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Methods in thermodynamic formalism for the Bergweiler family of transcendental entire maps

by

Adrián ESPARZA-AMADOR and Irene INOQUIO-RENTERIA

Presented by Feliks PRZYTYCKI

Summary. We study the family of transcendental entire functions $f_{\ell,c} : \mathbb{C} \to \mathbb{C}$ defined by

$$f_{\ell,c}(z) = c - (\ell - 1)\log c + \ell z - e^z, \quad \ell \geq 2, \quad |c - \ell| < 1,$$

which exhibits a rich dynamical behavior including attracting domains, wandering domains, and Baker domains of hyperbolic type that are positively separated from the post-singular set.

We show that the core techniques of thermodynamic formalism, such as the construction of conformal measures, the definition of pressure, and Bowen's formula, persist in this more intricate setting. In particular, we establish the existence and uniqueness of conformal measures for the associated map on the infinite cylinder. We also verify that the Hausdorff dimension of the radial Julia set is the unique zero of the pressure function. This case illustrates how thermodynamic methods remain robust even in the presence of multiple Fatou components and a more complex post-singular geometry.

1. Introduction. In the 1970s, the dynamical theory of thermodynamic formalism was introduced in mathematics, specifically to study expanding and hyperbolic dynamical systems. In holomorphic dynamics, this theory provides an exceptional framework for probabilistic characterization of Julia sets. Furthermore, by studying the topological pressure function of geometric potentials, this theory provides precise information about the fractal geometry of Julia sets.

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In [24], the authors developed a systematic account of thermodynamic formalism for hyperbolic rational functions and more general distance expanding maps. Many differences and novel phenomena emerge for transcendental entire and meromorphic functions that are absent for rational maps. The infinite degree of transcendental maps and the presence of the singularity at infinity demonstrate that the outlook of these classes is indeed different from the one of rational functions. It becomes necessary to focus on specific subclasses of maps to make progress in understanding their properties.

Barański, Kotus, Urbański and Zdunik were the first to overcome these difficulties and present a thermodynamic formalism, in particular for the class of periodic transcendental functions, such as exponential and trigonometric functions [2, 8, 18, 21, 26]. Mayer and Urbański [22, 23] have made significant contributions to the field, by developing the thermodynamic formalism for a comprehensive class of transcendental meromorphic dynamically regular functions that exhibit certain derivative growth. However, the application of their results to transcendental entire functions with Baker or wandering domains must be approached with caution.

In the transcendental context, a function is called *hyperbolic* if the post-singular set, denoted by $\mathcal{P}(f)$, is bounded and $\mathcal{P}(f)\cap\mathcal{J}(f)=\emptyset$. Stallard [25] introduced a generalized notion of hyperbolicity for entire maps described by the condition $\operatorname{dist}(\mathcal{J},\mathcal{P}(f))>0$, where $\operatorname{dist}(\cdot,\cdot)$ denotes the Euclidean distance. This notion is often referred to as *topologically hyperbolic* and E-*hyperbolic* in the literature. It is clear that a hyperbolic function is also E-hyperbolic, but the converse is not necessarily true.

Kotus and Urbański [19] successfully developed the thermodynamic formalism for the Fatou family of entire maps. In this work, we present an adapted version of this for the Bergweiler family $f_{\ell,c}$ (see Section 2). We focus on the existence and uniqueness of conformal measures and the study of the Hausdorff dimension of the radial Julia set. Instead of the *parabolic type* Baker domain of the Fatou family, the family here treated contains a *hyperbolic type* Baker domain, an attracting basin, and a couple of wandering domains. The reader interested in a deeper study of both types of components is invited to consult [4, 1, 3, 5, 12, 13] and the references therein.

Although this work builds on previous methods developed by Kotus and Urbański, the approach presented here extends the thermodynamic formalism to the Bergweiler family of transcendental entire maps. A closer study of the thermodynamic formalism of maps $f_{\ell,c}$ for the boundary parameter case $f_{1,0}$ would be an interesting direction for future research. A natural extension would be to apply the techniques to a subclass of general projectable meromorphic functions as discussed in [16].

In Section 2, we describe some dynamical properties of the family $f_{\ell,c}$. In Section 3, we gather preliminary results and prove the existence of a conformal measure. In Section 4, we present an adaptation of Bowen's formula to our setting, as detailed in [19, Section 5]. Finally, in Section 5 we provide graphical representations to give a more comprehensive understanding of the dynamics of the family under study.

2. Dynamics of the indexed family $f_{\ell,c}$. The aim of the present section is to determine the dynamical properties of each element of the family $f_{\ell,c}$ as defined in the introduction and the corresponding map $F_{\ell,c}$ in the quotient space $\mathbb{C}/\sim$. In particular, we classify the Fatou components.

2.1. The family $f_{\ell,c}$. Fix $\ell \in \mathbb{N}$ with $\ell \geq 2$. Given $c \in D(\ell,1) := \{z : |z-\ell| < 1\}$, consider the transcendental entire map $f_{\ell,c} : \mathbb{C} \to \mathbb{C}$ given by

$$f_{\ell,c}(z) = c - (\ell-1)\log c + \ell z - e^z, \tag{2.1}$$

with $f'_{\ell,c}(z) = \ell - e^z$.

Theorem 2.1. *The function $f_{\ell,c}$ given by* (2.1) *has a univalent invariant Baker domain $\mathcal{U}$ with* $\operatorname{dist}(\mathcal{U}, \mathcal{P}(f_{\ell,c})) > 0$. *Moreover, the Baker domain $\mathcal{U}$ is of hyperbolic type and is bounded by an analytic curve.*

Proof. First, note that $f_{\ell,c}$ is the *logarithmic lift*, under the projection map $\pi(z) = e^z$, of the analytic function $g : \mathbb{C} \to \mathbb{C}$ given by

$$g(z) = \frac{1}{c^{\ell-1}} z^\ell e^{c-z}.$$

In this situation, from [7] we have $\mathcal{J}(f) = \pi^{-1}(\mathcal{J}(g))$ and $\mathcal{F}(f) = \pi^{-1}(\mathcal{F}(g))$.

A direct calculation shows that $z = 0$ is a superattracting fixed point ($\ell \geq 2$), and $z = c$ is an attracting fixed point with multiplier satisfying $|\ell - c| < 1$. Hence, $z = c$ is superattracting if $c = \ell$. Since $\operatorname{Crit}(g) = \{0, \ell\}$, g is a hyperbolic entire map belonging to the Speiser class $\mathcal{S}$.

Now, if $V \subset \mathcal{F}(g)$ is the Böttcher component containing $z = 0$, it follows that $\mathcal{U} = \pi^{-1}(V)$ is an invariant Baker domain containing the half-plane $\{\operatorname{Re}(z) < -2\ell\}$, and $\partial\mathcal{U} = \pi^{-1}(\partial V)$, which is an analytic curve.

To prove $\operatorname{dist}(\mathcal{U}, \mathcal{P}(f_{\ell,c})) > 0$, note that $\operatorname{Crit}(f_{\ell,c}) = \{z_k = \log \ell + 2\pi i k : k \in \mathbb{Z}\}$. Since $f_{\ell,c}$ has no finite asymptotic values, it follows that $\operatorname{Sing}(f_{\ell,c}^{-1}) = f_{\ell,c}(\operatorname{Crit}(f_{\ell,c}))$. Moreover, $z = \log c$ is an attracting fixed point for $f_{\ell,c}$ with $f_{\ell,c}^n(\log \ell) \to \log c$ as $n \to \infty$. In general, we have

$$\operatorname{Re}(f_{\ell,c}^n(\log \ell + 2k\pi i)) \to \operatorname{Re}(\log c) \quad \text{as } n \to \infty,$$

and

$$\operatorname{dist}(\mathcal{P}(f_{\ell,c}), \mathcal{U}) \geq \operatorname{dist}(\log \ell, \partial\mathcal{U}_0) > 0,$$

where $\mathcal{U}_0$ denotes the immediate basin of attraction of the fixed point $z = \log c$.

Finally, we use [17, Theorem 3] to classify the Baker domain. To do this, notice that for every $z_0 \in \mathcal{U}$,

$$\mathrm{dist}(f_{\ell,c}^n(z_0), \partial\mathcal{U}) \geq |\mathrm{Re}(f_{\ell,c}^n(z_0)) + 2\ell|,$$

and

$$f_{\ell,c}^{n+1}(z_0) = \ell f_{\ell,c}^n(z_0) + o(|c - (\ell-1)\log c|) \quad \text{as } n \to \infty.$$

Hence

$$\frac{|f_{\ell,c}^{n+1}(z_0) - f_{\ell,c}^n(z_0)|}{\mathrm{dist}(f_{\ell,c}^n(z_0), \partial\mathcal{U})} > \ell > 0,$$

which implies that $f_{\ell,c}$ is locally conjugate to $z \mapsto \ell z$ near ∞, and hence $\mathcal{U}$ is of hyperbolic type. ∎

It is well-known that every attracting component will be *lifted* to an *escaping* wandering domain through the projection map. Together with the control of the post-singular set, this implies the following characterization of the Fatou set for each function in the family.

Corollary 2.2. *For $f_{\ell,c}$ as defined above, the Fatou set consists of the following sets:*

- *A univalent invariant Baker domain $\mathcal{U}_\ell$ and all of its preimages.*
- *A simply connected (Böttcher) Schroeder domain $\mathcal{U}_0$ for the (super)attracting fixed point $z_c = \log c$ and all of its preimages.*
- *$2(\ell-1)$ families of escaping wandering domains obtained by $2k\pi i$-translation of the attracting domain $\mathcal{U}_0$ for $k = \pm 1, \pm 2, \ldots, \pm(\ell-1)$, and all of its preimages.*

2.2. Dynamics in the quotient $\mathbb{C}/\sim$. We consider the open infinite strip

$$P := \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 2\pi\}.$$

Abusing notation, we will think of P as a subset of the infinite cylinder $Q = \mathbb{C}/\sim$, where the equivalence relation $\sim$ is defined on $\mathbb{C} \times \mathbb{C}$ by $w \sim z$ if and only if $w - z \in 2\pi i\mathbb{Z}$. The quotient space $\mathbb{C}/\sim$ is the infinite cylinder with the Riemann surface structure endowed by the canonical quotient mapping $\Pi : \mathbb{C} \to Q$. It follows directly that the map $f_{\ell,c}$ respects the equivalence relation $\sim$ and induces a unique map

$$F_{\ell,c} : Q \to Q$$

such that $F_{\ell,c} \circ \Pi = \Pi \circ f_{\ell,c}$. From now on, we will omit the subscripts on F and f whenever no misunderstanding may occur.

Given $M \geq 0$ and $E \subset Q$, we set $E_M = \{z \in Q : 0 \leq \mathrm{Re}(z) \leq M\}$ and $E_M^c = Q \setminus E_M$.

Lemma 2.3. *The map $f : P \to \mathbb{C}$ is a bijection.*

Proof. We proceed by cases according to a partition of P. Take $P_- := \{z \in P : 0 < \mathrm{Im}(z) < \pi\}$ and $P^- := \{z \in P : \pi < \mathrm{Im}(z) < 2\pi\}$.

If $z \in P_-$, then $\sin(\mathrm{Im}(z)) > 0$, hence

$$\begin{aligned}\mathrm{Im}(f(z)) &= \mathrm{Im}(c) - (\ell - 1)\arg c \\ &\quad + \ell\,\mathrm{Im}(z) - e^{\mathrm{Re}(z)}\sin(\mathrm{Im}(z)) < \ell\pi + \mathrm{Im}[c - (\ell-1)\log c].\end{aligned}$$

Similarly, if $z \in P^-$, then $\mathrm{Im}(f(z)) > \ell\pi + \mathrm{Im}(c-(\ell-1)\log c)$. Now, if $z \in P_-$, then $\mathrm{Im}(f'(z)) = -e^{\mathrm{Re}(z)}\sin(\mathrm{Im}(z)) < 0$. Since P_- is a convex subset, we conclude that $f|_{P_-}$ is injective. Analogously, if $z \in P^-$ then $\mathrm{Im}(f'(z)) > 0$ and $f|_{P^-}$ is injective. Therefore, $f|_{P_- \cup P^-}$ is injective.

Now, if $\mathrm{Im}(z) = \pi$, we have

$$\mathrm{Im}(f(z)) = \mathrm{Im}(c - (\ell-1)\log c) + \ell\pi,$$

and

$$f(x + \pi i) = c - (\ell - 1)\log c + \ell x + \ell\pi i + e^x.$$

Viewing f as a function $f : \mathbb{R} \to \mathbb{R}$ (up to $c - (\ell-1)\log c + \ell\pi i$), we obtain $f'(x + \pi i)) = \ell + e^x > 0$, so $f|_{\mathrm{Im}(z)=\pi}$ is also injective. It follows that $f : P \to \mathbb{C}$ is injective.

Finally, note that $f(x) = c - (\ell-1)\log c + \ell x - e^x \in f(P^-)$ and $f(x + 2\pi i) = c - (\ell-1)\log c + \ell x + 2\pi\ell i - e^x \in f(P_-)$ imply that $\partial(f(P)) \subset f(\partial P) \subset f(P)$ and then $\partial f(P) = \emptyset$. This way, $f(P) = \mathbb{C}$, which concludes the proof. ∎

We close this section with the *expanding* properties of F on its Julia set.

PROPOSITION 2.4.

- *If* $z \in \mathcal{J}(F)$, *then* $\limsup_{n\to\infty} |(F^n)'(z)| = +\infty$.
- *There exist* $L > 0$ *and* $\kappa > 1$ *such that for every* $z \in \mathcal{J}(F)$ *and every* $n \geq 1$, $|(F^n)'(z)| \geq L\kappa^n$.

Proof. The result follows verbatim as in [19]. ∎

3. Topological pressure and existence and uniqueness of conformal measures. In this section, we will prove the existence and uniqueness of ergodic conformal measures for the projected map $F_{\ell,c}$.

In order to simplify notation, we will sometimes write λ for $c-(\ell-1)\log c$. Since F (and f) has no finite asymptotic values, the post-singular set consists of the *post-critical set* of F, denoted by $\mathrm{PC}(F)$, given by

$$\mathrm{PC}(F) = \overline{\{F^n(\Pi(\log \ell)) : n \geq 0\}}.$$

In view of Lemma 2.3, the map $f : P \to \mathbb{C}$ is bijective, and we will denote by $f_*^{-1} : \mathbb{C} \to P$ its holomorphic inverse map.

Let $\mathcal{C}_b = C_b(\mathcal{J}(F))$ be the Banach space of all bounded continuous complex-valued functions on $\mathcal{J}(F)$.

For $t > 0$, let $\mathscr{L}_t : \mathcal{C}_b \to \mathcal{C}_b$ be the Perron–Frobenius operator, given by

$$\mathscr{L}_t g(z) = \sum_{x \in F^{-1}(z)} |F'(x)|^{-t} g(x). \tag{3.1}$$

For every $n \geq 1$, its iterates are given by

$$\mathscr{L}_t^n g(z) = \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t} g(x).$$

In particular, $\mathscr{L}_t^n \mathbb{1}(z) = \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t}$.

We will prove that $\|\mathscr{L}_t \mathbb{1}\|_\infty < \infty$ and since $|\mathscr{L}_t g(z)| \leq \|\mathscr{L}_t \mathbb{1}\|_\infty \|g\|_\infty$ the operator (3.1) is well-defined.

For every $t \geq 0$ and every $z \in Q \setminus \mathrm{PC}(F)$, define the *lower* and *upper topological pressure* respectively by

$$\underline{P}(t, z) = \liminf_{n\to\infty} \frac{1}{n} \log \mathscr{L}_t^n \mathbb{1}(z) \quad \text{and} \quad \overline{P}(t, z) = \limsup_{n\to\infty} \frac{1}{n} \log \mathscr{L}_t^n \mathbb{1}(z).$$

As in §2.2, we focus on the open infinite strip $P = \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 2\pi\}$. Given $M \geq 0$, we consider the subset of the cylinder Q, given by $E_M = \{z \in Q : 0 \leq \mathrm{Re}(z) \leq M\}$.

Proposition 3.1. *Let $t > 1$ and $z \in \mathcal{J}(F)$. Then the following hold:*

(1) *$\underline{P}(t, z)$ and $\overline{P}(t, z)$ do not depend on the choice of $z \in \mathcal{J}(F)$, so we can denote $\underline{P}(t, z)$ and $\overline{P}(t, z)$ by $\underline{P}(t)$ and $\overline{P}(t)$ respectively.*
(2) $\|\mathscr{L}_t \mathbb{1}\|_\infty := \sup\{\mathscr{L}_t \mathbb{1}(z) : z \in \mathcal{J}(F)\} < +\infty$.
(3) $\|\mathscr{L}_t^n \mathbb{1}(z)\|_\infty \leq \|\mathscr{L}_t \mathbb{1}(z)\|_\infty^n$ *and* $\overline{P}(t) \leq \log \|\mathscr{L}_t \mathbb{1}(z)\|_\infty$.
(4) *For every $t > 1$, $\overline{P}(t) < +\infty$.*
(5) *For $t > 1$, both functions $\underline{P}(t)$ and $\overline{P}(t)$ are convex, continuous, strictly decreasing and* $\lim_{t\to\infty} \overline{P}(t) = -\infty$.
(6) $\lim_{\mathrm{Re}(z)\to\infty} \mathscr{L}_t \mathbb{1}(z) = 0$.

Proof. (1) Since any two points in $\mathcal{J}(F)$ belong to an open simply connected set disjoint from $\mathrm{PC}(F)$, it follows from Koebe's distortion theorem that $\underline{P}(t, z)$ and $\overline{P}(t, z)$ are independent of z.

(2) For every $t > 1$ and $z \in Q \setminus \mathrm{PC}(F)$, we have

$$\mathscr{L}_t \mathbb{1}(z) = \sum_{x \in F^{-1}(z)} |F'(x)|^{-t} = \sum_{x \in F^{-1}(z)} |\ell - e^x|^{-t} = \sum_{k=-\infty}^{+\infty} |\ell - e^{z_k}|^{-t},$$

with $\tilde{z}$ being the only point in $\Pi^{-1}(z) \cap P$ and $z_k = f_*^{-1}(\tilde{z} + 2\pi i k)$, with $k \in \mathbb{Z}$, being the only point in P such that $f(z_k) = \tilde{z} + 2\pi i k$. Then $\ell - e^{z_k} = \ell - \ell z_k - \lambda + (\tilde{z} + 2\pi i k)$.

So,

$$\mathscr{L}_t \mathbb{1}(z) = \sum_{k=-\infty}^{+\infty} |\ell - \ell z_k - \lambda + (\tilde{z} + 2\pi i k)|^{-t}.$$

Let $\mathbb{Z}_\lambda := \{k \in \mathbb{Z} : \pi|k-\ell| \geq |\mathrm{Im}(\lambda)| + 2\ell\pi\}$. Then, for $z \in \mathcal{J}(F)$ and $k \in \mathbb{Z}_\lambda$, a careful calculation proves that

$$\left|\ell - \ell z_k - \lambda + (\tilde{z} + 2\pi i k)\right| \geq \pi|k-\ell|.$$

On the other hand, for $z = F(z_k)$, we have $f(z_k) = \tilde{z} + 2\pi i k$. Hence,

$$\mathrm{Re}(z) = \mathrm{Re}(\tilde{z}) = \mathrm{Re}(\lambda) + \ell\,\mathrm{Re}(z_k) - e^{\mathrm{Re}(z_k)}\cos(\mathrm{Im}(z_k)).$$

This way, we fix $T > 0$ so large that if $\mathrm{Re}(z) \geq T$, then $\mathrm{Re}(z_k) \geq \ell$ for all $k \in \mathbb{Z}$. So, for all such z and all $k \in \mathbb{Z}$, we have

$$|\ell - e^{z_k}| \geq |e^{z_k}| - \ell = e^{\mathrm{Re}(z_k)} - \ell \geq e^{\ell} - \ell > 1.$$

Since $\mathcal{J}(F)$ is a subset of Q without critical points, and $z_k \in \mathcal{J}(F)$, we have $\ell - e^{z_k} \neq 0$, and therefore $M = \inf\{|\ell - e^{z_k}| : z \in E_T,\ k \in \mathbb{Z}\setminus\mathbb{Z}_\lambda\} > 0$. It follows that for all $t > 1$ and all $z \in \mathcal{J}(F)$,

$$\begin{aligned}
\mathscr{L}_t\mathbb{1}(z) &= \sum_{k=-\infty}^{+\infty} |\ell - e^{z_k}|^{-t} \\
&= \sum_{k\in\mathbb{Z}_\lambda} |\ell - \ell z_k - \lambda + (\tilde{z} + 2\pi i k)|^{-t} + \sum_{k\in\mathbb{Z}\setminus\mathbb{Z}_\lambda} |\ell - e^{z_k}|^{-t} \\
&\leq \sum_{z\in\mathbb{Z}_\lambda} (\pi|k-\ell|)^{-t} + \#(\mathbb{Z}\setminus\mathbb{Z}_\lambda)(\max\{1, M\})^{-t} < +\infty.
\end{aligned}$$

Therefore, $\|\mathscr{L}_t\mathbb{1}\|_\infty < +\infty$.

(3) For every $n \geq 1$ and every $z \in \mathcal{J}(F)$,

$$\begin{aligned}
\mathscr{L}_t^n\mathbb{1}(z) &= \sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t} \\
&= \sum_{y\in F^{-(n-1)}(z)} |(F^{n-1})'(y)|^{-t} \sum_{x\in F^{-1}(y)} |F'(x)|^{-t} \\
&\leq \|\mathscr{L}_t\mathbb{1}\|_\infty \mathscr{L}_t^{n-1}\mathbb{1}(y).
\end{aligned}$$

So, by a direct inductive argument, we have $|\mathscr{L}_t^n\mathbb{1}(z)| \leq \|\mathscr{L}_t\mathbb{1}\|_\infty^n$ and consequently for all $t > 1$, $\overline{P}(t) = \overline{P}(t,z) \leq \log\|\mathscr{L}_t\mathbb{1}\|_\infty$.

Item (4) follows from (2) and (3).

Item (5) follows immediately from Hölder's inequality. Thus, thanks to convexity, the function $t \mapsto \overline{P}(t)$ with $t > 1$ is continuous. The fact that $P(t)$, for $t > 1$, is strictly decreasing and $\lim_{t\to+\infty} P(t) = -\infty$ follows from Proposition 2.4.

Finally, to prove (6), we consider $z \in \mathcal{J}(F)$ and $\tilde{z} \in \Pi^{-1}(z) \cap P$. By definition of $f_{\ell,c}$, it follows that, $\lim_{\mathrm{Re}(z)\to+\infty} \mathrm{Re}(z_k) = +\infty$, uniformly with respect to $k \in \mathbb{Z}$. Hence, $\lim_{\mathrm{Re}(z)\to+\infty} |\mathrm{Re}(\tilde{z} - \ell z_k)| = +\infty$ uniformly with respect to $k \in \mathbb{Z}$. Then

$$\begin{aligned}|\cdot| &= |\ell - \ell z_k - \lambda + (\tilde{z} + 2\pi i k)| \\ &\geq \tfrac{1}{2}\big(|\mathrm{Re}(\ell - \ell z_k - \lambda + (\tilde{z} + 2\pi i k))| + |\mathrm{Im}(\ell - \ell z_k - \lambda + (\tilde{z} + 2\pi i k))|\big) \\ &\geq \tfrac{1}{2}\left(\mathrm{Re}(\tilde{z} - \ell z_k)\right) + (\ell + \mathrm{Re}(\lambda)) + \tfrac{1}{2}(-2\ell\pi - \mathrm{Im}(\lambda) + 2\pi|k|) \\ &\geq M + \frac{\pi}{2}|k|.\end{aligned}$$

This implies $\mathscr{L}_t\mathbb{1}(z) \leq \sum_{k\in\mathbb{Z}}\big(M + \frac{\pi}{2}|k|\big)^{-t}$. Finally, letting $M \nearrow +\infty$, we get the desired result. ■

3.1. Conformal measure for the family $f_{\ell,c}$. Let $t, \alpha \in \mathbb{R}$. A measure ν supported on the Julia set $\mathcal{J}(F)$ is (t,α)-*conformal* if for every Borel set $A \subset \mathcal{J}(F)$ with $F|_A$ injective, $\nu(F(A)) = \int_A \alpha|F'|^t\, d\nu$.

Fix $n \geq 1$ and consider the vertical strip $E_n = \{z \in Q : 0 \leq \mathrm{Re}(z) \leq n\}$, and the compact, forward-invariant set $K_n = \bigcap_{j\geq 0} F^{-j}(E_n)$. Then, for each n, there exists a Borel probability measure m_n supported on K_n and a non-decreasing sequence $\{P_n(t)\}_{n=1}^{\infty}$ such that

$$m_n(F(A)) \geq e^{P_n(t)} \int\limits_A |F'|^t\, dm_n$$

for all Borel sets $A \subset E_n$ where $F|_A$ is injective. If in addition $A \cap \partial E_n = \emptyset$, then equality holds (see [9, Lemma 5.3], [24]).

The sequence $\{m_n\}_{n=1}^{\infty}$ of measures is *tight*, that is, for every $\epsilon > 0$, there exists a compact set C of $\mathcal{J}(F)$ such that $m_n(\mathcal{J}(F) \setminus C) < \epsilon$ for all n. To prove this, we estimate the measure of $E_M^c := Q \setminus E_M$ (restricted to $\mathrm{Re}(z) > 0$), splitting it into

$$E_1(M) := \{z \in \mathcal{J}(F) : \mathrm{Re}(F(z)) \geq M\}$$

and

$$E_2(M) := \{z \in \mathcal{J}(F) : \mathrm{Re}(z) \geq M,\ \mathrm{Re}(F(z)) < M\}.$$

Keeping the notation $\lambda = c - (\ell - 1)\log c$, we bound $m_n(E_1(M))$ and $m_n(E_2(M))$ separately, using the inverse branches $F_k^{-1}(z) := f_*^{-1}(z + 2\pi k i)$ of F.

For $E_1(M)$, we have

$$m_n(E_1(M)) \leq \sum_{k\in\mathbb{Z}} m_n(F_k^{-1}(E_M^c)) \leq \sum_{k\in\mathbb{Z}} e^{-P_n(t)} \sup_{z\in E_M^c} |(F_k^{-1})'(z)|^t.$$

Using $|(F_k^{-1})'(z)| = 1/|\ell - e^{z_k}|$ with $z_k = f_*^{-1}(z + 2\pi k i)$ and the relation $f(z_k) = \lambda + \ell z_k - e^{z_k} = z + 2\pi k i$, we estimate $\mathrm{Re}(z_k)$ by means of

$$\ell\,\mathrm{Re}(z_k) = \mathrm{Re}(z) + e^{\mathrm{Re}(z_k)}\cos(\mathrm{Im}(z_k)) - \mathrm{Re}(\lambda).$$

Then, using $|z + 2\pi k i| = |\lambda + \ell z_k - e^{z_k}| \geq \frac{1}{2}e^{\mathrm{Re}(z_k)}$, we get

$$\mathrm{Re}(z_k) \leq \log 2 + \log|z + 2\pi k i|.$$

Thus, $|\mathrm{Re}(z) - \ell\,\mathrm{Re}(z_k)| \geq \mathrm{Re}(z) - \ell\log 2 - \ell\log|z + 2\pi k i|$.

Letting $\hat{k} = \max\left\{k \geq 0 : \log|z + 2\pi li| \leq \frac{1}{3}\operatorname{Re}(z)\ \forall |l| \leq k\right\}$, it is easy to deduce the inequality $\hat{k} \geq e^{\operatorname{Re}(z)/4}$. For $M > 0$ large enough and $|k| \leq \hat{k}$, the reader may easily verify that

$$|z + 2k\pi i - \ell z_k - (\lambda - \ell)| \geq M/9 + \pi|k|.$$

For $|k| > \hat{k}$, the following inequality holds for M sufficiently large:

$$|z + 2k\pi i - \ell z_k - (\lambda - \ell)| \geq \pi|k|.$$

Thus, we get

$$\begin{aligned}
|\cdot| &= \sum_{k\in\mathbb{Z}} e^{-P_n(t)} \sup_{z\in E_M^c} |(F_k^{-1})'(z)|^t \\
&= e^{-P_n(t)} \sum_{|k|\leq\hat{k}} \sup_{z\in E_M^c} |(F_k^{-1})'(z)|^t + e^{-P_n(t)} \sum_{|k|>\hat{k}} \sup_{z\in E_M^c} |(F_k^{-1})'(z)|^t \\
&\leq e^{-P_n(t)} \sum_{|k|\leq\hat{k}} \left(\frac{M}{9} + \pi|k|\right)^{-t} + e^{-P_n(t)} \sum_{|k|>\hat{k}} (\pi|k|)^{-t} \\
&\leq 2e^{-P_n(t)} \sum_{k=0}^{\infty} \left(\frac{M}{9} + \pi|k|\right)^{-t} + \frac{2}{\pi^t} e^{-P_n(t)} \sum_{k>\hat{k}} k^{-t} \\
&\leq A_t^{(1)} e^{-P_n(t)} M^{1-t} + A_t^{(2)} e^{-P_n(t)} \hat{k}^{(1-t)} \\
&\leq A_t e^{-P_n(t)} \max\{M^{1-t}, e^{\frac{M}{4}(1-t)}\} = A_t e^{-P_n(t)} M^{1-t},
\end{aligned}$$

where $A_t^{(1)}, A_t^{(2)}, A_t$ are constants depending only on $t > 1$ and all this holds for M sufficiently large.

Since $\sup\{|F'(z)| : z \in K_n\} < \infty$, for n large enough we have $P_n(t) > -\infty$. Since $\{P_n(t)\}_{n=0}^{\infty}$ is non-decreasing, we put $\gamma(t) = \sup_n\{-P_n(t)\} < \overline{P(t)} < \infty$. Then, for M sufficiently large,

$$m_n(E_1(M)) \leq A_t e^{\gamma(t)} M^{1-t}. \tag{3.2}$$

For $E_2(M)$, note first that for $z \in E_2(M)$, $\operatorname{Re}(z) \geq M$ and $\operatorname{Re}(f(z)) < M$. Then

$$|f(z)| = |\lambda + \ell z - e^z| \geq |e^z| - |\ell z - \lambda| \geq |e^z| = e^{\operatorname{Re}(z)} \geq e^M.$$

Hence,

$$e^{2M} \leq |f(z)|^2 = |\operatorname{Re}(f(z))|^2 + |\operatorname{Im}(f(z))|^2 \leq M^2 + |\operatorname{Im}(f(z))|^2.$$

Thus,

$$|\operatorname{Im}(f(z))| \geq \sqrt{e^{2M} - M^2} \geq e^M/2.$$

For every $k \geq e^M/2$ and for M large enough we have

$$|\ell + z + 2\pi ki - \lambda - \ell f_*^{-1}(z + 2\pi ki)| \geq 2\pi k - 2(\ell + 1)\pi - |\lambda - \ell| \geq k,$$

proving $|(F_k^{-1})'(z)|^t \leq k^{-t}$.

Then for M large enough,

$$m_n(E_2(M)) \le e^{\gamma(t)} \sum_{k \ge e^M/2} k^{-t} \preceq e^{\gamma(t)} e^{M(1-t)}. \tag{3.3}$$

Combining (3.2) and (3.3), we see that the sequence $\{m_n\}_{n\ge1}$ is tight.

By Prokhorov's theorem, there exists a weak*-accumulation point m_t of $\{m_n\}$. Denote $\log \alpha_t := \lim_{n\to\infty} P_n(t)$. Then the limit m_t is a (t,α_t)-conformal measure.

THEOREM 3.2. *For every $t > 1$, there exists a (t,α_t)-conformal probability measure m_t on $\mathcal{J}(F)$, obtained as a weak*-limit of the family $\{m_n\}$, and $m_t(\mathcal{J}(F)) = 1$.*

3.2. Spectral and conformal properties of the normalized transfer operator. Let $\mathscr{L}_t^*$ denote the dual operator acting on the space of finite Borel measures on $\mathcal{J}(F)$, defined by

$$\int \phi \, d(\mathscr{L}_t^* \mu) = \int \mathscr{L}_t \phi \, d\mu, \quad \forall \phi \in C_b(\mathcal{J}(F)).$$

Define the normalized operator $\widehat{\mathscr{L}_t} := \alpha_t^{-1} \mathscr{L}_t$ and let $\widehat{\mathscr{L}_t^*}$ denote its dual.

PROPOSITION 3.3. *Let $t > 1$. Then the following properties hold:*

(i) *The conformal measure m_t satisfies $\widehat{\mathscr{L}_t^*} m_t = m_t$.*
(ii) $\sup_{n\ge0} \|\widehat{\mathscr{L}_t^n}(\mathbb{1})\|_\infty < \infty$.

Proof. Part (i) follows from standard arguments, as in [10]. The argument for (ii) is based on Koebe's distortion theorem and precise control of inverse branches. Fix $z, w \in Q \setminus \mathrm{PC}(F)$ and let $\gamma_{z,w}$ be a smooth arc in $Q \setminus B(\mathrm{PC}(F), 2\delta)$ with $\delta = \frac{1}{2}\min\left\{\frac{1}{2}, \mathrm{dist}(\mathcal{J}(F), \mathrm{PC}(F))\right\}$. There exists ℓ_M such that $\gamma_{z,w}$ can be covered by ℓ_M balls of radius δ. Let $U_{z,w}$ be their union.

If F^{-n} is holomorphic on $U_{z,w}$, Koebe distortion yields $K_M \ge 1$ such that

$$\frac{|(F_*^{-n})'(z)|}{|(F_*^{-n})'(w)|} \le K_M, \quad \text{so} \quad K_M^{-t} \le \frac{\widehat{\mathscr{L}_t^n}\mathbb{1}(z)}{\widehat{\mathscr{L}_t^n}\mathbb{1}(w)} \le K_M^t.$$

From Proposition 3.1(5), there exists M such that for $w \in E_M^c$, $\widehat{\mathscr{L}_t}\mathbb{1}(w) \le 1$. Then by induction, $\|\widehat{\mathscr{L}_t^n}\mathbb{1}\|_\infty \le K_M^t / m_t(E_M)$.

Suppose that the maximum norm above is achieved at $z^{n+1} \in Q$. If $z^{n+1} \in E_M$, then

$$\begin{aligned} 1 = \int \widehat{\mathscr{L}_t^{n+1}}\mathbb{1} \, dm_t &\ge \int_{E_M} \widehat{\mathscr{L}_t^{n+1}}\mathbb{1} \, dm_t \\ &\ge \inf_{E_M} \widehat{\mathscr{L}_t^{n+1}}\mathbb{1} \cdot m_t(E_M) \ge K_M^{-t} \|\widehat{\mathscr{L}_t^{n+1}}\mathbb{1}\|_\infty \cdot m_t(E_M), \end{aligned}$$

which gives the desired bound. If $z^{n+1} \in E_M^c$ then

$$\widehat{\mathscr{L}}_t^{n+1}\mathbb{1}(z^{n+1}) = \alpha_t^{-1} \sum_{y \in F^{-1}(z^{n+1})} |F'(y)|^{-t} \widehat{\mathscr{L}}_t^n \mathbb{1}(y)$$
$$\leq \alpha_t^{-1} \sum |F'(y)|^{-t} \|\widehat{\mathscr{L}}_t^n \mathbb{1}\|_\infty \leq \frac{K_M^t}{m_t(E_M)}. \quad \blacksquare$$

PROPOSITION 3.4. *The following hold:*

(i) *For every $\epsilon > 0$, there exists $M > 0$ such that*

$$\inf_{n \geq 0} \sup_{|\operatorname{Re}(z)| \leq M} \widehat{\mathscr{L}}_t^n \mathbb{1}(z) \geq 1 - \epsilon.$$

(ii) *There exists $M_0 > 0$ such that for all $M \leq M_0$,*

$$\inf_{n \geq 0} \inf_{|\operatorname{Re}(z)| \leq M} \widehat{\mathscr{L}}_t^n \mathbb{1}(z) \geq \frac{1}{4K_m^t} (\max\{K_M, K_{M_0}\})^{-1},$$

where K_M represents Koebe's distortion constant.

(iii) *The pressure satisfies $P(t) = \overline{P}(t) = \underline{P}(t) = \log \alpha_t$.*

Proof. For (i), assume the contrary. Then for some $\epsilon > 0$,

$$\inf_n \sup_{|\operatorname{Re}(z)| \leq M} \widehat{\mathscr{L}}_t^n \mathbb{1}(z) < 1 - \epsilon.$$

Let $\Theta = \sup_n \|\widehat{\mathscr{L}}_t^n \mathbb{1}\|_\infty$. Choose M so large that $m_t(E_M^c) \leq \epsilon/(4\Theta)$. Then

$$1 = \int \widehat{\mathscr{L}}_t^n \mathbb{1}\, dm_t = \int_{E_M} \widehat{\mathscr{L}}_t^n \mathbb{1}\, dm_t + \int_{E_M^c} \widehat{\mathscr{L}}_t^n \mathbb{1}\, dm_t$$
$$\leq (1 - \epsilon) m_t(E_M) + \Theta m_t(E_M^c) \leq (1 - \epsilon) + \epsilon/4 = 1 - 3\epsilon/4,$$

contradicting $\int = 1$. Item (ii) follows from (i) and Koebe's lower distortion bound.

To prove (iii), note that $\mathscr{L}_t^n \mathbb{1}(z) \leq \Theta \alpha_t^n$, so $\overline{P}(t) \leq \log \alpha_t$.

From (ii), for $x_0 \in Q$, $\mathscr{L}_t^n \mathbb{1}(x_0) \geq \frac{1}{4K_{M_0}} \alpha_t^n$ implies $\underline{P}(t) \geq \log \alpha_t$. Hence all pressures coincide. $\blacksquare$

3.3. Ergodicity. We consider the well-known escaping subset of the Julia set of F, defined by $I_\infty(F) = \{z \in \mathcal{J}(F) : \lim F^n(z) = +\infty\}$.

Note that the above set is well-defined since F acts on the quotient space Q and f has a Baker domain containing a left half-plane. So, the analogous set for f is given by $I_\infty(f) = \{z \in \mathcal{J}(f) : \lim_{n \to \infty} \operatorname{Re}(f^n(z)) = +\infty\}$.

It is clear that $I_\infty(f) = \Pi^{-1}(I_\infty(F))$, where $\Pi : \mathbb{C} \to Q$ is the quotient map. We denote by $\mathcal{J}_r(F)$ the corresponding complement of the escaping set, that is, $\mathcal{J}_r(F) = \mathcal{J}(F) \setminus I_\infty(F)$ and $\mathcal{J}_r(f) = \mathcal{J}(f) \setminus I_\infty(f)$.

We first prove a general result for conformal measures on F.

Lemma 3.5. *If ν is (t,β^t)-conformal for F^j, $t>1$, then for some $M>0$, $\nu(I_\infty(F))=0$ or equivalently $\nu(\mathcal{J}_r(F))=1$.*

Proof. Define inductively

$$\nu_0=\nu,\qquad \nu_{k+1}(A)=\int\limits_{F^k(A)}\beta^{-1}|((F^k|_{F^k(A)})^{-1})'|^t\,d\nu_k\quad\text{ for } k\ge 0,$$

for Borel sets A on which F^k is injective. For $B\subset\mathcal{J}(F)\cap E_M^c$, we get

$$\nu_{k+1}(F^{-1}(B))\le C\beta^{-1}M^{1-t}\nu_k(B),\ \text{so}\ \nu(F^{-j}(B))\le(C\beta^{-1}M^{1-t})^j\nu(B).$$

Then in a standard way we get, for $M>0$ sufficiently large,

$$\nu\Big(\bigcap_{n=0}^{\infty}F^{-n}(E_M^c)\Big)=0.$$

Consequently,

$$\nu\Big(\bigcup_{k=0}^{\infty}F^{-k}\Big(\bigcap_{n=0}^{\infty}F^{-n}(E_M^c)\Big)\Big)=\nu(I_\infty(F))=0.\ \blacksquare$$

From Theorem 3.2 with $j=1$, and $\beta=e^{P(t)}$, it follows that Lemma 3.5 applies for the limit measure m_t, and the next consequence follows analogously to [19, Corollary 4.12].

Corollary 3.6. *For all $M>0$ large enough and $t>1$, $m_t(E_M^c)\le Ce^{(1-t)M}$ for some constant C.*

Theorem 3.7. *For $t>1$, the probability measure m_t is the unique $(t,e^{P(t)})$-conformal measure for F. Moreover, it is ergodic with respect to all iterates of F.*

Proof. The proof has the same structure as in [19], where an analogous result is established for transcendental entire functions with Baker domains. In particular, the arguments concerning the vanishing of conformal measures on the escaping set, distortion estimates for inverse branches, and covering techniques carry over to our setting with minimal modifications.

We highlight that the family of maps $f_{\ell,c}$ considered here satisfies the same geometric properties, such as the existence of a Baker domain containing a left half-plane and appropriate control on distortion, which ensure that the proof remains valid. More precisely, Lemma 3.5 implies that $\nu(I_\infty(F))=0$ for any conformal measure ν, which allows the argument to be restricted to the radial Julia set $\mathcal{J}_r(F)$. Moreover, the inverse branches of F^n on appropriate subsets of $\mathcal{J}_r(F)$ admit uniform distortion bounds by Koebe's theorem. In addition, the measure m_t is regular, and the Besicovitch covering argument used in [19] applies also in our case.

Thus, the uniqueness of the conformal measure and its ergodicity with respect to all iterates of F remain invariant under this class of maps. $\blacksquare$

4. Bowen formula. We begin by observing that the analytical framework developed in [19, Section 5] for entire transcendental maps with Baker domains applies to our setting with functions $f_{\ell,c}(z) = \ell z + c - e^z$. As a result, the thermodynamic formalism for the induced quotient map F on the Julia set $\mathcal{J}(F)$ remains well-behaved.

Let us recall that $\mathcal{C}_b := C_b(\mathcal{J}(F))$ is the space of all bounded continuous complex-valued functions on $\mathcal{J}(F)$. Fix $\alpha \in (0,1]$, and define the *α-variation* of g by

$$v_\alpha(g) := \inf\{L \geq 0 : |g(x) - g(y)| \leq L|x-y|^\alpha,\ \forall x, y \in \mathcal{J}(F),\ |x-y| \leq \delta\},$$

where $\delta := \min\{1/2, \operatorname{dist}(\mathcal{J}(F), \operatorname{PC}(F))\}$. We then define the norm $\|g\|_\alpha := \|g\|_\infty + v_\alpha(g)$, and consider the Banach space

$$\mathrm{H}_\alpha := \{g \in \mathcal{C}_b : \|g\|_\alpha < +\infty\},$$

which is dense in $\mathcal{C}_b$ with respect to the uniform norm.

Now, fix $t \in \mathbb{C}$ with $\operatorname{Re}(t) \geq 0$. For all $x, y \in \mathcal{J}(F)$ with $|x-y| \leq \delta$, all $n \geq 1$, and all inverse branches F_v^{-n}, one has

$$\big||(F_v^{-n})'(x)|^t - |(F_v^{-n})'(y)|^t\big| \leq M_t |(F_v^{-n})'(x)|^{\operatorname{Re}(t)} |x-y|$$

for some constant $M_t > 0$. According to Proposition 3.1, the potential $\phi_t(z) := e^{-P(t)}|F'(z)|^{-t}$ satisfies certain regularity conditions: it is dynamically Hölder continuous and rapidly decreasing (see [19] for precise definitions). Consequently, the normalized transfer operator $\widehat{\mathscr{L}_t}$ acts on the Banach space H_α and admits a non-trivial fixed point.

Theorem 4.1. *Let $t > 1$. Then the following hold:*

(a) 1 *is a simple isolated eigenvalue of* $\widehat{\mathscr{L}_t} : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$.
(b) *The corresponding eigenspace is generated by a strictly positive function* $\psi_t \in \mathrm{H}_\alpha$ *with* $\int \psi_t\, dm_t = 1$ *and* $\lim_{\operatorname{Re}(z)\to\infty} \psi_t(z) = 0$.
(c) 1 *is the only eigenvalue of modulus* 1.

Sketch of proof. Combining Lemmas 5.1 and 5.2 of [19], we find that there exists a sequence $\{n_k\}$ such that

$$\frac{1}{n_k} \sum_{j=1}^{n_k} \widehat{\mathscr{L}_t}^j \mathbb{1} \to \psi_t \in \mathrm{H}_\alpha. \tag{4.1}$$

Since m_t is a fixed point of the dual operator $\widehat{\mathscr{L}_t^*}$, we have

$$1 = \int \frac{1}{n_k} \sum_{j=1}^{n_k} \widehat{\mathscr{L}_t}^j \mathbb{1}\, dm_t \to \int \psi_t\, dm_t \quad \text{as } n \to \infty.$$

From (4.1), $\psi_t = \widehat{\mathscr{L}_t}\psi_t$, and Proposition 3.1 shows that $\psi_t(z) \to 0$ as $\operatorname{Re}(z) \to \infty$. The ergodicity of m_t (Theorem 3.7) then implies that any other eigenfunction of modulus 1 is proportional to ψ_t, concluding the proof. ∎

We now state some finer statistical properties of the F-invariant measure $\mu_t := \psi_t m_t$ obtained from Theorem 4.1. The following result summarizes the consequences of the spectral theory developed in [19, Section 6], which remain valid in our setting due to the structure of the family $f_{\ell,c}$.

THEOREM 4.2. *Let $t > 1$ and define $\mu := \mu_t = \psi_t m_t$. Then the following hold:*

(1) *μ is F-invariant, ergodic for all iterates of F, and equivalent to m_t. In particular, $\mu(\mathcal{J}_r(F)) = 1$.*
(2) *The dynamical system (F, μ_t) is metrically exact.*
(3) *If $g \in \mathrm{H}_\alpha$, $\alpha \in (0, 1)$, and the asymptotic variance*

$$\sigma^2(g) = \lim_{n\to\infty} \frac{1}{n} \int \Big(\sum_{j=0}^{n-1} g \circ F^j - nEg \Big)^2 d\mu_t$$

exists, then $\{g \circ F^n\}_{n\geq 0}$ satisfies the Central Limit Theorem with respect to μ_t.

One of the sets relevant for geometric analysis is the radial Julia set $\mathcal{J}_r(F)$ (see [23]). Define

$$\mathcal{J}_{bd}(F) := \Big\{ z \in \mathcal{J}(F) : \inf_{n\geq 0} \mathrm{Re}(F^n(z)) < +\infty \Big\},$$

which clearly satisfies $\mathcal{J}_{bd}(F) \subset \mathcal{J}_r(F)$. The next result shows that this subset has large Hausdorff dimension; the proof uses construction adapted to the structure of $f_{\ell,c}$.

THEOREM 4.3. *We have $\mathrm{HD}(\mathcal{J}_{bd}(F)) > 1$.*

Proof. We construct an iterated function system (IFS) H_R and estimate from below the Hausdorff dimension of its limit set $J_R \subset \mathcal{J}_{bd}(F)$.

Fix $R > 1$ and define the strip

$$S_R := \{z \in P : R \leq \mathrm{Re}(z) \leq 4R,\ \varepsilon_R < \mathrm{Im}(z) < 2\pi - \varepsilon_R\},$$

where $\varepsilon_R > 0$ is small. For each $k \geq 1$, define $F_k^{-1} : S_R \to P$ by setting $F_k^{-1}(z) := f_*^{-1}(z + 2k\pi i)$, where f_*^{-1} is the inverse of $f_{\ell,c}|_P$. Since $f(z) = \ell z + \lambda - e^z$ with $\lambda = c - (\ell - 1)\log c$, we obtain

$$z + 2k\pi i = \ell F_k^{-1}(z) + \lambda - e^{F_k^{-1}(z)}.$$

Using this, we derive the following upper and lower bounds whose derivations are straightforward:

$$|z + 2k\pi i| \leq 2e^{\mathrm{Re}(F_k^{-1}(z))}, \qquad |z + 2k\pi i| \geq \tfrac{1}{2}e^{\mathrm{Re}(F_k^{-1}(z))},$$

valid for k large. Hence, if $k \in [e^{2R}, e^{3R}]$ with $R > 1$ sufficiently large, then

$$R \leq \mathrm{Re}(F_k^{-1}(z)) \leq 4R.$$

If we take $w = x + iy \in P \setminus (S_R \cap \{R \le \mathrm{Re}(z) \le 4R\})$, we have

$$|\mathrm{Im}(f(w))| \le e^x|\sin y| + 2|y| + |\mathrm{Im}(\lambda)| \le e^{4R}\sin\varepsilon_R + 4\pi + |\mathrm{Im}(\lambda)| \le \tfrac{1}{2}e^R$$

for $\varepsilon_R > 0$ sufficiently small. Hence, taking such $\varepsilon_R > 0$ and $k \in [e^R, e^{4R}]$, we have $F_k^{-1}(S_R) \subset S_R$.

Now consider $H_R := \{F_k^{-1} : k \in [e^R, e^{4R}]\}$, an IFS acting on S_R. As in [20], this IFS satisfies the open set condition, and its limit set J_R is contained in $\mathcal{J}_{bd}(F)$. We estimate the derivatives using the inverse of $f_{\ell,c}$: If $w = f(z) = \ell z + \lambda - e^z$, then

$$(f_*^{-1})'(w) = \frac{1}{w - \ell f_*^{-1}(w) - \lambda + \ell},$$

so for $z \in S_R$ and large k,

$$\begin{aligned}|(F_k^{-1})'(z)| &= \left|\frac{1}{z + 2k\pi i - \ell f_*^{-1}(z + 2k\pi i) + \ell - \lambda}\right| \\ &\ge \frac{1}{|z + 2k\pi i| + 8R + 2\ell\pi + |\lambda|} \\ &\ge \frac{1}{9k + 8R + 2\ell\pi + |\lambda|} \ge \frac{1}{10k}.\end{aligned}$$

Then the pressure function of the IFS satisfies

$$P_R(1) = \log\Big(\sum_{k=e^{2R}}^{e^{3R}} |(F_k^{-1})'(z)|\Big) \ge \log\Big(\sum_{k=e^{2R}}^{e^{3R}} \frac{1}{10k}\Big) = -\log 10 + R > 0.$$

Therefore, $\mathrm{HD}(J_R) > 1$. Since $J_R \subset \mathcal{J}_{bd}(F)$, we get $\mathrm{HD}(\mathcal{J}_{bd}(F)) > 1$. ∎

In compact settings, the common value of the limits $\overline{P}(t, F)$ and $\underline{P}(t, F)$, denoted simply by $P(t)$ (Proposition 3.1), coincides with the classical *geometric pressure function* for $t > 1$, linking ergodic theory and geometric properties such as Hausdorff dimension, entropy, and Lyapunov exponents. However, when F is transcendental and $\mathcal{J}(F)$ is non-compact, the classical thermodynamic theory must be adapted.

Despite the lack of compactness, the family $f_{\ell,c}(z) = \ell z + c - e^z$ retains enough structure to develop a geometric pressure theory. The pressure function $P(t)$ keeps the main properties necessary to establish a Bowen-type formula.

PROPOSITION 4.4. *The function $t \mapsto P(t)$, $t \ge 0$, satisfies the following:*

(1) *There exists $t \in (0, 1)$ such that $0 \le P(t) < +\infty$.*
(2) *There exists a unique $t > 1$ such that $P(t) = 0$.*

Proof. Using the IFS H_R constructed in Theorem 4.3, we find that $\mathrm{HD}(J_R) > 1$ for R large. Since $P(\mathrm{HD}(J_R)) \ge P_R(\mathrm{HD}(J_R)) = 0$, and by continuity and monotonicity of $P(t)$ (see Proposition 3.1), item (1) follows.

Item (2) follows from the convexity of $P(t)$ and the fact that $P(t) \to -\infty$ as $t \to +\infty$. ∎

Although $\mathcal{J}(F)$ is not compact, we can still define the Lyapunov exponent $\chi(F)$ as an integral with respect to $\mu_t = \psi_t m_t$. We use the auxiliary sets $A_n := \{z \in \mathcal{J}(F) : n \leq \mathrm{Re}(z) \leq n+1\}$ to deduce the following.

Lemma 4.5. *For $t > 1$, the Lyapunov exponent $\chi(F) = \int \log|F'|\, d\mu_t$ is finite.*

Proof. Since ψ_t is bounded and decays as $\mathrm{Re}(z) \to \infty$, and using Corollary 3.6, we have

$$\int \log|F'|\, d\mu_t \preceq \sum_{n=1}^{\infty} \int_{A_n} \log|F'|\, dm_t \leq \sum_{n=1}^{\infty} m_t(E_n^c) \log(1+e^n).$$

Hence,

$$\int \log|F'|\, d\mu_t \leq \sum_{n=1}^{\infty} Ce^{-tn}(1+n) < +\infty. \ ∎$$

We are now ready to formulate the Bowen formula, adapted to the radial Julia set.

Theorem 4.6 (Bowen's formula). *The Hausdorff dimension of the radial Julia set $\mathcal{J}_r(F)$ is the unique zero $\eta > 1$ of the pressure function $t \mapsto P(t)$.*

Proof. Let $\eta > 1$ be such that $P(\eta) = 0$. For $t > \eta$ we have $P(t) < 0$, so for $z \in \mathcal{J}(F)$ and large j,

$$\frac{1}{j} \log \sum_{x \in F^{-j}(z)} |(F^j)'(x)|^{-t} \leq \frac{1}{2} P(t) < 0.$$

Standard covering arguments (cf. [19]) yield $\mathrm{HD}(\mathcal{J}_r(F)) \leq t$, and taking $t \searrow \eta$ gives $\mathrm{HD}(\mathcal{J}_r(F)) \leq \eta$.

For the reverse inequality, let $\varepsilon > 0$. By Birkhoff's and Egorov's theorems applied to the ergodic measure μ_η, there exists a set $Y \subset \mathcal{J}_r(F)$ with $\mu_\eta(Y) \geq 1/2$ such that for all $x \in Y$ and large n,

$$\left| \frac{1}{n} \log|(F^n)'(x)| - \chi(F) \right| < \varepsilon.$$

Using Koebe's distortion theorem and the conformality of $\nu := m_\eta|_Y$, one obtains $\nu(B(x,r)) \preceq r^{\eta - \frac{2\varepsilon}{\chi-\varepsilon}}$. Thus $\mathrm{HD}(Y) \geq \eta - \frac{2\varepsilon}{\chi-\varepsilon}$, and letting $\varepsilon \to 0$ gives $\mathrm{HD}(\mathcal{J}_r(F)) \geq \eta$. ∎

Proposition 4.7 ([19, Prop. 7.4]). *The h-dimensional packing measure satisfies $P^h(\mathcal{J}_r(F)) = \infty$, where $h = \mathrm{HD}(\mathcal{J}_r(F))$. In fact, $P^h(G) = \infty$ for every non-empty open subset $G \subset \mathcal{J}_r(F)$.*

Combining Theorems 4.3 and 4.6, and Proposition 4.7, we conclude:

COROLLARY 4.8. *The radial Julia set satisfies* $1 < \mathrm{HD}(\mathcal{J}_r(F)) < 2$.

5. Graphical examples. In this section, we present a few examples from the family $f_{\ell,c}$ for different values of ℓ and corresponding parameters c as an illustration of the complexity of the Julia sets, the topology of the invariant Baker domain, the immediate basin of attraction, and the pair of sets of wandering domains.

5.1. Bergweiler's example. The first example is the function studied by Bergweiler [6]. For $\ell = c = 2$ we have $f_{2,2}(z) = 2 - \log 2 + 2z - e^z$.

For this function, the fixed point $z_c = \log 2$ is also a critical value, in other words, it is a *superattracting* fixed point. This way, the post-singular set not only coincides with the singular value, but attracts all the dynamics in the immediate basin of attraction U_0 and all of its vertical translates $U_n = U_0 + 2\pi n i$; see Figure 1.

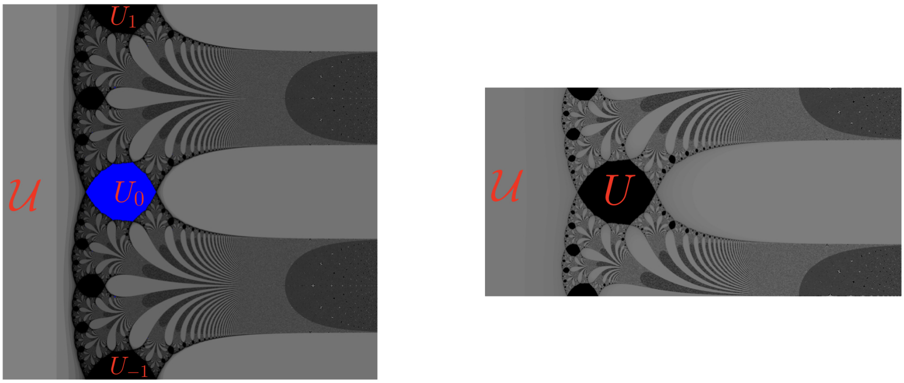

Fig. 1. Left: Dynamical plane of the function $f_{2,2}$. U_0 is the attracting domain (blue), $\mathcal{U}$ is the Baker domain (gray tones), while U_i represents the wandering domains (black). Right: The *simulated* dynamics in the quotient space. In this case, there are no wandering domains.

5.2. A general case. For this example we consider a bigger post-singular set. Take $\ell = 3$ and $c = 2.7 - 0.3i$, so

$$f_{3,c}(z) = 2.7 - 0.3i - 2\log(2.7 - 0.3i) + 3z - e^z.$$

Since the critical point $z_0 = \log 3$ is no longer a fixed point, the post-singular set is (much) bigger than the singular set. Here, the fixed point $z_c = \log(2.7 - 0.3i)$ is only *attracting*, but its immediate basin of attraction contains the critical point $z_0 = \log 3$, so the set $\{z_c + 2k\pi i : k \in \mathbb{Z}\}$ coincides with the derived set of the post-singular set; see Figure 2.

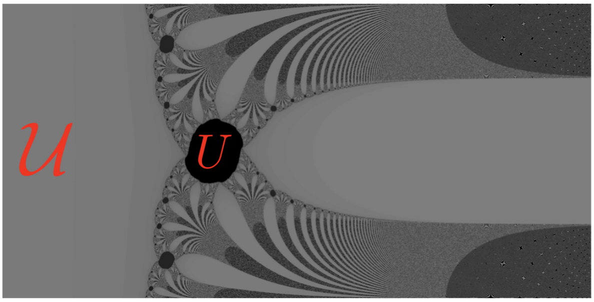

Fig. 2. The simulated dynamics of $F_{3,c}$ in the quotient space. Again, there are no wandering domains.

5.3. An example from the Fatou family. As mentioned in the Introduction, the family $f_{\ell,c}$ exhibits richer dynamical behavior than the Fatou family $f_\lambda(z) = z + \lambda - e^z$. In order for $f_{\ell,c}$ to belong to the Fatou family, we require that $\mathrm{Re}(c) < 0$. Then the left half-plane $\mathrm{Re}(z) < 0$ is contained in the invariant Baker domain, which is the only component of the Fatou set. For this example we take $\ell = 1$ and $c = -0.5$; see Figure 3.

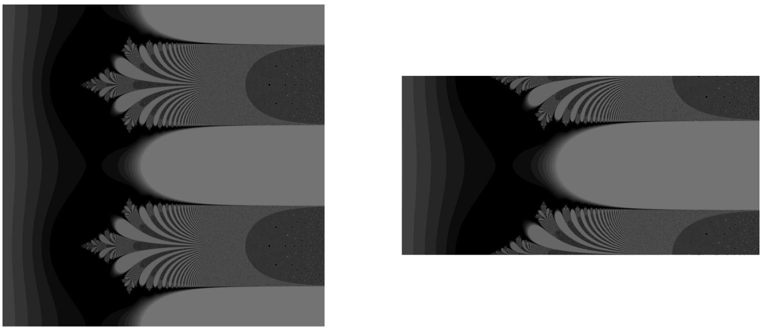

Fig. 3. Left: Dynamical plane of the function $f_{1,-1/2}$. Right: The simulated dynamics in the quotient space.

5.4. A boundary parameter case. In our setting, the Fatou family is $f_\lambda(z) = z + \lambda - e^z$, $\mathrm{Re}(\lambda) < 0$. And for $\ell = 1$, $c \in D(1, 1)$, the family $f_{1,c}$ is reduced to $f_{1,c}(z) = z + c - e^z$. It is not difficult to notice that there is one, and only one, common boundary point in the two families, corresponding to the parameter $c = 0$, giving the function $f(z) = z - e^z$. This function has been widely studied from geometric and measurable perspectives. In [14], the authors proved that this is a rigid map, that is, with a trivial deformation space. In [11], it was shown that this function can be obtained as a pinching process from the Fatou function $f_1(z) = z-1-e^z$. Unlike the functions in the Fatou family, $f_{1,0}$ possesses countably many Baker domains, each containing a unique singular value.

In terms of measure-theoretic aspects, it is well-known that the dynamics on the boundary of the Baker domains is ergodic. Moreover, the escaping set $\mathcal{I}(f)$ has zero harmonic measure. Recent results in [15] describe dynamically, topologically, and combinatorially the escaping points in the boundary of the Fatou set. It is natural to ask if the above machinery can be applied to this particular function.

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Adrián Esparza-Amador
Instituto de Ciencias Físicas y Matemáticas
Universidad Austral de Chile
Valdivia, Chile
E-mail: adrian.esparza@uach.cl

Irene Inoquio-Renteria
Departamento de Matemáticas
Universidad de La Serena
La Serena, Chile
E-mail: irene.inoquio@userena.cl