COLLOQUIUM MATHEMATICUM

Online First version

NON-HAUSDORFF T_1 PROPERTIES

BY

STEVEN CLONTZ

Abstract. Several weakenings of the T_2 property for topological spaces, including k-Hausdorff, KC, weakly Hausdorff, semi-Hausdorff, RC, and US, have been studied by mathematicians. Here we provide a complete survey of how these properties do or do not relate to one another, including several new results to fill in the gaps in the existing literature, motivated by the use of a community-maintained database of topological spaces and their properties.

1. Introduction. The following properties were introduced in Wilansky's "Between T_1 and T_2" [16].

DEFINITION 1.1 (P99 ([1])). A space is US ("Unique Sequential limits") provided that every convergent sequence has a unique limit.

DEFINITION 1.2 (P100). A space is KC ("Kompacts are Closed") provided that its compact subsets are closed.

The following fact is observed there.

PROPOSITION 1.3.

$$T_2 \Rightarrow KC \Rightarrow US \Rightarrow T_1$$

with no implications reversing.

In the process of modeling this result from 1967 in the π*-Base community database of topological counterexamples* [1] (an open-source web application serving as a modern adaptation of [15]) several other examples of properties in the literature properly implied by T_2 and properly implying T_1 were encountered. This paper aims to provide a modern update to Wilansky's 1967 article by surveying these non-Hausdorff T_1 properties.

2. k-Hausdorff. The following property was introduced in [8]; here we use an equivalent characterization. (We will revisit the original definition in

2020 *Mathematics Subject Classification*: Primary 54D10.
Key words and phrases: lower separation axioms, survey, database.
Received 30 December 2024; revised 9 March 2026.
Published online 11 June 2026.

([1]) Property ID as assigned in the π-Base [1].

Section 4, in addition to discussing the connection with "compactly generated" spaces.)

Definition 2.1 (P170). A space is *k_1-Hausdorff* or k_1H provided each compact subspace is T_2 ([2]).

Since T_2 is a hereditary property, it is immediate that all T_2 spaces are k_1H. In fact, we have the following.

Theorem 2.2.

$$T_2 \Rightarrow k_1H \Rightarrow KC.$$

Proof. This follows from [8, Theorem 2.1]. To see the second implication, let X be k_1H and let $K \subseteq X$ be compact. Then for each $x \in K$, the set $Y = K \cup \{x\}$ is compact, and thus T_2. Then K is a compact subset of the T_2 space Y, and thus closed in Y. It follows that $\{x\}$ is open in Y and not a limit point of K in X. ∎

Example 3.6 of [8] demonstrates that the first implication does not reverse, even when topologies are assumed to be generated by compact subspaces (P140), by taking a quotient of the tangent disc topology (S74 ([3])).

Without requiring spaces to be P140, an elementary counterexample is available.

Definition 2.3 (P136, [4]). A space is *anticompact* provided all compact subsets are finite.

Lemma 2.4. *All anticompact T_1 spaces are k_1H.*

Proof. Each compact subset is a finite T_1 space. All finite T_1 spaces are discrete and thus T_2. ∎

Definition 2.5 (P39, [6]). A space is *hyperconnected* (or anti-Hausdorff) provided every nonempty open set is dense.

Example 2.6 (S17). The co-countable topology on an uncountable set is k_1H but not T_2. To see this, first observe that the space is hyperconnected: every pair of nonempty open sets intersects. Therefore the space is not T_2, but the space is still k_1H, which follows from the fact that the space is anticompact and T_1, and from the preceding lemma.

Any example of a compact non-T_2 space provides an example of a non-k_1H space, such as the following.

Definition 2.7. The *one-point compactification* of a space X is $X \cup \{\infty\}$ where points of X have their usual neighborhoods, and neighborhoods of ∞ are complements of closed and compact subsets of X.

([2]) The subscript in k_1 will be explained in Section 4.

([3]) Space ID as assigned in the π-Base.

Example 2.8 (S29). The one-point compactification of the rationals is not T_2 (and therefore not k_1H) because it fails to be locally compact; see [16, Corollary 1]. This corollary also shows the space is KC as $\mathbb{Q}$ is a "k-space"; we will discuss the ambiguity of "k-space" in the absence of Hausdorff separation in Section 4, but accept the result here as $\mathbb{Q}$ is indeed Hausdorff.

Example 2.9 (S97). Counterexample #99 from [15] is the set $\omega^2 \cup \{\infty_x, \infty_y\}$ with ω^2 discrete. Neighborhoods of ∞_x contain all but finitely-many elements from each row of ω^2, and neighborhoods of ∞_y contain all but finitely-many entire rows of ω^2.

Compact and not-T_2 are immediate. To see that compacta are closed, note that a nonclosed set must have either ∞_x or ∞_y as a limit missing from the set. If ∞_x is the missing limit, the set must contain infinitely-many points from some row, and then the open cover of the set by a neighborhood of ∞_y missing that row and singletons of ω^2 otherwise has no finite subcover. If ∞_y is the missing limit, the set must contain at least one point from infinitely-many rows, and then the open cover of the set by a neighborhood of ∞_x missing those points and the singletons of ω^2 otherwise has no finite subcover.

3. Weakly Hausdorff. The following property was introduced in [9].

Definition 3.1 (P143). A space is *weakly Hausdorff* or wH provided the continuous image of any compact Hausdorff space into the space is closed.

In that paper it was observed that T_2 was sufficient and T_1 was necessary for this property. In fact, this can be tightened.

Theorem 3.2.

$$KC \Rightarrow wH \Rightarrow US.$$

Proof. For the first implication, note that the continuous image of compact is compact, so if compacts are closed, the continuous images of compact T_2 spaces are closed.

The second implication was originally shown in [11]. We will demonstrate a slightly different proof by showing the contrapositive. If a space Y is not US, then there must be some sequence a_n converging to two distinct points x and y.

Suppose $y \neq a_n$ for any $n < \omega$. Consider the compact Hausdorff space given by $X = \{0\} \cup \{2^{-n} : n < \omega\} \subseteq \mathbb{R}$. Consider $f : X \to Y$ where $f(0) = x$ and $f(2^{-n}) = a_n$. To prove continuity, we need only worry about preimages of the neighborhoods of x, but by definition any open neighborhood of x contains cofinitely-many points a_n, so its preimage in X is open. Then $f[X] = \{x\} \cup \{a_n : n < \omega\}$ is not closed as it is missing its limit point y.

If $y = a_N$, first consider the subsequence of a_n missing y. If this subsequence is infinite, then the above proof may be repeated using the subsequence instead. If the subsequence is finite, then $a_n = y$ for all but finitely-many n. Then $\{y\}$ is a compact Hausdorff space embedded in Y that is not closed, as x is a limit point of that set. ∎

Brian Scott showed in [14] that wH is productive, and in [13] that the square of a KC space need not be KC. So the square of such a space is wH but not KC. We provide the details here for the convenience of the reader.

Lemma 3.3. *The continuous image of a compact T_2 space into a wH space is T_2.*

Proof. Consider a compact T_2 space X and a continuous map $f : X \to Y$ with Y wH. Then f is a closed map by the definition of wH; and [7, Theorem 1.5.20] guarantees that the image of a T_4 space under a closed mapping is T_4. ∎

Lemma 3.4. *The product of wH spaces is wH.*

Proof. Let $\mathcal{Y}$ be a collection of wH spaces, X be compact T_2, and $f : X \to \prod \mathcal{Y}$ be continuous. For each $Y \in \mathcal{Y}$, the projection π_Y is continuous, so $\pi_Y \circ f : X \to Y$ is continuous. Since Y is wH, $\pi_Y \circ f[X]$ is closed, compact, and T_2.

It follows that $Z = \prod\{\pi_Y \circ f[X] : Y \in \mathcal{Y}\}$ is a closed, compact, and T_2 subspace of $\prod \mathcal{Y}$. Importantly, $f[X] \subseteq Z$ is a compact subspace of a T_2 space, and thus closed in Z. Therefore, $f[X]$ is closed in $\prod \mathcal{Y}$. ∎

Theorem 3.5. *The square of a compact non-T_2 space is not KC.*

Proof. The diagonal of a space is homeomorphic to the space, so the diagonal of a compact space is compact. But a space is T_2 if and only if its diagonal is closed, so the diagonal of a compact non-T_2 space is compact but not closed. ∎

Example 3.6 (S31). The square of the one-point compactification of the rationals is wH, because the one-point compactification of the rationals is KC and therefore wH.

But since the one-point compactification of the rationals is not T_2, its square is not KC by the previous theorem.

A slight modification of a common counterexample yields a space which is US but not wH.

Example 3.7 (S23). The *Arens–Fort space* is $\omega^2 \cup \{z\}$ where ω^2 is discrete, and neighborhoods of z contain all but a finite number of points from all but a finite number of columns. This space is T_2 and thus KC. It is also not locally compact: it is anticompact while ∞ has no finite neighborhoods.

Example 3.8 (S165). The one-point compactification $X = Y \cup \{\infty\}$ of the Arens–Fort space X is US but not wH. US is a result of Theorem 4 from [16]: the one-point compactification of any KC space is US.

The space is not wH because it contains a compact T_2 space which is not closed (since inclusion is a continuous function). Namely, $Y \setminus \{z\} = \omega^2 \cup \{\infty\}$ is a copy of the one-point compactification of a discrete countable space; equivalently, a copy of a nontrivial converging sequence in $\mathbb{R}$. But z is a limit point of this set, so it is not closed.

4. k-Hausdorff, revisited. We have established the existence of multiple weakly Hausdorff spaces that are not "k-Hausdorff" (S29, S31, S97). So the following quote from [12] may be surprising: "Every weak Hausdorff space is k-Hausdorff."

Regrettably, there are multiple inequivalent notions of "k-Hausdorff" in the literature. We will first establish the pattern they all appear to follow.

Definition 4.1. A space X is said to be *k_i-Hausdorff* or k_iH provided that the diagonal $\Delta_X = \{(x, x) : x \in X\}$ is k_i-closed in the product topology on X^2.

In turn, we are now obligated to define k_i-closed.

Definition 4.2. A subset C of a space is *k_1-closed* provided for every compact subset K of the space, the intersection $C \cap K$ is closed in the subspace topology for K.

Theorem 4.3 ([8, 2.1]). *The criteria for k_1-Hausdorff* (P170) *given in Definitions* 2.1 *and* 4.1 *are equivalent.*

(In fact, [8, 2.1] gives another characterization for k_1H: each compact subset is closed and locally compact.)

So the distinct notion of k-Hausdorff from [12] is what we would call k_2H (P171) according to the following definition.

Definition 4.4. A subset C of a space X is *k_2-closed* provided for every compact Hausdorff space K and continuous map $f : K \to X$, $f^{\leftarrow}[C]$ is closed in K (4).

The reader may recall a connection here with "compactly generated" topological spaces, or "k"-spaces:

Definition 4.5. A space X is said to be a *k_i-space* provided that every k_i-closed set is closed.

(4) A set is k_3-closed if its intersection with each compact Hausdorff subspace is closed in the subspace, but even indiscrete spaces have a k_3-closed diagonal, so we do not investigate this topic further here.

Here, k_1-space is P140 of π-Base, while k_2-space is P141, and k_2-space implies k_1-space. It is somewhat difficult to disentangle when each of these concepts was first introduced in the literature, given that most textbooks and articles introduce these notions in the context of (k_1-)Hausdorff spaces, where it may be immediately seen that both definitions are equivalent.

On the other hand, some authors (e.g. [12]) consider the category of weakly Hausdorff k_2-spaces (i.e. CGWH spaces, P148) to be ideal for the study of homotopy theory and algebraic topology. Here we see that such spaces need not be k_1-Hausdorff.

Example 4.6 (S29). The one-point compactification of the rationals was previously noted in Example 2.8 to be KC (and thus wH) but not k_1H. So let A be a set missing its limit point l.

If $l \in \mathbb{Q}$, choose $l_n \in A$ with $l_n \in (l - 2^{-n}, l + 2^n)$.

If $l = \infty$, then $A \subseteq \mathbb{Q}$. Since ∞ is a limit point, A is not contained in a compact subset of $\mathbb{Q}$. In particular, A contains a sequence l_n with no subsequence that converges within $\mathbb{Q}$ (if not, the closure over these subsequence limits would be a sequentially compact and thus compact subset of $\mathbb{Q}$), and therefore l_n converges to $l = \infty$.

Either way, $f(n) = l_n$ and $f(\omega) = l$ defines a continuous map $f : \omega+1 \to \mathbb{Q} \cup \{\infty\}$. Since $f^{\leftarrow}[A] = \omega$ is not closed, A is not k_2-closed. Therefore this space is a k_2-space and thus CGWH.

(The reader interested in further connections between separation axioms and generating families is directed towards the sequel [5] of this paper.)

We will find the following characterization for k_2H convenient, as like k_1H's Definition 2.1, it does not require considering the diagonal of the square.

Theorem 4.7 ([12, 4.2.4]). *A space X is k_2H if and only if for every compact Hausdorff space K, a continuous map $f : K \to X$, and points $k_0, k_1 \in K$ such that $f(k_0) \neq f(k_1)$, there exist open neighborhoods U_0, U_1 of k_0, k_1 such that $f[U_0] \cap f[U_1] = \emptyset$.*

We may now establish this notion of k-Hausdorff as strictly weaker than the property explored earlier.

Theorem 4.8.

$$wH \Rightarrow k_2H \Rightarrow US.$$

Proof. The first implication is Proposition 11.2 of [12]: given $f : K \to X$ and $k_0, k_1 \in K$ with $f(k_0) \neq f(k_1)$, by Lemma 3.3 we see that $f[K]$ is T_2. So there exist disjoint open neighborhoods V_0, V_1 of $f(k_0), f(k_1)$, and $U_0 = f^{\leftarrow}[V_0], U_1 = f^{\leftarrow}[V_1]$ are open neighborhoods of k_0, k_1 with $f[U_0] \cap f[U_1] = V_0 \cap V_1 = \emptyset$.

Now let X be k_2H and let l_0, l_1 be limits of $x_n \in X$. Let $K = (\omega+1)\times 2$, and let $f : K \to X$ be defined by $f(n,i) = x_n$, $f(\omega,0) = l_0$, and $f(\omega,1) = l_1$. To show this is continuous, we need only observe that inverse images of open subsets of X that contain (ω, i) contain a cofinite subset of $\omega \times \{i\}$. This follows as for an inverse open image to contain (ω, i), the open set must contain the limit l_i of x_n, and thus contain a final sequence of x_n, and thus the inverse open image contains a cofinite subset of $\omega \times 2$.

Finally, if $l_0 = f(\omega,0) \neq f(\omega,1) = l_1$, then there would exist open neighborhoods U_0, U_1 of $(\omega,0), (\omega,1)$ with $f[U_0] \cap f[U_1] = \emptyset$. But this is impossible as there exists $n < \omega$ with $(n,0) \in U_0$ and $(n,1) \in U_1$, and $f(n,0) = x_n = f(n,1)$. Thus $l_0 = l_1$, showing limits are unique and X is US. ∎

This allows us to show that weakly Hausdorff k_1-spaces need not be CGWH (k_2-spaces).

Example 4.9 (S31). The square X^2 of the one-point compactification of the rationals X was shown in Example 3.6 to be wH. Note that all compact spaces are k_1-spaces.

We know that X is wH and $wH \Rightarrow k_2H$, which means the diagonal $\Delta \subseteq X^2$ is k_2-closed. But since X is not T_2, Δ is not closed. Therefore X^2 is not a k_2-space as it contains a k_2-closed subset Δ which is not closed.

The arrows of Theorem 4.8 do not reverse. For the second:

Example 4.10 (S37). Consider the space $X = (\omega_1 + 1) \cup \{\omega_1'\}$, where $\omega_1 + 1$ has its order topology and ω_1' is a duplicate of ω_1.

To see that this space fails k_2H, consider the set $(\omega_1 + 1) \times 2$ and the map $(\alpha, i) \mapsto \alpha$ for $\alpha < \omega_1$, $(\omega_1, 0) \mapsto \omega_1$, and $(\omega_1, 1) \mapsto \omega_1'$. This map is continuous, $f(\omega_1,0) \neq f(\omega_1,1)$, but there are no open neighborhoods U_0, U_1 of $(\omega_1,0), (\omega_1,1)$ with $f[U_0] \cap f[U_1] = \emptyset$.

The fact that this space is US follows from the fact that $\omega_1 + 1$ is US, and no nontrivial sequences in $\omega_1 + 1$ converge to ω_1.

To reject the converse of $wH \Rightarrow k_2H$, we will use the following lemma, which strengthens [16, Theorem 4] used earlier.

Lemma 4.11. *The one-point compactification of a wH space is k_2H.*

Proof. Let $X^+ = X \cup \{\infty\}$ be the one-point compactification of X. Consider a compact Hausdorff K with continuous $f : K \to X^+$, and $k_0, k_1 \in K$ with $f(k_0) \neq f(k_1)$.

Suppose $f(k_i) \neq \infty$ for $t = 0, 1$. Note $f^{\leftarrow}[\{\infty\}]$ is a closed subset of K disjoint from $\{k_0, k_1\}$. So we may choose a closed neighborhood $K' \subseteq K$ of k_0, k_1 disjoint from the closed set $f^{\leftarrow}[\{\infty\}]$. Then $f{\restriction}K' : K' \to X$ witnesses the existence of neighborhoods N_0, N_1 of k_0, k_1 with $(f{\restriction}K')[N_0] \cap (f{\restriction}K')[N_1] = \emptyset$.

Thus we have

$$f[K' \cap N_0] \cap f[K' \cap N_1] = \emptyset$$

witnessing k_2H for X^+.

Now suppose for instance $f(k_0) = \infty$. Let N be a closed neighborhood of k_1 which misses the closed set $f^{\leftarrow}[\{\infty\}]$, and note $f[N]$ is compact and misses ∞; it is also closed by wH. Then $\infty \in X^+ \setminus f[N]$ open, and $k_0 \in f^{\leftarrow}[X^+ \setminus f[N]]$. Thus we have

$$f[f^{\leftarrow}[X^+ \setminus f[N]]] \cap f[N] = (X^+ \setminus f[N]) \cap f[N] = \emptyset$$

witnessing k_2H for X^+. ∎

Note that the above argument uses the full strength of wH.

Question 4.12. Is the one-point compactification of every k_2H space k_2H?

So let us now investigate a couple counter-examples.

Example 4.13 (S165). The one-point compactification of the Arens–Fort space was shown in Example 3.8 to fail wH, but satisfies k_2H by the preceding lemma: the Arens–Fort space is T_2.

Example 4.14 (S15). The cofinite topology on an infinite set is T_1 but not US.

We now have the following corollary.

Corollary 4.15.

$$T_2 \Rightarrow k_1H \Rightarrow KC \Rightarrow wH \Rightarrow k_2H \Rightarrow US \Rightarrow T_1$$

with no implications reversing.

5. Retracts are closed. In [18] the following property was introduced.

Definition 5.1. A *retract* R of a space X is a subspace for which a continuous $f : X \to R$ exists with $f(r) = r$ for all $r \in R$.

Definition 5.2 (P101). A space is RC provided each retract subspace is closed.

This provides another example of a property between T_1 and T_2.

Theorem 5.3.

$$T_2 \Rightarrow RC \Rightarrow T_1.$$

Proof. Given a T_2 space X, consider a retract $R \subseteq X$ and a witnessing map $f : X \to X$. Note $R = \{x \in X : x = f(x)\}$ is closed as $\{x \in X : g(x) = h(x)\}$ is closed for any Hausdorff space and continuous maps $g, h : X \to X$.

On the other hand, let X be RC. Then for each fixed $x \in X$, note $y \mapsto x$ is a retract, and thus $\{x\}$ is closed, showing X is T_1. ∎

However, this property cannot be placed into the chain of implications given in Corollary 4.15.

EXAMPLE 5.4. In [3], the authors provide an example of a space which is RC (in fact, "strongly rigid" as all continuous self-maps are either constant or the identity, see their Claim 4.9), but is not KC. An earlier version of this construction (arXiv:2211.12579v3) was in fact not US.

We do not delve into the details of the preceding construction as they are fairly technical. The author suspected that a more elementary example could be constructed, which was eventually provided by Marshall Williams in a response to the author's post to MathOverflow [17]:

EXAMPLE 5.5 (S192). Let $X = [0,\infty) \cup \{\infty_1, \infty_2\}$ where $[0,\infty)$ is an open subspace with its usual Euclidean topology, ∞_1 has neighborhoods of the form $\{\infty_1\} \cup (a,\infty) \setminus 2\mathbb{Z}$, and ∞_2 has neighborhoods of the form $\{\infty_2\} \cup (a,\infty) \setminus (2\mathbb{Z}+1)$. Observe that this space is connected, and it fails to be US as the sequence $a_n = n + \frac{1}{2}$ converges to both ∞_1 and ∞_2. It is also compact: for any open cover, choose any neighborhoods of ∞_1, ∞_2, and the difference is a closed and bounded subspace of $[0,\infty)$.

Note next that if A is a retract of X, then $\mathrm{cl}(A) \cap [0,\infty) = A \cap [0,\infty)$. If ∞_1 is a limit point of A, we have that $A \setminus 2\mathbb{Z}$ is unbounded. Furthermore, since A is the continuous image of connected X, we have $A \supseteq [a,\infty)$ for some a. Thus $\infty_1 \in \mathrm{cl}(A)$ implies $\infty_1 \in A$. A similar argument shows $\infty_2 \in \mathrm{cl}(A)$ implies $\infty_2 \in A$, and therefore $\mathrm{cl}(A) = A$.

We can see that k_1H is not sufficient to imply RC.

EXAMPLE 5.6 (S17). In Example 2.6 we noted that the co-countable topology on an uncountable set is k_1H but not T_2. In fact, the space fails to be RC: partition this space into pairs of points, and consider a map that sends each pair to a single point within their set. Then this image is uncountable and therefore not closed. But this image is a retract as the map is continuous: given a co-countable set, the inverse image is still co-countable.

It is important that S17 is not compact, given the following result.

PROPOSITION 5.7. *All compact KC spaces are RC.*

Proof. Every retract of a compact space is compact, so every retract of a compact KC space is closed. ∎

Proposition 5.7 cannot be reversed: as we have already noted in Example 5.5 the space S192 has closed retracts.

Additionally, KC cannot be weakened to wH here.

PROPOSITION 5.8. *The square of a non-T_2 space is not RC.*

Proof. The diagonal is a retract of the square by the continuous map $(x, y) \mapsto (x, x)$, and a space is T_2 if and only if its diagonal is closed. ∎

Note that the previous two propositions provide an alternative proof for Theorem 3.5.

EXAMPLE 5.9 (S31). The square of the one-point compactification of the rationals was noted in Example 3.6 to be wH, but fails to be RC by the preceding proposition.

6. Semi-Hausdorff. The semi-Hausdorff property was introduced in [2].

DEFINITION 6.1. An open set is *regular open* if it equals the interior of its closure.

DEFINITION 6.2 (P10). A space is *semiregular* if it has a basis of regular open sets.

DEFINITION 6.3 (P169). A space is *semi-Hausdorff* or sH if for each pair of distinct points x, y, there exists a regular open neighborhood of x missing y.

PROPOSITION 6.4. *All T_1 semiregular spaces are semi-Hausdorff.*

Proof. Given T_1, there exists an open neighborhood U of x missing y. Then by semiregularity, U contains a regular open neighborhood V of x missing y. ∎

THEOREM 6.5.

$$T_2 \Rightarrow sH \Rightarrow T_1.$$

Proof. Given points x, y, let U, V be their disjoint open neighborhoods. Then $\mathrm{cl}(U) \cap V = \emptyset$, so $\mathrm{int}(\mathrm{cl}(U))$ is a regular open neighborhood of x missing y. The second implication is immediate from the definitions. ∎

EXAMPLE 6.6. The RC-not-KC space constructed in [3] is sH. Likewise, its arXiv preprint 2211.12579v3 includes a construction which is RC, sH, but not US.

In this case we may return to a more straightforward example to show not all sH spaces are T_2.

EXAMPLE 6.7 (S97). In Example 2.9 this space was shown to be KC, and therefore T_1. The space is also compact and thus RC.

Singletons are regular open neighborhoods for points of ω^2. The open neighborhoods of ∞_y missing at least the first row of ω^2 are regular open, and the open neighborhoods of ∞_x missing at least the first column of ω^2 are regular open.

Since the space is semiregular and T_1, it is sH. (Note that the authors of [15] assume T_2 in their definition for semiregular, explaining why it is marked as false in their reference chart.)

Example 6.8 (e.g. S37, Example 4.10). Consider any T_2 space X with a nonisolated point $x \in X$. Let $X' = X \cup \{x'\}$ have the topology generated by the original open subsets of X, and for each open neighborhood U of x, let $\{x'\} \cup U \setminus \{x\}$ be a basic open neighborhood of x'.

This space remains T_1, but it can be seen immediately that x, x' cannot be separated by open sets. In fact, this space cannot be even sH as every regular open neighborhood of x contains x': given a neighborhood U of x, its closure contains the open neighborhood $\{x'\} \cup U$ of x'. Likewise, the space is not RC: the map $x' \mapsto x$ which is otherwise the identity shows X is a retract of X' which is not closed.

In Example 4.10 it was noted that S37 is US, but not k_2H. In fact, we have already seen an example that shows that k_1H is insufficient to imply sH.

Proposition 6.9. *The only nonempty hyperconnected sH space is the singleton.*

Proof. Let x, y be distinct points of a hyperconnected space. Then every regular open neighborhood of x is the entire space, and thus cannot be used to separate x from y. ∎

Example 6.10 (S17). In Examples 2.6 and 7.3 we noted that the co-countable topology on an uncountable set is k_1H but not RC. The space also fails to be sH by the preceding proposition.

We have now established that, like RC, sH cannot be placed into the chain of implications given in Corollary 4.15: S17 is k_1H but neither sH nor RC, while the example from arXiv:2211.12579v3 is sH and RC, but not even US. (We will introduce a more elementary example of an sH space that fails to be US as Example 6.14 below.)

So how might RC and sH be related? We again revisit a counter-example that will show us that RC need not imply sH, even when spaces are compact.

Definition 6.11. A space is *nowhere locally compact* if no point of the space has a compact neighborhood.

Theorem 6.12. *The one-point compactification of any nowhere locally compact nonempty space X is not sH.*

Proof. Consider a regular open neighborhood U of the compactifying point ∞. Since X is nowhere locally compact, every neighborhood of a point in x cannot miss U. It follows that $\mathrm{cl}(U) = X \cup \{\infty\}$, and thus

$U = \mathrm{int}(\mathrm{cl}(U)) = \mathrm{int}(X \cup \{\infty\}) = X \cup \{\infty\}$. Therefore ∞ cannot be separated from any other point by a regular open set. ∎

EXAMPLE 6.13 (S29). The one-point compactification of the rationals is KC (Example 2.8), and therefore RC. But the space fails to be sH by the previous theorem: compact subsets of the rationals are nowhere dense and therefore cannot contain an open set, showing the rationals are nowhere locally compact.

We may also obtain an example of a compact space which is sH but neither RC nor US.

EXAMPLE 6.14 (S185). We introduce a coarsening of the topology of S97 on $\omega^2 \cup \{\infty_x, \infty_y\}$. We continue to assume ω^2 is discrete, and neighborhoods of ∞_y contain all but finitely-many rows, but neighborhoods of ∞_x must contain all but finitely-many columns.

The argument that this space is sH is identical to the argument for S97, and it is also immediate to verify that the space is compact. However, unlike S97, this space is neither RC nor US. To see the latter, note that the sequence (n, n) converges to both ∞_x and ∞_y.

As for the former, we will show that the set $\{(n, n) : n < \omega\} \cup \{\infty_y\}$ (which is not closed) is a retract. This set is the image of the map $(a, b) \mapsto (\max\{a, b\}, \max\{a, b\})$, $\infty_x \mapsto \infty_y$, and $\infty_y \mapsto \infty_y$. To see that this map is continuous, we need only consider the preimage of basic neighborhoods of ∞_y. Such a neighborhood is of the form $\{(n, n) : N \leq n < \omega\} \cup \{\infty_y\}$, and its preimage is equal to $\{(n, m) : N \leq n < \omega, m < \omega\} \cup \{\infty_x\} \cup \{(m, n) : N \leq n < \omega,\ m < \omega\} \cup \{\infty_y\}$ and is thus open.

7. Locally Hausdorff. The final property we will investigate was studied in e.g. [10].

DEFINITION 7.1 (P84). A space is *locally Hausdorff* or lH provided every point has a T_2 neighborhood.

THEOREM 7.2.

$$T_2 \Rightarrow lH \Rightarrow T_1.$$

Proof. The first implication is immediate. For the second, take distinct points x, y and let U be an open T_2 neighborhood of x. If U misses y, we are done; otherwise use the T_2 property of U to obtain a smaller open neighborhood V of x missing y. ∎

We may quickly verify these do not reverse by returning to previously explored counterexamples.

Example 7.3 (S17). The co-countable topology on an uncountable set is k_1H (Example 2.6), but neither RC nor sH. It is also not lH as every nonempty open set is homeomorphic to the whole non-T_2 space.

Example 7.4 (S29). The one-point compactification of the rationals is KC and RC, and thus T_1, but not sH. It is also not lH as no neighborhood of ∞ is T_2.

Example 7.5 (S185). This space is sH, but neither US nor RC. It is lH, and in fact locally metrizable, as neighborhoods of ∞_x that miss ∞_y and vice versa are copies of the subspace

$$\{(0,0)\} \cup \left\{ \left(\frac{\cos(1/m)}{n}, \frac{\sin(1/m)}{n} \right) : 0 < m, n < \omega \right\} \subseteq \mathbb{R}^2.$$

Example 7.6 (e.g. S37). By doubling a nonisolated point of a T_2 space we obtain a space which is neither RC nor sH, but remains lH.

We have thus demonstrated that lH is incomparable with RC, and fails to slot into the chain of implications in Corollary 4.15. We have also shown that lH does not imply sH. So we proceed to construct a counterexample to the converse as well.

Example 7.7 (e.g. S186). Let X be an sH but non-T_2 space, and let ω have the discrete topology. Our space is $(X \times \omega) \cup \{\infty\}$, where $X \times \omega$ is open with the product topology. Neighborhoods of ∞ contain $X \times (\omega \setminus N)$ for some $N < \omega$.

Immediately we see that the space is not lH as every neighborhood of ∞ contains a copy of the non-T_2 space X.

To see that the space is sH, first consider a point $(x, n) \in X \times \omega$. Given $y \in X \setminus \{x\}$, choose regular open $U \subseteq X$ that contains x but misses y. Then $U \times \{n\}$ is a regular open set that contains (x, n) but misses (y, n). Given $y \in X$ and $m \in \omega \setminus \{n\}$, $X \times \{n\}$ is a regular open set that misses (y, m). Likewise, it misses ∞.

Finally, consider ∞. Given any $(x, n) \in X \times \omega$, $(X \times (\omega \setminus (n+1))) \cup \{\infty\}$ is a regular open set containing ∞ that misses (x, n).

8. Conclusion. The results of this paper are summarized in Figure 1; dotted lines indicate implications that hold only assuming spaces are compact. Note that this diagram is complete, as counter-examples exist for every pair not connected by an arrow; furthermore, these examples are compact in all possible cases:

- k_1H through US:
  - k_1, but not RC nor sH nor lH: S17 / Example 2.6
  - compact KC, but not k_1H nor sH nor lH: S29 / Example 2.8
  - compact wH, but not KC nor RC: S31 / Example 3.6

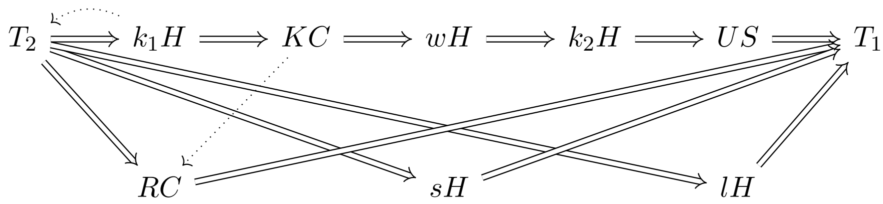

Fig. 1. Summary of results.

  - compact k_2H, but not wH: S165 / Example 3.8
  - compact US, but not k_2H: S37 / Example 4.10
- compact RC:
  - but not US nor sH: S192 / Example 5.5
  - but not lH nor sH: S29 / Example 2.8
- compact sH:
  - but not US nor RC: S185 / Example 6.14
  - but not lH: S186 / Example 7.7
- compact lH:
  - but not US nor RC: S185 / Example 6.14
  - but not US nor sH: S192 / Example 5.5
- compact T_1, but not US, nor sH, nor RC, nor lH: S15 / Example 4.14

To easily query these counter-examples (and to look up a few details not explicitly proven in this survey, e.g. why S15 fails to be semi-Hausdorff),

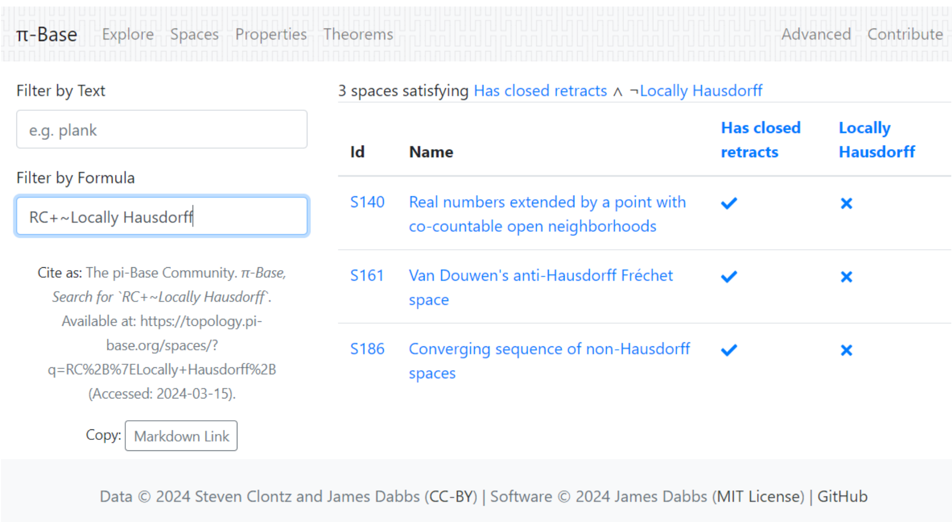

Fig. 2. The Explore feature of π-Base

the π-Base includes an Explore page (as shown in Figure 2) that allows for revealing which spaces in the database satisify or fail the user's desired properties. Future work of the π-Base will include the generation of diagrams such as Figure 1 automatically from its database, and the generation of open (to the π-Base) questions which we anticipate will be particularly helpful for developing projects appropriate for undergraduate and graduate students.

Acknowledgements. The author would like to acknowledge the numerous helpful suggestions of the anonymous referee, including improved proofs of Lemma 3.3 and Theorem 5.3, and correctly suggesting that the original hypothesis of Lemma 4.11 could be strengthened from KC to wH (and asking Question 4.12).

REFERENCES

[1] *π-Base community database of topological counterexamples*, https://topology.pi-base.org, accessed: 2023-08-20.
[2] T. Banakh and A. Ravsky, *Each regular paratopological group is completely regular*, Proc. Amer. Math. Soc. 145 (2017), 1373–1382.
[3] T. Banakh and Y. Stelmakh, *Examples of strongly rigid countable (semi)Hausdorff spaces*, Topology Proc. 62 (2023), 179–196.
[4] P. Bankston, *The total negation of a topological property*, Illinois J. Math. 23 (1979), 241–252.
[5] S. Clontz and M. Williams, *Separation axioms among us*, Topology Appl. 375 (2025), art. 109467, 11 pp.
[6] E. K. van Douwen, *An anti-Hausdorff Fréchet space in which convergent sequences have unique limits*, Topology Appl. 51 (1993), 147–158 (English).
[7] R. Engelking, *General Topology*, Sigma Ser. Pure Math. 6, Heldermann, Berlin, 1989.
[8] J. Lawson and B. Madison, *Quotients of k-semigroups*, Semigroup Forum 9 (1974), 1–18.
[9] M. C. McCord, *Classifying spaces and infinite symmetric products*, Trans. Amer. Math. Soc. 146 (1969), 273–298.
[10] S. B. Niefield, *A note on the locally hausdorff property*, Cahiers Topologie Géométrie Différentielle Catégor. 24 (1983), 87–95.
[11] P. Rabau, *Relationship between weak Hausdorff and US properties*, Math. Stack Exchange, https://math.stackexchange.com/q/4267169.
[12] C. Rezk, *Compactly generated spaces*, nLab, https://ncatlab.org/nlab/files/Rezk%5C_CompactlyGeneratedSpaces.pdf.
[13] B. M. Scott, *If a is compact and b is Lindelöf space, will be $a \cup b$ Lindelöf*, Math. Stack Exchange, https://math.stackexchange.com/q/466564.
[14] B. M. Scott, *Weak Hausdorff space not KC*, Math. Stack Exchange, https://math.stackexchange.com/q/964546.
[15] L. A. Steen, J. A. Seebach, and L. A. Steen, *Counterexamples in Topology*, Springer, 1978.
[16] A. Wilansky, *Between T_1 and T_2*, Amer. Math. Monthly 74 (1967), 261–266.

[17] M. Williams, *Answer to 'A "simple" space with closed retracts but non-unique sequential limits'*, MathOverflow, https://mathoverflow.net/q/460346.

[18] D. van der Zypen, *"All retracts are closed" as separation axiom*, MathOverflow, https://mathoverflow.net/q/191016.

Steven Clontz
Department of Mathematics and Statistics
University of South Alabama
Mobile, AL, USA
E-mail: sclontz@southalabama.edu