FUNDAMENTA
MATHEMATICAE
Online First version

Non-representable quantum measures

by

Alexandru Chirvasitu

Abstract. Grade-d measures on a σ-algebra $\mathcal{A} \subseteq 2^X$ over a set X are generalizations of measures satisfying one of a hierarchy of weak additivity-type conditions initially introduced as interference operators in quantum mechanics. Every signed polymeasure λ on $(X, \mathcal{A})^d$ produces a grade-d measure as its diagonal $\widetilde{\lambda}(A) := \lambda(A, \dots, A)$, and we prove that as soon as $d \geq 2$, measures (as opposed to polymeasures) do not suffice: the separate σ-additivity of a λ producing $\mu = \widetilde{\lambda}$ cannot, generally, be amplified to global σ-additivity. This amends a result in the literature, asserting the contrary in case $d = 2$.

Introduction. Recall the familiar [12, §8.3] two-slit experimental setup that has become emblematic of some of the puzzling aspects of quantum mechanics: two electrons hitting a sensitive receptor after first passing through two narrow openings in a barrier produce an interference pattern characteristic of the peak-and-trough interaction of two waves rather than the naively expected simple addition of the electrons' individual effects. [17, §2] provides a measure-theoretic interpretation of that and like phenomena, as the non-vanishing of the *interference*

$$I_1\mu(S_0, S_1) := \mu(S_0) + \mu(S_1) - \mu(S_0 \sqcup S_1) \qquad (\text{“}\sqcup\text{” denoting disjoint union})$$

of a set function $\mu \in \mathbb{R}^{\mathcal{A}}$ for a *measurable space* $(X, \mathcal{A})$ (i.e. [7, Remark 111B(c)] a set X equipped with a σ-*algebra* [7, Definition 111A] $\mathcal{A} \subseteq 2^X$).

This in turn suggests a chain of such *interference operators* ([17, (1)] or [16, §3], with a shift in indices)

$$I_d\mu(S_0, \dots, S_d) := \sum_{\ell=0}^{d} (-1)^\ell \sum_{(i_0 < \dots < i_\ell) \in \{0,\dots,d\}^{\ell+1}} \mu\Big(\bigsqcup_k S_{i_k}\Big).$$

2020 *Mathematics Subject Classification*: Primary 28A35; Secondary 28A60, 47A07, 81Q65.

Key words and phrases: bimeasure, countably additive, diagonally positive, higher-grade additivity, interference operator, measurable space, polymeasure, quantum measure.
Received 10 September 1015; revised 27 February 2026.
Published online 15 September 2026.

DOI: 10.4064/fm250910-3-4

Following [9, §5], a *grade-d measure* on a measurable space $(X, \mathcal{A})$ is a set function $\mu \in \mathbb{R}^{\mathcal{A}}$ which is both

- *grade-d additive* in the sense that $I_d\mu \equiv 0$, and
- *continuous* in the sense of [9, §2, p. 682]: $\mu(A) = \lim_n \mu(A_n)$ whenever A is the intersection (respectively union) of a non-increasing (respectively non-decreasing) sequence $(A_n)_n \subset \mathcal{A}$.

Since (by contrast to [9]) the material below does not assume grade-d measures non-negative, modifiers such as *positive* (i.e. taking non-negative values) and *signed* (real-valued) will occasionally qualify the phrase for clarity.

[9, §§4–5] revolve around tracing connections between grade-d measures and plain measures on the power $(X, \mathcal{A})^d = (X^d, \mathcal{A}^{\otimes d})$ in the usual sense [1, §3.3] of measure-space products. Write

$$\mathcal{A} \ni A \xmapsto{\widetilde{\lambda}} \lambda(A^d)$$

for the *diagonal* of a function defined on $\mathcal{A}^d$ (just the plain Cartesian product, consisting of the *cylinders* [1, Vol. 1, p. 188] $A_1 \times \cdots \times A_d$, $A_j \in \mathcal{A}$).

[9, Theorem 5.1] argues that $\widetilde{\lambda}$ is a positive grade-d measure whenever λ is a *diagonally-positive* (i.e. $\widetilde{\lambda} \geq 0$) measure on $(X, \mathcal{A})^d$.

Conversely, [9, Theorem 4.2] asserts that a grade-2 positive measure μ is uniquely of the form $\widetilde{\lambda}$ for a diagonally-positive *symmetric* (i.e. $\lambda(A \times B) = \lambda(B \times A)$) measure on $(X, \mathcal{A})^2$.

It is the latter statement that provided the motivation for the present note, for it appears to admit counterexamples (hence the paper's and Section 1's titles):

THEOREM 0.1. *For every $d \in \mathbb{Z}_{\geq 2}$ there is a positive grade-d measure μ on a measurable space $(X, \mathcal{A})$ not realizable as $\widetilde{\lambda}$ for a signed diagonally-positive measure λ on $(X, \mathcal{A})^d$.*

The gap in the proof of [9, Theorem 4.2] appears to be the jump (relegated in the text to an otherwise unspecified "standard argument") between

- the *separate* σ-additivity of the only candidate λ with $\widetilde{\lambda} = \mu$, in the sense that

$$\forall (A \in \mathcal{A}) : \lambda(A \times -), \lambda(- \times A) \text{ are } \sigma\text{-additive}, \tag{0.1}$$

and

- its (plain, global) σ-additivity on the *set algebra* [1, Definition 1.2.1]

$$\mathcal{A} \otimes_0 \mathcal{A} := \left\{ \bigcup_{i=1}^{n} A_i' \times A_i'' : A_i', A_i'' \in \mathcal{A} \right\}$$

of finite unions of cylinders.

In fact, (0.1) makes the set function λ what the literature refers to as a *bimeasure*: [18, Definition 4.1], [13, p. 234], [15, §1] or [3, §1.2] (and [4, Definition 1], for instance, for the more general d-factor *polymeasures*). The existence of non-σ-additive polymeasures (in addition to [14, §1, (ii)] see also [13, Example 2]) at the very least shows that the passage between the two types of countable additivity cannot be automatic.

What the proof of [9, Theorem 4.2] appears to (effectively) show is rather

THEOREM 0.2. *For a measurable space $(X, \mathcal{A})$ the map $\lambda \mapsto \widetilde{\lambda}$ restricts to a linear isomorphism between*

- *the space of finite symmetric signed bimeasures on $(X, \mathcal{A})^2$, and*
- *that of signed grade-2 measures on $(X, \mathcal{A})$.*

Under the said bijection the diagonal positivity of λ becomes equivalent to the positivity of the corresponding grade-2 measure.

This in turn relies on an ever so slightly enhanced [9, Theorem 5.1], reading as follows:

THEOREM 0.3. *For a measurable space $(X, \mathcal{A})$ and $d \in \mathbb{Z}_{\geq 1}$ the diagonal $\widetilde{\lambda}$ of a signed polymeasure on $(X, \mathcal{A})^d$ is a grade-d (signed) measure.*

The text below provides short arguments, indicating the few alterations that the existing proofs require. Higher-grade analogues of the above are the subject of future work [2].

1. Non-diagonal weakly additive measures. Example 1.1 below is a straightforward adaptation of [14, §1, (ii)] (which is its 2- rather than d-fold analogue) to arbitrary numbers of factors, and provides an example of

- probability spaces $(Y_i, \mathcal{B}_i, \nu_i)$, $i \in \{1, \ldots, d\}$, and
- a non-negative, additive, non-σ-additive, total-mass-1 function
$$\mathcal{B}_1 \otimes_0 \cdots \otimes_0 \mathcal{B}_d \xrightarrow{\pi} [0,1]$$
(the left-hand side denoting the algebra generated by *cylinders*, i.e. sets $\prod_{i=1}^d S_i,\ S_i \in \mathcal{B}_i$), with *marginals* [1, 9.12(vii)]

$$(1.1)\qquad \nu_j(S) := \pi(\mathrm{pr}_j^{-1}(S)), \quad \prod_{i=1}^d Y_i \xrightarrow[j\text{th projection}]{\mathrm{pr}_j} Y_j, \qquad 1 \leq j \leq d.$$

EXAMPLE 1.1. Set $Y_j := I := [0,1]$ for all $j \in \{1, \ldots, d\}$ and denote by ℓ the usual Lebesgue measure on I. Fix also a partition

$$I = \bigsqcup_{j=1}^d Z_j, \quad \ell^*(Z_j) = 1, \quad \ell^* := \text{Lebesgue } \textit{outer measure } [1, §1.5],$$

and define

$$\mathcal{B}_j := \{S \cap Z_j : S \in \mathrm{Borel}(I)\}, \qquad 1 \le j \le d.$$

Finally, for a finite disjoint union $S \in 2^{I^d}$ of Borel cylinders, π is defined by

$$\prod_{j=1}^{d} \mathcal{B}_j \ni T \cap \prod_j Z_j \xrightarrow{\pi} \ell_{\mathrm{diag}}(T) := \ell(\{t \in I : (t, \dots, t) \in T\}). \tag{1.2}$$

The argument in [14, §1, (ii)] plainly applies to show that π cannot be countably additive. Observe also that the marginals (1.1) of π are

$$\mathcal{B}_j \ni S \cap Z_j \xrightarrow{\nu_j} \ell(S),$$

indeed (countably additive) probability measures. ♦

Proof of Theorem 0.1. In the context of Example 1.1 (with its notation in place throughout the present discussion), μ is as follows.

Set $X := \bigsqcup_j Y_j$, with $\mathcal{A}|_{Y_j} = \mathcal{B}_j$, $j \in \{1, \dots, d\}$ (i.e. form the disjoint union of measurable spaces in the most straightforward fashion).

Now π can be regarded as defined on (a set algebra over) $\prod_j Y_j \subset X^d$, hence also on all of X^d, as the annihilating subsets of

$$\bigsqcup_{\substack{\text{permutations}\\ \sigma \ne \mathrm{id}}} \prod_{j=1}^{d} Y_{\sigma i} = X^d \setminus (Y_1 \times \cdots \times Y_d);$$

as such, we relabel π as λ.

Finally, take for μ the diagonal $\widetilde{\lambda}$:

$$\mu(A) := \widetilde{\lambda}(A) := \lambda(A \times \cdots \times A = A^d).$$

Grade-d additivity is immediate from the additivity of λ in each variable, so we will be done once we verify that μ is continuous in the sense recalled in the Introduction:

- [2, Lemma 1.1] (a higher-grade analogue of [9, Lemma 4.1]) recovers the *symmetrization*

  $$\frac{1}{d!} \sum_{\sigma \in \text{symmetric group } S_d} \lambda(A_{\sigma 1} \times \cdots \times A_{\sigma d})$$

  from $\widetilde{\lambda}$ for mutually disjoint $A_i \in \mathcal{A}$,
- *every*

  $$\prod_{j=1}^{d} A_j \subseteq \prod_{j=1}^{d} Y_j \subseteq X^d, \qquad A_j \in \mathcal{B}_j,$$

  has mutually disjoint components, and
- by construction the restriction of λ to $\prod_j Y_j$ is not σ-additive on $\mathcal{B}_1 \otimes_0 \cdots \otimes_0 \mathcal{B}_d$.

It is in verifying the continuity of μ that the details of how $(Y_j, \mathcal{B}_j, \nu_j)$ and π are constructed become relevant, and we refer the reader to Example 1.1 for notation.

I first claim that the set functions π of Example 1.1 are polymeasures (and hence so too is λ). Indeed, writing $A_{j'} = S_{j'} \cap Z_{j'}$ for $j' \neq j$, simply observe that

$$\pi(A_1, \dots, A_{j-1}, \bullet, A_{j+1}, \dots, A_d) = \Big(S \cap Z_j \mapsto \ell\Big(S \cap \bigcap_{j' \neq j} S_{j'}\Big)\Big) \in [0,1]^{\mathcal{B}_j},$$

plainly countably additive. [4, Theorem 1] now provides the continuity of λ in the even stronger-than-desired form

$$A_{jn} \xrightarrow[n]{\text{in } \mathcal{A}} A_j,\ 1 \leq j \leq d \implies \lambda\Big(\prod_j A_{jn}\Big) \xrightarrow[n]{} \lambda\Big(\prod_j A_i\Big), \tag{1.3}$$

where convergence $A_n \xrightarrow[n]{} A$ in a σ-algebra means

$$\bigcup_n \bigcap_{m \geq n} A_m =: \liminf_n A_n = A = \limsup_n A_n := \bigcap_n \bigcup_{m \geq n} A_m$$

(as in [10, Chapter 1, Section 2.1, Definition 2.1], say). ■

Remark 1.2. Note, however, that [9, Theorem 4.2] survives as long as one can ensure the generally absent countable additivity. One device for doing so is to have $(X, \mathcal{A})$ equipped with the marginals of λ sufficiently well-behaved: per [1, Exercise 9.12.70], countable additivity is automatic for positive grade-2 measures μ provided those marginals λ_i are *perfect* in the sense of [1, Definition 7.5.1]: for arbitrary measurable functions $(X, \mathcal{A}) \xrightarrow{f} \mathbb{R}$ and subsets $E \subseteq \mathbb{R}$ with $f^{-1}E \in \mathcal{A}$ there is a Borel $E' \subseteq E$ with $\lambda_i(f^{-1}E') = \lambda_i(f^{-1}E)$.

Proof of Theorem 0.3. We include a short(er) proof for the grade-d additivity of $\widetilde{\lambda}$ for polymeasures λ, to emphasize a useful combinatorial identity between interference operators. Writing

$$\Delta_S \nu := \nu - \nu(S \sqcup \bullet) \in \mathbb{R}^{\{T \in \mathcal{C} : T \cap S = \emptyset\}}$$

for real-valued set functions $\nu \in \mathbb{R}^{\mathcal{C}}$ defined on some class $\mathcal{C} \subseteq 2^X$, observe that

$$I_{d-1} \Delta_{S_0} \nu(S_1, \dots, S_d) = I_d \nu(S_0, S_1, \dots, S_d) - \nu(S_0).$$

In applying this to $\nu := \widetilde{\lambda}$, observe that

$$I_{d-1} \Delta_{S_0} \widetilde{\lambda}(S_1, \dots, S_d) + \widetilde{\lambda}(S_0) = I_{d-1}(\Delta_{S_0} \widetilde{\lambda} - \widetilde{\lambda}(S_0)) = 0$$

by induction (on the index d), given that the argument $\Delta_{S_0}\widetilde{\lambda} - \widetilde{\lambda}(S_0)$ of I_{d-1} (with the second term $\widetilde{\lambda}(S_0)$ regarded here as a constant set function) is a sum of polymeasures on $(X \setminus S_0, \mathcal{A}|_{X\setminus S_0})^{k\in\{1,\dots,d-1\}}$ obtained by fixing at least one and at most $d-1$ of the arguments of λ to S_0.

The only other issue on which the proof of [9, Theorem 5.1] needs an update is the fact that even though (generally) not σ-additive, polymeasures still have continuous diagonals. [4, Theorem 1] provides this and more (see (1.3) above), hence the conclusion. ∎

Proof of Theorem 0.2. Once we dispose of the first statement, the second, concerning positivity, is clearly a consequence. As to the former, Theorem 0.3 proves that $\widetilde{\bullet}$ does indeed take grade-2 signed values, its injectivity on symmetric finitely additive set functions on the algebra $\mathcal{A}\otimes_0\mathcal{A}$ of finite cylinder unions follows from the reconstruction formula [9, Lemma 4.1]

$$\mathcal{A}^2 \ni (A_1, A_2) \mapsto \tfrac{1}{2}\big(\mu(A\cup B)+\mu(A\cap B)-\mu(A\setminus B)-\mu(B\setminus A)\big), \tag{1.4}$$

and [9, proof of Theorem 4.2] already effectively handles the surjectivity of $\widetilde{\bullet}$, for it does address *separate* countable additivity. ∎

We record also the following immediate consequence of Theorem 0.2, which does not appear to be obvious a priori (especially in the signed case).

COROLLARY 1.3. *A grade-2 signed measure μ on a measurable space $(X,\mathcal{A})$ is bounded in the sense that*

$$\sup\{|\mu(A)| : A\in\mathcal{A}\} < \infty.$$

Proof. Immediate from Theorem 0.2, given that bimeasures are automatically bounded in this same sense [18, Theorem 4.5] (as, in fact, are polymeasures generally: [4, result (N) and the sentence following it]). ∎

REMARK 1.4. The countable additivity of λ defined by (1.4) on $\mathcal{A}\times\mathcal{A}$ and extended additively to the algebra $\mathcal{A}^{\otimes_0 2}$ of finite cylinder unions is reduced in [11, Théorème 4] over sufficiently well-behaved measurable spaces (*universally measurable* [8, §2] subsets of compact metrizable spaces equipped with the Borel σ-algebra) to a stronger form of boundedness than that provided by Corollary 1.3: one needs rather

$$\sup\sum_i |\lambda(A_i'\times A_i'')| < \infty, \tag{1.5}$$

the supremum being taken over all finite disjoint families $\{A_i'\times A_i''\}_i$ of cylinders. Now, a bimeasure can "misbehave" in at least two qualitatively distinct ways:

In Example 1.1, (1.5) plainly holds, since everything in sight is positive and finite, but the measurable spaces involved are not "sufficiently tame" in the sense alluded to above.

On the other hand, in [13, Example 2] that tameness does obtain: the bimeasure is of the form

$$2^{(\mathbb{Z}_{\ge 0})^2} \ni (S,T) \mapsto \sum_{\substack{s\in S\\ t\in T}} a(s,t) \in \mathbb{R},$$

where $(a(s,-))_{s\ge 0}$ is a sequence in $\ell^1(\mathbb{R})$ *unconditionally* [6, §1] but not *absolutely* convergent (such sequences always exist [6, Theorem 1] in infinite-dimensional Banach spaces). The tameness referred to above obtains here, so the bimeasures in question can only fail to be measures because (1.5) does not hold.

In the language of [4, Definition 3], say, the distinction is between finite bimeasures having finite *semivariation* (which they all do [4, Theorem 3(4)]) and their having finite (*total*) *variation*, which (by contrast to measures [5, Lemma III.1.5 and Corollary III.4.5]) some do not. ♦

Acknowledgements. I am grateful for the anonymous referee's careful reading and valuable suggestions for the improvement of the paper.

References

[1] V. I. Bogachev, *Measure Theory. Vols. I, II*, Springer, Berlin, 2007.
[2] A. Chirvasitu, *Polarized quantum measures*, 2025, in preparation.
[3] N. Dinculeanu and M. Muthiah, *Bimeasures in Banach spaces*, Ann. Mat. Pura Appl. (4) 178 (2000), 339–392.
[4] I. Dobrakov, *On integration in Banach spaces, VIII. Polymeasures*, Czechoslovak Math. J. 37 (112) (1987), 487–506.
[5] N. Dunford and J. Schwartz, *Linear operators. I. General theory*, Pure Appl. Math. 7, Interscience, New York, 1958.
[6] A. Dvoretzky and C. A. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A. 36 (1950), 192–197.
[7] D. H. Fremlin, *Measure theory. Vol. 1. The Irreducible Minimum*, Torres Fremlin, Colchester, 2004.
[8] R. K. Getoor, *On the construction of kernels*, in: Séminaire de Probabilités IX, Lecture Notes in Math. 465, Springer, Berlin, 1975, 443–463.
[9] S. Gudder, *Quantum measure theory*, Math. Slovaca 60 (2010), 681–700.
[10] A. Gut, *Probability: A Graduate Course*, 2nd ed., Springer Texts in Statist., Springer, New York, 2013.
[11] J. Horowitz, *Une remarque sur les bimesures*, in: Séminaire de Probabilités XI, Lecture Notes in Math. 581, Springer, Berlin, 1977, 59–64.
[12] R. I. G. Hughes, *The Structure and Interpretation of Quantum Mechanics*, Harvard Univ. Press, Cambridge, MA, 1989.
[13] I. Kluvánek, *Remarks on bimeasures*, Proc. Amer. Math. Soc. 81 (1981), 233–239.
[14] E. Marczewski and C. Ryll-Nardzewski, *Remarks on the compactness and non direct products of measures*, Fund. Math. 40 (1953), 165–170.
[15] M. Morse and W. Transue, *C-bimeasures Λ and their integral extensions*, Ann. of Math. (2) 64 (1956), 480–504.
[16] R. B. Salgado, *Some identities for the quantum measure and its generalizations*, Modern Phys. Lett. A 17 (2002), 711–728.

[17] R. D. Sorkin, *Quantum mechanics as quantum measure theory*, Modern Phys. Lett. A 9 (1994), 3119–3127.
[18] K. Ylinen, *On vector bimeasures*, Ann. Mat. Pura Appl. (4) 117 (1978), 115–138.

Alexandru Chirvasitu
Department of Mathematics
University at Buffalo
Buffalo, NY 14260-2900, USA
E-mail: achirvas@buffalo.edu