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On the membership of two-variable rational inner functions in spaces of Dirichlet type

by

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Abstract. We study membership of rational inner functions on the bidisk $\mathbb{D}^2$ in a scale of Dirichlet spaces considered by Bera, Chavan, and Ghara, and in higher-order variants of these spaces. We give a characterization for membership in terms of the geometric concept of contact order of a rational inner function at its singular points, and we record some consequences and variants of our main result.

1. Introduction

1.1. Preliminaries on rational inner functions.

Let $n \geq 1$ and let

$$\mathbb{D}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \colon |z_j| < 1,\ j = 1, \dots, n\}$$

denote the unit polydisk. A bounded holomorphic function $\varphi \colon \mathbb{D}^n \to \mathbb{C}$ is said to be *inner* if its non-tangential limits $\varphi^*(\zeta)$ exist at almost every point ζ belonging to the distinguished boundary

$$\mathbb{T}^n = \{\zeta = (\zeta_1, \dots, \zeta_n) \colon |\zeta_j| = 1,\ j = 1, \dots, n\},$$

and satisfy $|\varphi^*(\zeta)| = 1$ whenever the limit exists.

Inner functions play a central role in the function theory of the polydisk, as well as in the study of operators acting on spaces of holomorphic functions in $\mathbb{D}^n$ (see for instance [36, 1]). The simplest inner functions are the rational ones; in other words, the inner functions of the form $\varphi = q/p$ where $q, p \in \mathbb{C}[z_1, \dots, z_n]$. To avoid trivialities, we assume that p and q have no common factor, and that p is *stable* with respect to $\mathbb{D}^n$ meaning that its zero set $\mathcal{Z}_p$ satisfies $\mathcal{Z}_p \cap \mathbb{D}^n = \varnothing$. In fact, $\varphi \colon \mathbb{D}^n \to \mathbb{C}$ is a rational inner function if and only if φ admits the representation

$$\varphi(z) = e^{ia} z_1^{N_1} \cdots z_n^{N_n} \frac{\tilde{p}(z)}{p(z)},$$

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where $a \in \mathbb{R}$, $N = (N_1, \dots, N_n) \in \mathbb{N}$, and

$$\widetilde{p}(z_1, z_2) := z_1^{m_1} \cdots z_n^{m_n} \overline{p(1/\overline{z_1}, \dots, 1/\overline{z_n})}$$

is the reflection of a stable polynomial p (see [36, Chapter 5]).

When $n = 1$, the class of rational inner functions (abbreviated RIFs) coincides with finite Blaschke products in the unit disk $\mathbb{D}$, as can be seen by applying the fundamental theorem of algebra to the denominator polynomial p and writing it as a product of factors of the form $1 - \bar{\lambda} z$ where $\lambda \in \mathbb{D}$. In particular, a one-variable rational inner function can be written as

$$\phi(z_1) = e^{ia} z_1^N \prod_{k=1}^{n} \frac{|\lambda_k|}{\lambda_k} \frac{z_1 - \lambda_k}{1 - \bar{\lambda}_k z_1}.$$

Finite Blaschke products have been studied extensively by many authors, and they are in many ways the nicest inner functions in one variable. For instance, every finite Blaschke product continues holomorphically to a strictly larger disk which contains $\mathbb{D}$ and in particular is a smooth function on $\overline{\mathbb{D}}$. A further example of the smoothness of finite Blaschke products is provided by a well-known result of Beurling which states that if an inner function $\phi \colon \mathbb{D} \to \mathbb{C}$ has a finite Dirichlet integral, that is,

$$\int_{\mathbb{D}} |\varphi'(x + iy)|^2 \, dx \, dy < \infty,$$

then φ is a finite Blaschke product [19, Theorem 7.6.9]. We refer the reader to [21] for a comprehensive account of finite Blaschke products.

By contrast, an important feature of RIFs that distinguishes them from their one-dimensional counterparts is that they can have singularities on the distinguished boundary $\mathbb{T}^n$: these occur at points $\zeta \in \mathbb{T}^n$ where p and hence also $\tilde{p}$ vanish. Despite not being continuous on $\overline{\mathbb{D}^n}$ when singularities are present [34], RIFs do have some degree of regularity: by a remarkable theorem of Knese [28, Theorem C], they have a non-tangential limit at *every* point on the n-torus $\mathbb{T}^n$, and the value of this limit is unimodular. Thus, RIFs in two or more variables can be viewed as striking a balance between singular behavior and some residual smoothness. They exhibit a number of intimate connections with different branches of mathematics:

1. In operator theory in Hilbert function spaces (see for instance [1, 26, 13]), they serve as generators of certain classes of shift-invariant subspaces, give rise to tractable types of pluriharmonic measures [12, 4], and induce composition operators [10].
2. They feature as solutions to interpolation problems on the polydisk (see for example the celebrated Agler's Pick Interpolation Theorem [1], or the seminal work of Kosiński [32] on the three-point Nevanlinna–Pick interpolation problem on the polydisk).

3. They have connections with concepts from classical algebraic geometry, specifically, with Puiseux factorization theorems and the study of the zero sets of stable polynomials in two or more complex dimensions [16, 14, 17].
4. Lastly, RIFs feature in function theory in the guise of Hilbert and Banach spaces of holomorphic functions, and their boundary behavior has been studied via membership of RIFs in the Dirichlet spaces on the bidisk as well as the membership of their derivatives in Hardy spaces [15, 17, 9].

Our work is motivated mainly by (4), but the connections with the other directions above will be evident in what follows.

From now on, let us take $n = 2$; the two-variable theory is generally better understood. There are several reasons for this, some of which will be reviewed shortly. Two important examples of RIFs on the bidisk with boundary singularities are furnished by

$$\kappa(z_1, z_2) = \frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}$$

and

$$\psi(z_1, z_2) = \frac{4z_1^2z_2 - z_1^2 - 3z_1z_2 - z_1 + z_2}{4 - 3z_1 - z_2 - z_1z_2 + z_1^2},$$

where $(z_1, z_2) \in \mathbb{D}^2$.

The first of these functions has appeared frequently in the literature, in particular in work of Knese, whereas the second function (sometimes called the AMY example) appears in a paper by Agler, McCarthy, and Young [1]. The reader is invited to verify the discontinuous nature of these functions at the point $(1, 1)$ by examining them on the sets $\{(\zeta_1, 1)\colon \zeta_1 \in \mathbb{T}\}$ and $\{(\zeta_1, \bar{\zeta_1})\colon \zeta_1 \in \mathbb{T}\}$.

The papers [15, 16] study singularities of two-variable RIFs using several different approaches. One way of quantifying the singular nature of an RIF in $\mathbb{D}^2$ is via derivative integrability, for instance by determining the values of $\mathfrak{p} \geq 1$ such that $\frac{\partial \phi}{\partial z_j} \in L^{\mathfrak{p}}(\mathbb{T}^2)$. A main result of [15] is the complete characterization of derivative integrability in terms of a geometric quantity called contact order; see Section 3 for a formal definition. The paper [16] continues these investigations and proves, among other things, that the unimodular level sets of an RIF $\phi = \tilde{p}/p$ satisfy

$$\mathcal{C}_\alpha = \operatorname{clos}\{\zeta \in \mathbb{T}^2\colon \phi^*(\zeta) = \alpha\} = \{\zeta \in \mathbb{T}^2\colon \tilde{p}(\zeta) - \alpha p(\zeta) = 0\},$$

and can be parametrized by analytic functions. Refined local descriptions of the zero sets of the polynomials p and $\tilde{p}$ as well as $\tilde{p} - \alpha p$ have been developed in [14, 29], with a view towards applications including the analysis of RIF derivative singularities.

In higher dimensions, matters become more complicated as is explained in [17, 9]. For instance, different partial derivatives may exhibit different

derivative integrability; the level sets $\mathcal{C}_\alpha$ cannot in general be described as graphs of continuous functions; and cyclicity of a polynomial is determined not only by the size of its zero set but also its location within $\mathbb{T}^n$. In light of these facts, we restrict our attention to two-variable RIFs in the present work.

1.2. Dirichlet spaces on the bidisk. In this paper, we contribute to the study of singular behavior of RIFs by examining these functions in norms of Dirichlet type, which involve partial derivatives of an RIF. Before we present our results, we need to review several definitions of Dirichlet spaces.

Let $\vec{\alpha} \in \mathbb{R}^2$. The Dirichlet spaces $\mathfrak{D}_{\vec{\alpha}}$ on the bidisk can be defined using the Taylor series representation $f = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} a_{k\ell} z_1^k z_2^\ell$ of a holomorphic function as

$$\mathfrak{D}_{\vec{\alpha}}(\mathbb{D}^2) := \Big\{ f \in \mathcal{O}(\mathbb{D}^2) : \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} (k+1)^{\alpha_1} (\ell+1)^{\alpha_2} |a_{k\ell}|^2 < \infty \Big\}.$$

In the restricted range $\alpha_j \in (0,2)$ we have an equivalent definition of these spaces in terms of integrals, namely

$$\mathfrak{D}_{\vec{\alpha}}(\mathbb{D}^2) := \left\{ f \in \mathcal{O}(\mathbb{D}^2) : \int_{\mathbb{D}^2} \left| \frac{\partial^2 (z_1 z_2 f(z_1, z_2))}{\partial z_1 \partial z_2} \right|^2 dA_{\alpha_1}(z_1)\, dA_{\alpha_2}(z_2) < \infty \right\},$$

for a pair $\vec{\alpha} = (\alpha_1, \alpha_2) \in (0,2)^2$, where $dA_{\alpha_j}(z_j) = (1-|z_j|^2)^{1-\alpha_j} dA(z_j)$ and $dA(z_j) = dx_j dy_j/\pi$ is the normalized Lebesgue measure on the unit disk.

The spaces $\mathfrak{D}_{\vec{\alpha}}$ were introduced by Kaptanoğlu [25] and have subsequently been studied by several authors; see for instance [24, 35, 5] and the references provided there. These spaces are a natural extension of the standard scale of one-variable weighted Dirichlet spaces D_α consisting of holomorphic functions $f = \sum_{k=0}^{\infty} a_k z^k$ in $\mathbb{D}$ whose Taylor coefficients satisfy

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty.$$

The spaces $\mathfrak{D}_{\vec{\alpha}}$ can be viewed as tensor products of the one-variable spaces, $\mathfrak{D}_{\vec{\alpha}} = D_{\alpha_1} \otimes D_{\alpha_2}$, and $\mathfrak{D} = \mathfrak{D}_{(1,1)}$ is characterized in terms of its invariance with respect to automorphisms of $\mathbb{D}^2$. In particular, the papers [6, 7] provide a complete characterization of which polynomials are cyclic in $\mathfrak{D}_\alpha$. Moreover, Bickel et al. [15] studied the membership of RIFs in the Dirichlet-type spaces of the bidisk, and the membership of the partial derivatives of RIFs on the Hardy space of the bidisk. In the latter case, a complete characterization is obtained in [15] but the membership problem in $\mathfrak{D}_\alpha$ remains open for general two-variable RIFs. The following conjecture is implicit in [15], and is supported by [15, Theorem 10.2] and other results in that paper and in [9].

CONJECTURE 1. *Let $\varphi = \tilde{p}/p$ be a Rational Inner Function on the bidisk with at least one singularity. Then $\varphi \notin \mathfrak{D}(\mathbb{D}^2)$.*

A refined version of Conjecture 1 would ask for a characterization of the $\mathfrak{D}_{\vec{\alpha}}$-membership of an RIF, again in terms of contact order.

The main goal of this paper is to characterize RIF membership in a different but related scale of Dirichlet-type spaces, namely those studied by Bera, Chavan, and Ghara [8]. For $f = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} a_{kl} z_1^k z_2^\ell$ a holomorphic function on the bidisk, they consider the following spaces using the coefficient norm:

$$\mathcal{D}(\mathbb{D}^2) := \Big\{ f \in \mathcal{O}(\mathbb{D}^2) : \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} (k+\ell+1)|a_{k\ell}|^2 < +\infty \Big\}.$$

The space $\mathcal{D}(\mathbb{D}^2)$ is a Hilbert space, and possesses an equivalent integral norm

$$\begin{aligned}\|f\|^2_{\mathcal{D}(\mathbb{D}^2)} = \|f\|^2_{H^2(\mathbb{D}^2)} &+ \sup_{r<1} \int_{\mathbb{T}}\int_{\mathbb{D}} \left| \frac{\partial f}{\partial z_1}(z_1, re^{i\theta}) \right|^2 dA(z_1)\, d\theta \\ &+ \sup_{r<1} \int_{\mathbb{T}}\int_{\mathbb{D}} \left| \frac{\partial f}{\partial z_2}(re^{i\theta}, z_2) \right|^2 dA(z_2)\, d\theta.\end{aligned}$$

In our paper, we consider weighted versions, and study the membership of RIFs in these spaces. Fix $\alpha \in (0,2)$. The *mixed-norm Dirichlet-type spaces* $\mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)$ consist of holomorphic functions $f\colon \mathbb{D}^2 \to \mathbb{C}$ satisfying the norm boundedness condition

(1.1)
$$\begin{aligned}\|f\|^2_{\mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)} = \|f\|^2_{H^2(\mathbb{D}^2)} &+ \sup_{r<1} \int_{\mathbb{T}}\int_{\mathbb{D}} \left| \frac{\partial f}{\partial z_1}(z_1, re^{i\theta}) \right|^2 (1-|z_1|^2)^{1-\alpha}\, dA(z_1)\, d\theta \\ &+ \sup_{r<1} \int_{\mathbb{T}}\int_{\mathbb{D}} \left| \frac{\partial f}{\partial z_2}(re^{i\theta}, z_2) \right|^2 (1-|z_2|^2)^{1-\alpha}\, dA(z_2)\, d\theta.\end{aligned}$$

As is explained in [8], the space $\mathcal{D}$ is closely connected with cyclic analytic 2-isometries. Indeed, $\mathcal{D}$ together with its weighted counterparts features in a variant of Richter's representation theorem (see [19]) for pairs of operators [8, Theorem 2.4]. Weighted spaces corresponding to (1.1) were recently studied by Nailwal and Zalar [33], and by Torkinejad Ziarati [37], with a focus on polynomials cyclic with respect to shift operators.

The advantage of mixed-norm Dirichlet-type spaces in the present context is that in the integral representation of their norm, one integral is taken over the circle $\mathbb{T}$. If the integrand is an RIF, say $\varphi(z_1, z_2)$, then for each $\zeta_2 \in \mathbb{T}$ the slice function $z_1 \mapsto \varphi_{\zeta_2}(z_1) = \varphi(z_1, \zeta_2)$ is a finite Blaschke product, whose zeros depend on the ζ_2-variable. These mixed-norm spaces in

general differ from the mixed-partial $\mathfrak{D}_\alpha$ spaces but we will see that there do exist connections between $\mathcal{D}_\alpha$ and $\mathfrak{D}_\alpha$.

Naturally, further generalizations of $\mathcal{D}_{1,1,\alpha}$ are possible, for instance by introducing weight functions, or by considering derivatives beyond first partials. In the latter direction, we consider higher-order weighted Dirichlet spaces on the bidisk, denoted by $\mathcal{D}_{m,n}(\mathbb{D}^2)$, which are endowed with the norm

$$\|f\|^2_{\mathcal{D}_{m,n}(\mathbb{D}^2)} = \|f\|^2_{H^2(\mathbb{D}^2)} + \sup_{r<1} \int_{\mathbb{T}} \int_{\mathbb{D}} \left| \frac{\partial^m f}{\partial z_1^m}(z_1, re^{i\theta}) \right|^2 dA(z_1)\, d\theta + \sup_{r<1} \int_{\mathbb{T}} \int_{\mathbb{D}} \left| \frac{\partial^n f}{\partial z_2^n}(re^{i\theta}, z_2) \right|^2 dA(z_2)\, d\theta,$$

This norm is equivalent to the coefficient norm

$$\|f\|^2_{\mathcal{D}_{m,n}(\mathbb{D}^2)} \approx \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} (k^{2m-1} + \ell^{2n-1} + 1)|a_{k\ell}|^2.$$

For $n = m = 1$ we recover the space $\mathcal{D}(\mathbb{D}^2) = \mathcal{D}_{1,1}(\mathbb{D}^2)$ defined as above.

In the present paper we essentially resolve the problem of membership of RIFs in the spaces $\mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)$. As we will later observe, equivalently, we characterize the membership of RIFs in the intersection $\mathfrak{D}_{(\alpha,0)}(\mathbb{D}^2) \cap \mathfrak{D}_{(0,\alpha)}(\mathbb{D}^2)$ of anisotropic Dirichlet spaces. Moreover, we address the membership problem for a restricted class of RIFs in the higher-order derivative spaces $\mathcal{D}_{m,n}(\mathbb{D}^2)$, and thus obtain some further albeit somewhat circumstantial support for Conjecture 1.

2. Main results. In [15], the membership of the partial derivatives of RIFs in $H^{\mathfrak{p}}(\mathbb{D}^2)$ was completely characterized in terms of a geometric quantity called *contact order*, which informally measures the speed at which the zero sets of the slice functions $z_1 \mapsto \varphi(z_1, \zeta_2)$ and $z_2 \mapsto \varphi(\zeta_1, z_2)$ approach $\mathbb{T}$. We review this notion in detail in Section 3, but emphasize here that this is a characteristic of the underlying polynomial p.

The main result we obtain in the present paper is a characterization for the membership in $\mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)$ of RIFs. The statement is again formulated in terms of the contact order of an RIF, and is strikingly similar to the corresponding result [15, Theorem 4.1] concerning $H^{\mathfrak{p}}$-membership of partial derivatives.

THEOREM 2.1. *Let $p \in \mathbb{C}[z_1, z_2]$ be stable on $\mathbb{D}^2$. Then $\varphi = \tilde{p}/p \in \mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)$ if and only if $\alpha < 1 + 1/K$, where K is the contact order of φ.*

Let us point out that if φ has no singularities, then $K = 0$ and the statement of the theorem is trivial.

The techniques used to obtain Theorem 2.1 are reminiscent of those in [15], with some important differences in the part of the proof that establishes necessity.

One immediately deduces the following two corollaries.

COROLLARY 2.2. *Let φ be as in Theorem* 2.1. *Then $\varphi \in \mathfrak{D}_{(\alpha,0)}(\mathbb{D}^2) \cap \mathfrak{D}_{(0,\alpha)}(\mathbb{D}^2)$ if and only if $\alpha < 1 + 1/K$.*

The proof is an application of Proposition 4.5.

COROLLARY 2.3. *Let φ be an RIF. If $\alpha < \frac{1}{2} + \frac{1}{2K}$, then $\varphi \in \mathfrak{D}_\alpha(\mathbb{D}^2)$.*

Let us briefly comment on these corollaries. The sufficiency part of Corollary 2.2 is known from [15], but we underline that the necessity part was not proved in that paper. Moreover, Corollary 2.3 follows from the characterization of $H^{\mathfrak{p}}$-containment of partial derivatives. The interest here lies in the fact that our proof directly uses Dirichlet-type norms.

Another simple corollary we obtain is the following.

COROLLARY 2.4. *Let φ be an RIF. Then $\varphi \in \mathcal{D}_{1,1,1}(\mathbb{D}^2)$.*

In other words, all RIFs belong to the *mixed norm Dirichlet space* $\mathcal{D}_{1,1,1}(\mathbb{D}^2)$, whether they have singularities or not, and therefore all RIFs belong to $\mathfrak{D}_{1/2}(\mathbb{D}^2)$.

The main difficulty one faces when investigating the membership of an RIF in $\mathfrak{D}_\alpha$ is the presence of mixed partials. This leads to the question of what happens to the membership of RIFs in the higher-order Dirichlet-type spaces that we defined above. In this direction, we obtain the following result, which showcases an interesting phenomenon:

THEOREM 2.5. *Let $p \in \mathbb{C}[z_1, z_2]$ be of bidegree $(1,1)$, stable on $\mathbb{D}^2$ and vanishing only on $\tau \in \mathbb{T}^2$. Let also $\varphi = \widetilde{p}/p$. Then $\varphi \in \mathcal{D}_{n,m}(\mathbb{D}^2)$ if and only if $\max(m,n) < 2$. Moreover, the condition $n < 2$ is necessary whenever p is of bidegree $(1,k)$ and $m < 2$ is necessary whenever p is of bidegree $(k,1)$, for integers $k \geq 1$.*

2.1. Structure of the paper. Section 3 contains a brief review of concepts and results we use in our proofs, which are in turn given in Section 4. Section 5 collects some miscellaneous material, including two-variable weighted Douglas formulas and results that relate Dirichlet-space membership to properties of Agler decompositions. Finally, in Section 6 we present several examples that illustrate our results.

2.2. Notation. Throughout the paper, we shall use the notation $A \approx B$ for inequalities of the form $C_1 B \leq A \leq C_2 B$ for some positive constants $C_1, C_2 > 0$. Moreover, we denote by $A \lesssim B$ inequalities of the form $A \leq CB$ for some positive constant $C > 0$.

3. Contact order and past results. Before we discuss the proofs of the main results, we review some background and tools used in the study of RIFs. Let us first discuss the concept of *contact order*. We refer the reader to [15, 14] for in-depth treatments. Consider a rational inner function $\varphi(z_1, z_2) = \widetilde{p}(z_1, z_2)/p(z_1, z_2)$, and assume that it has a singularity $(\tau_1, \tau_2) \in \mathbb{T}^2$. In other words, we assume that $p(\tau_1, \tau_2) = 0$ and then necessarily also $\tilde{p}(\tau_1, \tau_2) = 0$.

For each $\zeta_2 \in \mathbb{T}$, the function

$$\mathbb{D} \ni z_1 \mapsto \varphi_{\zeta_2}(z_1)$$

is a finite Blaschke product, with discrete zero set $\mathcal{Z} = \{a_1(\zeta_2), \dots, a_N(\zeta_2)\} \subset \mathbb{D}$. Note that $N \leq \deg_{z_1} p$. The same holds if we fix the z_1-variable on the circle $\mathbb{T}$. As is explained in [15], the distance from the zero set of ϕ_{ζ_2} to the unit circle $\mathbb{T}$ is controlled by $\min_{1\leq j\leq N}\{1 - |a_j(\zeta_2)|\}$. As $\zeta_2 \to \tau_2 \in \mathbb{T}$, some of the zeros of the finite Blaschke product ϕ_{ζ_2} tend to τ_1. It is natural to ask at what speed this occurs. The answer is given in the following theorem.

THEOREM 3.1 ([15, Theorem 3.5]). *Let $\varphi = \tilde{p}/p$ be a rational inner function on $\mathbb{D}^2$ with $\deg \varphi = (m, n)$ and a singularity on $\mathbb{T}^2$ with z_2-coordinate τ_2. Then there exists a rational number $K > 0$ such that*

$$\epsilon(\varphi, \zeta_2) = \min_{1\leq j\leq n}\{1 - |a_j(\zeta_2)|\} \approx |\tau_2 - \zeta_2|^K,$$

*for all $\zeta_2 \in \mathbb{T}$ in a neighborhood U of τ_2. The number K is called the (z_1, τ_2)-*contact order *of φ.*

This result allows one to formally define the contact order of an RIF, even in the case of multiple singularities. The following definition makes this concrete.

DEFINITION 3.2 ([15, Definition 3.6]). Let $\varphi = \tilde{p}/p$ be a rational inner function on $\mathbb{D}^2$ with $\deg \varphi = (m, n)$ and let $\tau_{2,1}, \dots, \tau_{2,J}$ denote the distinct z_2-coordinates of the singularities of φ. Then the z_1-*contact order* of φ is defined by

$$K^1 := \max\{(z_1, \tau_{2,j})\text{-contact order of } \varphi : 1 \leq j \leq J\}.$$

The definition of the z_2-contact order K^2 of φ is analogous. The papers [16, 14] present more systematic studies of contact order. One basic result is that the z_1- and z_2-contact orders are equal, meaning that it makes sense to speak of the *contact order* $K = K^1 = K^2$ of φ. Another important property is that K is an even integer. Very roughly speaking, these result are consequences of Puiseux factorization theorems for stable polynomials.

The result that connects the contact order as an algebro-geometric quantity and the membership of partial derivatives of RIFs on the Hardy spaces $H^{\mathfrak{p}}(\mathbb{D}^2)$ is the following.

THEOREM 3.3 ([15, Theorem 4.1]). *Let $\varphi = \widetilde{p}/p$ be an RIF. Then $\frac{\partial \varphi}{\partial z_j}$, $j = 1, 2$, belong to $H^{\mathfrak{p}}(\mathbb{D}^2)$ if and only if $\mathfrak{p} < 1 + 1/K$.*

4. Proofs of the main results. First, we prove Theorem 2.1. Then, we prove Proposition 4.5. This proposition will allow us to obtain Corollaries 2.3 and 2.2. We finish this section with the proof of Theorem 2.5.

Proof of Theorem 2.1. First, we prove the sufficiency. Before we proceed, the following lemma is required. We state it for n-th order derivatives for convenience, as we will apply it to both of our proofs.

LEMMA 4.1. *For $z_1 \in \mathbb{D}, \varphi = \tilde{p}/p$ an RIF, and $n \in \mathbb{N}$ the following holds:*

$$\sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\partial^n \varphi}{\partial z_1^n}(z_1, re^{i\theta}) \right|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\partial^n \varphi}{\partial z_1^n}(z_1, e^{i\theta}) \right|^2 d\theta.$$

Proof. For $f \in H^2(\mathbb{D})$,

$$\sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta,$$

with the right-hand side interpreted in terms of non-tangential limits. We observe that

$$\frac{\partial^n \varphi}{\partial z_1^n}(z_1, e^{i\theta}) = \frac{Q_n(z_1, e^{i\theta})}{p^{n+1}(z_1, e^{i\theta})},$$

and note that the fraction is an analytic function of θ because the denominator is non-vanishing on $\mathbb{D}$, as a consequence of p being stable. Setting $f(re^{i\theta}) = \frac{\partial^n \varphi}{\partial z_1^n}(z_1, re^{i\theta})$ finishes the proof. ∎

We shall also recall the well-known Rudin–Forelli estimates (see [23, Theorem 1.7]) on the disk. For $\beta \in (-1, +\infty)$ and $\gamma \in (-\infty, \infty)$, we have

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\beta}{|1-\overline{w}z|^{2+\beta+\gamma}} \, dA(z) \approx \begin{cases} 1, & \gamma < 0, \\ -\log(1-|w|^2), & \gamma = 0, \\ (1-|w|^2)^{-\gamma}, & \gamma > 0, \end{cases}$$

as $|w| \to 1$. Now, we are ready to continue with the proof. Due to the symmetry of the RIFs, by the previous lemma and the definition of the $\mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)$-norm, it suffices to estimate from above the integral

$$I(\varphi) = \int_{\mathbb{T}} \int_{\mathbb{D}} \left| \frac{\partial \varphi}{\partial z_1}(z_1, e^{i\theta}) \right|^2 (1-|z_1|^2)^{1-\alpha} \, dA(z_1) \, d\theta.$$

Fixing the z_2-variable on the circle, we get $\varphi(z_1, \zeta_2) = \varphi_{\zeta_2}(z_1)$ where $\varphi_{\zeta_2}(z_1)$ is a finite Blaschke product in the z_1-variable. Therefore,

$$\frac{\partial \varphi}{\partial z_1}(z_1, \zeta_2) = \varphi'_{\zeta_2}(z_1), \quad z_1 \in \mathbb{D}.$$

By the formula for the derivative of a finite Blaschke product (see [15, Lemma 4.2] for details), we obtain

$$\left|\frac{\partial\varphi}{\partial z_1}(z_1,\zeta_2)\right|^2 \le \left|\sum_j |b'_{a_j(\zeta_2)}(z_1)|\right|^2 \lesssim \sum_j |b'_{a_j(\zeta_2)}(z_1)|^2,$$

where $b_{a_j}(z_1) = \frac{z_1 - a_j(\zeta_2)}{1-\overline{a_j(\zeta_2)}z_1}$ are the individual Blaschke factors making up φ_{ζ_2}. Note here that the last inequality is justified by applying the simple inequality $(a+b)^2 \le 2(a^2+b^2)$. Hence,

$$\begin{aligned}
&\iint_{\mathbb{T}\,\mathbb{D}} \left|\frac{\partial\varphi}{\partial z_1}(z_1,e^{i\theta})\right|^2 dA_\alpha(z_1)\,d\theta \\
&\qquad \le \sum_j \iint_{\mathbb{T}\,\mathbb{D}} |b'_{a_j(e^{i\theta})}(z_1)|^2 (1-|z_1|^2)^{1-\alpha}\,dA(z_1)\,d\theta \\
&\qquad = \sum_j \iint_{\mathbb{T}\,\mathbb{D}} \frac{(1-|a_j(e^{i\theta})|^2)^2(1-|z_1|^2)^{1-\alpha}}{|1-\overline{a_j(e^{i\theta})}z_1|^4}\,dA(z_1)\,d\theta \\
&\qquad = \sum_j \int_{\mathbb{T}} (1-|a_j(e^{i\theta})|^2)^2 \left(\int_{\mathbb{D}} \frac{(1-|z_1|^2)^{1-\alpha}}{|1-\overline{a_j(e^{i\theta})}z_1|^4}\,dA(z_1)\right) d\theta \\
&\qquad \approx \sum_j \int_{\mathbb{T}} (1-|a_j(e^{i\theta})|^2)^2 \cdot \frac{1}{(1-|a_j(e^{i\theta})|^2)^{1+\alpha}}\,d\theta
\end{aligned}$$

with the last asymptotic equality coming from the Rudin–Forelli estimates, after choosing $\beta = 1-\alpha$, $\gamma = 1+\alpha$ and writing $4 = 2+\beta+\gamma = 2+(1-\alpha)+(1+\alpha)$. Now assume that the singularity of the RIF that realizes its contact order is at the point $\tau = (\tau_1,\tau_2) \in \mathbb{T}^2$. From the results in [15] (in particular pp. 295–298), it is known that

$$\int_{\mathbb{T}} \sum_j (1-|a_j(e^{i\theta})|^2)^{1-\alpha}\,d\theta \approx \int_U |\tau_2-\zeta_2|^{K(1-\alpha)}\,|d\zeta_2|,$$

where U is a neighborhood of $\tau_2 \in \mathbb{T}$. The integral on the right-hand side converges if and only if $K(1-\alpha) > -1$, where K is the contact order of the RIF. This in turn, implies convergence of the integral provided $\alpha < 1+1/K$. These observations finish the sufficiency part of the proof.

For the necessity part, we need a lower estimate on the integrals in the $\mathcal{D}_{1,1,\alpha}$-norm that allows us to compare it with

$$\int_U |\tau_2-\zeta_2|^{K(1-\alpha)}\,|d\zeta_2|,$$

where U is again a small neighborhood around the τ_2-coordinate of the singularity. As before, we take a detour via one-variable functions, with the following lemma playing a crucial role.

LEMMA 4.2. *Let $B(z)$ be a finite Blaschke product with N distinct, simple zeros $a_1, \dots, a_N \in \mathbb{D}$. Then*

$$\int_{\mathbb{D}} |B'(z)|^2 (1-|z|^2)^{1-\alpha}\, dA(z) \geq (1-|a_j|^2)^{1-\alpha} \prod_{j \neq k} \left| \frac{a_k - a_j}{1 - \overline{a_k} a_j} \right|^2, \quad j = 1, \dots, N.$$

In particular,

$$\int_{\mathbb{D}} |B'(z)|^2 (1-|z|^2)^{1-\alpha}\, dA(z) \geq \max_{j=1,\dots,N} \left\{ (1-|a_j|^2)^{1-\alpha} \prod_{j \neq k} \left| \frac{a_k - a_j}{1 - \overline{a_k} a_j} \right|^2 \right\}.$$

Proof. The proof is essentially an application of the bounded point evaluation property for weighted Bergman spaces (see [23, Proposition 1.1]). Namely,

$$(1-|a_j|^2)^{3-\alpha} |B'(a_j)|^2 \leq \|B'(z)\|^2_{A^2_{1-\alpha}(\mathbb{D})}.$$

Substituting

$$|B'(a_j)| = \frac{1}{1-|a_j|^2} \prod_{j \neq k} \left| \frac{a_j - a_k}{1 - \overline{a_k} a_j} \right|,$$

we obtain the desired statements. ∎

LEMMA 4.3. *Let B be a finite Blaschke product and let B_* be a finite Blaschke product with the same zeros as B, but with each zero having multiplicity* 1. *Then, for each $\alpha \in (0,2)$,*

$$\int_{\mathbb{D}} |B'(z)|^2 (1-|z|^2)^{1-\alpha}\, dA(z) \geq \int_{\mathbb{D}} |B'_*(z)|^2 (1-|z|^2)^{1-\alpha}\, dA(z).$$

Proof. The initial finite Blaschke product can be factored as $B(z) = B_*(z) \cdot R(z)$, where R is another (potentially constant) finite Blaschke product, built using the repeated Blaschke factors of B. The *backwards shift operator*, defined as

$$S^*(f) := \frac{f(z) - f(0)}{z}, \quad z \in \mathbb{D},$$

is a contraction in $D_\alpha(\mathbb{D})$, the Dirichlet-type spaces on the unit disk. To see this, observe that

$$\begin{aligned} \|S^* f(z)\|^2_{D_\alpha(\mathbb{D})} &= \sum_{n=0}^{\infty} (n+1)^\alpha |a_{n+1}|^2 \\ &= \sum_{m=1}^{\infty} m^\alpha |a_m|^2 \leq \sum_{m=1}^{\infty} (m+1)^\alpha |a_m|^2 = \|f\|^2_{D_\alpha(\mathbb{D})}. \end{aligned}$$

Let us define the *co-analytic Toeplitz operator*

$$T_{\overline{R}} f = P_+(\overline{R} f) \colon D_\alpha(\mathbb{D}) \to D_\alpha(\mathbb{D})$$

with symbol $\overline{R}$; here P_+ denotes the orthogonal projection onto $D_\alpha(\mathbb{D})$. Since R is itself a finite Blaschke product, we can write

$$R(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

and as a consequence

$$T_{\overline{R}}(f) = \sum_{n=0}^{\infty} \overline{b_n} (S^*)^n(f).$$

By von Neumann's inequality for functions in the disk algebra, we obtain

$$\|T_{\overline{R}}(f)\|^2_{D_\alpha(\mathbb{D})} \leq \|R\|^2_{H^\infty(\mathbb{D})} \|f\|^2_{D_\alpha(\mathbb{D})}.$$

Moreover, on the circle $\mathbb{T}$ we observe that $\overline{R}B = B_*$ and therefore $T_{\overline{R}}(B) = B_*$. Setting $f = B$ finishes the proof. ∎

Remark 4.4. An earlier version of our paper imposed the additional assumption that p be locally square free, meaning that p has no repeated factors in its local Puiseux factorizations at zeros on $\mathbb{T}$. Lemma 4.3, which is likely well-known to specialists, was generously suggested to us by an anonymous referee, and has allowed us to remove this additional assumption to get a full characterization.

For the continuation of our proof, it suffices to work with only one Blaschke factor, as Lemma 4.3 allows us to use any subproduct of our initial Blaschke product. As before, we pick $j \in \{1, \ldots, n\}$ that witnesses the contact order of φ. Then the $\mathcal{D}_{1,1,\alpha}$-norm is bounded from below by

$$\begin{aligned}
&\int\!\!\int_{\mathbb{T}\,\mathbb{D}} \left| \frac{\partial \varphi}{\partial z_1}(z_1, e^{i\theta}) \right| dA_\alpha(z_1)\, d\theta \\
&\qquad \gtrsim \int\!\!\int_{\mathbb{T}\,\mathbb{D}} \left| \frac{(1 - |a_j(e^{i\theta})|^2)}{(1 - \overline{a_j(e^{i\theta})} z_1)^2} \right|^2 (1 - |z_1|^2)^{1-\alpha}\, dA(z_1)\, d\theta \\
&\qquad = \int_{\mathbb{T}} (1 - |a_j(e^{i\theta})|^2)^2 \left(\int_{\mathbb{D}} \frac{(1 - |z_1|^2)^{1-\alpha}\, dA(z_1)}{|1 - \overline{a_j(e^{i\theta})} z_1|^4} \right) d\theta \\
&\qquad \approx \int_{\mathbb{T}} (1 - |a_j(e^{i\theta})|^2)^{1-\alpha}\, d\theta \\
&\qquad \approx \int_{\mathbb{T}} |\tau_2 - e^{i\theta}|^{K(1-\alpha)}\, d\theta,
\end{aligned}$$

with the last integral converging if and only if $K(1-\alpha) > -1$, or equivalently $\alpha < 1 + 1/K$. This finishes the proof. ∎

In what follows, we will give the proofs of the two corollaries stated in the main results section, namely Corollaries 2.2 and 2.3. For this, we shall need the following.

PROPOSITION 4.5. *Let $f \in \mathcal{O}(\mathbb{D}^2)$. Then $f \in \mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)$ if and only if $f \in \mathfrak{D}_{(\alpha,0)}(\mathbb{D}^2) \cap \mathfrak{D}_{(0,\alpha)}(\mathbb{D}^2)$.*

Proof. One can observe that

$$\int_{\mathbb{D}} \sup_{r>0} \int_{\mathbb{T}} |\partial_{z_1} f(z_1, re^{i\theta})|^2 \, d\theta \, (1-|z_1|^2)^{1-\alpha} \, dA(z_1) \\ = \int_{\mathbb{D}} \|\partial_{z_1} f(z_1, re^{i\theta})\|^2_{H^2(\mathbb{D})} (1-|z_1|^2)^{1-\alpha} \, dA(z_1).$$

Due to the Littlewood–Paley formula (see e.g. [38]) and Fubini's Theorem, we obtain

$$\int_{\mathbb{D}} \|\partial_{z_1} f(z_1, re^{i\theta})\|^2_{H^2(\mathbb{D})} \, dA_\alpha(z_1) \\ = \int_{\mathbb{D}} \Big(\int_{\mathbb{D}} |\partial_{z_2} \partial_{z_1} f(z_1, z_2)|^2 (1-|z_1|^2) \, dA(z_2) \Big) \, dA_\alpha(z_1) \\ = \int_{\mathbb{D}^2} \left| \frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2) \right|^2 dA_0(z_1) \, dA_\alpha(z_2).$$

Repeating the same argument for the second integral, we arrive at

$$\int_{\mathbb{D}} \|\partial_{z_2} f(re^{i\theta}, z_2)\|^2_{H^2(\mathbb{D})} \, dA_\alpha(z_2) \\ = \int_{\mathbb{D}} \Big(\int_{\mathbb{D}} |\partial_{z_1} \partial_{z_2} f(z_1, z_2)|^2 (1-|z_1|^2) \, dA(z_1) \Big) \, dA_\alpha(z_2) \\ = \int_{\mathbb{D}^2} \left| \frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2) \right|^2 dA_0(z_2) \, dA_\alpha(z_1).$$

Summing up we obtain the desired equivalence. ∎

REMARK 4.6. One can immediately observe that if $f \in \mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)$ then $f \in \mathfrak{D}_{\alpha/2}(\mathbb{D}^2)$. To see this, we apply the geometric-arithmetic mean inequality to get

$$\|f\|^2_{\mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)} = \int_{\mathbb{D}^2} \left| \frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2) \right|^2 dA_0(z_1) \, dA_\alpha(z_2) \\ + \int_{\mathbb{D}^2} \left| \frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2) \right|^2 dA_\alpha(z_1) \, dA_1(z_2) \\ \gtrsim \int_{\mathbb{D}^2} \left| \frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2) \right|^2 (1-|z_1|^2)^{\frac{2-\alpha}{2}} (1-|z_2|^2)^{\frac{2-\alpha}{2}} \, dA(z_1) \, dA(z_2).$$

These two observations immediately yield Corollaries 2.2 and 2.3.

Proof of Theorem 2.5. From our assumption on the bidegree of p, the function $\varphi_{\zeta_2}(z_1)$ is a single Blaschke factor in the z_1-variable, that is

$$\varphi(z_1,\zeta_2)=\lambda(\zeta_2)\frac{z_1-a(\zeta_2)}{1-\overline{a(\zeta_2)}z_1},\quad z_1\in\mathbb{D},\ \zeta_2\in\mathbb{T},$$

for some $\lambda\in\mathbb{T}$. This, in turn, implies that

$$\frac{\partial^n\varphi}{\partial z_1^n}(z_1,\zeta_2)=\lambda(\zeta_2)\frac{\overline{a(\zeta_2)}^{n-1}(1-|a(\zeta_2)|^2)}{(1-\overline{a(\zeta_2)}z_1)^{n+1}}.$$

Now we will calculate the $\mathcal{D}_{n,m}$-norm of φ. By the Rudin–Forelli estimates,

$$\begin{aligned}\iint_{\mathbb{T}\,\mathbb{D}}\left|\frac{\partial^n\varphi}{\partial z_1^n}(z_1,e^{i\theta})\right|^2 dA(z_1)\,d\theta&=\iint_{\mathbb{T}\,\mathbb{D}}\left|\frac{\overline{a(e^{i\theta})}^{n-1}(1-|a(e^{i\theta})|^2)}{(1-\overline{a(e^{i\theta})}z_1)^{n+1}}\right|^2 dA(z_1)\,d\theta\\&=\int_{\mathbb{T}}|\overline{a(e^{i\theta})}|^{2n-2}(1-|a(e^{i\theta})|^2)^2\left(\int_{\mathbb{D}}\frac{dA(z_1)}{|1-\overline{a(e^{i\theta})}z_1|^{2n+2}}\right)d\theta\\&\approx\int_{\mathbb{T}}|\overline{a(e^{i\theta})}|^{2n-2}(1-|a(e^{i\theta})|^2)^{2-2n}\,d\theta\approx\int_{\mathbb{T}}(1-|a(e^{i\theta})|^2)^{2-2n}\,d\theta\\&\approx\int_U|\tau_2-e^{i\theta}|^{K(2-2n)}\,d\theta,\end{aligned}$$

with the last integral convergent if and only if $n<1+\frac{1}{2K}$. This implies that the integral diverges whenever we take derivatives of order $n\geq 2$. On the other hand, assuming that p is either of bidegree $(1,k)$ or $(k,1)$ for a positive integer $k>1$, and $\varphi\in\mathcal{D}_{n,m}$, the above estimates reveal that $n<2$ and $m<2$ respectively, making the condition necessary. This observation finishes the proof. ∎

5. An alternative approach using Agler kernels. In this section, we explore an alternative approach to the RIF membership problem by using Agler decompositions. In brief, membership of an RIF in a space of Dirichlet type is translated into integrability requirements for certain Agler kernels.

In 1988, Agler proved that every holomorphic function $f\colon\mathbb{D}^2\to\mathbb{D}$ possesses a decomposition of the form

$$1-f(z)\overline{f(w)}=(1-z_1\overline{w_1})K_1(z,w)+(1-z_2\overline{w_2})K_2(z,w),$$

where $K_1,K_2\colon\mathbb{D}^2\times\mathbb{D}^2\to\mathbb{C}$ are positive semi-definite kernel functions. For more on these kernels, see [1] and the references therein. Later, Knese [27] gave a *refined Agler decomposition* for holomorphic functions $f\colon\mathbb{D}^2\to\mathbb{D}$. Let us provide more details here. Every holomorphic function $f\colon\mathbb{D}^2\to\mathbb{D}$ has a decomposition of the form

$$f(z)-f(w)=(z_1-w_1)L_1(z,w)+(z_2-w_2)L_2(z,w)$$

for all $z,w\in\mathbb{D}^2$, where L_1,L_2 are again positive semi-definite kernel func-

tions which satisfy

$$|L_j(z,w)|^2 \le K_j(z,z)K_j(w,w) \quad \text{and} \quad L_j(z,z) = \frac{\partial f}{\partial z_j}(z).$$

The proof of [15, Theorem 10.2] combined the local Dirichlet integral, a two-variable Douglas formula, and Agler kernels. In the present article, we imitate their approach, utilizing the refined Agler kernels from the paper [27].

Let us recall here some facts about the classical Dirichlet space of the unit disk. For a function $f \in H^2(\mathbb{D})$, the Dirichlet integral $\int_{\mathbb{D}} |f'(z)|^2\, dA$ is equal to

$$\mathrm{Doug}(f) := \frac{1}{4\pi^2} \int\limits_{\mathbb{T}^2} \left| \frac{f(\zeta) - f(\eta)}{\zeta - \eta} \right|^2 |d\zeta|\, |d\eta|.$$

See [19, Chapter 1] for a proof of this fact.

In the case of the weighted Dirichlet-type spaces, a similar formula holds for the weighted Dirichlet integral, with the difference here being that we have an equivalence of semi-norms and not a strict equality. To be precise,

$$\mathcal{D}_\alpha(f) = \int\limits_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{1-\alpha}\, dA(z) \approx \int\limits_{\mathbb{T}^2} \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^{1+\alpha}}\, |d\zeta|\, |d\eta|.$$

A similar formula is established in [15] for the space $\mathfrak{D}(\mathbb{D}^2)$. Here we also prove a weighted Douglas formula for the bidisk.

Proposition 5.1. *Let $0 < \alpha \le 1$ and suppose $f \in \mathfrak{D}_\alpha(\mathbb{D}^2)$. Then*

$$\begin{aligned}\mathrm{Doug}_\alpha(f) = &\int\limits_{\mathbb{T}^2} \frac{|f(\zeta_1,0) - f(\eta_1,0)|^2}{|\zeta_1 - \eta_1|^{1+\alpha}}\, |d\zeta_1|\, |d\eta_1| \\ &+ \int\limits_{\mathbb{T}^2} \frac{|f(0,\zeta_1) - f(0,\eta_1)|^2}{|\zeta_2 - \eta_2|^{1+\alpha}}\, |d\zeta_2|\, |d\eta_2| \\ &+ \int\limits_{\mathbb{T}^4} \frac{|f(\zeta_1,\zeta_2) - f(\zeta_1,\eta_2) + f(\eta_1,\eta_2) - f(\eta_1,\zeta_2)|^2}{|\zeta_1 - \eta_1|^{1+\alpha} |\zeta_2 - \eta_2|^{1+\alpha}}\, |d\zeta|\, |d\eta|\end{aligned}$$

is an equivalent Dirichlet norm for the space $\mathfrak{D}_\alpha(\mathbb{D}^2)$.

Proof. Obviously $|f(0,0)| = |a_{00}|$. Fixing $z_2 = 0$ for the second integral, and along with the fact that $f(\cdot,0) \in \mathfrak{D}_\alpha(\mathbb{D})$, we obtain

$$\int\limits_{\mathbb{T}^2} \frac{|f(\zeta_1,0) - f(\eta_1,0)|^2}{|\zeta_1 - \eta_1|^{1+\alpha}}\, |d\zeta_1|\, |d\eta_1| \approx \sum_{k=1}^{\infty} k^\alpha |a_{k0}^2|.$$

We argue similarly for $f(0,\cdot)$. For the last term, we observe that after changing the variables to $\zeta_j = e^{i(s_j + t_j)}, \eta_j = e^{it_j}$ for $j = 1, 2$, we obtain

$$\int_{\mathbb{T}^4} \frac{|f(\zeta_1,\zeta_2)-f(\zeta_1,\eta_2)+f(\eta_1,\eta_2)-f(\eta_1,\zeta_2)|^2}{|\zeta_1-\eta_1|^{1+\alpha}|\zeta_2-\eta_2|^{1+\alpha}}\,|d\zeta|\,|d\eta|$$

$$= \int_{[0,2\pi]^2}\int_{[0,2\pi]^2} \frac{\begin{aligned}&|f(e^{i(s_1+t_1)},e^{i(s_2+t_2)})-f(e^{i(s_1+t_1)},e^{it_2})\\&\qquad -f(e^{it_1},e^{i(s_2+t_2)})+f(e^{it_1},e^{it_2})|^2\end{aligned}}{|e^{is_1}-1|^{1+\alpha}|e^{is_2}-1|^{1+\alpha}}\,dt\,ds.$$

Applying Parseval's formula twice, we obtain

$$\frac{1}{4\pi^2}\sum_{k,\ell=0}^{\infty}|a_{k\ell}|^2\int_{[0,2\pi]^2}\frac{|e^{is_1k}-1|^2|e^{is_2\ell}-1|^2}{|e^{is_1}-1|^{1+\alpha}|e^{is_2}-1|^{1+\alpha}}\,ds \approx \sum_{k,\ell=0}^{\infty}k^\alpha\ell^\alpha|a_{k\ell}|^2.$$

and the proof is complete. ∎

In what follows, we will refer to the kernel functions L_j as *refined Agler kernels*. To lighten notation, we introduce the notation $L_j((z_1,z_2),(z_1,w_2)) = L_j(z_1,z_2,w_2)$.

We lay out our approach in the following series of propositions.

PROPOSITION 5.2. *Let $\varphi=\tilde{p}/p$ be an RIF and L_j its refined Agler kernels. Then $\varphi\in\mathcal{D}_{1,1,\alpha}$ if and only if $|\zeta_j-\eta_j|^{\frac{1-\alpha}{2}}L_j(\zeta_1,\zeta_2,\eta_j)\in L^2(\mathbb{T}^3)$, $j=1,2$.*

Proof. The main tool for the proof is the one-dimensional weighted Douglas formula. Observe that

$$\int_{\mathbb{T}}\int_{\mathbb{D}}|\partial_{z_1}\varphi(z_1,e^{i\theta})|^2(1-|z_1|^2)^{1-\alpha}\,dA(z_1)\,d\theta=\int_{\mathbb{T}}\|\varphi_{\zeta_2}(z_1)\|^2_{\mathcal{D}_\alpha(\mathbb{D})}\,d\theta.$$

It is well known that

$$\|\varphi_{\zeta_2}(z_1)\|^2_{\mathcal{D}_\alpha(\mathbb{D})}\approx\int_{\mathbb{T}^2}\frac{|\varphi_{\zeta_2}(\zeta_1)-\varphi_{\zeta_2}(\eta_1)|^2}{|\zeta_1-\eta_1|^{1+\alpha}}\,|d\zeta_1|\,|d\eta_1|.$$

Now, by the refined Agler decomposition,

$$\varphi_{\zeta_2}(\zeta_1)-\varphi_{\zeta_2}(\eta_1)=(\zeta_1-\eta_1)L_1(\zeta_1,\zeta_2,\eta_1),\qquad \zeta_1,\zeta_2,\eta_1\in\mathbb{T}.$$

Substituting this into the one-dimensional Douglas formula, we obtain

$$\begin{aligned}\int_{\mathbb{T}}\int_{\mathbb{D}}&|\partial_{z_1}\varphi(z_1,e^{i\theta})|^2(1-|z_1|^2)^{1-\alpha}\,dA(z_1)\,d\theta\\&\approx\int_{\mathbb{T}}\left(\int_{\mathbb{T}^2}\frac{|\varphi_{\zeta_2}(\zeta_1)-\varphi_{\zeta_2}(\eta_1)|^2}{|\zeta_1-\eta_1|^{1+\alpha}}\,|d\zeta_1|\,|d\eta_1|\right)|d\zeta_2|\\&=\int_{\mathbb{T}}\left(\int_{\mathbb{T}^2}\frac{|\zeta_1-\eta_1|^2|L_1(\zeta_1,\zeta_2,\eta_1)|^2}{|\zeta_1-\eta_1|^{1+\alpha}}\right)|d\zeta_2|\\&=\int_{\mathbb{T}^3}|\zeta_1-\eta_1|^{1-\alpha}|L_1(\zeta_1,\zeta_2,\eta_1)|^2\,|d\zeta_1|\,|d\zeta_2|\,|d\eta_1|\end{aligned}$$

and the result follows. ∎

COROLLARY 5.3. *Let $\varphi = \widetilde{p}/p$ be an RIF and let L_j be its refined Agler kernels. Then $L_j(\zeta_1, \zeta_2, \eta_1) \in L^2(\mathbb{T}^3)$ for $j = 1, 2$.*

Proof. From Corollary 2.4, we know that every RIF belongs to the Dirichlet-type space $\mathcal{D}_{1,1,1}(\mathbb{D}^2)$. This in turn implies that the functions $L_j(\zeta_1, \zeta_2, \eta_1)$ are in $L^2(\mathbb{T}^2)$. ∎

Next we prove a similar proposition for the classical weighted Dirichlet spaces $\mathfrak{D}_\alpha(\mathbb{D}^2)$. This time we apply the weighted Douglas formula on the bidisk.

PROPOSITION 5.4. *Let φ be an RIF and L_j its refined Agler kernels. Then $\varphi \in \mathfrak{D}_\alpha(\mathbb{D}^2)$ if and only if*

$$\int\limits_{\mathbb{T}^4} \frac{|\zeta_2 - \eta_2|^{1-\alpha}}{|\zeta_1 - \eta_1|^{1+\alpha}} |L_2(\zeta_1, \zeta_2, \eta_2) - L_2(\eta_1, \eta_2, \zeta_2)|^2 \, |d\zeta| \, |d\eta| < +\infty.$$

Proof. The main tool of the proof is the refined Agler decomposition and the weighted Douglas formula for the weighted Dirichlet space of the bidisk we proved above. Set $(z_1, z_2) = (\zeta_1, \zeta_2) \in \mathbb{T}^2$ and $(w_1, w_2) = (\eta_1, \eta_2) \in \mathbb{T}^2$. From the refined Agler decomposition we obtain

$$\varphi(\zeta_1, \zeta_2) - \varphi(\zeta_1, \eta_2) = (\zeta_2 - \eta_2) L_2(\zeta_1, \zeta_2, \eta_1),$$

and

$$\varphi(\eta_1, \eta_2) - \varphi(\eta_1, \zeta_2) = (\eta_2 - \zeta_2) L_2(\eta_1, \eta_2, \zeta_2),$$

for all $\zeta, \eta \in \mathbb{T}^2$. Summing up,

$$\begin{aligned} \varphi(\zeta_1, \zeta_2) - \varphi(\zeta_1, \eta_2) + \varphi(\eta_1, \eta_2) - \varphi(\eta_1, \zeta_2) \\ = (\zeta_2 - \eta_2)(L_2(\zeta_1, \zeta_2, \eta_1) - L_2(\eta_1, \eta_2, \zeta_2)) \end{aligned}$$

for $\zeta_1, \zeta_2, \eta_1, \eta_2 \in \mathbb{T}$. The result follows after substituting into the weighted Douglas norm of $\mathfrak{D}_\alpha(\mathbb{D}^2)$. ∎

REMARK 5.5. In the weighted case, the local Dirichlet integral formula does not hold even in the one-dimensional setting. This poses the most significant obstacle in this method.

6. Examples. In this section we showcase that the standard example

$$\kappa(z_1, z_2) = \frac{2z_1 z_2 - z_1 - z_2}{2 - z_1 - z_2}, \quad z \in \mathbb{D}^2,$$

is in $\mathcal{D}(\mathbb{D}^2)$, but fails to belong to $\mathcal{D}_{m,n}(\mathbb{D}^2)$ whenever $\max(m, n) \geq 2$. Moreover, we prove that the AMY function belongs to $\mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)$ if and only if $\alpha < 5/4$. These examples are covered by our main results but we here illustrate how to recover membership by direct computations.

EXAMPLE 6.1. $\kappa \in \mathcal{D}(\mathbb{D}^2)$.

Proof. Due to symmetry of the RIF, it suffices to prove that

$$\lim_{r\to 1^-} I_r(\kappa) = \lim_{r\to 1^-} \int_{\mathbb{T}} \left(\int_{\mathbb{D}} \left| \frac{\partial \kappa}{\partial z_1}(z_1, re^{i\theta}) \right|^2 dA(z_1) \right) d\theta < +\infty.$$

Calculations show that

$$\frac{\partial \kappa}{\partial z_1}(z_1, z_2) = \frac{-2(z_2-1)^2}{(2-z_1-z_2)^2}$$

for all $(z_1, z_2) \in \mathbb{D}^2$. Hence

$$I_r(\kappa) = \int_{\mathbb{T}} |re^{i\theta} - 1|^4 \left(\int_{\mathbb{D}} \frac{1}{|2 - re^{i\theta} - z_1|^4}\, dA(z_1) \right) d\theta.$$

Set $w = 2 - re^{i\theta}$ and observe that $|w| \geq 1$, which implies that $1/|w| \leq 1$. Factoring out w from the denominator in the inner integral we get

$$I_r(\kappa) = \int_{\mathbb{T}} \frac{|re^{i\theta} - 1|^4}{|w|^4} \left(\int_{\mathbb{D}} \frac{1}{|1 - z_1/w|^4}\, dA(z_1) \right) d\theta.$$

We now observe that

$$\int_{\mathbb{D}} \frac{1}{|1 - z_1/w|^4}\, dA(z_1) = \langle k_{1/\overline{w}}(z_1), k_{1/\overline{w}}(z_1) \rangle = \frac{1}{(1 - |1/w|^2)^2} = \frac{|w|^4}{(|w|^2 - 1)^2}.$$

Substituting this inside the integral yields

$$I_r(\kappa) = \int_0^{2\pi} \frac{|re^{i\theta} - 1|^4}{(|2 - re^{i\theta}|^2 - 1)^2}\, d\theta = \int_0^{2\pi} \frac{(r^2 - 2r\cos\theta + 1)^2}{(r^2 - 4r\cos\theta + 3)^2}\, d\theta < +\infty$$

and the result follows. ∎

EXAMPLE 6.2. $\kappa \notin \mathcal{D}_{n,m}(\mathbb{D}^2)$ for $\max(n, m) \geq 2$.

Proof. We shall follow the approach and the calculations of [15, Example 12.1]. We will briefly navigate the reader through the calculations by writing out the important estimates. We will omit the majority of the details, as the interested reader can find them in [15]. The function κ has the Taylor series representation

$$\kappa(z_1, z_2) = a_{00} + \sum_{k,\ell=0}^{\infty} a_{(k+1)(\ell+1)} z_1^k z_2^\ell,$$

and its coefficients satisfy

$$a_{(k+1)(\ell+1)} = \binom{k+\ell}{k} 2^{-(k+\ell)} - \binom{k+\ell+2}{k+1} 2^{-(k+\ell+2)} \approx \frac{1}{(k+\ell)^{3/2}},$$

with the last equality being an application of Stirling's approximation. For sufficient large indices k, ℓ, we take the tail of the sum and we find that

$$\begin{aligned}\|\kappa\|^2_{\mathcal{D}_{n,m}(\mathbb{D}^2)} &\geq \sum_{k\geq N}^{\infty}(k^{2n-1}+k^{2m-1})\frac{1}{k^{5/2}}\\ &\approx \sum_{k\geq N}^{\infty}\frac{1}{k^{-2n+1+5/2}}+\sum_{k\geq N}^{\infty}\frac{1}{k^{-2m+1+5/2}},\end{aligned}$$

for N sufficiently large. Immediately, one observes that we lose convergence whenever at least one of n, m is greater than $5/4$. Restricting to integer values, we see that we lose membership whenever we differentiate κ the second time in either z_1 or z_2. ∎

EXAMPLE 6.3. Consider $\kappa = \frac{2z_1z_2-z_1-z_2}{2-z_1-z_2}$. If $\alpha < 1/2$, then $\varphi \in \mathfrak{D}_\alpha(\mathbb{D}^2)$.

In this example we will consider the refined Agler decomposition of κ, following the approach of Proposition 5.2. We will prove that for all $2\alpha < 1$, the RIF κ belongs to $\mathcal{D}_{1,1,2\alpha}(\mathbb{D}^2)$. We have

$$L_1(\zeta_1,\zeta_2,\eta_1) = \frac{2(\zeta_2-1)(1-\zeta_2)}{(2-\zeta_1-\zeta_2)(2-\eta_1-\zeta_2)} = \frac{-2(1-\zeta_2)^2}{(2-\zeta_1-\zeta_2)(2-\eta_1-\zeta_2)}.$$

Now consider $\alpha < 1/2$. Then $1-2\alpha > 0$ and $|\zeta_1-\eta_1|^{1-2\alpha} \leq 2^{1-2\alpha}$ by the triangle inequality. Hence,

$$\begin{aligned}&\int_{\mathbb{T}^3}|\zeta_1-\eta_1|^{1-2a}|L_1(\zeta_1,\zeta_2,\eta_1)|^2\,|d\zeta_1|\,|d\zeta_2|\,|d\zeta_3|\\ &\qquad= \int_{\mathbb{T}^3}\frac{4|\zeta_1-\eta_1|^{1-2\alpha}|1-\zeta_2|^4}{|2-\zeta_1-\zeta_2|^2|2-\eta_1-\zeta_2|^2}\,|d\zeta_1|\,|d\zeta_2|\,|d\zeta_3|\\ &\qquad\lesssim \int_{\mathbb{T}^3}\frac{|1-\zeta_2|^4\,|d\zeta_1|\,|d\zeta_2|\,|d\eta_1|}{|2-\zeta_1-\zeta_2|^2|2-\eta_1-\zeta_2|^2}\\ &\qquad\approx \int_{\mathbb{T}}|1-\zeta_2|^4\left(\int_{\mathbb{T}}\frac{|d\zeta_1|}{|2-\zeta_1-\zeta_2|^2}\right)^2|d\zeta_2|\\ &\qquad\approx \int_{\mathbb{T}}\frac{|1-\zeta_2|^4}{(|2-\zeta_2|^2-1)^2}\,|d\zeta_2|.\end{aligned}$$

Note that $|1-\zeta_2|^4 = |1-e^{i\theta}|^4 = (2|\sin(\theta/2)|)^4 = 16\sin^4(\theta/2)$. Hence,

$$\int_{\mathbb{T}}\frac{|1-\zeta_2|^4}{(|2-\zeta_2|^2-1)^2}\,|d\zeta_2| = \int_0^{2\pi}\frac{16\sin^4(\theta/2)}{64\sin^4(\theta/2)}\,d\theta = \frac{1}{4} < \infty.$$

This means that $\kappa \in \mathcal{D}_{1,1,2\alpha}(\mathbb{D}^2)$ for all $\alpha < 1$. By Proposition 5.4, we then find that $\kappa \in \mathfrak{D}_\alpha(\mathbb{D})$ for all $\alpha < 1/2$. It is worth noting here that the sharp constant α for which we have membership of κ in the spaces $\mathfrak{D}_\alpha$ is $\alpha = 3/4$

(see [15, Example 12.1]). Nevertheless, this method aligns with the fact that all RIFs belong to $\mathcal{D}_{1,1}(\mathbb{D}^2)$ and, as a consequence, to $\mathfrak{D}_{1/2}(\mathbb{D}^2)$.

Example 6.4 (AMY example). Let

$$\psi(z_1, z_2) = \frac{4z_1^2 z_2 - z_1^2 - 3z_1 z_2 - z_1 + z_2}{4 - 3z_1 - z_2 - z_1 z_2 + z_1^2}.$$

Then $\psi \in \mathcal{D}_{1,1,\alpha}(\mathbb{D}^2)$ if and only if $\alpha < 5/4$.

We need to estimate, from above and below, the integrals

$$I_1 = \int\int_{\mathbb{T}\,\mathbb{D}} \left| \frac{\partial \psi}{\partial z_1}(z_1, \zeta_2) \right|^2 dA_\alpha(z_1)\, d\theta,$$

$$I_2 = \int\int_{\mathbb{T}\,\mathbb{D}} \left| \frac{\partial \psi}{\partial z_1}(\zeta_1, z_2) \right|^2 dA_\alpha(z_2)\, d\theta.$$

The sufficiency part is just an application of the sufficiency part of Theorem 2.1. For the necessity part, one needs to study the branches of the zero set of ψ for both cases, i.e. whenever the z_1-variable is fixed on the circle and whenever the z_2-variable is fixed on the circle.

The branches of the zero set of the slice $\psi_{\zeta_2}(z_1)$ are

$$z_1 = a_1(z_2) = \frac{3z_2 + 1 + \sqrt{-7z_2^2 + 10z_2 + 1}}{2(4z_2 - 1)}$$

and

$$z_1 = a_2(z_2) = \frac{3z_2 + 1 - \sqrt{-7z_2^2 + 10z_2 + 1}}{2(4z_2 - 1)};$$

see Figure 1 for a visualization. The contact order of the singularity $(1, 1)$ for the AMY function is well known to be $K = 4$ [15, Example 2].

We estimate the integral I_1 from below and obtain

$$\begin{aligned} &\int\int_{\mathbb{T}\,\mathbb{D}} \left| \frac{\partial \psi}{\partial z_1}(z_1, \zeta_2) \right|^2 dA_\alpha(z_1)\, d\theta \\ &\qquad \geq \max_{j=1,2} \int_{\mathbb{T}} (1 - |a_j(\zeta_2)|^2)^{1-\alpha} \rho_{\mathbb{D}}(a_1(\zeta_2), a_2(\zeta_2))\, |d\zeta_2| \\ &\qquad \approx \int_U |1 - \zeta_2|^{4(1-\alpha)}\, |d\zeta_2|, \end{aligned}$$

where U is a neighborhood around 1. This integral evidently converges if and only if $\alpha < 5/4$.

The integral I_2 is associated with a one-branch zero set, corresponding to having only one Blaschke factor. The reader is invited to check that the integral I_2 also converges if and only if $\alpha < 5/4$. Combining these with the

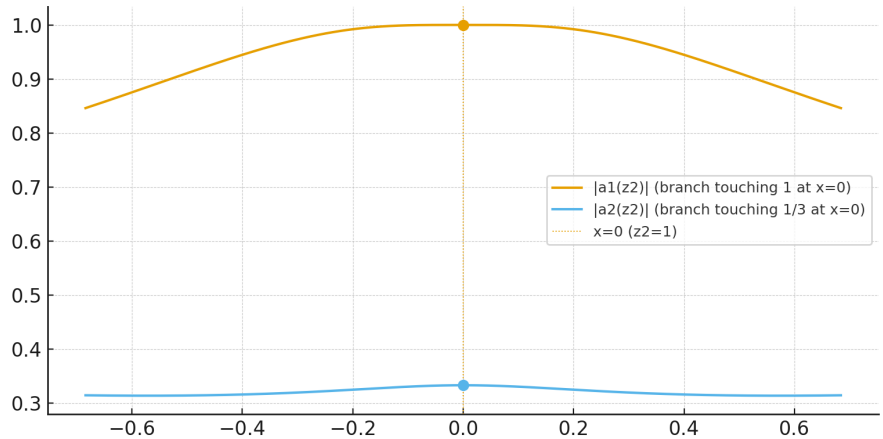


Fig. 1. Graphs of the functions $|a_1(\zeta_2)|$ and $|a_2(\zeta_2)|$ on a neighborhood of the origin after mapping the ζ_2-variable on the real line. We observe that as $z_2 \to 1$, we have $\rho_{\mathbb{D}}(a_1(\zeta_2), a_2(\zeta_2)) \to 1$ because $a_1(1) = 1$ and $a_2(1) = 1/3$.

sufficiency estimates, we get a sharp cut-off for α and membership of the AMY function in $\mathcal{D}_{1,1,\alpha}$.

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