APPLICATIONES MATHEMATICAE
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DRAZIN'S PSEUDO-INVERSE OF T-MATRICES

Abstract. T-matrices are singular $n \times n$ real matrices, with $n = 2k + 1$, $k \in \mathbb{N}$, in which apart from the elements of the first row and the middle column, all other elements are zero. We construct Drazin's pseudo-inverse for each T-matrix.

1. Introduction. Penrose [6] gave the definition of a *generalised* inverse for quadratic matrices and its main properties [1]. Drazin [2] defined a pseudo-inverse for elements of any associative ring. We will apply this definition to the ring of so called T-matrices, and construct Drazin's pseudo-inverse for any T-matrix.

2. The construction of Drazin's pseudo-inverse for T-matrices

Definition 2.1 ([4]). A matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & 0 & \dots & a_{2m} & \cdots & 0 & 0 \\ 0 & 0 & \dots & a_{3m} & \ddots & 0 & 0 \\ \dots & & & & & & \\ 0 & 0 & \dots & a_{nm} & \cdots & 0 & 0 \end{bmatrix}_{n\times n} \tag{2.1}$$

where $a_{ij} \in \mathbb{R}$ and $n = 2m - 1$, $m \in \mathbb{N}_{\geq 2}$, will be called a *real T-matrix of order n*.

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Let $\mathcal{A}$ be the set of real T-matrices of order $n \geq 3$, $n = 2m-1$, $m \in \mathbb{N}_{\geq 2}$. We can easily verify that $(\mathcal{A}, +, \cdot)$ is a subring of $M_n(\mathbb{R})$, the ring of $n \times n$ real matrices. Matrices in $\mathcal{A}$ are clearly *singular*, so their inverse matrix does not exist.

Definition 2.2 ([4, 3]). Let $A \in \mathcal{A}$, $k \in \mathbb{N}$. We denote by X a matrix from $\mathcal{A}$ which satisfies the conditions:

(i) $A^{k+1}X = A^k$,
(ii) $XAX = X$,
(iii) $AX = XA$.

The matrix X will be called *Drazin's pseudo-inverse* of A.

Next, we find a formula for the matrix X, which turns out to be unique. From (i) for $k = 1$, that is, $A^2 \cdot X = A$, comparing the elements in respective positions, we get

$$x_{i1} = \frac{a_{i1}}{a_{11}}, \qquad i \in \{2, \dots, n\} \setminus \{m\}, \qquad x_{mn} = \frac{1}{a_{mn}}. \tag{2.2}$$

From (iii), we similarly obtain

$$x_{im} = \frac{a_{im}}{a_{mm}^2}, \qquad i = 2, \dots, n. \tag{2.3}$$

Considering conditions (2.2) and (2.3), from (ii) we get

$$x_{1m} = \frac{-a_{11} \sum_{i=2}^{n} a_{1i}a_{i,m} - a_{mm} \sum_{i=2,\, i\neq m} a_{1i}a_{im}}{a_{11}a_{mm}^2}. \tag{2.4}$$

Theorem 2.3. *For every $n = 2m-1$, $m \in \mathbb{N}_{\geq 2}$, and $a_{11}, a_{mm} \neq 0$ the matrix*

(2.5)
$$X = \begin{bmatrix} \frac{1}{a_{11}} & \frac{a_{12}}{a_{11}^2} & \cdots & \frac{-a_{11}\sum_{i=2}^{n} a_{1i}a_{im} - a_{mm}\sum_{i=2,i\neq m}^{n} a_{1i}a_{im}}{a_{11}a_{mm}^2} & \cdots & \frac{a_{1n}}{a_{11}^2} \\ 0 & 0 & \cdots & \frac{a_{2m}}{a_{mm}^2} & \cdots & 0 \\ 0 & 0 & \cdots & \frac{a_{3m}}{a_{mm}^2} & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & \cdots & \frac{a_{nm}}{a_{mm}^2} & \cdots & 0 \end{bmatrix}_n$$

is Drazin's pseudo-inverse T-matrix of the singular matrix (2.1).

3. Proof of Theorem 2.3. It suffices to verify that the three conditions of Definition 2.2 hold.

(i) Let us first prove that $A^2 \cdot X = A$. As the computations are straightforward, we will only check that the $(1, m)$-entries of the matrices $A^2 \cdot X$ and

A are equal. Indeed, the $(1, m)$-entry of $A^2 \cdot X$ equals

$$a_{11} \sum_{i=1, i \neq m}^{n} a_{1i} x_{im} + x_{mm} \sum_{i=1}^{n} a_{1i} a_{im}. \tag{3.1}$$

By substituting (2.2)–(2.4), and performing straightforward algebraic manipulations, this expression reduces to a_{1m}.

To establish that $A^{k+1}X = A^k$ for all $k \in \mathbb{N}$, we proceed as follows:

$$A^{k+1}X = A^{k-1}A^2X = A^{k-1}A = A^k, \tag{3.2}$$

where we have used the case $A^2 \cdot X = A$.

(ii) *Proof that* $XAX = X$. It suffices to show that the $(1, m)$-entries of XAX and X coincide, as the remaining entries can be verified directly. Let P denote the $(1, m)$-entry of XAX:

$$P = x_{11} \sum_{i=1}^{n} a_{1i} x_{im} + x_{mm} \sum_{i=1}^{n} x_{1i} a_{im}. \tag{3.3}$$

We aim to prove that $P = x_{1m}$. We first isolate the term x_{1m} in the two sums:

$$P = x_{11} \Big(a_{11} x_{1m} + \sum_{i=2}^{n} a_{1i} x_{im} \Big) + x_{mm} \Big(a_{mm} x_{1m} + \sum_{\substack{i=2 \\ i \neq m}}^{n} x_{1i} a_{im} \Big).$$

From (2.2) and (2.3), we have

$$x_{11} = \frac{1}{a_{11}}, \qquad x_{mm} = \frac{1}{a_{mm}}, \qquad x_{im} = \frac{a_{im}}{a_{mm}^2} \quad \text{for } i \geq 2.$$

Substituting these results into the expression for P gives

$$P = \frac{1}{a_{11}} \Big(a_{11} x_{1m} + \frac{1}{a_{mm}^2} \sum_{i=2}^{n} a_{1i} a_{im} \Big) + \frac{1}{a_{mm}} \Big(a_{mm} x_{1m} + \frac{1}{a_{11}^2} \sum_{\substack{i=2 \\ i \neq m}}^{n} a_{1i} a_{im} \Big)$$

$$= 2x_{1m} + \frac{a_{11} \sum_{i=2}^{n} a_{1i} a_{im} + a_{mm} \sum_{\substack{i=2 \\ i \neq m}}^{n} a_{1i} a_{im}}{a_{11}^2 a_{mm}^2} = 2x_{1m} - x_{1m} = x_{1m}.$$

(iii) *Proof that* $AX = XA$. It suffices to show that the $(1, m)$-entries of AX and XA coincide, or equivalently that

$$\sum_{i=1}^{n} a_{1i} x_{im} = \sum_{i=1}^{n} x_{1i} a_{im}.$$

This is proved by demonstrating that the difference between the left-hand side and the right-hand side is zero. Substituting (2.2)–(2.4) we get

$$\sum_{i=1}^{n} a_{1i}x_{im} - \sum_{i=1}^{n} x_{1i}a_{im}$$
$$= a_{11}x_{1m} + \frac{1}{a_{mm}^2}\sum_{i=1}^{n} a_{1i}a_{im} - \left(\frac{a_{1m}}{a_{11}^2} + \frac{1}{a_{11}^2}\sum_{\substack{i=1\\ i\neq m}}^{n} a_{1i}a_{im} + \frac{a_{mm}}{a_{11}}x_{1m}\right).$$

Expanding, regrouping the terms, and substituting the explicit expression for x_{1m} shows that this equals

$$-\frac{1}{a_{mm}^2}\sum_{i=1}^{n} a_{1i}a_{im} - \frac{1}{a_{11}a_{mm}}\sum_{\substack{i=1\\ i\neq m}}^{n} a_{1i}a_{im} + \frac{1}{a_{mm}^2}\sum_{i=1}^{n} a_{1i}a_{im} = 0.$$

REMARK 3.1. The conditions $a_{11} \neq 0$ and $a_{m,m} \neq 0$ are necessary, because when $a_{11} = 0$, x_i cannot be determined for $i \in \{1, \dots, n\}$; if $a_{m,m} = 0$, then $x_{j,n}$ cannot be determined for $j \in \{1, \dots, n\}$.

COROLLARY 3.2. *For any T-matrix and its Drazin's pseudo-inverse T-matrix X, we have* $AXA = A$.

Proof. See Theorem 2.3 and Definition 2.2. ∎

4. Application of T-matrices in steady-state solution of dynamical systems. Consider the linear dynamical system of the form $x'(t) = Ax(t) + b$, where $x(t) = (x_1(t), \dots, x_n(t))^T$, $A \in \mathbb{R}^{n\times n}$ is a singular matrix and $b \in \mathbb{R}^n$ is an arbitrary constant vector. To determine the steady-state solution, we set $x'(t) = 0$, which leads to the algebraic system $Ax(t) = -b$. Since A is singular, the system may have no solution or infinitely many solutions depending on whether $-b$ belongs to the column space of A.

CASE 1: *Consistent system.* If $-b$ lies in the column space of A, let X be the T-pseudo-inverse of A. Then one solution is given by

$$x_p = X(-b), \qquad x = X(-b) + (I - XA)z, \qquad z \in \mathbb{R}^n.$$

CASE 2: *Inconsistent system.* If $-b$ is not in the column space of A, then the system $Ax(t) = -b$ has no exact solution. In this case, an approximate equilibrium vector can be considered. Let X be the T-pseudo-inverse of A. A candidate vector associated with the system can then be written as $x = X(-b)$.

EXAMPLE 4.1. The system

$$\begin{aligned} x_1'(t) &= 3x_1 + 4x_2 + 2x_3 + 1,\\ x_2'(t) &= x_2(t),\\ x_3'(t) &= 5x_2(t) \end{aligned} \tag{4.1}$$

can be written in the matrix form $x'(t) = Ax(t) + b$, where

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix} \quad \text{and} \quad b = [1 \quad 0 \quad 0]^T.$$

For A we find the pseudo-inverse

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & 0 \\ 0 & x_{23} & 0 \end{bmatrix}$$

Based on (2.2)–(2.4) we obtain

$$x_{11} = \frac{1}{3}, \quad x_{12} = -\frac{52}{9}, \quad x_{13} = \frac{2}{9}, \quad x_{22} = 1, \quad x_{23} = 5,$$

and therefore

$$X = \begin{bmatrix} \frac{1}{3} & -\frac{52}{9} & \frac{2}{9} \\ 0 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix}.$$

For the specified $b = [1 \ \ 0 \ \ 0]^T$, we compute the solution of the system (4.1):

$$x = -\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & 0 \\ 0 & x_{23} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix} \tag{4.2}$$

The results were verified using the Euler method and compared with (4.2) implemented in Mathematica. The system has the solution

$$\begin{aligned} x_1(t) &= \frac{1}{3}(-1 + e^{3t}), \\ x_2(t) &= 0, \\ x_3(t) &= 0. \end{aligned}$$

Further comparisons and extensions will be considered in future work, particularly regarding the application of Drazin's pseudo-inverse to fractional-order dynamical systems involving singular matrices with unknown parameters.

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