(k, x)$, *then*

$$\langle L_\kappa, \in\rangle \models \mathsf{Sat}_{\Sigma_1}(k, x).$$

Proof. Work in the theory $\mathsf{M} + \Pi_1$-$\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. Let $\kappa > \omega$ be a cardinal. Let $\phi(k, x, y)$ be a Δ_0-formula such that

$$\mathsf{KPI} \vdash \forall k \forall x(\mathsf{Sat}_{\Sigma_1}(k, x) \leftrightarrow \exists y \phi(k, x, y)).$$

Let $k \in \omega$ and let $x \in L_\kappa$. Suppose that $\mathsf{Sat}_{\Sigma_1}(k, x)$ holds. Let y be such that $\phi(k, x, y)$ holds. Let $\lambda > \kappa$ be a limit ordinal such that $y \in L_\lambda$. So, $\langle L_\lambda, \in\rangle \models \exists y \phi(k, x, y)$. Let $\gamma \in \kappa$ be such that $\omega, x \in L_\gamma$. Note that $|L_\gamma| < \kappa$. Taking the Skolem hull of $\langle L_\lambda, \in\rangle$, let $\langle X, \in\rangle \prec \langle L_\lambda, \in\rangle$ be such that $L_\gamma \subseteq X$ and $|X| < \kappa$. Taking the collapse $\langle X, \in\rangle$, we obtain $L_\beta \in L_\kappa$ and an isomorphism between $\langle X, \in\rangle$ and $\langle L_\beta, \in\rangle$ that is the identity on x and k. Therefore, $\langle L_\beta, \in\rangle \models \exists y \phi(k, x, y)$. So, since ϕ is Δ_0, $\langle L_\kappa, \in\rangle \models \exists y \phi(k, x, y)$. Therefore, $\langle L_\kappa, \in\rangle \models \mathsf{Sat}_{\Sigma_1}(k, x)$. $\blacksquare$

The proofs of Lemmas 4.3–4.5 make use of the fact that the theory $\mathsf{M} + \Pi_1$-$\mathsf{Collection} + \mathsf{V} = \mathsf{L}$ proves the Mostowski Collapsing Lemma, a result that is not provable in $\mathsf{KPI} + \Sigma_1$-$\mathsf{Foundation} + \mathsf{V} = \mathsf{L}$. Despite this, if $\mathcal{M}$ satisfies $\mathsf{M} + \Pi_1$-$\mathsf{Collection} + \mathsf{V} = \mathsf{L}$, then we can exploit elementarity to show that if $\mathcal{M} \prec_{e,2} \mathcal{N}$ and $\mathcal{N}$ satisfies Δ_0-$\mathsf{Collection}$, then the properties of L_κ obtained from Lemmas 4.3–4.5 also hold for L_κ in $\mathcal{N}$ where κ is a cardinal in $\mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$.

Lemma 4.6. *Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. Let $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be such that $\mathcal{M} \prec_{e,2} \mathcal{N}$ and $\mathcal{N} \models \Delta_0\text{-}\mathsf{Collection}$. If $\kappa \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ is such that $\mathcal{N} \models (\kappa$ is a cardinal$)$, then*

$$\langle (L_\kappa^{\mathcal{N}})^*, \in^{\mathcal{N}} \rangle \models \mathsf{KPI} + \mathsf{Separation} + \mathsf{V} = \mathsf{L}.$$

Proof. By Theorem 3.5, $\mathcal{N} \models \mathsf{KPI} + \Sigma_1\text{-}\mathsf{Foundation} + \mathsf{V} = \mathsf{L}$. Let $\kappa \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ be such that $\mathcal{N} \models (\kappa$ is a cardinal$)$. Let $\sigma_0(z)$ be the formula

$$\forall \lambda((z \in \lambda \wedge \lambda \text{ is a cardinal}) \Rightarrow (\langle L_\lambda, \in \rangle \models \mathsf{KPI} + \mathsf{Separation} + \mathsf{V} = \mathsf{L})).$$

Note that $\sigma_0(z)$ is Π_2^{KPI}. Now, Lemma 4.4 shows that $\mathcal{M} \models \sigma_0(\omega)$. So, since $\mathcal{M} \prec_{e,2} \mathcal{N}$ and $\mathcal{N} \models \mathsf{KPI}$, $\mathcal{N} \models \sigma_0(\omega)$. This yields the conclusion. ∎

Lemma 4.7. *Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. Let $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be such that $\mathcal{M} \prec_{e,2} \mathcal{N}$ and $\mathcal{N} \models \Delta_0\text{-}\mathsf{Collection}$. If $\kappa \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ is such that $\mathcal{N} \models (\kappa$ is a cardinal$)$, then*

$$\mathcal{M} \prec_{e,2} \langle (L_\kappa^{\mathcal{N}})^*, \in^{\mathcal{N}} \rangle.$$

Proof. By Theorem 3.5, $\mathcal{N} \models \mathsf{KPI} + \Sigma_1\text{-}\mathsf{Foundation} + \mathsf{V} = \mathsf{L}$. Let $\kappa \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ be such that $\mathcal{N} \models (\kappa$ is a cardinal$)$. Let $\sigma_1(z)$ be the formula

$$\forall \lambda \left(\begin{array}{c} (z \in \lambda \wedge \lambda \text{ is a cardinal}) \Rightarrow \\ (\forall k \in \omega)(\forall x \in L_\lambda)(\mathsf{Sat}_{\Sigma_1}(k, x) \Rightarrow \langle L_\lambda, \in \rangle \models \mathsf{Sat}_{\Sigma_1}(k, x)) \end{array} \right).$$

Note that $\sigma_1(z)$ is Π_2^{KPI}. Lemma 4.5 shows that $\mathcal{M} \models \sigma_1(\omega)$. So, since $\mathcal{M} \prec_{e,2} \mathcal{N}$ and $\mathcal{N} \models \mathsf{KPI}$, $\mathcal{N} \models \sigma_1(\omega)$. In particular,

$$\mathcal{M} \prec_{e,1} \langle (L_\kappa^{\mathcal{N}})^*, \in^{\mathcal{N}} \rangle \prec_{e,1} \mathcal{N}.$$

Now, let $\phi(x, \vec{z})$ be a Π_1-formula. Let $\vec{a} \in M$ and suppose that $\langle (L_\kappa^{\mathcal{N}})^*, \in^{\mathcal{N}} \rangle \models \exists x \phi(x, \vec{a})$. Let $b \in (L_\kappa^{\mathcal{N}})^*$ be such that $\langle (L_\kappa^{\mathcal{N}})^*, \in^{\mathcal{N}} \rangle \models \phi(b, \vec{a})$. Since $\langle (L_\kappa^{\mathcal{N}})^*, \in^{\mathcal{N}} \rangle \prec_{e,1} \mathcal{N}$, $\mathcal{N} \models \phi(b, \vec{a})$. In particular, $\mathcal{N} \models \exists x \phi(x, \vec{a})$. Therefore, since $\mathcal{M} \prec_{e,2} \mathcal{N}$, $\mathcal{M} \models \exists x \phi(x, \vec{a})$. This shows that $\mathcal{M} \prec_{e,2} \langle (L_\kappa^{\mathcal{N}})^*, \in^{\mathcal{N}} \rangle$. ∎

This allows us to prove the base case ($n = 1$) of the Main Theorem (Theorem 1.4).

Theorem 4.8. *Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{M} + \Pi_1\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. If $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is such that $\mathcal{M} \prec_{\text{topless},2} \mathcal{N}$ and $\mathcal{N} \models \Delta_0\text{-}\mathsf{Collection}$, then $\mathcal{M} \models \Pi_2\text{-}\mathsf{Collection}$.*

Proof. Let $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ with $\mathcal{M} \prec_{\text{topless},2} \mathcal{N}$ and $\mathcal{N} \models \Delta_0\text{-}\mathsf{Collection}$. By Theorem 3.5, $\mathcal{N} \models \mathsf{KPI} + \Sigma_1\text{-}\mathsf{Foundation} + \mathsf{V} = \mathsf{L}$. Using Lemma 4.2, let $\kappa \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ be such that $\mathcal{N} \models (\kappa$ is a cardinal$)$. Therefore, by Lemma 4.7, $\mathcal{M} \prec_{\text{topless},2} \langle (L_\kappa^{\mathcal{N}})^*, \in^{\mathcal{N}} \rangle$. Moreover, by Lemma 4.6, $\langle (L_\kappa^{\mathcal{N}})^*, \in^{\mathcal{N}} \rangle \models \mathsf{KPI} + \mathsf{Separation}$. Therefore, by Theorem 2.7, $\mathcal{M} \models \Pi_2\text{-}\mathsf{Collection}$. ∎

Using this result we show that the minimum model of $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection}$ has no proper Σ_2-elementary end extension that satisfies $\Delta_0\text{-}\mathsf{Collection}$.

THEOREM 4.9. *Let $\alpha \in \omega_1$ be such that L_α is the minimum model of* $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection}$. *There is no $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ such that $\langle L_\alpha, \in \rangle \prec_{e,2} \mathcal{N}$, $L_\alpha \neq N$ and $\mathcal{N} \models \Delta_0\text{-}\mathsf{Collection}$.*

Proof. Note that, by Theorem 2.2, there exists $\alpha \in \omega_1$ such that L_α is the minimum model of $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection}$. Suppose, for a contradiction, that $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is such that $\langle L_\alpha, \in \rangle \prec_{e,2} \mathcal{N}$, $N \neq L_\alpha$ and $\mathcal{N} \models \Delta_0\text{-}\mathsf{Collection}$. By Theorem 3.5,

$$\mathcal{N} \models \mathsf{KPI} + \Sigma_1\text{-}\mathsf{Foundation} + \mathsf{V} = \mathsf{L}.$$

Since $N \neq L_\alpha$, $\mathrm{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\langle L_\alpha, \in \rangle}$ is nonempty. Suppose, for a contradiction, that γ is the least element of $\mathrm{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\langle L_\alpha, \in \rangle}$. Therefore, $(L_\gamma^{\mathcal{N}})^* = L_\alpha$, and

$$\mathcal{N} \models \exists x((x \text{ is transitive}) \wedge (\langle x, \in \rangle \models \mathsf{M} + \Pi_1\text{-}\mathsf{Collection})).$$

Since $\langle L_\alpha, \in \rangle \prec_{e,1} \mathcal{N}$,

$$\langle L_\alpha, \in \rangle \models \exists x((x \text{ is transitive}) \wedge (\langle x, \in \rangle \models \mathsf{M} + \Pi_1\text{-}\mathsf{Collection})),$$

but this contradicts L_α being the minimum model of $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection}$. Therefore, $\langle L_\alpha, \in \rangle \prec_{\text{topless},2} \mathcal{N}$. So, by Theorem 4.8, $\langle L_\alpha, \in \rangle \models \Pi_2\text{-}\mathsf{Collection}$, which contradicts Corollary 2.4. ∎

This result shows that complexity of the scheme $\Delta_0\text{-}\mathsf{Collection}$ computed by Lemma 3.1 is best possible.

COROLLARY 4.10. *Let $\alpha \in \omega_1$ be such that L_α is the minimum model of* $\mathsf{M} + \Pi_1\text{-}\mathsf{Collection}$. *There is no collection Γ of Σ_3-sentences such that*

$$\langle L_\alpha, \in \rangle \models \Gamma \text{ and } \mathsf{M}^- + \Gamma \vdash \Delta_0\text{-}\mathsf{Collection}.$$

5. Σ_{n+1}-elementary end extensions that satisfy Π_{n-1}-Collection. Here we generalise the results of the preceding section to prove that for $n > 1$, if $\mathcal{M}$ satisfies $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$ and $\mathcal{N}$ is such that

$$\mathcal{M} \prec_{\text{topless},n+1} \mathcal{N}$$

and $\mathcal{N}$ satisfies $\Pi_{n-1}\text{-}\mathsf{Collection}$, then $\Pi_{n+1}\text{-}\mathsf{Collection}$ holds in $\mathcal{M}$. This allows us to show that for all $n > 1$, the minimal model of $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection}$ has no proper Σ_{n+1}-elementary end extension that satisfies $\Pi_{n-1}\text{-}\mathsf{Collection}$.

We begin by showing that if $\mathcal{M}$ satisfies $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$ and $\mathcal{N}$ is a proper extension of $\mathcal{M}$ such that $\mathcal{M} \prec_{e,n+1} \mathcal{N}$ and $\mathcal{N}$ satisfies $\Pi_{n-1}\text{-}\mathsf{Collection}$, then $\Sigma_{n+1}\text{-}\mathsf{Separation}$ must hold in $\mathcal{M}$.

THEOREM 5.1. *Let $n \in \omega$ with $n \geq 1$. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. If $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is such that $\mathcal{M} \prec_{e,n+1} \mathcal{N}$, $M \neq N$ and $\mathcal{N} \models \Pi_{n-1}\text{-}\mathsf{Collection}$, then $\mathcal{M} \models \Sigma_{n+1}\text{-}\mathsf{Separation}$.*

Proof. Let $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be such that $\mathcal{M} \prec_{e,n+1} \mathcal{N}$, $M \neq N$ and $\mathcal{N} \models \Pi_{n-1}\text{-}\mathsf{Collection}$. By Theorems 3.5 and 3.7,

$$\mathcal{N} \models \mathsf{KPI} + \Pi_n \cup \Sigma_n\text{-}\mathsf{Foundation} + \mathsf{V} = \mathsf{L}.$$

Using Lemma 4.2, let $\kappa \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ be such that $\mathcal{N} \models (\kappa \text{ is a cardinal})$. Note that, by Theorem 3.6, we need to show that $\Pi_{n+1}\text{-}\mathsf{Foundation}$ holds in $\mathcal{M}$. Towards this end, let $\phi(y, x, \vec{z})$ be a Σ_n-formula and let $\vec{a} \in M$. Using the fact that $\mathcal{M} \prec_{e,n+1} \mathcal{N}$, for all $x \in M$,

$$(5.1) \qquad \mathcal{M} \models \forall y \phi(y, x, \vec{a}) \quad \text{if and only if} \quad \mathcal{N} \models (\forall y \in L_\kappa)\phi(y, x, \vec{a}).$$

Now, the formula $(\forall y \in L_\kappa)\phi(y, x, \vec{z})$ is equivalent to a Σ_n-formula in $\mathcal{N}$. Suppose that $b \in M$ is an element of the class $A = \{x \in M \mid \mathcal{M} \models \forall y \phi(y, x, \vec{a})\}$ of $\mathcal{M}$. Consider the class

$$B = \{x \in N \mid \mathcal{N} \models (x \in b) \wedge (\forall y \in L_\kappa)\phi(y, x, \vec{a})\}.$$

By (5.1), if B is empty, then b is an $\in$-minimal element of A. Otherwise, (5.1) implies that any $\in$-minimal element of B is also an $\in$-minimal element of A, and $\Sigma_n\text{-}\mathsf{Foundation}$ in $\mathcal{N}$ ensures that B must have an $\in$-minimal element. This shows that $\Pi_{n+1}\text{-}\mathsf{Foundation}$ and thus $\Sigma_{n+1}\text{-}\mathsf{Separation}$ holds in $\mathcal{M}$. ∎

Remark 5.2. By Theorem 2.5, there exists $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ such that $\mathcal{M} \models \mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$ and $\Pi_{n+1}\text{-}\mathsf{Foundation}$ fails in $\mathcal{M}$. Theorem 5.1 shows that such an $\mathcal{M}$ does not have a proper Σ_{n+1}-elementary end extension that satisfies $\Pi_{n-1}\text{-}\mathsf{Collection}$. Therefore, Theorem 5.1 already shows that there exist nonstandard models of $\mathsf{M} + \Pi_n\text{-}\mathsf{Collection}$ that have no proper Σ_{n+1}-elementary end extension satisfying $\Pi_{n-1}\text{-}\mathsf{Collection}$.

The following is a weakening of [Res, Corollary 1.27], which is attributed to R. Pino, obtained by noting that, in the theory KPI, $\Pi_{n+1}\text{-}\mathsf{Foundation}$ implies the principle of induction on the natural numbers for Σ_{n+1}-formulae.

Lemma 5.3 (Pino). *Let $n \in \omega$ with $n \geq 1$. The theory*

$$\mathsf{KPI} + \Pi_n\text{-}\mathsf{Collection} + \Pi_{n+1}\text{-}\mathsf{Foundation} + \mathsf{V} = \mathsf{L}$$

proves that for all ordinals α, there exists an ordinal $\beta > \alpha$ such that $L_\beta \prec_n \mathbb{V}$.

This allows us to prove the Main Theorem in the cases where $n \geq 2$.

Theorem 5.4. *Let $n \in \omega$ with $n \geq 2$. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be such that $\mathcal{M} \models \mathsf{M} + \Pi_n\text{-}\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. If $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ with $\mathcal{M} \prec_{\text{topless},n+1} \mathcal{N}$ and $\mathcal{N} \models \Pi_{n-1}\text{-}\mathsf{Collection}$, then $\mathcal{M} \models \Pi_{n+1}\text{-}\mathsf{Collection}$.*

Proof. Let $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be such that $\mathcal{M} \prec_{\text{topless},n+1} \mathcal{N}$ and $\mathcal{N} \models \Pi_{n-1}\text{-}\mathsf{Collection}$. By Theorem 3.7,

$$\mathcal{N} \models \mathsf{M} + \Pi_{n-1}\text{-}\mathsf{Collection} + \Sigma_n\text{-}\mathsf{Separation} + \mathsf{V} = \mathsf{L}.$$

Using Lemma 4.2, let $\kappa \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ be such that $\mathcal{N} \models (\kappa \text{ is a cardinal})$.

Work inside $\mathcal{N}$. Consider

$$A = \{X \in L_\kappa \mid \exists\beta(X = L_\beta) \wedge (X \prec_n \mathbb{V})\},$$

which is a set by Σ_n-$\mathsf{Separation}$. Let

$$B = \{\xi \in \kappa \mid (\exists x \in A)(\exists\beta \in x)(\beta \in \kappa \wedge \xi \leq \beta)\}.$$

Let $\eta = \sup(B)$.

Work in the meta-theory again. By Lemma 5.3, $\mathsf{Ord}^{\mathcal{M}} \subseteq B^*$. Furthermore, $B^* \cap (\mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}) \neq \emptyset$, otherwise η would be the least element of $\mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$. Let $\alpha \in \mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ be such that $L_\alpha^{\mathcal{N}} \in A^*$. Therefore,

$$\mathcal{M} \prec_{\mathrm{topless},n} \langle (L_\alpha^{\mathcal{N}})^*, \in^{\mathcal{N}}\rangle \prec_{e,n} \mathcal{N}.$$

Let $\phi(x,\vec{z})$ be a Π_n-formula and let $\vec{a} \in M$. Suppose that $\langle (L_\alpha^{\mathcal{N}})^*, \in^{\mathcal{N}}\rangle \models \exists x\phi(x,\vec{a})$. Let $b \in (L_\alpha^{\mathcal{N}})$ be such that $\langle (L_\alpha^{\mathcal{N}})^*, \in^{\mathcal{N}}\rangle \models \phi(b,\vec{a})$. So, $\mathcal{N} \models \exists x\phi(x,\vec{a})$. And, since $\mathcal{M} \prec_{e,n+1} \mathcal{N}$, $\mathcal{M} \models \exists x\phi(x,\vec{a})$. This shows that $\mathcal{M} \prec_{\mathrm{topless},n+1} \langle (L_\alpha^{\mathcal{N}})^*, \in^{\mathcal{N}}\rangle$. Note that, since $n \geq 2$ and $L_\alpha^{\mathcal{N}}$ is a set in $\mathcal{N}$,

$$\langle (L_\alpha^{\mathcal{N}})^*, \in^{\mathcal{N}}\rangle \models \mathsf{KPI} + \mathsf{Separation}.$$

Therefore, by Theorem 2.7, $\mathcal{M} \models \Pi_{n+1}$-$\mathsf{Collection}$. ■

Theorem 5.4 allows us to generalise Theorem 4.9 and provide a negative answer to Question 1.3 by showing that for $n \geq 2$, the minimum model of $\mathsf{M}+\Pi_n$-$\mathsf{Collection}$ has no proper Σ_{n+1}-elementary end extension that satisfies Π_{n-1}-$\mathsf{Collection}$.

THEOREM 5.5. *Let $n \in \omega$ with $n \geq 2$. Let $\alpha \in \omega_1$ be such that L_α is the minimum model of* $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$. *There is no $\mathcal{N} = \langle N, \in^{\mathcal{N}}\rangle$ such that $\langle L_\alpha, \in\rangle \prec_{e,n+1} \mathcal{N}$, $N \neq L_\alpha$ and $\mathcal{N} \models \Pi_{n-1}$-*$\mathsf{Collection}$.

Proof. Suppose, for a contradiction, that $\mathcal{N} = \langle N, \in^{\mathcal{N}}\rangle$ is such that $\langle L_\alpha, \in\rangle \prec_{e,n+1} \mathcal{N}$, $N \neq L_\alpha$ and $\mathcal{N} \models \Pi_{n-1}$-$\mathsf{Collection}$. By Theorem 3.7,

$$\mathcal{N} \models \mathsf{M} + \Pi_{n-1}\text{-}\mathsf{Collection} + \Sigma_n\text{-}\mathsf{Separation} + \mathsf{V} = \mathsf{L}.$$

Since $N \neq L_\alpha$, $\mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\langle L_\alpha, \in\rangle}$ is nonempty. To be able to apply Theorem 2.7, we need to show that $\mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\langle L_\alpha, \in\rangle}$ has no least element. Suppose, for a contradiction, that γ is the least element of $\mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\langle L_\alpha, \in\rangle}$. But then $(L_\gamma^{\mathcal{N}})^* = L_\alpha$ and

$$\mathcal{N} \models \exists x((x \text{ is transitive}) \wedge (\langle x, \in\rangle \models \mathsf{M} + \Pi_n\text{-}\mathsf{Collection})).$$

Since $\langle L_\alpha, \in\rangle \prec_{e,1} \mathcal{N}$,

$$\langle L_\alpha, \in\rangle \models \exists x((x \text{ is transitive}) \wedge (\langle x, \in\rangle \models \mathsf{M} + \Pi_n\text{-}\mathsf{Collection})),$$

which contradicts the fact that L_α is the minimum model of $\mathsf{M}+\Pi_n$-$\mathsf{Collection}$. This shows that $\langle L_\alpha, \in\rangle \prec_{\mathrm{topless},n+1} \mathcal{N}$. Therefore, by Theorem 2.7, $\langle L_\alpha, \in\rangle \models \Pi_{n+1}$-$\mathsf{Collection}$, which contradicts Corollary 2.4. ■

Theorem 5.5 shows that the complexity of Π_n-$\mathsf{Collection}$ in the Lévy hierarchy computed by Lemma 3.1 is best possible.

COROLLARY 5.6. *Let $n \in \omega$ with $n \geq 1$. Let $\alpha \in \omega_1$ be such that L_α is the minimum model of* $\mathsf{M} + \Pi_{n+1}$-$\mathsf{Collection}$. *Then there is no collection Γ of Σ_{n+3}-sentences such that*

$$\langle L_\alpha, \in\rangle \models \Gamma \quad \textit{and} \quad \mathsf{M} + \Gamma \vdash \Pi_n\text{-}\mathsf{Collection}.$$

[Kau, Remark 2] observes that there is a countable ordinal α such that $\langle L_\alpha, \in\rangle$ has proper Σ_1-elementary end extension that satisfies KPI, but Π_1-$\mathsf{Collection}$ does not hold in $\langle L_\alpha, \in\rangle$. This observation can be generalised to show that the assumption that the end extension contains no least new ordinal cannot be eliminated from Theorem 1.4.

OBSERVATION 5.7. *There exist $\alpha, \beta \in \omega_1$ such that $\alpha < \beta$, $\langle L_\alpha, \in\rangle \prec_{e,n+1} \langle L_\beta, \in\rangle$, $\langle L_\beta, \in\rangle \models \Pi_{n+1}$-$\mathsf{Collection}$ and Π_{n+1}-$\mathsf{Collection}$ does not hold in $\langle L_\alpha, \in\rangle$.*

Proof. Let $\beta \in \omega_1$ be such that $\langle L_\beta, \in\rangle \models \mathsf{M} + \Pi_{n+1}$-$\mathsf{Collection}$. By Lemma 5.3, the set $A = \{\gamma \in \omega_1 \mid \langle L_\gamma, \in\rangle \prec_{e,n+1} \langle L_\beta, \in\rangle\}$ is nonempty. Let α be the least element of A. Then, by Lemma 5.3, $\langle L_\alpha, \in\rangle$ does not satisfy Π_{n+1}-$\mathsf{Collection}$. ■

Finally, we observe that Theorem 1.4 yields a characterisation of the countable models of $\mathsf{MOST} + \mathsf{V} = \mathsf{L}$ that have a topless Σ_{n+1}-elementary end extension satisfying Π_{n-1}-$\mathsf{Collection}$. The following strengthening of [M25, Theorem 4.15] can be obtained by replacing the crude compactness argument used in the proof [M25, Theorem 4.5] with a more sophisticated Henkin model construction, such as the argument used in the proof of [Fri, Theorem 2.2]:

THEOREM 5.8. *Let $n \in \omega$ with $n \geq 1$. Let S be a recursively enumerable $\mathcal{L}$-theory such that*

$$S \vdash \mathsf{KP} + \Pi_n\text{-}\mathsf{Collection} + \Sigma_{n+1}\text{-}\mathsf{Foundation},$$

and let $\mathcal{M} = \langle M, \in^{\mathcal{M}}\rangle$ be a countable model of S. Then there exists $\mathcal{N} = \langle N, \in^{\mathcal{N}}\rangle$ such that $\mathcal{M} \prec_{\text{topless},n} \mathcal{M}$ and $\mathcal{N} \models S$.

This allows us to show that a countable model $\mathcal{M}$ of $\mathsf{MOST} + \mathsf{V} = \mathsf{L}$ has a topless Σ_n-elementary end extension that satisfies Π_{n-1}-$\mathsf{Collection}$ if and only if $\mathcal{M}$ satisfies Π_{n+1}-$\mathsf{Collection}$.

THEOREM 5.9. *Let $n \in \omega$ with $n \geq 1$. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}}\rangle$ be a countable model of* $\mathsf{MOST} + \mathsf{V} = \mathsf{L}$. *Then the following are equivalent:*

(I) $\mathcal{M} \models \Pi_{n+1}$-$\mathsf{Collection}$*;*
(II) *there exists $\mathcal{N} = \langle N, \in^{\mathcal{N}}\rangle$ such that*

$$\mathcal{M} \prec_{\text{topless},n+1} \mathcal{N} \quad \textit{and} \quad \mathcal{N} \models \Pi_{n-1}\text{-}\mathsf{Collection}.$$

Proof. (II)⇒(I) follows immediately from Theorems 1.1 and 1.4. To see that (I)⇒(II), using Theorem 1.1, let $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ be such that $\mathcal{M} \prec_{e,n+2} \mathcal{N}$ and $M \neq N$. By Theorem 3.4, $\mathcal{N} \models \Pi_{n-1}$-$\mathsf{Collection}$. Therefore, if $\mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ contains no least element, then we are done. So, assume that $\mathsf{Ord}^{\mathcal{N}} \setminus \mathsf{Ord}^{\mathcal{M}}$ has a least element. In this case, $\mathcal{M} \models \mathsf{Foundation}$ and, using Theorem 5.8, there exists $\mathcal{N}_0 = \langle N_0, \in^{\mathcal{N}_0} \rangle$ such that $\mathcal{M} \prec_{\mathrm{topless},n+1} \mathcal{N}_0$ and $\mathcal{N}_0 \models \Pi_{n+1}$-$\mathsf{Collection}$. ■

6. Questions. The Main Theorem assumes that $\mathsf{Powerset}$ holds in the structure being end extended, and the proof presented here uses this assumption.

PROBLEM 6.1. *Let $n \in \omega$ with $n \geq 1$. Let $\mathcal{M} \models \mathsf{KPI} + \Pi_n$-$\mathsf{Collection} + \mathsf{V} = \mathsf{L}$. If $\mathcal{N}$ is a topless Σ_{n+1}-elementary end extension of $\mathcal{M}$ that satisfies Π_{n-1}-$\mathsf{Collection}$, does Π_{n+1}-$\mathsf{Collection}$ necessarily hold in $\mathcal{M}$?*

If Question 6.1 has a negative answer, can a counterexample be found amongst the minimum models?

PROBLEM 6.2. *Let $n \in \omega$ with $n \geq 1$. Does the minimum model of $\mathsf{KPI} + \Pi_n$-$\mathsf{Collection}$ have a proper Σ_{n+1}-elementary end extension that satisfies Π_{n-1}-$\mathsf{Collection}$?*

In [M25], it is shown that every countable model of $\mathsf{KP} + \Pi_n$-$\mathsf{Collection} + \Sigma_{n+1}$-$\mathsf{Foundation}$ has a proper elementary end extension that satisfies $\mathsf{KP} + \Pi_n$-$\mathsf{Collection} + \Sigma_{n+1}$-$\mathsf{Foundation}$. The following two questions from [M25] remain open:

PROBLEM 6.3. *Does every countable model of $\mathsf{KPI} + \Pi_n$-$\mathsf{Collection}$ have a proper Σ_n-elementary end extension that satisfies $\mathsf{KPI} + \Pi_n$-$\mathsf{Collection}$?*

And, if Question 6.3 has a negative answer:

PROBLEM 6.4. *Does every countable model of $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$ have a proper Σ_n-elementary end extension that satisfies $\mathsf{M} + \Pi_n$-$\mathsf{Collection}$?*

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