ANNALES
POLONICI MATHEMATICI
Online First version

On the connectedness of the boundary of q-complete domains

Rafael B. Andrist

Abstract. The boundary of every relatively compact Stein domain in a complex manifold of dimension at least 2 is connected. No assumptions on the boundary regularity are necessary. The same proofs hold also for q-complete domains, and in the context of almost complex manifolds as well.

1. Introduction. When we refer to a *domain*, we always mean an open and connected subset of a manifold. It seems to have been well-known to the experts in the 1980s that every bounded strictly pseudoconvex domain with $\mathcal{C}^2$-smooth boundary in $\mathbb{C}^n, n \geq 2$, has connected boundary. In fact, already in 1953 Serre pointed out that every Stein manifold of dimension at least 2 has only one end (see Section 5). For a relatively compact Stein domain that admits a collar, this would already imply the connectedness of the boundary. However, to the best knowledge of the author, the earliest publication mentioning that every bounded strictly pseudoconvex domain with $\mathcal{C}^2$-smooth boundary in $\mathbb{C}^n, n \geq 2$, has connected boundary, is due to Rosay and Stout in 1989 [21, Corollary, p. 1018], who actually prove a stronger result. Again, the connectedness of the boundary was noted by Balogh and Bonk [1, p. 513] under the same assumptions. In the monograph of Stout [25, Corollary 2.4.7] it was established that every relatively compact, strictly pseudoconvex domain with $\mathcal{C}^2$-smooth boundary in a Stein manifold of dimension at least 2 has connected boundary. The proof given there makes use of a theorem of Forstnerič [7] about complements of Runge domains.

The almost complex case, but with a $\mathcal{C}^\infty$-smooth defining J-plurisubharmonic function on a neighborhood of the closure of the relatively compact domain, is treated by Bertrand and Gaussier [2].

2020 *Mathematics Subject Classification*: Primary 32F27; Secondary 32F10, 32Q60.
Key words and phrases: pseudoconvex domain, ends of a manifold, Morse theory, almost complex manifold.
Received 27 September 2025; revised 24 February 2026.
Published online 18 May 2026.

For the special case of bounded pseudoconvex domains in $\mathbb{C}^n$, a proof without any assumptions on the boundary regularity can be found in the second edition of the textbook of Jarnicki and Pflug [16, Corollary 2.6.10]. It relies mainly on a topological result that was provided by Czarnecki, Kulczycki and Lubawski [4] with an elementary proof: for a bounded domain in $\mathbb{R}^n$ the connectedness of its complement is equivalent to the connectedness of its boundary. A very similar proof is given by Izzo [15], who uses an elegant homology argument that relies only on $H_1(\mathbb{R}^n; \mathbb{Z}) = 0$ to obtain the above-mentioned topological fact.

We will prove the natural generalization of these results without any assumption on boundary regularity and on the ambient space.

Definition 1.1 ([8, Definition 3.1.3]). Let X be a complex manifold. We say that a domain $\Omega \subset X$ is *q-complete* if there exists a $\mathcal{C}^2$-smooth exhaustion function $\varphi\colon \Omega \to [0, +\infty)$ that is *q-convex*, i.e. its Levi form has at most $q-1$ negative or zero eigenvalues at each critical point in Ω.

Theorem 1.2. *Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$ and let $\Omega \subset X$ be a relatively compact q-complete domain with $n > q$. Then the boundary $\partial\Omega$ is connected.*

By a result of Grauert [12], Ω is Stein if and only if it is 1-complete, and thus we obtain the following corollary.

Corollary 1.3. *Let X be a complex manifold with $\dim_{\mathbb{C}} X = n > 1$ and let $\Omega \subset X$ be a relatively compact Stein domain. Then the boundary $\partial\Omega$ is connected.*

Since a domain $\Omega \subset X$ in a Stein manifold X is a domain of holomorphy if and only if it is pseudoconvex, we also arrive at the next corollary.

Corollary 1.4. *Let X be a Stein manifold with $\dim_{\mathbb{C}} X = n > 1$ and let $\Omega \subset X$ be a relatively compact domain of holomorphy. Then the boundary $\partial\Omega$ is connected.*

The proof also extends to the almost complex situation.

Theorem 1.5. *Let (X, J) be an almost complex manifold with $\dim_{\mathbb{R}} X \geq 4$, and let $\Omega \subset X$ be a relatively compact domain with a $\mathcal{C}^2$-smooth strictly J-plurisubharmonic exhaustion function. Then the boundary $\partial\Omega$ is connected.*

Remark 1.6. The result is sharp in the following sense:

(1) In one dimension, every bounded domain of $\mathbb{C}$ is pseudoconvex – and in fact a domain of holomorphy – but obviously need not have connected boundary. If we assume in addition that the domain is simply connected, its boundary will be connected (by the result in [4] mentioned above and

the fact that the complement has only one connected component), but in general not path-connected; e.g. consider

$$\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1,\ \sin(1/\operatorname{Re} z) - 2 < \operatorname{Im} z < \sin(1/\operatorname{Re} z) + 2\}.$$

(2) Unbounded domains, even when simply connected, need not have connected boundary; e.g. an infinite strip in $\mathbb{C}$. By taking direct products, this yields also counterexamples in higher dimensions.

REMARK 1.7. For symplectic manifolds with contact type boundaries, the boundary need not be connected; see McDuff [18] and Geiges [10].

This short note is organized as follows: In Section 2 we present the basic notions of Morse theory, and in Section 3 we first give the proof for relatively compact, strictly pseudoconvex domains with $\mathcal{C}^2$-smooth boundary in a Stein manifold. We could not find a reference for this proof which was communicated to us by Franc Forstnerič.

In Section 4 we provide the topological background for the theory of ends and of continua. In Section 5 we give the proofs for the general situation.

2. Morse theory. This section serves as a very brief overview of the basics of Morse theory and is based on the monograph of Nicolaescu [20].

DEFINITION 2.1. Let X be a locally compact Hausdorff space. An *exhaustion* (*by compacts*) of X is a sequence of compacts $K_j \subseteq X$, $j \in \mathbb{N}$, such that $\bigcup_{j\in\mathbb{N}} K_j = X$ and $K_j \subset (K_{j+1})^\circ$ for every $j \in \mathbb{N}$.

DEFINITION 2.2. Let X be a locally compact Hausdorff space. A continuous function $\varphi\colon X \to \mathbb{R}$ is called *exhaustive*, or an *exhaustion function*, if the closed sublevel sets

$$X^c := \{x \in X : \varphi(x) \le c\}$$

are compact for every $c < \sup_{x\in X} \varphi(x)$.

For any strictly increasing sequence $(c_j)_j \subset \mathbb{R}$ such that $\lim_{j\to\infty} c_j = \sup_{x\in X} \varphi(x)$, the closed sublevel sets X^{c_j} of the exhaustion function φ form an exhaustion by compacts of X.

DEFINITION 2.3. Let X be a connected smooth manifold. A function $\varphi\colon X \to \mathbb{R}$ of class $\mathcal{C}^2$ is called a *Morse function* if all of its critical points are *non-degenerate*, i.e. the Hessian of φ is non-degenerate at all of its critical points. The *Morse index* of a critical point p of φ is the index of its Hessian at p, i.e. the number of negative eigenvalues of the Hessian at p.

On a smooth manifold, Morse functions are abundant. In fact, small perturbations of functions of class $\mathcal{C}^2$ are Morse, see [20, Section 1.2]. The topology of a manifold can be reconstructed from its Morse function as shown in Theorems 2.4 and 2.6 below.

Theorem 2.4 ([20, Theorem 2.6]). *Suppose that the interval* $[a,b] \subset \mathbb{R}$ *contains no critical values of* φ. *Then the sublevel sets* X^a *and* X^b *are diffeomorphic. Furthermore,* X^a *is a deformation retract of* X^b, *so that the inclusion* $X^a \hookrightarrow X^b$ *is a homotopy equivalence.*

A *k-handle* of dimension n is the manifold with corners $H_{k,n} = \overline{B}^k \times \overline{B}^{n-k}$, where $\overline{B}^k$ is the closed unit ball in $\mathbb{R}^k$. The closed ball $\overline{B}^k \times \{0\} \subset H_{k,n}$ is called the *core*, while the closed ball $\{0\} \times \overline{B}^{n-k} \subset H_{k,n}$ is called the *cocore*.

Attaching a k-handle to $X^{c-\varepsilon}$ is done be identifying $\partial B^k \times \overline{B}^{n-k}$ with an embedded copy of $\partial B^k \times \overline{B}^{n-k}$ in $\partial X^{c-\varepsilon}$ such that the cocore is mapped to the normal direction of the embedded boundary of the core.

Remark 2.5. Note that the boundary of the core has two components if $k = 1$, but has only one component if $k > 1$. Hence, for $k > 1$, the k-handle is necessarily glued to one connected component of $X^{c-\varepsilon}$. See Figures 2 and 1.

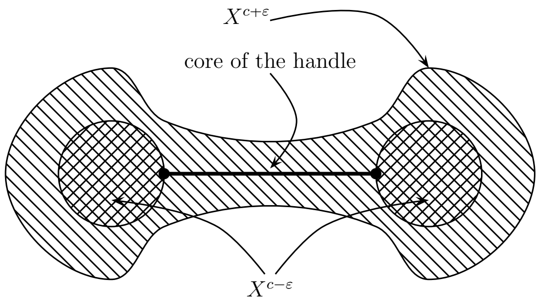

Fig. 1. Glueing a 1-handle to two different connected components: the complement remains connected.

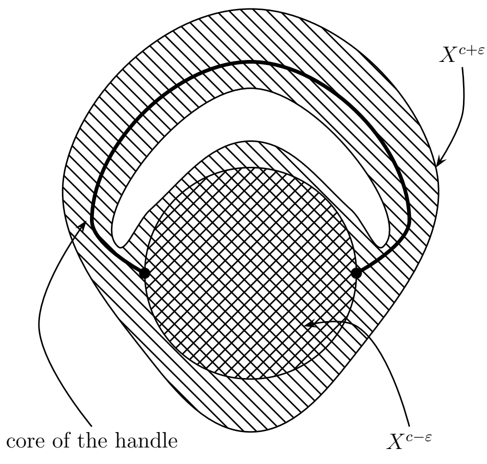

Fig. 2. Glueing a 1-handle to B^2: the complement is dissected into two connected components.

The following is the fundamental structure theorem of Morse theory. It describes the change of the topology at the critical point of a Morse function caused by glueing a k-handle. The type of the handle is determined by the Morse index at a critical point.

THEOREM 2.6 ([20, Theorem 2.7]). *Suppose c is a critical value of φ containing a single critical point p of Morse index k. Then for every $\varepsilon > 0$ sufficiently small, the sublevel set $\{\varphi \le c+\varepsilon\}$ is diffeomorphic to $\{\varphi \le c-\varepsilon\}$ with a k-handle of dimension m attached. Moreover, $\{\varphi \le c+\varepsilon\}$ is homotopy-equivalent to $\{\varphi \le c-\varepsilon\}$ with a closed ball $\overline{B}^k$ attached.*

Note that if φ is a Morse function, then so is $-\varphi$. We can therefore apply Theorems 2.4 and 2.6 also to superlevel sets of φ, but we need to take into account that instead of glueing a k-handle at a critical point p of φ, we glue an $(n-k)$-handle at the critical point p of $-\varphi$.

Although the following corollary is well-known, we include a proof due to the lack of a reference.

COROLLARY 2.7. *If a connected smooth manifold X of dimension n admits an exhaustive Morse function $\varphi\colon X \to \mathbb{R}$ with Morse indices $< n-1$, then $X \setminus X^c$ is connected for every $c \in \mathbb{R}$.*

Proof. Let $c_0 < c_1 < \cdots$ be the (finite or countably infinite) list of critical values of φ. Since φ is exhaustive, it has a global minimum at c_0, i.e. Morse index 0. When passing a critical point c_j of Morse index 0, we add a disjoint closed ball, and the complement in X remains connected. When passing a critical point c_j of Morse index > 0, we glue a k-handle to $X^{c_j-\varepsilon}$ for $\varepsilon > 0$ small enough and obtain $X^{c_j+\varepsilon}$. The complement $X^{c_j+\varepsilon} \setminus X^{c_j-\varepsilon}$ is homotopy-equivalent to its strong deformation retract which is obtained by glueing just the k-dimensional core of the handle. Since $k < n-1$, the connectedness of the complement remains unchanged. (Alternatively, we could also consider $-\varphi$ and note that the connectivity of the complement only changes if a 1-handle with respect to $-\varphi$ is attached; see Remark 2.5.) ■

3. A classical proof for smooth boundary. The following proof for the situation where Ω is a relatively compact, strongly pseudoconvex domain with $\mathcal{C}^2$-smooth boundary in a Stein manifold was communicated to us by Franc Forstnerič. This is likely the classical proof that has been known at least since the 1980s.

Consider a relatively compact domain $\Omega = \{\rho < c\}$ in a Stein manifold X of complex dimension $n > 1$, and assume that ρ is a strongly plurisubharmonic Morse function on a neighborhood of $\overline{\Omega}$, with $d\rho \neq 0$ on $\partial\Omega$. The topology of Ω^t and $\partial\Omega^t$ only changes when t passes a critical level set of ρ. (The local normal form of a strongly plurisubharmonic Morse function at an isolated critical point was known in principle since 1924 due to a result of Takagi [26] which was recovered by Schur [22] and by Harvey and Wells [13]: its Morse index is at most n.) When passing a local minimum, a new connected component of Ω^t appears, which is not a concern. The only other type of point which can change the connectivity of Ω^t is a critical point of index 1. At such a point, we add a 1-handle to Ω^t. There are two possibilities – either this handle is attached with both ends to the same component of Ω^t, or it joins two distinct components. In both cases we see by inspection that the boundary of any connected component remains connected.

Handles of higher index up to $2n-2$ do not change the connectivity of the domain or its boundary: the core of a handle of index k is $\overline{B}^k$ which is attached with its boundary $(k-1)$-sphere to $\partial\Omega^t$. Removing a submanifold of real codimension ≥ 2 from a manifold does not disconnect the manifold: in our case, we apply this argument to the boundary sphere of $\overline{B}^k$ as a submanifold of $\partial\Omega^t$.

The connectivity of $\partial\Omega^t$ would change by attaching a handle of index $2n-1$ (i.e. of real codimension 1), but this is not allowed since ρ is plurisubharmonic and $n \geq 2$.

4. Ends of topological spaces. The definition of *ends* of a topological space goes back to Freudenthal [9] and has led to a well-developed theory; see the textbook of Hughes and Ranicki [14].

Definition 4.1.

(1) A *neighborhood of an end* in a non-compact topological space X is a subspace $U \subset X$ which contains a component of $X \setminus K$ for a non-empty compact subspace $K \subset X$.
(2) An *end* e of X is an equivalence class of sequences $X \supset U_1 \supset U_2 \supset \cdots$ of connected open neighborhoods of an end such that

$$\bigcap_{i=1}^{\infty} \overline{U_i} = \emptyset$$

subject to the equivalence relation

$$(X \supset U_1 \supset U_2 \supset \cdots) \sim (X \supset V_1 \supset V_2 \supset \cdots)$$

if for each U_i there exists j with $U_i \supseteq V_j$, and for each V_j there exists i with $V_j \supseteq U_i$.

Remark 4.2 ([14, Example 3(i) in the Introduction]). Let X be a topological space with a proper map $\varphi\colon X \to [0,+\infty)$ which is onto, and such that the inverse images $U_t = \varphi^{-1}(t,+\infty) \subseteq X$, $t \geq 1$, are connected. Then X has one end.

LEMMA 4.3. *Let X be a connected, locally connected, locally compact and σ-compact Hausdorff space. Then X has one end if and only if there exists an exhaustion by compacts $(K_j)_{j\in\mathbb{N}}$ of X such that $X \setminus K_j$ is connected for every $j \in \mathbb{N}$.*

Proof. This follows from [14, Proposition 1.19 and Remark 1.15]. ∎

DEFINITION 4.4 (Kuratowski [17, §47]). A *continuum* is a non-empty, compact, connected Hausdorff space.

There are different notions of continua considered in the literature, and sometimes a continuum is required to be a metric space, e.g. in Nadler [19]. For our application, only the following lemma is needed, which holds for the definition given above.

LEMMA 4.5 ([17, §47], [19, Theorem 1.8]). *The intersection of a decreasing sequence of continua is a continuum.*

PROPOSITION 4.6. *Let X be a manifold with countable basis of topology and let $\Omega \subset X$ be a relatively compact domain with one end. Then Ω has connected boundary.*

Proof. We apply Lemma 4.3 to obtain an exhaustion by compacts K_j such that $\Omega \setminus K_j$ is connected for every $j \in \mathbb{N}$. Then $(\overline{\Omega \setminus K_j})$ constitutes a decreasing sequence of continua. Thus its limit, which is the boundary $\partial\Omega$, is also a continuum by Lemma 4.5. ∎

REMARK 4.7. If the boundary $\partial\Omega$ of the relatively compact domain Ω admits a *collar*, i.e. a neighborhood of $\partial\Omega$ with a homeomorphism to $\partial\Omega \times [0,1)$ that takes $\partial\Omega$ to $\partial\Omega \times \{0\}$, then we have a one-to-one correspondence between ends and boundary components.

5. Proofs. It was already noted by Serre that a Stein manifold of complex dimension at least 2 has only one end [23, p. 59]; see also [24]. His short cohomological argument for this fact is given with more details by Gilligan and Huckleberry [11, p. 186].

Another way of seeing this is to consider a strictly plurisubharmonic Morse exhaustion function, which is the approach taken by Forstnerič [7].

Proof of Theorem 1.2. Let $\dim_{\mathbb{C}} X = n$. Since Ω is relatively compact, by a small perturbation we may assume that the $\mathcal{C}^2$-smooth exhaustion function $\varphi\colon \Omega \to [0,+\infty)$ is a Morse function and still q-complete. The Morse index at a critical point is at most $n+q-1$ (see the monograph of Forstnerič [8, Sections 3.10 and 3.11] for more details). Note that $2n-1 > n+q-1 \Leftrightarrow n > q$ is satisfied by assumption. Since we glue only handles of index $\leq n+q-1$, and Ω has real dimension $2n$, the complement of any sublevel set of φ is connected by Corollary 2.7. By Lemma 4.3 the domain Ω has only one end. Proposition 4.6 now gives the conclusion. ∎

Proof of Theorem 1.5. Let $\dim_{\mathbb{R}} X = 2n$. Let $\varphi\colon \Omega \to \mathbb{R}$ be a strictly J-plurisubharmonic exhaustion function. Since the domain Ω is relatively compact, we may slightly disturb φ if necessary to obtain a strictly J-plurisubharmonic Morse exhaustion function. The Morse index of φ at a critical point in an almost complex manifold is at most n by [3, Corollary 3.4]. Now the conclusion is the same as in the proof of Theorem 1.2 above. ∎

REMARK 5.1. The proofs of Theorems 1.2 and 1.5 rely only on differential topology and the bound on the Morse index of the critical points. Hence, for any relatively compact domain Ω in a smooth manifold of real dimension n and a Morse exhaustion function $\varphi\colon \Omega \to \mathbb{R}$ of class $\mathcal{C}^2$ whose Morse indices at critical points are bounded by $n-2$, the boundary $\partial\Omega$ is connected.

Acknowledgements. The author would like to thank Matteo Fiacchi, Franc Forstnerič, Tobias Harz, Gerrit Hermann, Hendrik Hermann, Alexander Izzo, and Jaka Smrekar for interesting and helpful discussions and suggestions. Finally, the author thanks the anonymous reviewers for comments that improved the presentation of this manuscript.

Funding. The author was supported by the European Union (ERC Advanced grant HPDR, 101053085 to Franc Forstnerič) and grant N1-0237 from ARRS, Republic of Slovenia.

References

[1] Z. M. Balogh and M. Bonk, *Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains*, Comment. Math. Helv. 75 (2000), 504–533.

[2] F. Bertrand and H. Gaussier, *Gromov hyperbolicity of strongly pseudoconvex almost complex manifolds*, Proc. Amer. Math. Soc. 143 (2015), 3901–3913.

[3] K. Cieliebak and Y. Eliashberg, *From Stein to Weinstein and Back: Symplectic Geometry of Affine Complex Manifolds*, Amer. Math. Soc. Colloq. Publ. 59, Amer. Math. Soc., Providence, RI, 2012.

[4] A. Czarnecki, M. Kulczycki and W. Lubawski, *On the connectedness of boundary and complement for domains*, Ann. Polon. Math. 103 (2012), 189–191.

[5] K. Diederich and J. E. Fornæss, *A smooth pseudoconvex domain without pseudoconvex exhaustion*, Manuscripta Math. 39 (1982), 119–123.

[6] K. Diederich and A. Sukhov, *Plurisubharmonic exhaustion functions and almost complex Stein structures*, Michigan Math. J. 56 (2008), 331–355.

[7] F. Forstnerič, *Complements of Runge domains and holomorphic hulls*, Michigan Math. J. 41 (1994), 297–308.

[8] F. Forstnerič, *Stein Manifolds and Holomorphic Mappings: The Homotopy Principle in Complex Analysis*, Ergeb. Math. Grenzgeb. (3) 56, Springer, Cham, 2017.

[9] H. Freudenthal, *Über die Enden topologischer Räume und Gruppen*, Math. Z. 33 (1931), 692–713.
[10] H. Geiges, *Examples of symplectic 4-manifolds with disconnected boundary of contact type*, Bull. London Math. Soc. 27 (1995), 278–280.
[11] B. Gilligan and A. T. Huckleberry, *Complex homogeneous manifolds with two ends*, Michigan Math. J. 28 (1981), 183–198.
[12] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. (2) 68 (1958), 460–472.
[13] F. R. Harvey and R. O. Wells, Jr., *Zero sets of non-negative strictly plurisubharmonic functions*, Math. Ann. 201 (1973), 165–170.
[14] B. Hughes and A. Ranicki, *Ends of Complexes*, Cambridge Tracts in Math. 123, Cambridge Univ. Press, Cambridge, 1996.
[15] A. J. Izzo, *Every connected bounded domain of holomorphy has connected boundary*, arXiv:2308.08551v1 (2023).
[16] M. Jarnicki and P. Pflug, *Extension of Holomorphic Functions*, 2nd ed., De Gruyter Exp. Math. 34, De Gruyter, Berlin, 2020.
[17] K. Kuratowski, *Topology. Vol. II*, Academic Press, New York, and Państwowe Wydawnictwo Naukowe [Polish Scientific Publishers], Warszawa, 1968.
[18] D. McDuff, *Symplectic manifolds with contact type boundaries*, Invent. Math. 103 (1991), 651–671.
[19] S. B. Nadler, Jr., *Continuum Theory: An Introduction*, Monogr. Textbooks Pure Appl. Math. 158, Dekker, New York, 1992.
[20] L. Nicolaescu, *An Invitation to Morse Theory*, 2nd ed., Universitext, Springer, New York, 2011.
[21] J.-P. Rosay and E. L. Stout, *Radó's theorem for CR-functions*, Proc. Amer. Math. Soc. 106 (1989), 1017–1026.
[22] I. Schur, *Ein Satz ueber quadratische Formen mit komplexen Koeffizienten*, Amer. J. Math. 67 (1945), 472–480.
[23] J.-P. Serre, *Quelques problèmes globaux relatifs aux variétés de Stein*, in: Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, Georges Thone, Liège, 1953, 57–68.
[24] J.-P. Serre, *Oeuvres – Collected Papers I. 1949–1959*, Springer Collected Works in Math., Springer, Heidelberg, 2013.
[25] E. L. Stout, *Polynomial Convexity*, Progr. Math. 261, Birkhäuser Boston, Boston, MA, 2007.
[26] T. Takagi, *On an algebraic problem related to an analytic theorem of Carathéodory and Fejér and on an allied theorem of Landau*, Japan. J. Math. 1 (1924), 83–93 [= Proc. Phys.-Math. Soc. Japan (3) 6 (1924). 130–140].

Rafael B. Andrist
Faculty of Mathematics and Physics
University of Ljubljana
Ljubljana, Slovenia
E-mail: rafael-benedikt.andrist@fmf.uni-lj.si