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Exploring multifractal moment measures and scaling functions in Moran structures

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Contents

Abstract

The L^q-spectrum of a Borel measure is a fundamental concept in multifractal analysis. It is widely recognized that the L^q-spectrum associated with a fractal measure provides significant insights into its underlying dynamics and geometry. Consequently, the study of the L^q-spectrum is crucial for understanding dynamical systems and fractal measures. Our objective in this paper is to determine the exact rate of convergence of the L^q-spectra for Moran measures satisfying the Set Strong Separation Condition. As an application, we demonstrate that the empirical multifractal moment measures converge weakly to the normalized multifractal measures. Finally, we reexamine the analysis using tube formulas, and we try to show that the multifractal and fractal dimensions of the overlaps in a Moran set satisfying the Strong Open Set Condition are strictly smaller than the dimension of the set itself.

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Notation	Description	Page
$\mathscr{H}^t$	t-dimensional Hausdorff measure	9
$\dim_{\mathscr{H}}$	Hausdorff dimension	9
$\mathscr{P}^t$	t-dimensional packing measure	10
$\dim_{\mathscr{P}}$	packing dimension	10
$\mathscr{H}_\nu^{q,t}$	multifractal Hausdorff measure with respect to the measure ν	10
$\dim_\nu^q$	multifractal Hausdorff dimension	10
$\mathscr{P}_\nu^{q,t}$	multifractal packing measure with respect to the measure ν	10
Dim_ν^q	multifractal packing dimension	10
$\mathscr{B}(x,\rho)$	the closed ball with center x and radius ρ	11
$\mathscr{M}_{\nu,\rho}^{q,c}$	qth covering moment with respect to the measure ν	11
$\mathscr{M}_{\nu,\rho}^{q,p}$	qth packing moment with respect to the measure ν	11
$\overline{d}_{\nu,c}^q$	upper multifractal covering box-dimension	11
$\underline{d}_{\nu,c}^q$	lower multifractal covering box-dimension	11
$\overline{d}_{\nu,p}^q$	upper multifractal packing box-dimension	11
$\underline{d}_{\nu,p}^q$	lower multifractal packing box-dimension	11
H^t	lower t-dimensional Hewitt–Stromberg measure	11
$\underline{\dim}_{MB}$	lower Hewitt–Stromberg dimension	11
P^t	upper t-dimensional Hewitt–Stromberg measure	12
$\overline{\dim}_{MB}$	upper Hewitt–Stromberg dimension	12
$H_\nu^{q,t}$	lower multifractal Hewitt–Stromberg measure with respect to the measure ν	12
b_ν^q	lower multifractal Hewitt–Stromberg dimension	12
$P_\nu^{q,t}$	upper multifractal Hewitt–Stromberg measure with respect to the measure ν	12
B_ν^q	upper multifractal Hewitt–Stromberg dimension	12
$\mathscr{T}_n$	set of (admissible) words of length n	13
$\mathscr{T}$	set of (admissible) words of finite length	13
$\mathscr{M}(I,\{a_n\},\{b_{n,j}\})$	collection of Moran sets associated with the set I, a sequence $\{a_n\}$ of positive integers and a sequence $\{b_{n,j}\}$ of positive numbers	13
Δ	$\inf_{i\neq j\ \boldsymbol{\alpha}\in\mathscr{T}} \mathrm{dist}(I_{\boldsymbol{\alpha}*i}, I_{\boldsymbol{\alpha}*j})/\lvert I_{\boldsymbol{\alpha}}\rvert$	14
OSC	Open Set Condition	14
SOSC	Strong Open Set Condition	15
SSC	Strong Separation Condition	15
$\beta_n(q)$	the unique function that satisfies $\sum_{\boldsymbol{\alpha}\in\mathscr{T}_n} p_{\boldsymbol{\alpha}}^q \lvert I_{\boldsymbol{\alpha}}\rvert^{\beta_n(q)} = 1$	15
$\underline{\beta}(q)$	the lower limit of $\beta_n(q)$ as n tends to infinity	15
$\overline{\beta}(q)$	the upper limit of $\beta_n(q)$ as n tends to infinity	15
$b_\nu(q)$	$b_\nu^q(\operatorname{supp}\nu)$	16
$B_\nu(q)$	$B_\nu^q(\operatorname{supp}\nu)$	16
$\nu\llcorner\mathcal{A}$	the restriction of the measure ν to the set $\mathcal{A}$	24
δ_x	the Dirac measure concentrated at x	24

1. Introduction

Self-similar fractals arise as the unique attractors of iterated function systems (IFS) [1, 3] composed of a finite number of contracting similarity mappings that satisfy the open set condition. Every point x in such a fractal F, generated by an IFS $\mathscr{S}$, can be viewed as the outcome of an infinite encoding process $\mathscr{T}$ (which may not always be uniquely defined) over the alphabet $\Sigma_k = \{1, \ldots, k\}$ [22], where k denotes the number of contraction mappings in the system $\mathscr{S}$. A classical result by Moran [20] and Falconer states that the Hausdorff and packing dimensions of a self-similar fractal coincide with its similarity dimension, which depends solely on the contraction factors of the similarity mappings. The class of self-similar fractals encompasses iconic examples such as the Cantor set, the von Koch curve, and the Sierpiński triangle, along with a wide variety of other complex and intriguing sets in Euclidean space.

Fractal geometry and multifractal analysis [11, 16, 18, 19, 33] offer powerful mathematical frameworks to investigate objects and phenomena characterized by self-similarity, irregularity, and complexity across multiple scales [6, 12]. These topics mainly focus on Moran measures and Moran sets, named after the mathematician P. A. Moran [20], who pioneered a class of geometric constructions rooted in probabilistic methods. These concepts have become essential for studying both deterministic and random structures with fractal attributes. Moran sets emerge from iterative geometric processes that extend the notion of self-similarity [13–15]. They are constructed through a recursive procedure, where a base geometric shape is repeatedly subdivided according to specific rules (see [12]). These rules determine the scaling ratios, relative positions, and the number of subdivisions at each step. Unlike classical self-similar sets, such as the Cantor set or the Sierpiński triangle, Moran sets introduce greater flexibility by allowing scaling ratios and subset arrangements to vary across iterations. This adaptability makes Moran sets particularly relevant for modeling real-world fractal structures, which often display variable scaling properties.

Moran measures are probability measures defined on Moran sets, capturing the distribution of mass within these intricate constructions. Built iteratively alongside the geometric framework of Moran sets, these measures enable precise quantification of how mass or probability density is allocated across scales. By bridging geometry and measure theory, Moran measures facilitate the computation of key fractal characteristics such as the Hausdorff dimension and the multifractal spectrum. Their versatility makes them indispensable in multifractal analysis, where they help characterize the heterogeneity of singularities within a measure. In particular, Moran measures provide a robust foundation for studying multifractality, a phenomenon where different regions of a fractal exhibit distinct scaling behaviors; see, for example, [5]. In multifractal analysis, Moran measures play a crucial role in understanding the scaling laws that govern mass distributions at different scales. The multifractal spectrum, derived from Moran measures, provides a detailed profile of singularity strengths and their prevalence within the measure (see [4, 11, 16, 27–31, 34, 35, 38]). This spectrum offers valuable insights into the complexity and heterogeneity of the underlying fractal object, enabling applications in fields such as physics, biology, and beyond [2, 10, 17].

Among various concepts associated with these measures, we focus on analyzing their L^q-spectrum, with particular emphasis on the rate of convergence of the L^q-spectra for Moran measures. The foundational result in this area was established by Olsen in [24], where he conducted an in-depth analysis of the asymptotic behavior of the qth moments of self-similar measures under the Open Set Condition (OSC). This result was subsequently extended by several authors [36, 37]. For $q \in \mathbb{R}$, our goal is to determine the exact rate of convergence of the L^q-spectra for Moran measures that satisfy the Set Strong Separation Condition. As an application, we demonstrate that the empirical multifractal moment measures converge weakly to the normalized multifractal measures. By addressing these questions, we aim to deepen the understanding of multifractal behavior and provide new insights into the dynamics of Moran measures, and we try to establish that the multifractal and fractal dimensions of the overlap regions in a Moran set satisfying the Strong Open Set Condition are strictly less than the overall dimension of the set.

Consider $f_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \dots, n$ as contracting similarity transformations, and let $(p_1, \dots, p_n)$ represent a probability vector. We define $\mathscr{K}$ to be the Moran set and ν to be the Moran measure (see Section 2.7) associated with the pairs (f_i, p_i). For a real number q and a positive number ρ, we introduce the definitions of the qth covering moment and the qth packing moment of the measure ν as follows:

$$\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K}) = \inf \Big\{ \sum_{x\in E} \nu(\mathscr{B}(x,\rho))^q \;\Big|\; E \text{ is a } \rho\text{-spanning subset of } \mathscr{K} \Big\},$$

and

$$\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K}) = \sup \Big\{ \sum_{x\in E} \nu(\mathscr{B}(x,\rho))^q \;\Big|\; E \text{ is a } \rho\text{-separated subset of } \mathscr{K} \Big\}.$$

It is well established that if the Moran set $\mathscr{K}$ fulfills the Set Strong Separation Condition then

$$\liminf_{\rho\to 0} \frac{\log \mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}{-\log\rho} = \liminf_{\rho\to 0} \frac{\log \mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{-\log\rho} = \underline{\beta}(q) \tag{1.1}$$

and

$$\limsup_{\rho\to 0} \frac{\log \mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}{-\log\rho} = \limsup_{\rho\to 0} \frac{\log \mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{-\log\rho} = \overline{\beta}(q), \tag{1.2}$$

where $\underline{\beta}(q) = \liminf_{n\to+\infty} \beta_n(q)$ and $\overline{\beta}(q) = \limsup_{n\to+\infty} \beta_n(q)$ such that $\beta_n(q)$ is defined by

$$\sum_{\boldsymbol{\alpha}\in\mathscr{T}_n} p^q_{\boldsymbol{\alpha}} |I_{\boldsymbol{\alpha}}|^{\beta_n(q)} = 1,$$

here $|I_{\boldsymbol{\alpha}}|$ represents the product of the Lipschitz constant associated with the similarity transformation $f_{\boldsymbol{\alpha}}$. Under the assumption that the Set Strong Separation Condition is satisfied, we can precisely determine in Theorem 3.1 the rate of convergence in (1.1) and (1.2). Specifically, there exist multiplicatively periodic functions $\overline{\pi}_q, \underline{\pi}_q, \overline{\Pi}_q, \underline{\Pi}_q : (0,\infty) \to \mathbb{R}$ such that:

$$\frac{\mathscr{M}_{\nu,\rho}^{q,c}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}} = \overline{\pi}_q(\rho) + \epsilon(\rho), \qquad \frac{\mathscr{M}_{\nu,\rho}^{q,p}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}} = \overline{\Pi}_q(\rho) + \epsilon(\rho), \tag{1.3}$$

$$\frac{\mathscr{M}_{\nu,\rho}^{q,c}(\mathscr{K})}{\rho^{-\underline{\beta}(q)}} = \underline{\pi}_q(\rho) + \epsilon(\rho), \qquad \frac{\mathscr{M}_{\nu,\rho}^{q,p}(\mathscr{K})}{\rho^{-\underline{\beta}(q)}} = \underline{\Pi}_q(\rho) + \epsilon(\rho), \tag{1.4}$$

where $\epsilon(\rho) \to 0$ as $\rho \to 0$.

Utilizing the results from (1.3) and (1.4), we establish in Theorem 4.1 that the empirical multifractal moment measures converge weakly as ρ approaches 0, indicating that

$$\frac{1}{\mathscr{M}_{\nu,\rho}^{q,c}(\mathscr{K})} \sum_{x \in E_\rho} \nu(\mathscr{B}(x,\rho))^q \delta_x \to \frac{\mathscr{H}_\nu^{q,\underline{\beta}(q)} \llcorner \mathscr{K}}{\mathscr{H}_\nu^{q,\underline{\beta}(q)}(\mathscr{K})} \quad \text{weakly as } \rho \to 0$$

and

$$\frac{1}{\mathscr{M}_{\nu,\rho}^{q,p}(\mathscr{K})} \sum_{x \in F_\rho} \nu(\mathscr{B}(x,\rho))^q \delta_x \to \frac{\mathscr{P}_\nu^{q,\overline{\beta}(q)} \llcorner \mathscr{K}}{\mathscr{P}_\nu^{q,\overline{\beta}(q)}(\mathscr{K})} \quad \text{weakly as } \rho \to 0,$$

where δ_x denotes the Dirac measure concentrated at $x \in \mathbb{R}^d$ and for each positive ρ, E_ρ is defined as a (suitable) minimal ρ-spanning subset of $\mathscr{K}$, while F_ρ serves as a (suitable) maximal ρ-separated subset of $\mathscr{K}$. Furthermore, $\mathscr{P}_\nu^{q,\overline{\beta}(q)}(\mathscr{K})$ and $\mathscr{H}_\nu^{q,\underline{\beta}(q)}(\mathscr{K})$ represent, respectively, the multifractal packing measure and the multifractal Hausdorff measure associated with the measure ν.

Tube formulas pertain to the analysis of the volumes of ρ-neighborhoods surrounding sets. Over the last two decades, Lapidus has spearheaded a rigorous and systematic exploration of tube formulas in the context of fractal geometry. Building upon this trajectory, a natural question arises: to what extent can one formulate a comprehensive theory of multifractal tube formulas for multifractal measures? The objective of Section 5 is to lay the groundwork for such a theoretical framework. We begin by formally defining multifractal tube formulas and, in a broader sense, multifractal tube measures applicable to general multifractal measures. Subsequently, we provide a full description of the asymptotic behavior of the multifractal tube formulas, specifically for Moran measures that satisfy the Set Strong Separation Condition, which is practically a result equivalent to Theorem 3.1 using tube formulas. In addition, as an application, we will explicitly determine the weak limits of the multifractal tube measures for Moran measures ν (Theorems 5.3 and 5.4).

We establish the main results of this paper under the Set Strong Separation Condition. Nevertheless, verifying the Strong Open Set Condition (SOSC) or the Open Set Condition (OSC) remains a difficult task. We therefore conjecture that the conclusions of Theorem 2.10 may also hold under (SOSC) or (OSC). In this direction, we provide partial supporting results at the end of the paper.

The structure of the paper is as follows. In Section 2, we review the essential preliminary definitions. Section 3 presents one of the main results: the exact rate of convergence of the L^q-spectra for Moran measures satisfying the Set Strong Separation Condition. Section 4 provides an illustrative example demonstrating the convergence of empirical multifractal moment measures to the normalized multifractal measure. In Section 5, we revisit this analysis using tube formulas. Finally, in Section 6, we offer several remarks and discuss an open problem that arises when relaxing the separation condition—recall that our main results rely on the Strong Open Set Condition (SOSC).

2. Technical preliminaries

Before presenting the main results, we provide the necessary technical background. In Section 2.1, we recall the definitions of the Hausdorff and packing measures and dimensions. Sections 2.2–2.3 introduce the multifractal Hausdorff and packing measures and dimensions, while Section 2.4 is devoted to the multifractal box dimensions. The multifractal and fractal Hewitt–Stromberg measures and dimensions are discussed in Sections 2.5–2.6. Section 2.7 recalls the definitions of Moran sets and measures. Finally, Section 2.8 presents several separation conditions.

2.1. Hausdorff and packing measures and dimensions. For a subset $\mathscr{A} \subseteq \mathbb{R}^d$, the diameter of $\mathscr{A}$ is defined by

$$|\mathscr{A}| = \sup\{|x-y| : x, y \in \mathscr{A}\}.$$

We refer to the countable set $(U_i)_i$ as a *δ-covering* of $\mathscr{A}$ if

$$\mathscr{A} \subset \bigcup_i U_i \quad \text{and} \quad 0 < |U_i| < \delta.$$

For $t \geq 0$ and $\delta > 0$, set

$$\mathscr{H}^t_\delta(\mathscr{A}) = \inf\Big\{\sum_i |U_i|^t \;\Big|\; (U_i)_i \text{ is a } \delta\text{-covering of } \mathscr{A}\Big\}.$$

Then the *t-dimensional Hausdorff measure* of $\mathscr{A}$ is

$$\mathscr{H}^t(\mathscr{A}) = \lim_{\delta\to 0} \mathscr{H}^t_\delta(\mathscr{A}) = \sup_{\delta>0} \mathscr{H}^t_\delta(\mathscr{A}).$$

Next, the *Hausdorff dimension* of $\mathscr{A}$ is given by

$$\dim_{\mathscr{H}}(\mathscr{A}) = \inf\{t \geq 0 \mid \mathscr{H}^t(\mathscr{A}) = 0\} = \sup\{t \geq 0 \mid \mathscr{H}^t(\mathscr{A}) = +\infty\}.$$

Now, for $t \geq 0$ and $\delta > 0$, we introduce the *t-dimensional packing pre-measure* as follows:

$$\overline{\mathscr{P}}^t_\delta(\mathscr{A}) = \sup\Big\{\sum_i (2\rho_i)^t \;\Big|\; (\mathscr{B}(x_i,\rho_i))_i \text{ is a } \delta\text{-packing of } \mathscr{A}\Big\},$$

$$\overline{\mathscr{P}}^t(\mathscr{A}) = \lim_{\delta\to 0} \overline{\mathscr{P}}^t_\delta(\mathscr{A}) = \inf_{\delta>0} \overline{\mathscr{P}}^t_\delta(\mathscr{A}).$$

This enables us to define the *t-dimensional packing measure* as

$$\mathscr{P}^t(\mathscr{A}) = \inf\Big\{\sum_i \overline{\mathscr{P}}^t(E_i) \;\Big|\; \mathscr{A} \subset \bigcup_n E_i\Big\}.$$

Likewise, we define the *packing dimension* of $\mathscr{A}$ as

$$\dim_{\mathscr{P}}(\mathscr{A}) = \inf\{t \geq 0 \mid \mathscr{P}^t(\mathscr{A}) = 0\} = \sup\{t \geq 0 \mid \mathscr{P}^t(\mathscr{A}) = +\infty\}.$$

2.2. Multifractal Hausdorff measures and dimensions. Let ν be a probability measure on $\mathbb{R}^d$. Let $\mathscr{A}$ be a non-empty subset of $\mathbb{R}^d$, $\delta > 0$ and let $q, t \in \mathbb{R}$. We define the multifractal Hausdorff pre-measure as

$$\overline{\mathscr{H}}^{q,t}_{\nu,\delta}(\mathscr{A}) = \inf\left\{\sum_i \nu(\mathscr{B}(x_i,\rho_i))^q(2\rho_i)^t \;\middle|\; (\mathscr{B}(x_i,\rho_i))_i \text{ is a centered } \delta\text{-covering of } \mathscr{A}\right\}.$$

The function $\overline{\mathscr{H}}^{q,t}_{\nu,\delta}$ is σ-subadditive, but it is not increasing, so we modify the definition as follows:

$$\overline{\mathscr{H}}^{q,t}_{\nu}(\mathscr{A}) = \lim_{\delta\to 0} \overline{\mathscr{H}}^{q,t}_{\nu,\delta}(\mathscr{A}) = \sup_{\delta>0} \overline{\mathscr{H}}^{q,t}_{\nu,\delta}(\mathscr{A}).$$

Finally,

$$\mathscr{H}^{q,t}_{\nu}(\mathscr{A}) = \sup_{E\subseteq\mathscr{A}} \overline{\mathscr{H}}^{q,t}_{\nu}(E).$$

The multifractal Hausdorff measure $\mathscr{H}^{q,t}_{\nu}$ is considered as an outer metric measure on $\mathbb{R}^d$.

There is a unique number $\dim^q_\nu(\mathscr{A}) \in [-\infty, +\infty]$ such that

$$\mathscr{H}^{q,t}_{\nu}(\mathscr{A}) = \begin{cases} +\infty & \text{if } t < \dim^q_\nu(\mathscr{A}), \\ 0 & \text{if } t > \dim^{\boldsymbol{q}}_\nu(\mathscr{A}). \end{cases}$$

2.3. Multifractal packing measures and dimensions. Let ν be a probability measure on $\mathbb{R}^d$. Consider a non-empty set $\mathscr{A}$ of $\mathbb{R}^d$ and let $\delta > 0$ and $q, t \in \mathbb{R}$. We define the multifractal packing pre-measure as

$$\overline{\mathscr{P}}^{q,t}_{\nu,\delta}(\mathscr{A}) = \sup\left\{\sum_i \nu(\mathscr{B}(x_i,\rho_i))^q(2\rho_i)^t \;\middle|\; (\mathscr{B}(x_i,\rho_i))_i \text{ is a } \delta\text{-packing of } \mathscr{A}\right\}.$$

While the function $\overline{\mathscr{P}}^{q,t}_{\nu,\delta}$ increases, it is not σ-subadditive, so we can define

$$\overline{\mathscr{P}}^{q,t}_{\nu}(\mathscr{A}) = \lim_{\delta\to 0} \overline{\mathscr{P}}^{q,t}_{\nu,\delta}(\mathscr{A}) = \inf_{\delta>0} \overline{\mathscr{P}}^{q,t}_{\nu,\delta}(\mathscr{A}),$$

and

$$\mathscr{P}^{q,t}_{\nu}(\mathscr{A}) = \inf_{\mathscr{A}\subseteq\bigcup_i E_i} \sum_i \overline{\mathscr{P}}^{q,t}_{\nu}(E_i).$$

On $\mathbb{R}^d$, the mixed multifractal packing measure $\mathscr{P}^{q,t}_{\nu}$ is considered as an outer metric measure.

There is a unique number $\mathrm{Dim}^q_\nu(\mathscr{A}) \in [-\infty, +\infty]$ such that

$$\mathscr{P}^{q,t}_{\nu}(\mathscr{A}) = \begin{cases} +\infty & \text{if } t < \mathrm{Dim}^q_\nu(\mathscr{A}), \\ 0 & \text{if } t > \mathrm{Dim}^q_\nu(\mathscr{A}). \end{cases}$$

2.4. Multifractal box-dimensions. Let $\mathscr{A} \subseteq \mathbb{R}^d$ and $\rho > 0$. A subset E of $\mathscr{A}$ is said to be *ρ-separated* if

$$|x - y| > 2\rho \quad \text{for all } x, y \in E \text{ with } x \neq y.$$

And E is called *ρ-spanning* if

$$\mathscr{A} \subseteq \bigcup_{x\in E} \mathscr{B}(x,\rho),$$

where $\mathscr{B}(x,\rho)$ is the closed ball with center x and radius ρ.

For a probability measure ν of $\mathbb{R}^d$, $q \in \mathbb{R}$ and $\rho > 0$ we define the *covering moment scaling function* of $\mathscr{A}$ with respect to ν by

$$\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{A}) = \inf\Big\{\sum_i \nu(\mathscr{B}(x_i,\rho))^q \;\Big|\; (\mathscr{B}(x_i,\rho))_i \text{ is a centered cover of } \mathscr{A}\Big\},$$

and the *packing moment scaling function* of $\mathscr{A}$ with respect to ν by

$$\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{A}) = \sup\Big\{\sum_i \nu(\mathscr{B}(x_i,\rho))^q \;\Big|\; (\mathscr{B}(x_i,\rho))_i \text{ is a centered packing of } \mathscr{A}\Big\}.$$

In other words,

$$\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{A}) = \inf\Big\{\sum_{x\in\mathscr{A}} \nu(\mathscr{B}(x,\rho))^q \;\Big|\; E \text{ is a } \rho\text{-spanning subset of } \mathscr{A}\Big\},$$

$$\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{A}) = \sup\Big\{\sum_{x\in\mathscr{A}} \nu(\mathscr{B}(x,\rho))^q \;\Big|\; E \text{ is a } \rho\text{-separated subset of } \mathscr{A}\Big\}.$$

Now, the *upper* and *lower multifractal covering box-dimensions* denoted by $\overline{d}^{\,q}_{\nu,c}(\mathscr{A})$ and $\underline{d}^{q}_{\nu,c}(\mathscr{A})$ and the *upper* and *lower multifractal packing box-dimensions* denoted by $\overline{d}^{\,q}_{\nu,p}(\mathscr{A})$ and $\underline{d}^{q}_{\nu,p}(\mathscr{A})$ are introduced as follows:

$$\overline{d}^{\,q}_{\nu,c}(\mathscr{A}) = \limsup_{\rho\to 0} \frac{\log \mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{A})}{-\log\rho}, \qquad \underline{d}^{q}_{\nu,c}(\mathscr{A}) = \liminf_{\rho\to 0} \frac{\log \mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{A})}{-\log\rho},$$

and

$$\overline{d}^{\,q}_{\nu,p}(\mathscr{A}) = \limsup_{\rho\to 0} \frac{\log \mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{A})}{-\log\rho}, \qquad \underline{d}^{q}_{\nu,p}(\mathscr{A}) = \liminf_{\rho\to 0} \frac{\log \mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{A})}{-\log\rho}.$$

2.5. Hewitt–Stromberg measures and dimensions. Let $t,\rho > 0$. For $\mathscr{A} \subseteq \mathbb{R}^d$, set

$$\mathscr{N}_\rho(\mathscr{A}) = \inf\{\sharp\{J\} \mid (\mathscr{B}(x_i,\rho))_{i\in J} \text{ is a centered cover of } \mathscr{A}\}.$$

The *lower Hewitt–Stromberg pre-measure* of $\mathscr{A}$ is defined as follows:

$$D^t(\mathscr{A}) = \liminf_{\rho\to 0} \mathscr{N}_\rho(\mathscr{A})(2\rho)^t$$

and

$$\overline{H}^{t}(\mathscr{A}) = \sup_{F\subseteq\mathscr{A}} D^t(E).$$

Next, we introduce the *lower t-dimensional Hewitt–Stromberg measure* of $\mathscr{A}$ as

$$H^t(\mathscr{A}) = \inf_{\mathscr{A}\subseteq\bigcup_i E_i} \sum_i \overline{H}^{t}(E_i).$$

The *lower Hewitt–Stromberg dimension* $\mathscr{A}$ is defined by

$$\underline{\dim}_{MB}(\mathscr{A}) = \inf\{s \ge 0 \mid H^t(\mathscr{A}) = 0\} = \sup\{s \ge 0 \mid H^t(\mathscr{A}) = +\infty\}.$$

Similarly, we define

$$\mathscr{M}_\rho(\mathscr{A}) = \sup\{\sharp\{J\} \mid (\mathscr{B}(x_i,\rho))_{i\in J} \text{ is a centered packing of } \mathscr{A}\}.$$

The *upper Hewitt–Stromberg pre-measure of* $\mathscr{A}$ is defined as follows:

$$\overline{P}^t(\mathscr{A}) = \limsup_{\rho\to 0} \mathscr{M}_\rho(\mathscr{A})(2\rho)^t.$$

The *upper t-dimensional Hewitt–Stromberg measure* of $\mathscr{A}$ is defined by

$$P^t(\mathscr{A}) = \inf_{\mathscr{A}\subseteq\bigcup_i E_i} \sum_i \overline{P}^t(E_i).$$

The *upper Hewitt–Stromberg dimension* of $\mathscr{A}$ is defined by

$$\overline{\dim}_{MB}(\mathscr{A}) = \inf\{t\geq 0 \mid P^t(\mathscr{A}) = 0\} = \sup\{t\geq 0 \mid P^t(\mathscr{A}) = +\infty\}.$$

2.6. Multifractal Hewitt–Stromberg measures and dimensions. Let $q, t \in \mathbb{R}$. For $\mathscr{A} \subseteq \mathbb{R}^d$, we define a pre-measure of $\mathscr{A}$ by

$$D_\nu^{q,t}(\mathscr{A}) = \liminf_{\rho\to 0} \mathscr{M}_{\nu,\rho}^{q,c}(\mathscr{A})(2\rho)^t.$$

$D_\nu^{q,t}$ neither increases nor is countably subadditive. A standard adjustment is required to obtain an outer measure. Consequently, we consider

$$\overline{H}_\nu^{q,t}(\mathscr{A}) = \inf_{\mathscr{A}\subseteq\bigcup_i \mathscr{A}_i} \sum_i D_\nu^{q,t}(\mathscr{A}_i), \qquad H_\nu^{q,t}(\mathscr{A}) = \sup_{E\subseteq\mathscr{A}} \overline{H}_\nu^{q,t}(E).$$

There is a unique number $b_\nu^q(\mathscr{A}) \in [-\infty, +\infty]$ such that

$$H_\nu^{q,t}(\mathscr{A}) = \begin{cases} +\infty & \text{if } t < b_\nu^q(\mathscr{A}), \\ 0 & \text{if } t > b_\nu^q(\mathscr{A}). \end{cases}$$

In the same manner, we introduce a pre-measure of $\mathscr{A}$ by

$$\overline{P}_\nu^{q,t}(\mathscr{A}) = \limsup_{\rho\to 0} \mathscr{M}_{\nu,\rho}^{q,p}(\mathscr{A})(2\rho)^t.$$

The function $\overline{P}_\nu^{q,t}$ is increasing but not σ-additive, thus giving an outer measure as follows:

$$P_\nu^{q,t}(\mathscr{A}) = \inf_{\mathscr{A}\subseteq\bigcup_i E_i} \sum_i \overline{P}_\nu^{q,t}(E_i).$$

There is a unique number $B_\nu^q(\mathscr{A}) \in [-\infty, +\infty]$ such that

$$P_\nu^{q,t}(\mathscr{A}) = \begin{cases} +\infty & \text{if } t < B_\nu^q(\mathscr{A}), \\ 0 & \text{if } t > B_\nu^q(\mathscr{A}). \end{cases}$$

Similarly, there is a unique number $\Delta_\nu^q(\mathscr{A}) \in [-\infty, +\infty]$ such that

$$\overline{P}_\nu^{q,t}(\mathscr{A}) = \begin{cases} +\infty & \text{if } t < \Delta_\nu^q(\mathscr{A}), \\ 0 & \text{if } t > \Delta_\nu^q(\mathscr{A}). \end{cases}$$

REMARK 2.1. It is evident that, for $t > 0$, we have

$$\mathscr{H}_\nu^{0,t} = \mathscr{H}^t, \qquad \mathscr{P}_\nu^{0,t} = \mathscr{P}^t,$$
$$H_\nu^{0,t} = H^t, \qquad P_\nu^{0,t} = P^t.$$

2.7. Moran sets and measures. Before establishing our main results, it is necessary to revisit the category of homogeneous Moran sets, as documented in [34, 35]. Assume $\{a_n\}_{n\geq 1}$ represents a sequence of positive integers, and $\{b_{n,j}\}_{n\geq 1, 1\leq j\leq a_n}$ denotes a sequence of positive numbers that meet the following criteria:

$$a_n \geq 2, \quad 0 < b_{n,j} < 1 \quad \text{for } n \geq 1,\, 1 \leq j \leq a_n.$$

Let $\mathscr{T}_0 = \emptyset$. For $n, m \in \mathbb{N}$ such that $m \leq n$, we consider

$$\mathscr{T}_{m,n} = \{(i_m, i_{m+1}, \dots, i_n) \mid 1 \leq i_j \leq a_j,\, m \leq j \leq n\},$$

$$\mathscr{T}_n = \mathscr{T}_{1,n} \quad \text{and} \quad \mathscr{T} = \bigcup_{n\geq 0} \mathscr{T}_n.$$

For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathscr{T}_n$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_m) \in \mathscr{T}_{n+1,m}$, we denote the juxtaposition of $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ as

$$\boldsymbol{\alpha}\boldsymbol{\sigma} = \boldsymbol{\alpha} * \boldsymbol{\sigma} = (\alpha_1, \dots, \alpha_n, \sigma_1, \dots, \sigma_m).$$

Definition 2.2. Let I denote a compact subset of $\mathbb{R}^d$ such that the interior of I is dense in I, and for convenience, we assume the diameter of I is 1. The following conditions provide us with a framework to describe a Moran structure, which represents a collection $\mathscr{F} = \{I_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathscr{T}\}$ of closed subsets of I:

($H1$) $I_\emptyset = I$.

($H2$) For any $\boldsymbol{\alpha} \in \mathscr{T}$ there exists a similarity mapping $f_{\boldsymbol{\alpha}} : \mathbb{R}^d \to \mathbb{R}^d$ such that $I_{\boldsymbol{\alpha}} = f_{\boldsymbol{\alpha}}(I)$, meaning that the set $I_{\boldsymbol{\alpha}}$ is geometrically similar to I.

($H3$) For any $n \geq 0$ and $\boldsymbol{\alpha} \in \mathscr{T}_n$, $I_{\boldsymbol{\alpha}*1}, \dots, I_{\boldsymbol{\alpha}*a_{n+1}}$ are subsets of $I_{\boldsymbol{\alpha}}$. Moreover,

$$I^\circ_{\boldsymbol{\alpha}*i} \cap I^\circ_{\boldsymbol{\alpha}*j} = \emptyset \quad \text{for } i \neq j,$$

where J° signifies the interior of J.

($H4$) For any $n \geq 1$ and $\boldsymbol{\alpha} \in \mathscr{T}_{n-1}$,

$$\frac{|I_{\boldsymbol{\alpha}*j}|}{|I_{\boldsymbol{\alpha}}|} = b_{n,j} \quad \text{for } 1 \leq j \leq a_n.$$

Let $\mathscr{F}$ represent a collection of closed subsets of I that have the properties of a Moran structure. We define the *Moran set* associated with $\mathscr{F}$ by

$$\mathscr{K} := \mathscr{K}(\mathscr{F}) = \bigcap_{n\geq 1} \bigcup_{\boldsymbol{\alpha}\in\mathscr{T}_n} I_{\boldsymbol{\alpha}}.$$

The collection of Moran sets associated with I, $\{a_n\}$ and $\{b_{n,j}\}$ is denoted by $\mathscr{M}(I, \{a_n\}, \{b_{n,j}\})$.

Remark 2.3. If

$$\lim_{n\to+\infty} \sup_{\boldsymbol{\alpha}\in T_n} |I_{\boldsymbol{\alpha}}| > 0,$$

then there are interior points in $\mathscr{K}$. Consequently, properties associated with measure and dimension become trivial by default. Therefore, we consider the case where

$$\lim_{n\to+\infty} \sup_{\boldsymbol{\alpha}\in T_n} |I_{\boldsymbol{\alpha}}| = 0.$$

Definition 2.4. Let $\{p_{n,j}\}_{j=1}^{a_n}$ be a probability vector (i.e., $p_{n,j} > 0$ and $\sum_{j=1}^{a_n} p_{n,j} = 1$ for $n \geq 1$) and let p_0 represent the infimum of the set $\{p_{n,j}\}$, assuming $p_0 > 0$. Let ν

be a mass distribution on $\mathscr{K}$. We say that ν is a *Moran measure* on $\mathscr{K} = \operatorname{supp}\nu$ if for any $I_{\boldsymbol{\alpha}}$ (where $\boldsymbol{\alpha}$ belongs to $\mathscr{T}$),

$$\nu(I_{\boldsymbol{\alpha}}) = p_{1,\alpha_1}p_{2,\alpha_2}\cdots p_{n,\alpha_n}.$$

For $\boldsymbol{\alpha} \in \mathscr{T}_n$, we write $|\boldsymbol{\alpha}| = n$. Also if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathscr{T}_n$, define

$$\begin{aligned} f_{\boldsymbol{\alpha}} &= f_{1,\alpha_1} \circ f_{2,\alpha_2} \circ \ldots \circ f_{n,\alpha_n}, \\ p_{\boldsymbol{\alpha}} &= p_{1,\alpha_1}p_{2,\alpha_2}\cdots p_{n,\alpha_n}, \\ |I_{\boldsymbol{\alpha}}| &= b_{1,\alpha_1}b_{2,\alpha_2}\cdots b_{n,\alpha_n}. \end{aligned}$$

Additionally, we can truncate $\boldsymbol{\alpha} \in \mathscr{T}$ to its nth place by using $\boldsymbol{\alpha}|n = (\alpha_1, \ldots, \alpha_n)$. Furthermore, if $\boldsymbol{l}, \boldsymbol{\alpha} \in \mathscr{T}$, we write

$$\boldsymbol{l} \nsubseteq \boldsymbol{\alpha} \quad \text{if and only if} \quad \boldsymbol{\alpha} \neq \boldsymbol{ulv} \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in \mathscr{T}.$$

If $\Phi \subseteq \mathscr{T}$ and $\boldsymbol{\alpha} \in \mathscr{T}$, we write

$$\Phi \nsubseteq \boldsymbol{\alpha} \quad \text{if and only if} \quad \boldsymbol{l} \nsubseteq \boldsymbol{\alpha} \text{ for all } \boldsymbol{l} \in \Phi.$$

Remark 2.5. (1) It is important to note that the set $\mathscr{K}$ (Moran set) satisfies the invariance equality

$$\mathscr{K} = \bigcup_{j=1}^{a_n} f_{n,j}(\mathscr{K}),$$

where $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ is a list of contracting similarities. And ν is the unique Moran measure such that

$$\nu = \sum_{j=1}^{a_n} p_{n,j}\, \nu \circ f_{n,j}^{-1}.$$

(2) A Moran set and a Moran measure are, respectively, an extension of the self-similar set and the self-similar measure proposed by [12]. So we say in this case that the corresponding Moran set (resp. measure) is a generalized self-similar set if there is a collection $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ of contracting similarities satisfying $I_{\boldsymbol{\alpha}} = f_{\boldsymbol{\alpha}}(I)$ for $\boldsymbol{\alpha} \in \mathscr{T}_n$ (resp. measure, if it is supported by a generalized self-similar set).

2.8. Separation conditions

Definition 2.6. We say that the Moran set $\mathscr{K}$ fulfills the *Set Strong Separation Condition* if the condition $(H3)$ in Definition 2.2 is reinforced by

$$\Delta = \inf_{\substack{\boldsymbol{\alpha} \in \mathscr{T} \\ i \neq j}} \frac{\operatorname{dist}(I_{\boldsymbol{\alpha}*i}, I_{\boldsymbol{\alpha}*j})}{|I_{\boldsymbol{\alpha}}|} > 0,$$

where $\operatorname{dist}(\mathscr{A}, \mathscr{B}) = \inf_{l \in \mathscr{A},\, t \in \mathscr{B}} \operatorname{dist}(l, t)$ for any pair of sets $\mathscr{A}$ and $\mathscr{B}$.

Definition 2.7. We say that the list $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ of contracting similarities satisfies the *Open Set Condition* (OSC) if there exists an open, non-empty, bounded subset $O \subset I$ such that, for all $n \geq 1$,

$$\bigcup_{j=1}^{a_n} f_{n,j}(O) \subset O \quad \text{and} \quad f_{n,i}(O) \cap f_{n,j}(O) = \emptyset \text{ for } i \neq j.$$

Definition 2.8. We say that the list $\{f_{n,j} \mid n\geq 1,\ 1\leq j\leq a_n\}$ of contracting similarities satisfies the *Strong Open Set Condition* (SOSC) if there exists an open, non-empty, bounded subset $O \subset I$ such that

$$\bigcup_{j=1}^{a_n} f_{n,j}(O) \subset O, \quad f_{n,i}(O) \cap f_{n,j}(O) = \emptyset \text{ for } i \neq j \quad \text{and} \quad O \cap \mathscr{K} \neq \emptyset,$$

for all $n \geq 1$.

Definition 2.9. We say that the list $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ of contracting similarities satisfies the *Strong Separation Condition* (SSC) if there exists an open, non-empty, bounded subset $O \subset I$ such that

$$\bigcup_{j=1}^{a_n} f_{n,j}(O) \subset O \quad \text{and} \quad \overline{f_{n,i}(O)} \cap \overline{f_{n,j}(O)} = \emptyset \text{ for } i \neq j,$$

for all $n \geq 1$.

If the list $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ of contracting similarities satisfies the (SOSC) condition, there exists an open, non-empty, bounded subset O such that $O \cap \mathscr{K} \neq \emptyset$ and then, we can select an element $\boldsymbol{l}$ from $\mathscr{T}$ such that $f_{\boldsymbol{l}}(\mathscr{K}) \subseteq O$. Thus, there exists $n \geq 1$ such that

$$\Phi_n = \{\boldsymbol{l} \in \mathscr{T}_n \mid f_{\boldsymbol{l}}(\mathscr{K}) \subseteq O\} \neq \emptyset.$$

For $q \in \mathbb{R}$ and a positive integer n with $\Phi_n \neq \emptyset$, we write

$$E_n(t) = \sum_{\substack{|\boldsymbol{\alpha}|=n \\ \boldsymbol{\alpha}\notin\Phi_n}} p_{\boldsymbol{\alpha}}^q |I_{\boldsymbol{\alpha}}|^t \quad \text{for } t \in \mathbb{R}.$$

Since the function $E_n : \mathbb{R} \to \mathbb{R}$ is continuous and strictly decreasing with

$$\lim_{t\to-\infty} E_n(t) = +\infty \quad \text{and} \quad \lim_{t\to+\infty} E_n(t) = 0,$$

employing the intermediate value theorem we find $\varphi_n(q) \in \mathbb{R}$ such that

$$\sum_{\substack{|\boldsymbol{\alpha}|=n \\ \boldsymbol{\alpha}\notin\Phi_n}} p_{\boldsymbol{\alpha}}^q |I_{\boldsymbol{\alpha}}|^{\varphi_n(q)} = 1.$$

Additionally, we define $\beta_n : \mathbb{R} \to \mathbb{R}$ to be the unique function that satisfies

$$\sum_{\boldsymbol{\alpha}\in\mathscr{T}_n} p_{\boldsymbol{\alpha}}^q |I_{\boldsymbol{\alpha}}|^{\beta_n(q)} = 1.$$

For $q \in \mathbb{R}$, we define

$$\underline{\beta}(q) = \liminf_{n\to+\infty} \beta_n(q), \qquad \overline{\beta}(q) = \limsup_{n\to+\infty} \beta_n(q),$$

and

$$\varphi(q) = \inf_{\Phi_n\neq\emptyset} \varphi_n(q).$$

Since

$$\sum_{\substack{|\boldsymbol{\alpha}|=n \\ \boldsymbol{\alpha}\notin\Phi_n}} p_{\boldsymbol{\alpha}}^q |I_{\boldsymbol{\alpha}}|^{\varphi_n(q)} = 1 = \sum_{|\boldsymbol{\alpha}|=n} p_{\boldsymbol{\alpha}}^q |I_{\boldsymbol{\alpha}}|^{\beta_n(q)},$$

we have

$$\varphi_n(q) < \beta_n(q).$$

So, we have usually

$$\varphi(q) < \overline{\beta}(q).$$

And if there exists n_0 such that $\beta_{n_0}(q) \leq \underline{\beta}(q)$, we have

$$\varphi(q) < \underline{\beta}(q).$$

The compactness of $f_{\boldsymbol{l}}(\mathscr{K})$ for $\boldsymbol{l} \in \mathscr{T}$ implies that

$$c_n = \min_{\boldsymbol{l} \in \Phi_n} \operatorname{dist}(f_{\boldsymbol{l}}(\mathscr{K}), \mathbb{R}^d \setminus O) > 0 \quad \text{for all } n \text{ with } \Phi_n \neq \emptyset, \tag{2.1}$$

where $\operatorname{dist}(E,F) = \inf_{x \in E,\, y \in F} \operatorname{dist}(x,y)$ for any sets E and F.

Also, it is easy to see that

$$f_{\boldsymbol{\alpha}}(\mathscr{K}) \subseteq \overline{f_{\boldsymbol{\alpha}}(O)} \quad \text{for all } \boldsymbol{\alpha} \in \mathscr{T}$$

and

$$f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(O) = \emptyset \quad \text{for all } \boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T} \text{ with } |\boldsymbol{\alpha}| = |\boldsymbol{\sigma}| \ (\boldsymbol{\alpha} \neq \boldsymbol{\sigma}).$$

The next result states that, for a Moran measure, the aforementioned dimensions of the Moran set $\mathscr{K}$ all coincide and are identical to $\underline{\beta}(q)$ and $\overline{\beta}(q)$.

For $\mathscr{A} \subseteq \operatorname{supp} \nu =: \mathscr{K}$, we denote

$$b_\nu(q) = b_\nu^q(\operatorname{supp} \nu), \qquad B_\nu(q) = B_\nu^q(\operatorname{supp} \nu), \qquad \Lambda_\nu(q) = \Delta_\nu^q(\operatorname{supp} \nu).$$

Theorem 2.10. *Let ν be a Moran measure on $\mathscr{K}$. If $\Delta > 0$, then for all $q \in \mathbb{R}$,*

$$\dim_\nu^q(\mathscr{K}) = b_\nu(q) = \underline{d}_{\nu,c}^q(\mathscr{K}) = \underline{d}_{\nu,p}^q(\mathscr{K}) = \underline{\beta}(q),$$
$$\operatorname{Dim}_\nu^q(\mathscr{K}) = B_\nu(q) = \overline{d}_{\nu,c}^q(\mathscr{K}) = \overline{d}_{\nu,p}^q(\mathscr{K}) = \overline{\beta}(q).$$

Proof. This can be deduced from [9] and [34]. ∎

3. The exact rate of convergence of the qth moments of the Moran measures

Fractal geometry, a mathematical discipline designed to study intricate structures that display self-similarity and irregularity at all scales, has become a vital tool for analyzing both natural phenomena and mathematical objects. Traditional geometric constructs are insufficient to capture the complexity of certain sets or measures that exhibit fractal-like behavior, thereby necessitating more advanced frameworks. One such framework is multifractal analysis, which extends classical fractal geometry to accommodate measures with varying singularities across different scales. A fundamental element of multifractal analysis involves the investigation of qth moments of a measure, which provide valuable insights into the distribution and concentration of mass at different scales within the fractal structure. The qth moments of a measure ν are defined by

$$\nu_q(\rho) = \int_A |f(x)|^q \, d\nu(x),$$

where $f(x)$ represents a function of interest (such as a scaling function or a density function), and ρ is the scale parameter of the partition used in the analysis. These moments help quantify the local properties of the measure, along with its fractal dimension, which plays a crucial role in describing the overall geometry of the fractal [25, 26, 32].

In multifractal analysis, the rate at which these moments converge as the partition size ρ tends to zero is vital for understanding the scaling properties of the measure. The scaling behavior of the qth moments is commonly described by the multifractal spectrum $\tau(q)$, which illustrates how the moments decay with respect to ρ. Specifically, for a measure ν, the moments typically follow a scaling law of the form

$$\nu_q(\rho) \sim \rho^{\tau(q)},$$

where $\tau(q)$ denotes the singularity spectrum that governs the scaling of the qth moments across various regions of the fractal. The function $\tau(q)$ provides a comprehensive characterization of the fractal measure, revealing both the distribution of singularities and the underlying geometry of the fractal set (see for example [21, 23, 38]).

The precise rate of convergence of the qth moments is crucial for determining the fractal dimension and multifractal properties of the measure. It reflects how rapidly the moments approach a limiting value as the scale parameter ρ decreases, and its analysis enables the exact quantification of the irregularities inherent in the fractal structure. Understanding this convergence rate is not merely a theoretical concern but also has practical implications in fields such as signal processing, image analysis, and financial modeling, where fractal and multifractal phenomena are frequently encountered. In this context, examining the convergence rates of the qth moments provides valuable insights into the scaling behavior of fractals and the underlying dynamics of multifractal systems. Through the analysis of these rates, we gain a deeper understanding of how fractal measures evolve and how they can be applied to model complex, irregular patterns found in both nature and mathematical theory.

The next key result is the first main theorem of this paper, which is the precise determination of the convergence rate of the qth moments of a Moran measure, specifically the covering moment scaling function and the packing moment scaling function.

THEOREM 3.1. *Let $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ be a list of contracting similarities. Assume that $\Delta > 0$ and let $q \in \mathbb{R}$. For $\boldsymbol{\alpha} \in \mathscr{T}$, we have:*

(1) *The arithmetic case: If $\{\log(\frac{1}{b_{1,\alpha_1}}), \ldots, \log(\frac{1}{b_{n,\alpha_n}})\}$ is contained in a discrete (additive) subgroup of $\mathbb{R}$ (for $\theta > 0$, $\theta\mathbb{Z}$ is the smallest such subgroup), then there exist multiplicatively periodic functions $\overline{\pi}_q, \underline{\pi}_q, \overline{\Pi}_q, \underline{\Pi}_q : (0, \infty) \to \mathbb{R}$ with period e^θ satisfying*

$$\overline{\pi}_q(e^{\pm\theta}\rho) = \overline{\pi}_q(\rho), \qquad \underline{\pi}_q(e^{\pm\theta}\rho) = \underline{\pi}_q(\rho),$$
$$\overline{\Pi}_q(e^{\pm\theta}\rho) = \overline{\Pi}_q(\rho), \qquad \underline{\Pi}_q(e^{\pm\theta}\rho) = \underline{\Pi}_q(\rho)$$

for all $\rho > 0$, such that

$$\frac{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}} = \overline{\pi}_q(\rho) + \epsilon(\rho), \qquad \frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}} = \overline{\Pi}_q(\rho) + \epsilon(\rho),$$
$$\frac{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}{\rho^{-\underline{\beta}(q)}} = \underline{\pi}_q(\rho) + \epsilon(\rho), \qquad \frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\underline{\beta}(q)}} = \underline{\Pi}_q(\rho) + \epsilon(\rho),$$

where $\epsilon(\rho) \to 0$ as $\rho \to 0$.

(2) *The non-arithmetic case: If* $\{\log(\frac{1}{b_{1,\alpha_1}}),\ldots,\log(\frac{1}{b_{n,\alpha_n}})\}$ *is not contained in a discrete (additive) subgroup of* $\mathbb{R}$, *then there exist constants* $\overline{c},\underline{c},\overline{s},\underline{s}\in\mathbb{R}$ *such that*

$$\frac{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}}=\overline{c}+\epsilon(\rho),\qquad \frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}}=\overline{s}+\epsilon(\rho),$$
$$\frac{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}}=\underline{c}+\epsilon(\rho),\qquad \frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}}=\underline{s}+\epsilon(\rho),$$

where $\epsilon(\rho)\to 0$ *as* $\rho\to 0$.

To substantiate Theorem 3.1, we need several technical lemmas and the Renewal Theorem.

REMARK 3.2. If $\{f_{n,j}\mid n\geq 1,\, 1\leq j\leq a_n\}$ satisfies the (SSC) then clearly

$$\Delta_m=\min_{|\boldsymbol{\alpha}|=|\boldsymbol{\sigma}|=m,\,\boldsymbol{\alpha}\neq\boldsymbol{\sigma}}\operatorname{dist}(f_{\boldsymbol{\alpha}}(\mathscr{K}),f_{\boldsymbol{\sigma}}(\mathscr{K}))>0$$

for all $m\in\mathbb{N}$.

LEMMA 3.3. *Suppose that* $\{f_{n,j}\mid n\geq 1,\, 1\leq j\leq a_n\}$ *satisfies the* (SSC). *Let* $q\in\mathbb{R}$, $m\in\mathbb{N}$ *and* $\boldsymbol{\alpha}\in\mathscr{T}$ *with* $|\boldsymbol{\alpha}|=m$. *For all* $0<\rho<\Delta_m$ *we have*

$$\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})=\sum_{|\boldsymbol{\alpha}|=m}\mathscr{M}^{q,c}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})),$$
$$\mathscr{M}^{q,c}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K}))=p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,c}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}),$$
$$\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})=\sum_{|\boldsymbol{\alpha}|=m}\mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})),$$
$$\mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K}))=p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,p}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}).$$

Proof. Since $\operatorname{dist}(f_{\boldsymbol{\alpha}}(\mathscr{K}),f_{\boldsymbol{\sigma}}(\mathscr{K}))\geq\Delta_m$ for all $\boldsymbol{\sigma}\in\mathscr{T}$ with $|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|$ and $\boldsymbol{\sigma}\neq\boldsymbol{\alpha}$, this conclusion can be readily derived from the definitions, reasoning as in the proofs of Lemmas 3.4 and 3.5 below. ∎

For $q\in\mathbb{R}$ and $\boldsymbol{\alpha},\boldsymbol{\sigma}\in\mathscr{T}$ with $|\boldsymbol{\alpha}|=|\boldsymbol{\sigma}|$, we define

$$\mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho)=\mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})\cap\mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho)),$$
$$\mathscr{P}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho)=\mathscr{M}^{q,c}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})\cap\mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho)),$$

where $\mathscr{B}(\mathscr{A},\rho)=\{x\mid\operatorname{dist}(x,\mathscr{A})\leq\rho\}$ for $\mathscr{A}\subseteq\mathbb{R}^d$.

LEMMA 3.4. *Suppose that* $\{f_{n,j}\mid n\geq 1,\, 1\leq j\leq a_n\}$ *satisfies the* (SSC). *Fix* $q\in\mathbb{R}$ *and* $n\geq 1$.

(1) *For* $\rho>0$,

$$\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})\leq\sum_{|\boldsymbol{\alpha}|=n}\mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})).$$

(2) *For* $\rho>0$,

$$\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})\leq\sum_{|\boldsymbol{\alpha}|=n}\mathscr{M}^{q,c}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})).$$

(3) *For* $\rho > 0$,

$$-\sum_{|\boldsymbol{\alpha}|=n}\sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho) + \sum_{|\boldsymbol{\alpha}|=n} \mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}\mathscr{K}) \leq \mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K}).$$

(4) *For* $\rho > 0$,

$$-\sum_{|\boldsymbol{\alpha}|=n}\sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{P}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho) + \sum_{|\boldsymbol{\alpha}|=n} \mathscr{M}^{q,c}_{\nu,\rho}(f_{\boldsymbol{\alpha}}\mathscr{K}) \leq \mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K}).$$

Proof. (1) Let $\rho > 0$. Consider a ρ-separated subset F of $\mathscr{K}$. By making use of $\mathscr{K} = \bigcup_{|\boldsymbol{\alpha}|=n} f_{\boldsymbol{\alpha}}(\mathscr{K})$ we have

$$\sum_{x\in F}\nu(\mathscr{B}(x,\rho))^q \leq \sum_{|\boldsymbol{\alpha}|=n}\sum_{x\in F\cap f_{\boldsymbol{\alpha}}(\mathscr{K})}\nu(\mathscr{B}(x,\rho))^q \leq \sum_{|\boldsymbol{\alpha}|=n}\mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})).$$

Taking the supremum over all ρ-separated subsets F leads to the desired conclusion.

(2) The proof is identical to that of (1).

(3) Let $\rho > 0$. For each $\alpha \in \mathscr{T}$ with $|\boldsymbol{\alpha}| = n$, let $F_{\boldsymbol{\alpha}}$ be a ρ-separated subset of $f_{\boldsymbol{\alpha}}(\mathscr{K})$. Define

$$R_{\boldsymbol{\alpha}} = F_{\boldsymbol{\alpha}} \setminus \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho),$$

$$H_{\boldsymbol{\alpha},\boldsymbol{\sigma}} = F_{\boldsymbol{\alpha}} \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho) \qquad \text{for all } \boldsymbol{\sigma}\in\mathscr{T} \text{ with } |\boldsymbol{\sigma}| = |\boldsymbol{\alpha}| \text{ and } \boldsymbol{\sigma}\neq\boldsymbol{\alpha}.$$

It can be deduced that

$$\begin{aligned}\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K}) &\geq \sum_{|\boldsymbol{\alpha}|=n}\sum_{x\in R_{\boldsymbol{\alpha}}}\nu(\mathscr{B}(x,\rho))^q\\ &\geq \sum_{|\boldsymbol{\alpha}|=n}\Big(\sum_{x\in F_{\boldsymbol{\alpha}}}\nu(\mathscr{B}(x,\rho))^q - \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}}\sum_{x\in H_{\boldsymbol{\alpha},\boldsymbol{\sigma}}}\nu(\mathscr{B}(x,\rho))^q\Big)\\ &\geq \sum_{|\boldsymbol{\alpha}|=n}\Big(\sum_{x\in F_{\boldsymbol{\alpha}}}\nu(\mathscr{B}(x,\rho))^q - \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}}\mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho)\Big).\end{aligned}$$

Taking the supremum over all ρ-separated subsets $F_{\boldsymbol{\alpha}}$ leads to the desired result.

(4) The proof is identical to that of (3). ∎

LEMMA 3.5. *Suppose that* $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ *satisfies the* (SSC). *Let* $q \in \mathbb{R}$ *and* $\boldsymbol{\alpha} \in \mathscr{T}$.

(1) *For* $\rho > 0$,

(i)
$$\mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})) \leq p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,p}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}) + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}}\mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho),$$

(ii)
$$\mathscr{M}^{q,c}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})) \leq p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,c}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}) + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}}\mathscr{P}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho).$$

(2) *For* $q \geq 0$ *and* $\rho > 0$,

$$p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,p}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}) \leq \mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})).$$

Proof. (1) (i) Let $\rho > 0$ and let F be a ρ-separated subset of $f_{\boldsymbol{\alpha}}(\mathscr{K})$. Write

$$R = F \setminus \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho),$$

$$H_{\boldsymbol{\sigma}} = F \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho) \quad \text{for all } \boldsymbol{\sigma} \in \mathscr{T} \text{ with } |\boldsymbol{\sigma}| = |\boldsymbol{\alpha}| \text{ and } \boldsymbol{\sigma} \neq \boldsymbol{\alpha}.$$

Note that $f_{\boldsymbol{\sigma}}^{-1}\mathscr{B}(x,\rho) = \emptyset$ for all $x \in R$ and all $\boldsymbol{\sigma} \in \mathscr{T}$ with $|\boldsymbol{\sigma}| = |\boldsymbol{\alpha}|$ and $\boldsymbol{\sigma} \neq \boldsymbol{\alpha}$. As a result,

$$\begin{aligned}
\sum_{x\in F} \nu(\mathscr{B}(x,\rho))^q &\leq \sum_{x\in R} \nu(\mathscr{B}(x,\rho))^q + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \sum_{x\in H_{\boldsymbol{\sigma}}} \nu(\mathscr{B}(x,\rho))^q \\
&= \sum_{x\in R} \Big(\sum_{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|} p_{\boldsymbol{\sigma}} \nu(f_{\boldsymbol{\sigma}}^{-1}\mathscr{B}(x,\rho)) \Big)^q + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \sum_{x\in H_{\boldsymbol{\sigma}}} \nu(\mathscr{B}(x,\rho))^q \\
&\leq \sum_{x\in R} (p_{\boldsymbol{\alpha}} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho)))^q + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{M}_{\nu,\rho}^{q,p}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho)) \\
&= \sum_{x\in f_{\boldsymbol{\alpha}}^{-1}R} p_{\boldsymbol{\alpha}}^q \nu(\mathscr{B}(x,\rho|I_{\boldsymbol{\alpha}}|^{-1}))^q + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{Q}_{\boldsymbol{\alpha},\boldsymbol{\sigma}}^q(\rho) \\
&\leq p_{\boldsymbol{\alpha}}^q \mathscr{M}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}^{q,p}(\mathscr{K}) + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{Q}_{\boldsymbol{\alpha},\boldsymbol{\sigma}}^q(\rho).
\end{aligned}$$

By taking the supremum over all ρ-separated subsets F, the desired conclusion is reached.

(ii) The proof is identical to that of (1).

(2) Let $\rho > 0$ and let F be an $|I_{\boldsymbol{\alpha}}|^{-1}\rho$-separated subset of $\mathscr{K}$. Write

$$R = f_{\boldsymbol{\alpha}}F \setminus \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho),$$

$$H = f_{\boldsymbol{\alpha}}F \cap \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho).$$

Note that $f_{\boldsymbol{\sigma}}^{-1}\mathscr{B}(x,\rho) = \emptyset$ for all $x \in R$ and all $\boldsymbol{\sigma} \in \mathscr{T}$ with $|\boldsymbol{\sigma}| = |\boldsymbol{\alpha}|$ and $\boldsymbol{\sigma} \neq \boldsymbol{\alpha}$. Also, since $q \geq 0$, we obtain

$$\nu(\mathscr{B}(x,\rho))^q = \Big(\sum_{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|} p_{\boldsymbol{\sigma}} \nu(f_{\boldsymbol{\sigma}}^{-1}\mathscr{B}(x,\rho)) \Big)^q \geq p_{\boldsymbol{\alpha}}^q \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q$$

for all x and all $\rho > 0$. This implies that

$$\begin{aligned}
\mathscr{M}_{\nu,\rho}^{q,p}(f_{\boldsymbol{\alpha}}(\mathscr{K})) &\geq \sum_{x\in R} \nu(\mathscr{B}(x,\rho))^q + \sum_{x\in H} \nu(\mathscr{B}(x,\rho))^q \\
&= \sum_{x\in R} \Big(\sum_{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|} p_{\boldsymbol{\sigma}} \nu(f_{\boldsymbol{\sigma}}^{-1}\mathscr{B}(x,\rho)) \Big)^q + \sum_{x\in H} \nu(\mathscr{B}(x,\rho))^q
\end{aligned}$$

$$\begin{aligned}
&= \sum_{x\in R} (p_{\boldsymbol{\alpha}} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho)))^q + \sum_{x\in H} \nu(\mathscr{B}(x,\rho))^q \\
&= \sum_{x\in R\cup H} p_{\boldsymbol{\alpha}}^q \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q - \sum_{x\in H} p_{\boldsymbol{\alpha}}^q \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q + \sum_{x\in H} \nu(\mathscr{B}(x,\rho))^q \\
&= p_{\boldsymbol{\alpha}}^q \sum_{x\in f_{\boldsymbol{\alpha}}F} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q + \sum_{x\in H} (\nu(\mathscr{B}(x,\rho))^q - p_{\boldsymbol{\alpha}}^q \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q) \\
&\geq p_{\boldsymbol{\alpha}}^q \sum_{x\in F} \nu(\mathscr{B}(x,\rho|I_{\boldsymbol{\alpha}}|^{-1}))^q = p_{\boldsymbol{\alpha}}^q \mathscr{M}^{q,p}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}).
\end{aligned}$$

Taking the supremum over all ρ-separated subsets F yields the conclusion. ■

REMARK 3.6. The statements of Lemmas 3.4 and 3.5 are still true under the (SOSC).

LEMMA 3.7. *Suppose that* $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ *satisfies the* (SSC). *Let* $q \in \mathbb{R}$. *Define* $h_{1,n}, h_{2,n} : [0,\infty) \to \mathbb{R}$ *by*

$$h_{1,n}(Y) = e^{-Y\beta_n(q)}\Big(\mathscr{M}^{q,c}_{\nu,e^{-Y}}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p_{\boldsymbol{\alpha}}{}^q \mathscr{M}^{q,c}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}e^{-Y}}(\mathscr{K})\Big),$$

$$h_{2,n}(Y) = e^{-Y\beta_n(q)}\Big(\mathscr{M}^{q,p}_{\nu,e^{-Y}}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p_{\boldsymbol{\alpha}}{}^q \mathscr{M}^{q,p}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}e^{-Y}}(\mathscr{K})\Big).$$

Then $h_{1,n}(Y) = h_{2,n}(Y) = 0$ *for all* $Y > \log(1/\Delta_1)$. *Consequently,* $h_{1,n}$ *and* $h_{2,n}$ *are directly Riemann integrable.*

Proof. This follows directly from Lemma 3.3. ■

To introduce the renewal theorem, let us start with the following definition. A function $f : [0,\infty) \to \mathbb{R}$ is said to be *directly Riemann integrable* with integral I if

$$\sum_n |M_n(\delta)| < \infty \quad \text{for all } \delta > 0, \qquad \sum_n |m_n(\delta)| < \infty \quad \text{for all } \delta > 0,$$

and

$$\sum_n M_n(\delta)\delta \to I \quad \text{as } \delta \to 0, \qquad \sum_n m_n(\delta)\delta \to I \quad \text{as } \delta \to 0,$$

where

$$M_n(\delta) = \sup_{n\delta\leq x\leq(n+1)\delta} f(x), \quad m_n(\delta) = \inf_{n\delta\leq x\leq(n+1)\delta} f(x) \quad \text{for } n = 0,1,2,\ldots.$$

If $f : [0,\infty) \to \mathbb{R}$ is directly Riemann integrable, then the integral of f is denoted by $\int_0^\infty f(y)\,dy$. This notation emphasizes that the Riemann integral over $[0,\infty)$ is calculated directly, without relying on limits of integrals over $[0,a]$. It is evident that if $f : [0,\infty) \to \mathbb{R}$ is Riemann integrable on all compact subintervals and there exist positive constants $c_1, c_2 > 0$ such that

$$|f(x)| \leq c_1 e^{-c_2 x} \quad \text{for all } x \geq 0,$$

then f is directly Riemann integrable. The proof of Theorem 3.1 depends on the following theorem from Renewal Theory.

THEOREM 3.8 (The Renewal Theorem). *Assuming* P *is a probability measure on* $[0,\infty)$ *with* $P(\{0\}) = 0$ *and* $\int_0^\infty y\,dP(y) < \infty$, *let* $h : [0,\infty) \to \mathbb{R}$ *be a measurable and directly*

Riemann integrable function. Assume $H \in L^1(\mathbb{R}, P)$ satisfies the Renewal Equation

$$H(x) = h(x) + \int_0^\infty H(x-y)\,dP(y) \quad \text{for all } x \geq 0.$$

(1) *In the arithmetic case where $\operatorname{supp} P$ is within a discrete (additive) subgroup of $\mathbb{R}$, with $\lambda\mathbb{Z}$ being the smallest subgroup with $\lambda > 0$, we have*

$$H(x) = \Pi(x) + \epsilon(x),$$

where $\Pi : [0,\infty) \to \mathbb{R}$ is a λ-periodic function and $\epsilon(x) \to 0$ as $x \to \infty$.

(2) *In the non-arithmetic case where $\operatorname{supp} P$ is not within any discrete (additive) subgroup of $\mathbb{R}$, we have*

$$H(x) = c + \epsilon(x),$$

where $c \in \mathbb{R}$ and $\epsilon(x) \to 0$ as $x \to \infty$.

Proof. For information on the arithmetic case, refer to [7], and for the non-arithmetic case, consult [8]. ∎

Remark 3.9. It is worth noting that the function Π and the constant c in the Renewal Theorem can be explicitly determined. In fact, as shown in [7, 8],

$$\Pi(x) = \frac{\sum_{n\in\mathbb{Z}} h(x+\lambda n)}{\int_0^\infty y\,dP(y)} \quad \text{and} \quad c = \frac{\int_0^\infty h(y)\,dy}{\int_0^\infty y\,dP(y)}.$$

Proof of Theorem 3.1. (1) For $q \in \mathbb{R}$, let $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ satisfy the (SSC). Define $H_n : \mathbb{R} \to \mathbb{R}$ and $h_n : [0,\infty) \to \mathbb{R}$ by

$$\begin{aligned} H_n(Y) &= e^{-Y\beta_n(q)} \mathscr{M}^{q,p}_{\nu,e^{-Y}}(\mathscr{K}),\\ h_n(Y) &= e^{-Y\beta_n(q)} \Big(\mathscr{M}^{q,p}_{\nu,e^{-Y}}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}} \mathscr{M}^{q,p}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}e^{-Y}}(\mathscr{K}) \Big). \end{aligned}$$

Define a probability measure by

$$P = \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} |I_{\boldsymbol{\alpha}}|^{\beta_n(q)} \delta_{\log \frac{1}{|I_{\boldsymbol{\alpha}}|}}.$$

It is evident that $H_n \in L^1(\mathbb{R}, P)$, and by Lemma 3.7, h_n is directly Riemann integrable on $[0,\infty)$. Next, for all $Y > 0$,

$$\begin{aligned} H_n(Y) &= e^{-Y\beta_n(q)} \mathscr{M}^{q}_{\nu,e^{-Y}}(\mathscr{K})\\ &= e^{-Y\beta_n(q)} \Big(\sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}} \mathscr{M}^{q,p}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}e^{-Y}}(\mathscr{K}) + e^{Y\beta_n(q)} h(Y) \Big)\\ &= \sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}} |I_{\boldsymbol{\alpha}}|^{\beta_n(q)} H_n\Big(Y - \log \frac{1}{|I_{\boldsymbol{\alpha}}|} \Big) + h_n(Y)\\ &= \int_0^\infty H_n(Y-y)\,dP(y) + h_n(Y). \end{aligned}$$

By making use of the Renewal Theorem we deduce that in the arithmetic case,

$$\frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\beta_n(q)}} = \Pi_{q,n}(\rho) + \epsilon(\rho),$$

where $\Pi_{q,n}$ is a multiplicatively periodic function, while in the non-arithmetic case

$$\frac{\mathscr{M}_{\nu,\rho}^{q,p}(\mathscr{K})}{\rho^{-\beta_n(q)}} = s_n + \epsilon(\rho).$$

The desired result is now obtained by taking

$$\overline{\Pi}_q = \limsup_{n\to 0} \Pi_{q,n}, \quad \underline{\Pi}_q = \liminf_{n\to 0} \Pi_{q,n}, \quad \overline{s} = \limsup_{n\to 0} s_n, \quad \underline{s} = \liminf_{n\to 0} s_n.$$

(2) The analogous result for $\mathscr{N}_{\nu,\rho}^{q}(\mathscr{K})$ is demonstrated in a similar manner. ∎

4. Application of the main theorems to empirical multifractal moment measures

As an application of Theorem 3.1, we demonstrate the next main result, which shows that the empirical multifractal moment measures converge weakly. According to Theorem 2.10, the normalized restrictions $\mathscr{H}_\nu^{q,\underline{\beta}(q)} \llcorner \mathscr{K} / \mathscr{H}_\nu^{q,\underline{\beta}(q)}(\mathscr{K})$ and $\mathscr{P}_\nu^{q,\overline{\beta}(q)} \llcorner \mathscr{K} / \mathscr{P}_\nu^{q,\overline{\beta}(q)}(\mathscr{K})$ of the multifractal Hausdorff measure and the multifractal packing measure to $\mathscr{K}$ are well defined.

We proceed to define the empirical multifractal qth moment measures. For $q \in \mathbb{R}$ define

$$\Gamma^q = \Big\{(F,\rho) \;\Big|\; F \text{ is a } \rho\text{-spanning subset of } \mathscr{K},\ \sum_{x\in F} \nu(\mathscr{B}(x,\rho))^q = \mathscr{M}_{\nu,\rho}^{q,c}(\mathscr{K})\Big\},$$

$$\Lambda^q = \Big\{(F,\rho) \;\Big|\; F \text{ is a } \rho\text{-separated subset of } \mathscr{K},\ \sum_{x\in F} \nu(\mathscr{B}(x,\rho))^q = \mathscr{M}_{\nu,\rho}^{q,p}(\mathscr{K})\Big\}.$$

We define an order $\prec^q$ in Γ^q and Λ^q by $(E_1,\rho_1) \prec^q (E_2,\rho_2)$ if and only if $\rho_2 \le \rho_1$. It is evident that for every positive $\rho > 0$, there exists a ρ-spanning subset E of $\mathscr{K}$ and a ρ-separated subset F of $\mathscr{K}$ such that $(E,\rho) \in \Gamma^q$ and $(F,\rho) \in \Lambda^q$. Let δ_x denote the Dirac measure concentrated at $x \in \mathbb{R}^d$. A probability measure λ on $\mathscr{K}$ is termed a qth *order covering equidistribution* of ν if there exists a sequence $(F_n,\rho_n)_n \in \Gamma^q$ such that

$$\frac{1}{\mathscr{M}_{\nu,\rho_n}^{q,c}(\mathscr{K})} \sum_{x\in F_n} \nu(\mathscr{B}(x,\rho_n))^q \delta_x \to \lambda \quad \text{weakly;}$$

and it is called a qth *order packing equidistribution* of ν if there exists a sequence $(F_n,\rho_n)_n \in \Lambda^q$ such that

$$\frac{1}{\mathscr{M}_{\nu,\rho_n}^{q,p}(\mathscr{K})} \sum_{x\in F_n} \nu(\mathscr{B}(x,\rho_n))^q \delta_x \to \lambda \quad \text{weakly.}$$

Since the family of probability measures on a compact subset of $\mathbb{R}^d$ is compact in the weak topology, every compact subset of $\mathbb{R}^d$ necessarily has a qth order equidistribution of ν. For $q \ge 0$, Olsen [24] established that a self-similar set satisfying the Open Set Condition (OSC) possesses a unique qth order equidistribution, specifically identified as the normalized multifractal Hausdorff measure, and in the following theorem, we prove that this result is still valid when we take a Moran set.

Given a measure space $(\mathcal{X},\nu,\mathcal{E})$ and $\mathcal{A}\in\mathcal{E}$, the symbol $\nu\llcorner\mathcal{A}$ denotes the restriction of ν to $\mathcal{A}$, i.e.,

$$(\nu\llcorner\mathcal{A})(\mathcal{F})=\nu(\mathcal{A}\cap\mathcal{F})\quad\forall\mathcal{F}\in\mathcal{E}.$$

THEOREM 4.1. *Given a list $\{f_{n,j}\mid n\geq 1,\,1\leq j\leq a_n\}$ of contracting similarities, suppose that $\Delta>0$ and let $q\in\mathbb{R}$. Denote $\dim_\nu^q(\mathscr{K})=\underline{\beta}(q)=t$ and $\mathrm{Dim}_\nu^q(\mathscr{K})=\overline{\beta}(q)=s$. Define probability measures $\mu^q_{(F,\rho)}$ and $\nu^q_{(F,\rho)}$ on $\mathbb{R}^d$ by*

$$\mu^q_{(F,\rho)}=\frac{1}{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}\sum_{x\in F}\nu(\mathscr{B}(x,\rho))^q\delta_x\qquad\text{for }(F,\rho)\in\Gamma_q,$$
$$\nu^q_{(F,\rho)}=\frac{1}{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}\sum_{x\in F}\nu(\mathscr{B}(x,\rho))^q\delta_x\qquad\text{for }(F,\rho)\in\Lambda_q,$$

where δ_x denote the Dirac measure concentrated at $x\in\mathbb{R}^d$.

(1) *We have*

$$\mu^q_{(F,\rho)}\to\frac{\mathscr{P}^{q,s}_\nu\llcorner\mathscr{K}}{\mathscr{P}^{q,s}_\nu(\mathscr{K})}\qquad\text{weakly},$$
$$\nu^q_{(F,\rho)}\to\frac{\mathscr{P}^{q,s}_\nu\llcorner\mathscr{K}}{\mathscr{P}^{q,s}_\nu(\mathscr{K})}\qquad\text{weakly}.$$

(2) *If there exists n_0 such that $\beta_{n_0}(q)\leq\underline{\beta}(q)$, then*

$$\mu^q_{(F,\rho)}\to\frac{\mathscr{H}^{q,t}_\nu\llcorner\mathscr{K}}{\mathscr{H}^{q,t}_\nu(\mathscr{K})}\qquad\text{weakly},$$
$$\nu^q_{(F,\rho)}\to\frac{\mathscr{H}^{q,t}_\nu\llcorner\mathscr{K}}{\mathscr{H}^{q,t}_\nu(\mathscr{K})}\qquad\text{weakly}.$$

REMARK 4.2. The results of Theorem 4.1 remain valid also when the multifractal Hausdorff and packing measures are replaced by the lower and upper multifractal Hewitt–Stromberg measures.

To prove Theorem 4.1 we need the following lemmas.

LEMMA 4.3. *Assume that $\{f_{n,j}\mid n\geq 1,\,1\leq j\leq a_n\}$ satisfies the* (SSC). *Let $q\in\mathbb{R}$ and $\boldsymbol{\alpha}\in\mathscr{T}$. Then*

$$\underline{J}^q_\nu(f_{\boldsymbol{\alpha}},\mathscr{K})=\overline{J}^q_\nu(f_{\boldsymbol{\alpha}},\mathscr{K})=p^q_{\boldsymbol{\alpha}}.$$

Proof. Denote $|\boldsymbol{\alpha}|=m$. Given $x\in\mathscr{K}$. Then

$$f_{\boldsymbol{\alpha}}\mathscr{B}(x,\rho)\cap\mathscr{K}\subseteq f_{\boldsymbol{\alpha}}(\mathscr{K})\setminus\bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}}f_{\boldsymbol{\alpha}}(\mathscr{K})$$

for $0<\rho<\Delta_m$, and consequently

$$\frac{\nu f_{\boldsymbol{\alpha}}\mathscr{B}(x,\rho)}{\nu\mathscr{B}(x,\rho)}=\frac{\sum_{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|}p_{\boldsymbol{\sigma}}\nu(f_{\boldsymbol{\sigma}}^{-1}f_{\boldsymbol{\alpha}}\mathscr{B}(x,\rho))}{\nu\mathscr{B}(x,\rho)}=\frac{p_{\boldsymbol{\alpha}}\nu\mathscr{B}(x,\rho)}{\nu\mathscr{B}(x,\rho)}=p_{\boldsymbol{\alpha}}$$

for all $0<\rho<\Delta_m$. ∎

LEMMA 4.4. *Assume that $\{f_{n,j}\mid n\geq 1,\,1\leq j\leq a_n\}$ satisfies the* (SSC). *Let $q\in\mathbb{R}$, $m\in\mathbb{N}$ and $\boldsymbol{\alpha}\in\mathscr{T}$ with $|\boldsymbol{\alpha}|=m$.*

(1) *For* $(F,\rho)\in\Gamma^q$ *with* $0<\rho<\Delta_m$ *we have*

$$\mu^q_{(F,\rho)}(f_{\boldsymbol\alpha}(\mathscr K))=\frac{1}{\mathscr M^{q,c}_{\nu,\rho}(\mathscr K)}p^q_{\boldsymbol\alpha}\mathscr M^{q,c}_{\nu,\rho|I_{\boldsymbol\alpha}|^{-1}}(\mathscr K).$$

(2) *For* $(F,\rho)\in\Lambda^q$ *with* $0<\rho<\Delta_m$ *we have*

$$\nu^q_{(F,\rho)}(f_{\boldsymbol\alpha}(\mathscr K))=\frac{1}{\mathscr M^{q}_{\nu,\rho}(\mathscr K)}p^q_{\boldsymbol\alpha}\mathscr M^{q}_{\nu,\rho|I_{\boldsymbol\alpha}|^{-1}}(\mathscr K).$$

Proof. (1) Since $0<\rho<\Delta_m$, we deduce that $F\cap f_{\boldsymbol\sigma}(\mathscr K)$ is a ρ-spanning subset of $f_{\boldsymbol\sigma}(\mathscr K)$ for all $\boldsymbol\sigma\in\mathscr T$ with $|\boldsymbol\sigma|=m$, then

$$\begin{aligned}\mu^q_{(F,\rho)}(f_{\boldsymbol\sigma}(\mathscr K))&=\frac{1}{\mathscr M^{q,c}_{\nu,\rho}(\mathscr K)}\sum_{x\in F\cap f_{\boldsymbol\alpha}(\mathscr K)}\nu(\mathscr B(x,\rho))^q\\&\ge\frac{1}{\mathscr M^{q,c}_{\nu,\rho}(\mathscr K)}\mathscr M^{q,c}_{\nu,\rho}(f_{\boldsymbol\sigma}(\mathscr K)).\end{aligned}\tag{4.1}$$

By making use of Lemma 3.3 and (4.1), we have

$$\begin{aligned}1=\mu^q_{(F,\rho)}\Big(\bigcup_{|\boldsymbol\sigma|=m}f_{\boldsymbol\alpha}(\mathscr K)\Big)&=\sum_{|\boldsymbol\sigma|=m}\mu^q_{(F,\rho)}(f_{\boldsymbol\sigma}(\mathscr K))\\&\ge\sum_{|\boldsymbol\sigma|=m}\frac{1}{\mathscr M^{q,c}_{\nu,\rho}(\mathscr K)}\mathscr M^{q,c}_{\nu,\rho}(f_{\boldsymbol\sigma}(\mathscr K))=1.\end{aligned}\tag{4.2}$$

Combining (4.1) and (4.2) yields

$$\mu^q_{(F,\rho)}(f_{\boldsymbol\sigma}(\mathscr K))=\frac{1}{\mathscr M^{q,c}_{\nu,\rho}(\mathscr K)}\mathscr M^{q,c}_{\nu,\rho}(f_{\boldsymbol\sigma}(\mathscr K))=\frac{1}{\mathscr M^{q,c}_{\nu,\rho}(\mathscr K)}p^q_{\boldsymbol\sigma}\mathscr M^{q,c}_{\nu,\rho|I_{\boldsymbol\sigma}|^{-1}}(\mathscr K),$$

for all $\boldsymbol\sigma\in\mathscr T$ with $|\boldsymbol\sigma|=m$ by another application of Lemma 3.3.

(2) The proof of (1) closely parallels that of (2). ■

Proof of Theorem 4.1. Let $\{f_{n,j}\mid n\ge1,\ 1\le j\le a_n\}$ satisfy the (SSC) and $q\in\mathbb R$. For $\boldsymbol\alpha\in\mathscr T$ define

$$h_{\boldsymbol\alpha}(\rho)=\frac{\mathscr M^{q,p}_{\nu,\rho|I_{\boldsymbol\alpha}|^{-1}}(\mathscr K)}{(\rho|I_{\boldsymbol\alpha}|^{-1})^{-\overline\beta(q)}}\Big/\frac{\mathscr M^{q,p}_{\nu,\rho}(\mathscr K)}{\rho^{-\overline\beta(q)}}.$$

It is clear from Theorem 3.1 that

$$h_{\boldsymbol\alpha}(\rho)\to1\quad\text{as }\rho\to0.\tag{4.3}$$

Then for $(F,\rho)\in\Lambda^q(\mathscr K)$ and $0<\rho<\Delta_m$, from Lemma 4.4 we get

$$\begin{aligned}\nu^q_{(F,\rho)}(f_{\boldsymbol\alpha}(\mathscr K))&\le\frac{1}{\mathscr M^{q,p}_{\nu,\rho}(\mathscr K)}(p^q_{\boldsymbol\alpha}\mathscr M^{q,p}_{\nu,\rho/|I_{\boldsymbol\alpha}|}(\mathscr K))\le p^q_{\boldsymbol\alpha}|I_{\boldsymbol\alpha}|^{\overline\beta(q)}h_{\boldsymbol\alpha}(\rho)\\&=p^q_{\boldsymbol\alpha}|I_{\boldsymbol\alpha}|^sh_{\boldsymbol\alpha}(\rho)=g_{\boldsymbol\alpha}(\rho).\end{aligned}\tag{4.4}$$

It follows from (4.3) and (4.4) that

$$g_{\boldsymbol\alpha}(\rho)\to p^q_{\boldsymbol\alpha}|I_{\boldsymbol\alpha}|^s\quad\text{ as }\rho\to0\text{ for all }\boldsymbol\alpha\in\mathscr T.\tag{4.5}$$

Consequently, for $m \in \mathbb{N}$ we have

$$\begin{aligned} 1 = \nu^q_{(F,\rho)}(\mathscr{K}) &\le \sum_{|\boldsymbol{\alpha}|=m} \nu^q_{(F,r)}(f_{\boldsymbol{\alpha}}(\mathscr{K})) \\ &\le \sum_{|\boldsymbol{\alpha}|=m} g_{\boldsymbol{\alpha}}(\rho) \to \sum_{|\boldsymbol{\alpha}|=m} p^q_{\boldsymbol{\alpha}} |I_{\boldsymbol{\alpha}}|^s \le 1 \quad \text{as } \rho \to 0. \end{aligned} \tag{4.6}$$

We conclude now that

$$\nu^q_{(F,\rho)}(f_{\boldsymbol{\alpha}}(\mathscr{K})) \to p^q_{\boldsymbol{\alpha}} |I_{\boldsymbol{\alpha}}|^s \quad \text{as } \rho \to 0 \text{ for all } \boldsymbol{\alpha} \in \mathscr{T}. \tag{4.7}$$

For brevity, define $H = \mathscr{P}^{q,s}_{\nu} \llcorner \mathscr{K} / \mathscr{P}^{q,s}_{\nu}(\mathscr{K})$. Next, for $\delta > 0$ and $\boldsymbol{\alpha} \in \mathscr{T}$, it now follows straightforwardly from Lemmas 6.9 and 4.3 that

$$H(f_{\boldsymbol{\alpha}}(\mathscr{K})) = \frac{\mathscr{P}^{q,s}_{\nu}(f_{\boldsymbol{\alpha}}(\mathscr{K}))}{\mathscr{P}^{q,s}_{\nu}(\mathscr{K})} = |I_{\boldsymbol{\alpha}}|^s J^q_{\nu}(f_{\boldsymbol{\alpha}}, \mathscr{K})) \frac{\mathscr{P}^{q,s}_{\nu}(\mathscr{K})}{\mathscr{P}^{q,s}_{\nu}(\mathscr{K})} = |I_{\boldsymbol{\alpha}}|^s p^q_{\boldsymbol{\alpha}}. \tag{4.8}$$

By combining (4.7) and (4.8) we have

$$\nu^q_{(F,\rho)}(f_{\boldsymbol{\alpha}}(\mathscr{K})) \to H(f_{\boldsymbol{\alpha}}(\mathscr{K})) \quad \text{as } \rho \to 0 \text{ for all } \boldsymbol{\alpha} \in \mathscr{T}. \tag{4.9}$$

Next, consider a continuous function $f : \mathscr{K} \to \mathbb{R}$, and let $\varepsilon > 0$. Since f is uniformly continuous and $\max_{|\boldsymbol{\alpha}|=m} |f_{\boldsymbol{\alpha}}(\mathscr{K})| \le (b_{\max})^m |\mathscr{K}| \to 0$ as $m \to \infty$, there exists $M \in \mathbb{N}$ such that $|f(x) - f(y)| \le \varepsilon/2$ for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = M$ and for $x, y \in f_{\boldsymbol{\alpha}}(\mathscr{K})$. Furthermore, it follows from (4.9) that for each $\boldsymbol{\alpha} \in \mathscr{T}$ with $|\boldsymbol{\alpha}| = M$ there exists $\rho_{\boldsymbol{\alpha}}$ such that

$$|\nu^q_{(F,\rho)}(f_{\boldsymbol{\alpha}}(\mathscr{K})) - H(f_{\boldsymbol{\alpha}}(\mathscr{K}))| \le \varepsilon / (2n^M \|f\|_\infty)$$

for all $(F, \rho) \in \Lambda^q$ with $0 < \rho < \rho_{\boldsymbol{\alpha}}$.

Then, for each $(F, \rho) \in \Lambda^q$ with $0 < \rho < \min_{\boldsymbol{\alpha}=M} \rho_{\boldsymbol{\alpha}}$ we obtain

$$\begin{aligned} &\left| \int f \, d\nu^q_{(F,\rho)} - \int f \, dH \right| \\ &\qquad \le \sum_{|\boldsymbol{\alpha}|=M} \left| \int_{f_{\boldsymbol{\alpha}}(\mathscr{K})} f \, d\nu^q_{(F,\rho)} - \int_{f_{\boldsymbol{\alpha}}(\mathscr{K})} f \, dH \right| \\ &\qquad \le \sum_{|\boldsymbol{\alpha}|=M} \left(H(f_{\boldsymbol{\alpha}}(\mathscr{K})) \sup_{x,y \in f_{\boldsymbol{\alpha}}(\mathscr{K})} |f(x) - f(y)| + \|f\|_\infty \frac{\varepsilon}{2n^M \|f\|_\infty} \right) \\ &\qquad \le \frac{\varepsilon}{2} \sum_{|\boldsymbol{\alpha}|=M} H(f_{\boldsymbol{\alpha}}(\mathscr{K})) + \frac{\varepsilon}{2}. \end{aligned}$$

Finally, since $\mathscr{P}^s(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) = 0$ for all $\boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T}$ where $|\boldsymbol{\alpha}| = |\boldsymbol{\sigma}|$ and $\boldsymbol{\alpha} \neq \boldsymbol{\sigma}$ (refer to Proposition 6.10), it follows that

$$\sum_{|\boldsymbol{\alpha}|=M} H(f_{\boldsymbol{\alpha}}(\mathscr{K})) = H\Big(\bigcup_{|\boldsymbol{\alpha}|=M} f_{\boldsymbol{\alpha}}(\mathscr{K}) \Big) = H(\mathscr{K}) = 1,$$

and consequently

$$\left| \int f \, d\nu^q_{(F,\rho)} - \int f \, dH \right| \le \varepsilon.$$

This concludes the proof (the analogous result for $\nu^q_{(F,\rho)}$ is established in a similar manner).

(2) We just add the condition that there exists n_0 satisfying $\beta_{n_0}(q) \leq \underline{\beta}(q)$, and then we use the same arguments as in the last statement. ■

5. Equivalent results based on tube formulas

A multifractal tube is a concept in fractal theory, particularly within multifractal analysis. It describes a structure or object displaying multifractal characteristics, where both the geometry of the set or space and the behavior of measures or functions on it are complex. These tubes are often used to analyze fractals that exhibit self-similarity or self-affinity at multiple scales, representing objects in both mathematical and real-world contexts. Technically, a multifractal tube refers to a collection of sets whose size, behavior, or dimension varies depending on the observation scale. This scale-dependent behavior is often explored using multifractal formalism, which employs measures to capture the fractal dimensions at different points or regions within the set. Each region of the tube may exhibit distinct scaling exponents, reflecting the fractal nature at various levels.

In this section, we provide a full description of the asymptotic behavior of the multifractal tube formulas for Moran measures that satisfy the Set Strong Separation Condition. Specifically, we demonstrate that if the set $\{\log(\frac{1}{b_{1,\alpha_1}}), \dots, \log(\frac{1}{b_{n,\alpha_n}})\}$ is not contained within a discrete additive subgroup of $\mathbb{R}$, then $\mathscr{K}$ is (q, β_n)-multifractal Minkowski measurable with respect to ν. Conversely, if this set is contained in such a subgroup, then $\mathscr{K}$ is (q, β_n)-averagely multifractal Minkowski measurable with respect to ν. This result is summarized in Theorem 5.2 below. Also, as an application, we explicitly determine the weak limits of the multifractal tube measures for Moran measures ν (Theorems 5.3 and 5.4). But first, let us introduce some definitions to be used in this section.

5.1. Minkowski dimensions. Let $\mathscr{F} \subseteq \mathbb{R}^d$ and $\rho > 0$, we define the *Minkowski volume* of $\mathscr{F}$ by

$$\mathscr{V}_\rho(\mathscr{F}) = \frac{1}{\rho^d} \lambda^d(\mathscr{B}(\mathscr{F}, \rho)),$$

where $\mathscr{B}(\mathscr{F}, \rho) = \{x \in \mathbb{R}^d \mid \mathrm{dist}(x, \mathscr{F}) < \rho\}$ and λ^d represents the d-dimensional Lebesgue measure in $\mathbb{R}^d$.

By making use of the Minkowski volume, we can define the *upper* and *lower Minkowski dimension* of $\mathscr{F}$ by

$$\overline{d}_M(\mathscr{F}) = \limsup_{\rho \to 0} \frac{\log \mathscr{V}_\rho(\mathscr{F})}{-\log \rho}, \qquad \underline{d}_M(\mathscr{F}) = \liminf_{\rho \to 0} \frac{\log \mathscr{V}_\rho(\mathscr{F})}{-\log \rho}.$$

If $\underline{d}_M(\mathscr{F}) = \overline{d}_M(\mathscr{F})$ we define the *Minkowski dimension* of $\mathscr{F}$ by

$$d_M(\mathscr{F}) = \lim_{\rho \to 0} \frac{\log \mathscr{V}_\rho(\mathscr{F})}{-\log \rho}.$$

For $t \in \mathbb{R}$, we introduce the *upper* and *lower Minkowski content* of $\mathscr{F}$ as follows:

$$\overline{\mathscr{M}}^t(\mathscr{F}) = \limsup_{\rho \to 0} \frac{1}{\rho^{-t}} \mathscr{V}_\rho(\mathscr{F}), \qquad \underline{\mathscr{M}}^t(\mathscr{F}) = \liminf_{\rho \to 0} \frac{1}{\rho^{-t}} \mathscr{V}_\rho(\mathscr{F}).$$

If $\underline{\mathscr{M}}^t(\mathscr{F}) = \overline{\mathscr{M}}^t(\mathscr{F})$ we define the *Minkowski content* of $\mathscr{F}$ by

$$\mathscr{M}^t(\mathscr{F}) = \lim_{\rho\to 0} \frac{1}{\rho^{-t}} \mathscr{V}_\rho(\mathscr{F}).$$

Naturally, a set $\mathscr{F}$ may fail to be Minkowski measurable, meaning that the limit

$$\lim_{\rho\to 0} \frac{1}{\rho^{-t}} \mathscr{V}_\rho(\mathscr{F})$$

could fail to exist. In such cases, it is reasonable to examine the limiting behavior of appropriately defined averages of $\frac{1}{\rho^{-t}} \mathscr{V}_\rho(\mathscr{F})$. Thus, we introduce the *upper* and *lower average Minkowski content* of $\mathscr{F}$ by

$$\overline{\mathscr{M}}^t_a(\mathscr{F}) = \limsup_{\rho\to 0} \frac{1}{-\log\rho} \int_\rho^1 \frac{1}{s^{-t}} \mathscr{V}_s(\mathscr{F}) \frac{ds}{s},$$

$$\underline{\mathscr{M}}^t_a(\mathscr{F}) = \liminf_{\rho\to 0} \frac{1}{-\log\rho} \int_\rho^1 \frac{1}{s^{-t}} \mathscr{V}_s(\mathscr{F}) \frac{ds}{s}.$$

If $\underline{\mathscr{M}}^t_a(\mathscr{F}) = \overline{\mathscr{M}}^t_a(\mathscr{F})$ we define the *average Minkowski content* of $\mathscr{F}$, denoted by $\mathscr{M}^t_a(\mathscr{F})$, to be their common value.

5.2. Multifractal Minkowski dimensions. Let $\mathscr{F} \subseteq \mathbb{R}^d$ and $\rho > 0$. Given a real number q and Borel measure ν on $\mathbb{R}^d$, we introduce the *multifractal Minkowski volume* of $\mathscr{F}$ with respect to the measure ν by

$$\mathscr{V}^q_{\nu,\rho}(\mathscr{F}) = \frac{1}{\rho^d} \int_{\mathscr{B}(\mathscr{F},\rho)} \nu(\mathscr{B}(x,\rho))^q \, d\lambda^d(x),$$

where $\mathscr{B}(\mathscr{F},\rho) = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x,\mathscr{F}) < \rho\}$.

By making use of the multifractal Minkowski volume, we define the *upper* and *lower multifractal Minkowski dimension* of $\mathscr{F}$ by

$$\overline{d}^q_{M,\nu}(\mathscr{F}) = \limsup_{\rho\to 0} \frac{\log \mathscr{V}^q_{\nu,\rho}(\mathscr{F})}{-\log\rho},$$

$$\underline{d}^q_{M,\nu}(\mathscr{F}) = \liminf_{\rho\to 0} \frac{\log \mathscr{V}^q_{\nu,\rho}(\mathscr{F})}{-\log\rho}.$$

If $\underline{d}^q_{M,\nu}(\mathscr{F}) = \overline{d}^q_{M,\nu}(\mathscr{F})$ we define the *multifractal Minkowski dimension* of $\mathscr{F}$ to be

$$d^q_{M,\nu}(\mathscr{F}) = \lim_{\rho\to 0} \frac{\log \mathscr{V}^q_{\nu,\rho}(\mathscr{F})}{-\log\rho}.$$

For $q,t \in \mathbb{R}$, we introduce the *upper* and *lower multifractal Minkowski content* of $\mathscr{F}$ with respect to ν as follows:

$$\overline{\mathscr{M}}^{q,t}_\nu(\mathscr{F}) = \limsup_{\rho\to 0} \frac{1}{\rho^{-t}} \mathscr{V}^q_{\nu,\rho}(\mathscr{F}),$$

$$\underline{\mathscr{M}}^{q,t}_\nu(\mathscr{F}) = \liminf_{\rho\to 0} \frac{1}{\rho^{-t}} \mathscr{V}^q_{\nu,\rho}(\mathscr{F}).$$

If $\underline{\mathscr{M}}^{q,t}_\nu(\mathscr{A}) = \overline{\mathscr{M}}^{q,t}_\nu(\mathscr{F})$ we define the *multifractal Minkowski content* of $\mathscr{F}$ with respect to ν by

$$\mathscr{M}_{\nu}^{q,t}(\mathscr{F}) = \lim_{\rho\to 0} \frac{1}{\rho^{-t}} \mathscr{V}_{\nu,r}^{q}(\mathscr{F}).$$

Naturally, a set $\mathscr{F}$ may not be Minkowski measurable, meaning that the limit

$$\lim_{\rho\to 0} \frac{1}{\rho^{-t}} \mathscr{V}_{\nu,r}^{q}(\mathscr{F})$$

may not exist. In such cases, it is reasonable to examine the limiting behavior of appropriately defined averages of $\frac{1}{\rho^{-t}} \mathscr{V}_{\nu,\rho}^{q}(\mathscr{F})$. Thus, we define the *lower* and *upper average multifractal Minkowski content* of $\mathscr{F}$ with respect to ν as follows:

$$\underline{\mathscr{M}}_{\nu,a}^{q,t}(\mathscr{F}) = \liminf_{\rho\to 0} \frac{1}{-\log\rho} \int_{\rho}^{1} \frac{1}{s^{-t}} \mathscr{V}_{\nu,s}^{q}(\mathscr{F}) \frac{ds}{s},$$

$$\overline{\mathscr{M}}_{\nu,a}^{q,t}(\mathscr{F}) = \limsup_{\rho\to 0} \frac{1}{-\log\rho} \int_{\rho}^{1} \frac{1}{s^{-t}} \mathscr{V}_{\nu,s}^{q}(\mathscr{F}) \frac{ds}{s}.$$

If $\underline{\mathscr{M}}_{\nu,a}^{q,t}(\mathscr{F}) = \overline{\mathscr{M}}_{\nu,a}^{q,t}(\mathscr{F})$ we define the *average multifractal Minkowski content* of $\mathscr{F}$ with respect to ν, denoted by $\mathscr{M}_{\nu,a}^{q,t}(\mathscr{F})$, to be their common value.

5.3. Multifractal tube measures. Let ν be a Borel measure on $\mathbb{R}^d$, and let $\rho > 0$. For any real number q, we define the *multifractal Minkowski tube measure* $\mathscr{I}_{\nu,\rho}^{q}$ by

$$\mathscr{I}_{\nu,\rho}^{q}(\mathscr{F}) = \frac{1}{\rho^d} \int_{\mathscr{F}\cap\mathscr{B}(\operatorname{supp}\nu,\rho)} \nu(\mathscr{B}(x,\rho))^q \, d\lambda^d(x)$$

for Borel subsets $\mathscr{F}$ of $\mathbb{R}^d$.

Naturally, the measures $\mathscr{I}_{\nu,\rho}^{q}$ generally do not exhibit weak convergence as $\rho\to 0$ (in fact, $\mathscr{I}_{\nu,\rho}^{q}(\mathbb{R}^d) = \mathscr{V}_{\nu,\rho}^{q}(\mathscr{K})$ does not converge as $\rho\to 0$). Therefore, the weak convergence of $\mathscr{I}_{\nu,\rho}^{q}$ as $\rho\to 0$ can be ensured by normalizing these measures, and two natural options exist for normalization. The first approach is to normalize by volume. Specifically, we define the *volume-normalized multifractal tube measure* $V_{\nu,\rho}^{q}$ as

$$V_{\nu,\rho}^{q} = \frac{1}{\mathscr{I}_{\nu,\rho}^{q}(\mathbb{R}^d)} \mathscr{I}_{\nu,\rho}^{q}.$$

Secondly, normalization can be achieved through scaling. Specifically, we define the *lower* and *upper scaling-normalized multifractal tube measures* by

$$\underline{\mathscr{S}}_{\nu,\rho}^{q} = \frac{1}{\rho^{-\underline{d}_{M,\nu}^{q}(\operatorname{supp}\nu)}} \mathscr{I}_{\nu,\rho}^{q},$$

$$\overline{\mathscr{S}}_{\nu,\rho}^{q} = \frac{1}{\rho^{-\overline{d}_{M,\nu}^{q}(\operatorname{supp}\nu)}} \mathscr{I}_{\nu,\rho}^{q}.$$

THEOREM 5.1. *Let ν be a Moran measure on $\mathscr{K}$. If $\Delta > 0$, then for all $q\in\mathbb{R}$,*

$$\underline{d}_{M,\nu}^{q}(\mathscr{K}) = \underline{\beta}(q), \qquad \overline{d}_{M,\nu}^{q}(\mathscr{K}) = \overline{\beta}(q).$$

Proof. The proof is similar to the proof of Theorem 2.10. ■

THEOREM 5.2. *Given a list $\{f_{n,j} \mid n\ge 1,\ 1\le j\le a_n\}$ of contracting similarities, assume that $\Delta > 0$ and let $q\in\mathbb{R}$. For $\boldsymbol{\alpha}\in\mathscr{T}$, we have:*

(1) *The arithmetic case: If* $\{\log(\frac{1}{b_{1,\alpha_1}}),\dots,\log(\frac{1}{b_{n,\alpha_n}})\}$ *is contained in a discrete (additive) subgroup of* $\mathbb{R}$ *(for* $\theta>0$, $\theta\mathbb{Z}$ *is the smallest such subgroup), then there exist multiplicatively periodic functions* $\overline{\pi}_q$, $\underline{\pi}_q:(0,\infty)\to\mathbb{R}$ *with period* e^θ *satisfying* $\overline{\pi}_q(e^{\pm\theta}\rho)=\overline{\pi}_q(\rho)$, $\underline{\pi}_q(e^{\pm\theta}\rho)=\underline{\pi}_q(\rho)$ *for all* $\rho>0$ *and such that*

$$\frac{\mathscr{V}^q_{\nu,\rho}(\mathscr{K})}{\rho^{-\underline{\beta}(q)}}=\underline{\pi}_q(\rho)+\epsilon(\rho),\qquad \frac{\mathscr{V}^q_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}}=\overline{\pi}_q(\rho)+\epsilon(\rho),$$

where $\epsilon(\rho)\to0$ *as* $\rho\to0$.

(2) *The non-arithmetic case: If* $\{\log(\frac{1}{b_{1,\alpha_1}}),\dots,\log(\frac{1}{b_{n,\alpha_n}})\}$ *is not contained in a discrete (additive) subgroup of* $\mathbb{R}$, *then there exist constants* $\overline{c},\underline{c}\in\mathbb{R}$ *such that*

$$\frac{\mathscr{V}^q_{\nu,\rho}(\mathscr{K})}{\rho^{-\underline{\beta}(q)}}=\underline{c}+\epsilon(\rho),\qquad \frac{\mathscr{V}^q_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}}=\overline{c}+\epsilon(\rho),$$

where $\epsilon(\rho)\to0$ *as* $\rho\to0$.

Proof. The proof is identical to the proof of Theorem 3.1. In this case we just take, for $\boldsymbol{\alpha}\in\mathscr{T}$,

$$s_n=\frac{1}{-\sum_{\boldsymbol{\alpha}}p^q_{\boldsymbol{\alpha}}|I_{\boldsymbol{\alpha}}|^{\beta_n(q)}\log|I_{\boldsymbol{\alpha}}|}\int_0^1\rho^{\beta_n(q)}\theta_q(\rho)\,\frac{d\rho}{\rho},$$

where $\theta_q:(0,\infty)\to\mathbb{R}$ is expressed as

$$\theta_q(\rho)=\mathscr{V}^q_{\nu,\rho}(\mathscr{K})-\sum_{\boldsymbol{\alpha}}p^q_{\boldsymbol{\alpha}}\mathbb{1}_{(0,|I_{\boldsymbol{\alpha}}|]}(\rho)\mathscr{V}^q_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}\rho}(\mathscr{K}).$$

Also $\mathscr{K}$ is $(q,\beta_n(q))$ multifractal Minkowski measurable with respect to ν with

$$\mathscr{M}^{q,\beta_n(q)}_\nu(\mathscr{K})=\frac{1}{-\sum_{\boldsymbol{\alpha}}p^q_{\boldsymbol{\alpha}}|I_{\boldsymbol{\alpha}}|^{\beta_n(q)}\log|I_{\boldsymbol{\alpha}}|}\int_0^1\rho^{\beta_n(q)}\theta_q(\rho)\,\frac{d\rho}{\rho}$$

and

$$\Pi_{q,n}(\rho)=\frac{1}{-\sum_{\boldsymbol{\alpha}}p^q_{\boldsymbol{\alpha}}|I_{\boldsymbol{\alpha}}|^{\beta_n(q)}\log|I_{\boldsymbol{\alpha}}|}\sum_{\sigma\in\mathcal{Z},\,\rho e^{\sigma v}\le1}(\rho e^{v\sigma})^{\beta_n(q)}\theta_q(\rho e^{v\sigma})v.$$

Furthermore, $\mathscr{K}$ is $(q,\beta_n(q))$-averagely multifractal Minkowski measurable with respect to ν with

$$\mathscr{M}^{q,\beta_n(q)}_\nu(\mathscr{K})=\frac{1}{-\sum_{\boldsymbol{\alpha}}p^q_{\boldsymbol{\alpha}}|I_{\boldsymbol{\alpha}}|^{\beta_n(q)}\log|I_{\boldsymbol{\alpha}}|}\int_0^1\rho^{\beta_n(q)}\theta_q(\rho)\,\frac{d\rho}{\rho}.\ \blacksquare$$

As an application, we establish that the volume-normalized multifractal tube measures converge weakly as $\rho\to0$.

THEOREM 5.3. *Given a list* $\{f_{n,j}\mid n\ge1,\,1\le j\le a_n\}$ *of contracting similarities, suppose that* $\Delta>0$, *and denote* $\dim^q_\nu(\mathscr{K})=\underline{\beta}(q)=t$. *If there exists* n_0 *such that* $\beta_{n_0}(q)\le\underline{\beta}(q)$, *then*

$$V^q_{\nu,\rho}\to\frac{\mathscr{H}^{q,t}_\nu\llcorner\mathscr{K}}{\mathscr{H}^{q,t}_\nu(\mathscr{K})}\quad\textit{weakly.}$$

Proof. The proof is similar to that of Theorem 4.1, and is omitted. ■

We now proceed to investigate the limiting behavior of $\underline{\mathscr{S}}^q_{\nu,\rho}$ and $\overline{\mathscr{S}}^q_{\nu,\rho}$ as $\rho\to0$ for Moran measures ν.

THEOREM 5.4. *Given a list $\{f_{n,j} \mid n \geq 1,\, 1 \leq j \leq a_n\}$ of contracting similarities, suppose that $\Delta > 0$ and let $q \in \mathbb{R}$. Then*

$$\underline{\mathscr{S}}^q_{\nu,\rho} = \frac{1}{\rho^{-\underline{\beta}(q)}} \mathscr{I}^q_{\nu,\rho}, \qquad \overline{\mathscr{S}}^q_{\nu,\rho} = \frac{1}{\rho^{-\overline{\beta}(q)}} \mathscr{I}^q_{\nu,\rho}.$$

Define the average measure

$$\mathscr{S}^q_{\nu,\rho,a} = \frac{1}{-\log \rho} \int_\rho^1 \frac{1}{s^{-\beta_n(q)}} \mathscr{I}^q_{\nu,s} \frac{ds}{s}.$$

(1) *The non-arithmetic case: If $\left\{\log\left(\frac{1}{b_{1,\alpha_1}}\right), \dots, \log\left(\frac{1}{b_{n,\alpha_n}}\right)\right\}$ is not contained in a discrete (additive) subgroup of $\mathbb{R}$ and there exists n_0 such that $\beta_{n_0}(q) \leq \underline{\beta}(q)$, then*

$$\mathscr{S}^q_{\nu,\rho} \to \mathscr{M}^{q,\underline{\beta}(q)}_\nu(\mathscr{K}) \frac{\mathscr{H}^{q,\underline{\beta}(q)}_\nu \llcorner \mathscr{K}}{\mathscr{H}^{q,\underline{\beta}(q)}_\nu(\mathscr{K})} \quad \textit{weakly},$$

$$\mathscr{S}^q_{\nu,\rho,a} \to \mathscr{M}^{q,\underline{\beta}(q)}_{\nu,a}(\mathscr{K}) \frac{\mathscr{H}^{q,\underline{\beta}(q)}_\nu \llcorner \mathscr{K}}{\mathscr{H}^{q,\underline{\beta}(q)}_\nu(\mathscr{K})} \quad \textit{weakly}.$$

(2) *The arithmetic case: If $\left\{\log\left(\frac{1}{b_{1,\alpha_1}}\right), \dots, \log\left(\frac{1}{b_{n,\alpha_n}}\right)\right\}$ is contained in a discrete (additive) subgroup of $\mathbb{R}$ and there exists n_0 such that $\beta_{n_0}(q) \leq \underline{\beta}(q)$, then*

$$\mathscr{S}^q_{\nu,\rho,a} \to \mathscr{M}^{q,\underline{\beta}(q)}_{\nu,a}(\mathscr{K}) \frac{\mathscr{H}^{q,\underline{\beta}(q)}_\nu \llcorner \mathscr{K}}{\mathscr{H}^{q,\underline{\beta}(q)}_\nu(\mathscr{K})} \quad \textit{weakly}.$$

REMARK 5.5. The conclusions of Theorem 5.4 continue to hold upon substituting the multifractal Hausdorff measure with the lower multifractal Hewitt–Stromberg measure.

We need the following lemma to prove Theorem 5.4.

LEMMA 5.6 (see [21]). *Consider measurable functions $\gamma, g : (0,1) \to (0,\infty)$ satisfying*

$$\int_\rho^1 g(s) \frac{ds}{s} < \infty$$

for every ρ and $\int_\rho^1 \gamma(s) g(s)\, ds < \infty$ for all ρ. Let $m, M \geq 0$. Suppose that

$$\frac{1}{-\log \rho} \int_\rho^1 g(s) \frac{ds}{s} \to M \qquad \textit{as } \rho \to 0,$$

$$\gamma(\rho) \to m \qquad \textit{as } \rho \to 0.$$

Then

$$\frac{1}{-\log \rho} \int_\rho^1 \gamma(s) g(s) \frac{ds}{s} \to mM \qquad \textit{as } \rho \to 0, \tag{1}$$

$$\frac{1}{-\log \rho} \int_\rho^1 \gamma(s) \frac{ds}{s} \to m \qquad \textit{as } \rho \to 0. \tag{2}$$

REMARK 5.7. If $f : \mathbb{R}^d \to \mathbb{R}$ is a continuous function with compact support, then

$$\int f \, d\mathscr{S}^q_{\nu,\rho,a} = \frac{1}{-\log \rho} \int_\rho^1 \left(\int f \, d\mathscr{S}^q_{\nu,s} \right) \frac{ds}{s}.$$

Proof of Theorem 5.4. From Theorem 5.1, $\underline{d}^q_{M,\nu}(\mathscr{K}) = \underline{\beta}(q)$ and $\overline{d}^q_{M,\nu}(\mathscr{K}) = \overline{\beta}(q)$, which directly leads to the conclusion of the desired result.

(1) To simplify notation, let $\mathcal{H} = \mathscr{H}_\nu^{q,\underline{\beta}(q)} \llcorner \mathscr{K} / \mathscr{H}_\nu^{q,\underline{\beta}(q)}(\mathscr{K})$. Now consider a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support. Since it is straightforward that

$$\mathscr{S}^q_{\nu,\rho} = \frac{1}{\rho^{-\beta_n(q)}} \mathscr{V}^q_{\nu,\rho}(\mathscr{K}) V^q_{\nu,\rho},$$

it directly follows from Theorems 5.2 and 5.3 that

$$\int f \, d\mathscr{S}^q_{\nu,\rho} = \frac{1}{\rho^{-\beta_n(q)}} \mathscr{V}^q_{\nu,\rho}(\mathscr{K}) \int f \, dV^q_{\nu,\rho} \to \mathscr{M}_\nu^{q,\underline{\beta}(q)}(\mathscr{K}) \int f \, d\mathcal{H}.$$

By Lemma 5.6(2), applied to the function $\gamma : (0, \infty) \to (0, \infty)$ given by $\gamma(\rho) = \int f d\mathscr{S}^q_{\nu,\rho}$, along with the last result and Remark 5.7, it follows that

$$\begin{aligned}\int f \, d\mathscr{S}^q_{\nu,\rho,a} &= \frac{1}{-\log \rho} \int_\rho^1 \left(\int f \, d\mathscr{S}^q_{\nu,s} \right) \frac{ds}{s} \\ &\to \mathscr{M}_\nu^{q,\beta_n(q)}(\mathscr{K}) \int f \, d\mathcal{H} = \mathscr{M}_{\nu,a}^{q,\beta_n(q)}(\mathscr{K}) \int f \, d\mathcal{H}.\end{aligned}$$

Taking the lower limit as $n \to \infty$ completes the proof.

(2) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function with compact support. As previously mentioned, since $\mathscr{S}^q_{\nu,\rho} = \frac{1}{\rho^{-\beta_n(q)}} \mathscr{V}^q_{\nu,\rho}(\mathscr{K}) V^q_{\nu,\rho}$, it follows directly from Lemma 5.6 (applied to the functions $\gamma, g : (0, 1) \to (0, \infty)$ with $\gamma(\rho) = \int f \, dV^q_{\nu,\rho}$ and $g(\rho) = \frac{1}{\rho^{-\beta_n}} \mathscr{V}^q_{\nu,\rho}(\mathscr{K})$), as well as from Theorem 5.2, Theorem 5.3 and Remark 5.7, that

$$\begin{aligned}\int f \, d\mathscr{S}^q_{\nu,\rho,a} &= \frac{1}{-\log \rho} \int_\rho^1 \left(\int f d\mathscr{S}^q_{\nu,s} \right) \frac{ds}{s} \\ &= \frac{1}{-\log \rho} \int_\rho^1 \left(\frac{1}{\rho^{-\beta_n(q)}} \mathscr{V}^q_{\nu,\rho}(\mathscr{K}) \int f \, dV^q_{\nu,s} \right) \frac{ds}{s} \\ &\to \mathscr{M}_{\nu,a}^{q,\beta_n(q)}(\mathscr{K}) \int f \, d\mathcal{H}.\end{aligned}$$

Finally, we take the lower limit as $n \to \infty$. ∎

6. Some remarks and open problems

We have established the main results of this paper under the Set Strong Separation Condition, i.e., $\Delta > 0$. However, verifying the Strong Open Set Condition (SOSC) or the Open Set Condition (OSC) remains a challenging problem. We therefore conjecture that the conclusions of Theorem 2.10 also hold under (SOSC) or (OSC). In the following section, we provide partial results supporting this direction. In particular, we assume that the conclusions of Theorem 2.10 remain valid under (SOSC) in order to establish the following results. We were unable to deduce (SOSC) from (SSC) or from the condition $\Delta > 0$ as in the case of self-similar sets. However, the converse implication clearly fails, as demonstrated by the following example:

EXAMPLE 6.1. Consider a list $\{S_{1,1}, S_{1,2}, S_{2,1}, S_{2,2}, S_{2,3}\}$ of contracting similarities with respective Lipschitz constants $b_{1,1}, b_{1,2}, b_{2,1}, b_{2,2}, b_{2,3}$ (see Figure 1 and 2). $\{S_{1,1}, S_{1,2}\}$ are called Type 1, while $\{S_{2,1}, S_{2,2}, S_{2,3}\}$ are Type 2. It can be readily verified that both Types 1 and 2 satisfy the classical Strong Open Set Condition. However, while Type 1 also satisfies the classical Strong Separation Condition, Type 2 does not.

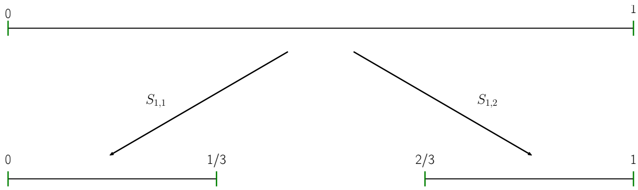

Fig. 1. Type 1 ($S_{1,1}$ and $S_{1,2}$)

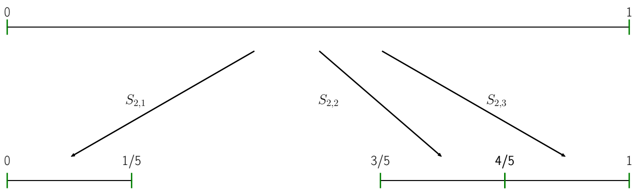

Fig. 2. Type 2 ($S_{2,1}, S_{2,2}$ and $S_{2,3}$)

Given a sequence $\{l_n\}_{n\geq 1}$ of integers such that

$$l_1 = 1, \qquad l_n < l_{n+1}, \qquad \lim_{n\to\infty} \frac{l_{n+1}}{l_n} = +\infty.$$

We set

$$a_i = \begin{cases} 2 & \text{if } l_{2n-1} \leq i < l_{2n}, \\ 3 & \text{if } l_{2n} \leq i < l_{2n+1} \end{cases}$$

and

$$b_{i,j} = \begin{cases} b_{1,j} = r_1 = \frac{1}{3} & \text{if } l_{2n-1} \leq i < l_{2n},\ 1 \leq j \leq 2, \\ b_{2,j} = r_2 = \frac{1}{5} & \text{if } l_{2n} \leq i < l_{2n+1},\ 1 \leq j \leq 3. \end{cases}$$

We will find that when $l_{2n-1} \leq i < l_{2n}$ we will use Type 1, and when $l_{2n} \leq i < l_{2n+1}$ we will use Type 2 in the system. (See Figure 3 for details. Here $l_1 = 1$, $l_2 = 2$, $l_3 = 4, \dots$, $I_{21} = (S_{1,2} \circ S_{2,1})([0,1])$ and $I_{122} = (S_{1,1} \circ S_{2,2} \circ S_{12})([0,1])$.) Such a system satisfies the (SOSC) but does not satisfy the (SSC).

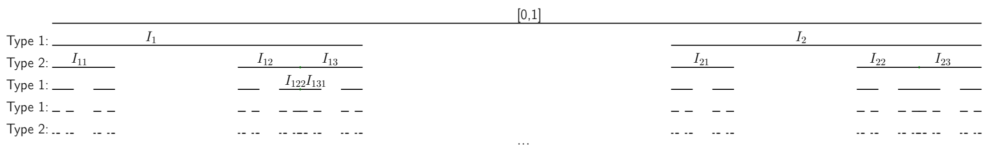

Fig. 3. A model

6.1. Dimensions of overlaps. In this section we demonstrate that the upper multifractal box-dimension, pertaining specifically to the overlaps, exhibits a clearly inferior value when compared to the multifractal box-dimension of the set.

THEOREM 6.2. *Suppose a list* $\{f_{n,j} \mid n \geq 1,\, 1 \leq j \leq a_n\}$ *of contracting similarities satisfies the* (SOSC) *and let* $q \in \mathbb{R}$.

(1) (a) *For* $\boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T}$ *with* $|\boldsymbol{\alpha}| = |\boldsymbol{\sigma}|$ *and* $\boldsymbol{\alpha} \neq \boldsymbol{\sigma}$,

$$\overline{d}^q_{\nu,p}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) \leq \varphi(q) < \overline{\beta}(q) = \overline{d}^q_{\nu,p}(\mathscr{K}).$$

(b) $$\operatorname{Dim}^q_\nu\Big(\bigcup_s \bigcup_{\substack{|\boldsymbol{\alpha}|=|\boldsymbol{\sigma}|=s \\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}} f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})\Big) \leq \varphi(q) < \overline{\beta}(q) = \operatorname{Dim}^q_\nu(\mathscr{K}).$$

(c) $$B^q_\nu\Big(\bigcup_s \bigcup_{\substack{|\boldsymbol{\alpha}|=|\boldsymbol{\sigma}|=s \\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}} f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})\Big) \leq \varphi(q) < \overline{\beta}(q) = B^q_\nu(\mathscr{K}).$$

(2) *If there exists* n_0 *such that* $\beta_{n_0}(q) \leq \underline{\beta}(q)$, *then*

(a) $$\operatorname{dim}^q_\nu\Big(\bigcup_s \bigcup_{\substack{|\boldsymbol{\alpha}|=|\boldsymbol{\sigma}|=s \\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}} f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})\Big) \leq \varphi(q) < \underline{\beta}(q) = \operatorname{dim}^q_\nu(\mathscr{K}).$$

(b) $$b^q_\nu\Big(\bigcup_s \bigcup_{\substack{|\boldsymbol{\alpha}|=|\boldsymbol{\sigma}|=s \\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}} f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})\Big) \leq \varphi(q) < \underline{\beta}(q) = b^q_\nu(\mathscr{K}).$$

In order to prove Theorem 6.2, we need some technical lemmas and two propositions.

LEMMA 6.3. *Assume that* $\{f_{n,j} \mid n \geq 1,\, 1 \leq j \leq a_n\}$ *fulfills the* (SOSC). *Let* $n \geq 1$ *be such that* $\Phi_n \neq \emptyset$. *Let* c_n *be as defined in* (2.1). *If* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{|\boldsymbol{\alpha}|})$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{|\boldsymbol{\sigma}|}) \in \mathscr{T}$ *with* $\alpha_1 \neq \sigma_1$ *and* $\operatorname{dist}(f_{\boldsymbol{\alpha}}(\mathscr{K}),\ f_{\boldsymbol{\sigma}}(\mathscr{K})) \leq c_n |I_{\boldsymbol{\alpha}}|$, *then*

$$\Phi_n \not\subseteq (\alpha_2, \ldots, \alpha_{|\boldsymbol{\alpha}|}).$$

Proof. Consider the open set O as in Definition 2.8. Assume, towards a contradiction, that there exists an element $\boldsymbol{l}$ in Φ_n which can be represented as a substring of $(\alpha_2, \ldots, \alpha_{|\boldsymbol{\alpha}|})$. In other words, there are $\boldsymbol{u}$ and $\boldsymbol{v}$ belonging to $\mathscr{T}$, where $\boldsymbol{u} = (u_1, ..., u_{|\boldsymbol{u}|}) \neq \emptyset$ and $\boldsymbol{\alpha}$ can be written as $\boldsymbol{\alpha} = \boldsymbol{ulv}$. Consequently,

$$\operatorname{dist}(f_{\boldsymbol{\alpha}}(\mathscr{K}),\ \mathbb{R}^d \setminus f_{\boldsymbol{u}}(O)) \geq \operatorname{dist}(f_{\boldsymbol{ul}}(\mathscr{K}),\ \mathbb{R}^d \setminus f_{\boldsymbol{u}}(O)) \geq c_n|I_{\boldsymbol{u}}| > c_n|I_{\boldsymbol{\alpha}}|. \tag{6.1}$$

Since $\boldsymbol{u} \neq \emptyset$ and $u_1 = k_1 \neq j_1$, this implies $f_{\boldsymbol{j}}(\mathscr{K}) \cap f_{\boldsymbol{u}}(O) = \emptyset$, i.e. $f_{\boldsymbol{j}}(\mathscr{K}) \subseteq \mathbb{R}^d \setminus f_{\boldsymbol{u}}(O)$. Therefore,

$$\operatorname{dist}(f_{\boldsymbol{\alpha}}(\mathscr{K}), \mathbb{R}^d \setminus f_{\boldsymbol{u}}(O)) \leq \operatorname{dist}(f_{\boldsymbol{\alpha}}(\mathscr{K}), f_{\boldsymbol{j}}(\mathscr{K})) \leq c_n|I_{\boldsymbol{\alpha}}|, \tag{6.2}$$

contradicting (6.1). ∎

LEMMA 6.4. *Let* ν *be a Moran measure on* $\mathscr{K}$. *If* $\{f_{n,j} \mid n \geq 1,\, 1 \leq j \leq a_n\}$ *fulfills the* (SOSC), *then*

$$\nu(f_{\boldsymbol{\alpha}}(\mathscr{K})) = p_{1,\alpha_1} \ldots p_{s,\alpha_n} = p_{\boldsymbol{\alpha}} \quad \textit{for all } \boldsymbol{\alpha} \in \mathscr{T}.$$

Proof. The proof is immediate from

$$\nu(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) = 0 \quad \text{for all } \boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T}. \ ∎$$

LEMMA 6.5 ([12]). *Given* $c_1, c_2 > 0$ *and a family* $(V_i)_i$ *of subsets of* $\mathbb{R}^d$, *assume that each set* V_i *includes a closed ball* B_i *with a radius of* $c_1\rho$ *and is contained in a closed ball of radius* $c_2\rho$. *Note that* $B_i \cap B_j = \emptyset$, $i \neq j$. *Then for all* $m > 0$ *and* $x \in \mathbb{R}^d$,

$$|\{i \mid V_i \cap \mathscr{B}(x, m\rho) \neq \emptyset\}| \leq (m + 2c_2)^d c_1^{-d}.$$

LEMMA 6.6 ([12]). *Fix* $q \in \mathbb{R}$. *Given* $\mathscr{A} > 0$ *and* $d_1, \dots, d_m \geq 0$ *with* $m \leq \mathscr{A}$. *Then*

$$\Big(\sum_i d_i\Big)^q \leq \max(1, \mathscr{A}^{q-1}) \sum_i d_i^q.$$

Consider the set

$$\mathscr{T}(\rho) = \{\boldsymbol{\lambda} \in \mathscr{T} \mid |I_{\boldsymbol{\lambda}}|\,|\mathscr{K}| < \rho \leq |I_{\boldsymbol{\lambda}|(|\boldsymbol{\lambda}|-1)}|\,|\mathscr{K}|\}.$$

PROPOSITION 6.7. *Suppose that* $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ *satisfies the* (SOSC). *Consider* $q \in \mathbb{R}$. *Take* $n \geq 1$ *such that* $\Phi_n \neq \emptyset$. *Fix* $\boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T}$ *with* $\boldsymbol{\alpha} \neq \boldsymbol{\sigma}$ *and* $|\boldsymbol{\alpha}| = |\boldsymbol{\sigma}| = n$. *Then there exists* $C > 0$ *such that*

$$\mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho) \leq C \sum_{\substack{\rho/|\mathscr{K}| \leq |I_{\boldsymbol{\lambda}}| \\ \Phi_n \not\subseteq \boldsymbol{\lambda}}} p^q_{1,\lambda_1} \cdots p^q_{n,\lambda_n}$$

for all $\rho > 0$.

Proof. Let

$$D = |\mathscr{K}|, \quad b_{\min} = \min_{n \geq 1, 1 \leq j \leq a_n} b_{n,j}, \quad b_{\max} = \max_{n \geq 1, 1 \leq j \leq a_n} b_{n,j}.$$

Fix $\rho > 0$ and let F be a ρ-separated subset of $f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho)$. For each $x \in F$ we may choose $\boldsymbol{\lambda}(x) \in \mathscr{T}(\rho)$ such that $x \in f_{\boldsymbol{\lambda}(x)}(\mathscr{K})$, so it is clear that

$$f_{\boldsymbol{\lambda}(x)}(\mathscr{K}) \subseteq \mathscr{B}(x, \rho) \cap \mathscr{K} \subseteq \bigcup_{\substack{\boldsymbol{\lambda} \in \mathscr{T}(\rho) \\ \operatorname{dist}(x,\, f_{\boldsymbol{\lambda}}(\mathscr{K})) \leq |I_{\boldsymbol{\lambda}}|}} f_{\boldsymbol{\lambda}}(\mathscr{K})$$

for all $x \in F$. This implies that

$$\nu(\mathscr{B}(x,\rho))^q \leq \begin{cases} \nu(f_{\boldsymbol{\lambda}(x)}(\mathscr{K}))^q & \text{if } q \leq 0, \\ \nu\Big(\bigcup\limits_{\substack{\boldsymbol{\lambda} \in \mathscr{T}(\rho) \\ \operatorname{dist}(x, f_{\boldsymbol{\lambda}}(\mathscr{K})) \leq |I_{\boldsymbol{\lambda}}|}} f_{\boldsymbol{\lambda}}(\mathscr{K})\Big)^q \leq \Big(\sum\limits_{\substack{\boldsymbol{\lambda} \in \mathscr{T}(\rho) \\ \operatorname{dist}(x, f_{\boldsymbol{\lambda}}(\mathscr{K})) \leq |I_{\boldsymbol{\lambda}}|}} \nu(f_{\boldsymbol{\lambda}}(\mathscr{K}))\Big)^q & \text{if } q \geq 0. \end{cases} \tag{6.3}$$

Let O be the open set in Definition 2.8. It contains a ball of radius ρ_1 and is contained in a ball of radius ρ_2. It follows that $f_{\boldsymbol{\lambda}}(O)$ contains a ball with radius $\rho_1|I_{\boldsymbol{\lambda}}|$ and is contained in a ball of radius $\rho_2|I_{\boldsymbol{\lambda}}|$. Furthermore, as $(f_{\boldsymbol{\lambda}}(O))_{\boldsymbol{\lambda} \in \mathscr{T}(\rho)}$ is a pairwise disjoint family of sets with $f_{\boldsymbol{\lambda}}(\mathscr{K}) \subseteq \overline{f_{\boldsymbol{\lambda}}(O)}$, Lemma 6.5 implies that

$$|\{\boldsymbol{\lambda} \in \mathscr{T}(\rho) \mid \operatorname{dist}(x, f_{\boldsymbol{\lambda}}(\mathscr{K})) \leq \rho\}| \leq \left|\left\{\boldsymbol{\lambda} \in \mathscr{T}(\rho) \;\middle|\; \overline{f_{\boldsymbol{\lambda}}(O)} \cap B\Big(x, \frac{D}{b_{\min}}|I_{\boldsymbol{\lambda}}|\Big) \neq \emptyset\right\}\right| \leq C_0,$$

where $C_0 = ((D/b_{\min} + 2\rho_2)/\rho_1)^d$. By using Lemma 6.6, we obtain

$$\Big(\sum_{\substack{\boldsymbol{\lambda} \in \mathscr{T}(\rho) \\ \operatorname{dist}(x, f_{\boldsymbol{\lambda}}(\mathscr{K})) \leq \rho}} \nu(f_{\boldsymbol{\lambda}}(\mathscr{K}))\Big)^q \leq C_1 \sum_{\substack{\boldsymbol{\lambda} \in \mathscr{T}(\rho) \\ \operatorname{dist}(x, f_{\boldsymbol{\lambda}}(\mathscr{K})) \leq \rho}} \nu(f_{\boldsymbol{\lambda}}(\mathscr{K}))^q, \tag{6.4}$$

where $C_1 = \max(1, C_0^{q-1})$.

Let c_n be as defined in (2.1). Select a natural number N such that $b_{\max}^{-(N-1)} \geq 2Dc_n^{-1}b_{\min}^{-(n-1)}$. It can be observed through the use of [22] that

$$\begin{aligned}&\text{if } x \in F,\ \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{|\boldsymbol{\lambda}|}) \in \mathscr{T}(\rho) \text{ and } \operatorname{dist}(x, f_{\boldsymbol{\lambda}}(\mathscr{K})) \leq \rho,\\ &\text{then } \Phi_n \not\subseteq (\lambda_{n+1}, \ldots, \lambda_{|\boldsymbol{\lambda}|-N}).\end{aligned} \tag{6.5}$$

To prove (6.5), we will consider two cases.

Case 1: $(\lambda_1, \ldots, \lambda_{|\boldsymbol{j}|}) \neq \boldsymbol{j}$. In this case, $\{1 \leq v \leq n \mid \lambda_v \neq j_v\} \neq \emptyset$. Let s be the minimum element in $\{1 \leq v \leq n \mid \lambda_v \neq j_v\}$. Consequently,

$$\begin{aligned}\operatorname{dist}(f_{\lambda_s,\ldots,\lambda_{(|\boldsymbol{\lambda}|-N)}}(\mathscr{K}), f_{j_s,\ldots,j_{|\boldsymbol{j}|}}(\mathscr{K})) &= |I_{\boldsymbol{\lambda}|(s-1)}| \operatorname{dist}(f_{\boldsymbol{\lambda}|(|\boldsymbol{\lambda}|-N)}(\mathscr{K}), f_{\boldsymbol{j}}(\mathscr{K}))\\ &\leq b_{\min}^{-(n-1)}(\operatorname{dist}(f_{\boldsymbol{\lambda}|(|\boldsymbol{\lambda}|-N)}(\mathscr{K}), x) + \operatorname{dist}(x, f_{\boldsymbol{j}}(\mathscr{K}))\\ &\leq 2\rho b_{\min}^{-(n-1)}.\end{aligned}$$

Moreover,

$$c_n|I_{\lambda_s,\ldots,\lambda_{|\boldsymbol{\lambda}|-N}}| \geq c_n|I_{\boldsymbol{\lambda}|(|\boldsymbol{\lambda}|-N)}| \geq c_n\frac{\rho}{Db_{\max}^{(N-1)}} \geq 2\rho b_{\min}^{-(n-1)}.$$

Utilizing Lemma 6.3, we conclude that $\Phi_n \not\subseteq (\lambda_{s+1}, \ldots, \lambda_{|\boldsymbol{\lambda}|-N})$. Hence

$$\Phi_n \not\subseteq (\lambda_{n+1}, \ldots, \lambda_{|\boldsymbol{\lambda}|-N}).$$

Case 2: $(\lambda_1, \ldots, \lambda_{|\boldsymbol{j}|}) = \boldsymbol{j}$. Since $\boldsymbol{i} \neq \boldsymbol{j}$, we infer that $\{1 \leq v \leq n \mid \lambda_v \neq i_v\} \neq \emptyset$. Let t denote the minimal value in $\{1 \leq v \leq n \mid \lambda_v \neq i_v\}$. As a result,

$$\begin{aligned}\operatorname{dist}(f_{\lambda_t,\ldots,\lambda_{|\boldsymbol{\lambda}|-N}}(\mathscr{K}), f_{i_t,\ldots,i_{|\boldsymbol{i}|}}(\mathscr{K})) &= \frac{1}{|I_{\boldsymbol{\lambda}|(t-1)}|}\operatorname{dist}(f_{\boldsymbol{\lambda}|(|\boldsymbol{\lambda}|-N)}(\mathscr{K}), f_{\boldsymbol{i}}(\mathscr{K}))\\ &\leq b_{\min}^{-(n-1)}(\operatorname{dist}(f_{\boldsymbol{\lambda}|(|\boldsymbol{\lambda}|-N)}(\mathscr{K}), x) + \operatorname{dist}(x, f_{\boldsymbol{i}}(\mathscr{K}))\\ &\leq 2\rho b_{\min}^{-(n-1)}.\end{aligned}$$

Similarly to Case 1, we have

$$c_n|I_{\lambda_t,\ldots,\lambda_{|\boldsymbol{\lambda}|-N}}| \geq c_n|I_{\boldsymbol{\lambda}|(|\boldsymbol{\lambda}|-N)}| \geq c_n\frac{\rho}{Db_{\max}^{(N-1)}} \geq 2\rho b_{\min}^{-(n-1)}.$$

Utilizing Lemma 6.3 again, we find that $\Phi_n \not\subseteq (\lambda_{t+1}, \ldots, \lambda_{|\boldsymbol{\lambda}|-N})$, and consequently $\Phi_n \not\subseteq (\lambda_{n+1}, \ldots, \lambda_{|\boldsymbol{\lambda}|-N})$. This completes the proof of (6.5).

Given that O contains a ball with a radius of ρ_1 and F is ρ-separated, we deduce the existence of a collection of pairwise disjoint balls denoted as (B_y) for $y \in F$. These balls have radii $\min((b_{\min}/D)\rho_1, 1)\rho$ so that the set

$$\bigcup_{\substack{\boldsymbol{\lambda}\in\mathscr{T}(\rho)\\ \operatorname{dist}(y, f_{\boldsymbol{\lambda}}(\mathscr{K}))\leq\rho}} \overline{f_{\boldsymbol{\lambda}}(O)}$$

contains B_y. Furthermore, this set is contained in a ball of radius

$$\rho + \max_{\boldsymbol{\lambda}\in\mathscr{T}(\rho)} |\overline{f_{\boldsymbol{\lambda}}(O)}| \leq (1 + \rho_2/D)\rho.$$

Since $f_{\boldsymbol\lambda}(\mathscr{K}) \subseteq \overline{f_{\boldsymbol\lambda}(O)}$, we can apply Lemma 6.5 to conclude that

$$\Big|\Big\{y \in F \;\Big|\; \Big(\bigcup_{\substack{\boldsymbol\lambda\in\mathscr{T}(\rho)\\ \operatorname{dist}(y,f_{\boldsymbol\lambda}(\mathscr{K}))\le\rho}} f_{\boldsymbol\lambda}(\mathscr{K})\Big) \cap \Big(\bigcup_{\substack{\boldsymbol\lambda\in\mathscr{T}(\rho)\\ \operatorname{dist}(x,f_{\boldsymbol\lambda}(\mathscr{K}))\le\rho}} f_{\boldsymbol\lambda}(\mathscr{K})\Big) \neq \emptyset\Big\}\Big|$$
$$\le \Big|\Big\{y \in F \;\Big|\; \Big(\bigcup_{\substack{\boldsymbol\lambda\in\mathscr{T}(\rho)\\ \operatorname{dist}(y,f_{\boldsymbol\lambda}(\mathscr{K}))\le\rho}} \overline{f_{\boldsymbol\lambda}(O)}\Big) \cap B(x,(1+\rho_2/D)\rho) \neq \emptyset\Big\}\Big| \le C_2$$

for all $x \in F$, where $C_2 = (((1+\rho_2/D) + 2(1+\rho_2/D))/\min(b_{\min}/D\rho_1, 1))^d$. Hence by (6.5),

$$\sum_{x\in F} \sum_{\substack{\boldsymbol\lambda\in\mathscr{T}(\rho)\\ \operatorname{dist}(x,\, f_{\boldsymbol\lambda}(\mathscr{K}))\le\rho}} \nu(f_{\boldsymbol\lambda}(\mathscr{K}))^q \le C_2 \sum_{\substack{\boldsymbol\lambda\in\mathscr{T}(\rho)\\ \Phi_n\not\subseteq(\lambda_{n+1},\dots,\lambda_{|\boldsymbol\lambda|-N})}} \nu(f_{\boldsymbol\lambda}(\mathscr{K}))^q \tag{6.6}$$

for $q \in \mathbb{R}$. By combining (6.1), (6.4), (6.6) and Lemma 6.4, we obtain

$$\begin{aligned}
\sum_{x\in F} \nu(\mathscr{B}(x,\rho))^q &\le \begin{cases} \displaystyle\sum_{x\in F} \nu(f_{\boldsymbol\lambda(x)}(\mathscr{K}))^q & \text{if } q \le 0,\\ \displaystyle\sum_{x\in F}\Big(\sum_{\substack{\boldsymbol\lambda\in\mathscr{T}(\rho)\\ \operatorname{dist}(x,f_{\boldsymbol\lambda}(\mathscr{K}))\le\rho}} \nu(f_{\boldsymbol\lambda}(\mathscr{K}))\Big)^q & \text{if } q \ge 0.\end{cases}\\
&\le \begin{cases} \displaystyle\sum_{x\in F} \sum_{\substack{\boldsymbol\lambda\in\mathscr{T}(\rho)\\ \operatorname{dist}(x,f_{\boldsymbol\lambda}(\mathscr{K}))\le\rho}} \nu(f_{\boldsymbol\lambda}(\mathscr{K}))^q & \text{if } q \le 0,\\ \displaystyle C_1\sum_{x\in F} \sum_{\substack{\boldsymbol\lambda\in\mathscr{T}(\rho)\\ \operatorname{dist}(x,f_{\boldsymbol\lambda}(\mathscr{K}))\le\rho}} \nu(f_{\boldsymbol\lambda}(\mathscr{K}))^q & \text{if } q \ge 0.\end{cases}\\
&\le C_1C_2 \sum_{\substack{\boldsymbol\lambda\in\mathscr{T}(\rho)\\ \Phi_n\not\subseteq(\lambda_{n+1},\dots,\lambda_{|\boldsymbol\lambda|-N})}} \nu(f_{\boldsymbol\lambda}(\mathscr{K}))^q\\
&\le C_1C_2 \sum_{\substack{\rho/|\mathscr{K}|\le|I_{\lambda_{n+1}\cdots\lambda_{|\boldsymbol\lambda|-N}}|\\ \Phi_n\not\subseteq(\lambda_{n+1},\dots,\lambda_{|\boldsymbol\lambda|-N})}} p_{1,\lambda_1}^q\cdots p_{n,\lambda_n}^q\\
&= C_1C_2 \sum_{\substack{\rho/|\mathscr{K}|\le|I_{\lambda_{n+1}\cdots\lambda_{|\boldsymbol\lambda|-N}}|\\ \Phi_n\not\subseteq(\lambda_{n+1},\dots,\lambda_{|\boldsymbol\lambda|-N})}} p_{\boldsymbol\lambda}^q. \quad \blacksquare
\end{aligned}$$

PROPOSITION 6.8. *Assume that* $\{f_{n,j} \mid n \ge 1,\ 1 \le j \le a_n\}$ *satisfies the* (SOSC). *Let* $q \in \mathbb{R}$ *and let* $n \ge 1$ *such that* $\Phi_n \neq \emptyset$. *Then there exists* $M > 0$ *such that*

$$\sum_{\substack{\rho\le|I_{\boldsymbol\lambda}|\\ \Phi_n\not\subseteq\boldsymbol\lambda}} p_{\boldsymbol\lambda}^q \le M\rho^{-\varphi_n(q)}.$$

Proof. We start by defining functions $g, h : (0,\infty) \to [0,\infty)$ as follows:

$$g(\rho) = \sum_{\substack{\rho\le|I_{\boldsymbol\lambda}|\\ \Phi_n\not\subseteq\boldsymbol\lambda}} p_{\boldsymbol\lambda}^q, \qquad h(\rho) = \rho^{\varphi_n(q)} g(\rho).$$

Let $\delta = (b_{\min})^n$. If $0 < \rho < \delta$, then

$$\begin{aligned} g(\rho) &= \sum_{\substack{|\boldsymbol{\lambda}|=n \\ \boldsymbol{\lambda}\notin\Phi_n}} \sum_{\substack{\rho\le|I_{\boldsymbol{\lambda\alpha}}| \\ \Phi_n\not\subseteq\boldsymbol{\lambda\alpha}}} p^q_{\boldsymbol{\lambda\alpha}} = \sum_{\substack{|\boldsymbol{\lambda}|=n \\ \boldsymbol{\lambda}\notin\Phi_n}} \sum_{\substack{\rho\le|I_{\boldsymbol{\lambda\alpha}}| \\ \Phi_n\not\subseteq\boldsymbol{\lambda\alpha}}} p^q_{1,\lambda_1\alpha_1}\cdots p^q_{n,\lambda_n\alpha_n} \\ &\le \sum_{\substack{|\boldsymbol{\lambda}|=n \\ \boldsymbol{\lambda}\notin\Phi_n}} p^q_{1,\lambda_1}\cdots p^q_{s,\lambda_n} \sum_{\substack{|I_{\boldsymbol{\lambda}}|^{-1}\rho\le|I_{\boldsymbol{\alpha}}| \\ \Phi_n\not\subseteq\boldsymbol{\alpha}}} p^q_{1,\alpha_1}\cdots p^q_{s,\alpha_n} \\ &= \sum_{\substack{|\boldsymbol{\lambda}|=n \\ \boldsymbol{\lambda}\notin\Phi_n}} p^q_{\boldsymbol{\lambda}} \sum_{\substack{|I_{\boldsymbol{\lambda}}|^{-1}\rho\le|I_{\boldsymbol{\alpha}}| \\ \Phi_n\not\subseteq\boldsymbol{\alpha}}} p^q_{\boldsymbol{\alpha}} = \sum_{\substack{|\boldsymbol{\lambda}|=n \\ \boldsymbol{\lambda}\notin\Phi_n}} p^q_{\boldsymbol{\lambda}}\, g(|I_{\boldsymbol{\lambda}}|^{-1}\rho). \end{aligned}$$

Thus, for all $0 < \rho < \delta$ we have

$$h(\rho) \le \sum_{\substack{|\boldsymbol{\lambda}|=n \\ \boldsymbol{\lambda}\notin\Phi_n}} p^q_{\boldsymbol{\lambda}} |I_{\boldsymbol{\lambda}}|^{\varphi_n(q)} (|I_{\boldsymbol{\lambda}}|^{-1}\rho)^{\varphi_n(q)} g(|I_{\boldsymbol{\lambda}}|^{-1}\rho) = \sum_{\substack{|\boldsymbol{\lambda}|=n \\ \boldsymbol{\lambda}\notin\Phi_n}} p^q_{\boldsymbol{\lambda}} |I_{\boldsymbol{\lambda}}|^{\varphi_n(q)} h(|I_{\boldsymbol{\lambda}}|^{-1}\rho).$$

Let $\theta = (b_{\max})^n$. By making use of the previous inequality and the definition of $\varphi_n(q)$ we find that if $0 < a < \delta$, then

$$\sup_{a\theta\le\rho<\delta} h(\rho) \le \sup_{a\theta\le\rho<\delta} \sum_{\substack{|\boldsymbol{\lambda}|=n \\ \boldsymbol{\lambda}\notin\Phi_n}} p^q_{\boldsymbol{\lambda}} |I_{\boldsymbol{\lambda}}|^{\varphi_n(q)} h(|I_{\boldsymbol{\lambda}}|^{-1}\rho) \le \sum_{\substack{|\boldsymbol{\lambda}|=n \\ \boldsymbol{\lambda}\notin\Phi_n}} p^q_{\boldsymbol{\lambda}} |I_{\boldsymbol{\lambda}}|^{\varphi_n(q)} \sup_{a\le t} h(t) = \sup_{a\le t} h(t).$$

Finally, we get

$$\sup_{a\theta\le\rho} h(\rho) \le \sup_{a\le t} h(t).$$

Then there exists a constant $M > 0$ such that

$$\rho^{\varphi_n(q)} g(\rho) = h(\rho) \le M \quad \text{for all } \rho > 0. \ \blacksquare$$

Proof of Theorem 6.2. (1) (a) Let $n \ge 1$ be such that $\Phi_n \ne \emptyset$. Fix $\boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T}$ with $|\boldsymbol{\alpha}| = |\boldsymbol{\sigma}| = n$ and $\boldsymbol{\alpha} \ne \boldsymbol{\sigma}$. Propositions 6.7 and 6.8 yield $M > 0$ such that

$$\mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho) \le M\rho^{-\varphi_n(q)} \quad \text{for all } \rho > 0. \tag{6.7}$$

Since $\mathscr{M}^q_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) \le \mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho)$, by making use of (6.7) we obtain

$$\overline{d}^q_{\nu,p}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) \le \limsup_{\rho\to 0} \frac{\mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho)}{-\log\rho} \le \limsup_{\rho\to 0} \frac{\log(M\rho^{-\varphi_n(q)})}{-\log\rho} = \varphi_n(q).$$

(b) Let $\xi \in \mathbb{N}$ and let $\rho > 0$ be as in Besicovitch's Covering Theorem [22]. Let $\varepsilon > 0$. Fix $0 < \delta < \rho$. Let $(\mathscr{B}(x_i, \rho_i))_i$ represent a centered δ-packing of $f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})$. Additionally, denote $\rho_s = \delta/2^{s-1}$ for $s \in \mathbb{N}$. Suppose i and s in $\mathbb{N}$ satisfy $\rho_{s+1} \le \rho_i < \rho_s$. Based on the definition of $(\rho_s)_s$, we can deduce that

$$(2\rho_i)^{\varphi_n(q)+\varepsilon} \le c\rho_s^{\varphi_n(q)+\varepsilon}, \quad (2\rho_i)^{\varphi_n(q)+\varepsilon} \le c\rho_{s+1}^{\varphi_n(q)+\varepsilon}, \tag{6.8}$$

where $c = 2^{\varphi_n(q)+\varepsilon}\max(1, 2^{\pm(\varphi_n(q)+\varepsilon)})$. Fix $s \in \mathbb{N}$. It is evident that $|x_i - x_j| > \rho_i + \rho_j \ge 2\rho_{s+1} = \rho_s$ for all $i \ne j$ with $\rho_{s+1} \le \rho_i < \rho_s$ and $\rho_{s+1} \le \rho_j < \rho_s$. By utilizing Besicovitch's Covering Theorem for the collection of balls $\{\mathscr{B}(x_i, \rho_s) \mid \rho_{s+1} \le \rho_i < \rho_s\}$, we can establish the existence of ξ subfamilies $I_{s1}, \dots, I_{s\xi}$ of $\{i \mid \rho_{s+1} \le \rho_i < \rho_s\}$ such that $\{i \mid \rho_{s+1} \le \rho_i < \rho_s\} = \bigcup_l I_{sl}$. Furthermore, for any $i, j \in I_{sl}$ with $i \ne j$, we have

$\mathscr{B}(x_i, \rho_s) \cap \mathscr{B}(x_j, \rho_s) = \emptyset$. Then

$$\sum_{\rho_{s+1} \le \rho_i < \rho_s} \nu(\mathscr{B}(x_i, \rho_s))^q \le \sum_{l=1}^{\xi} \sum_{i \in I_{sl}} \nu(\mathscr{B}(x_i, \rho_s))^q \le \sum_{l=1}^{\xi} \mathscr{M}^{q,p}_{\nu,\rho_s}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K})))$$
$$= \xi \mathscr{M}^{q,p}_{\nu,\rho_s}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K})))$$

for all $s \in \mathbb{N}$ and $q \in \mathbb{R}$.

It now follows from (6.7), (6.8) and the previous inequality that

$$\begin{aligned}
\sum_i \nu(\mathscr{B}(x_i, \rho_i))^q (2\rho_i)^{\varphi_n(q)+\varepsilon} &= \sum_s \sum_{\rho_{s+1} \le \rho_i < \rho_s} \nu(\mathscr{B}(x_i, \rho_i))^q (2\rho_i)^{\varphi_n(q)+\varepsilon} \\
&\le \begin{cases} c \sum_s \sum_{\rho_{s+1} \le \rho_i < \rho_s} \nu(\mathscr{B}(x_i, \rho_{s+1}))^q \rho_{s+1}^{\varphi_n(q)+\varepsilon} & \text{if } q \le 0, \\ c \sum_s \sum_{\rho_{s+1} \le \rho_i < \rho_s} \nu(\mathscr{B}(x_i, \rho_s))^q \rho_s^{\varphi_n(q)+\varepsilon} & \text{if } q \ge 0, \end{cases} \\
&\le \begin{cases} c \sum_s \mathscr{M}^{q,p}_{\nu,\rho_{s+1}}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho_{s+1})) \rho_{s+1}^{\varphi_n(q)+\varepsilon} & \text{if } q \le 0, \\ c \sum_s \xi \mathscr{M}^{q,p}_{\nu,\rho_s}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho_s)) \rho_s^{\varphi_n(q)+\varepsilon} & \text{if } q \ge 0, \end{cases} \\
&\le c\xi \sum_s \rho_s^{\varphi_n(q)+\varepsilon} Q^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho_s) \le c\xi M \sum_s \rho_s^{\varphi_n(q)+\varepsilon} \rho_s^{-\varphi_n(q)} \\
&= c\xi M \sum_s \frac{\delta^\varepsilon}{2^{\varepsilon(s-1)}} = c\xi M \frac{2^\varepsilon}{2^\varepsilon - 1} \delta^\varepsilon = \mathscr{A} \delta^\varepsilon,
\end{aligned}$$

where $\mathscr{A} = c\xi M \frac{2^\varepsilon}{2^\varepsilon - 1} < \infty$. We therefore deduce that

$$\overline{\mathscr{P}}^{q,\varphi_n(q)+\varepsilon}_{\nu,\delta}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) \le \mathscr{A} \delta^\varepsilon,$$

and

$$\mathscr{P}^{q,\varphi_n(q)+\varepsilon}_{\nu}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) = 0.$$

This completes the proof of (1)(b).

(c) For all $q \in \mathbb{R}$, we have

$$P^{q,\varphi_n(q)+\varepsilon}_{\nu}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) \le \mathscr{P}^{q,\varphi_n(q)+\varepsilon}_{\nu}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})).$$

In addition, from the second statement of Theorem 2.10, we have $\mathrm{Dim}^q_\nu(\mathscr{K}) = B_\nu(q)$. Thus we get the desired result.

(2) (a) Suppose that there exists n_0 such that $\beta_{n_0}(q) \le \underline{\beta}(q)$. Let $\xi \in \mathbb{N}$ and let $\rho > 0$ be as in Besicovitch's Covering Theorem [22]. Consider $\varepsilon > 0$. Fix $0 < \delta < \rho$. Let $(\mathscr{B}(x_i, \rho_i))_i$ represent a centered δ-covering of $f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})$. By making use of the other version of Besicovitch's Covering Theorem [23], there exist $\theta \in \mathbb{N}$ countable or finite subfamilies $\{\mathscr{B}(x_{ij}, \rho_{ij}) \mid 1 \le j \le \theta\}$ such that for each i, $\{\mathscr{B}(x_{ij}, \rho_{ij}) \mid 1 \le j \le \theta\}$ is a δ-packing of $f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})$. Denote $\rho_s = \delta/2^{s-1}$ for $s \in \mathbb{N}$. Suppose $i, s \in \mathbb{N}$ satisfy $\rho_{s+1} \le \rho_{ij} < \rho_s$. From the definition of $(\rho_s)_s$, we can deduce that

$$(2\rho_{ij})^{\varphi_n(q)+\varepsilon} \le c\rho_s^{\varphi_n(q)+\varepsilon}, \qquad (2\rho_{ij})^{\varphi_n(q)+\varepsilon} \le c\rho_{s+1}^{\varphi_n(q)+\varepsilon}, \tag{6.9}$$

where $c = 2^{\varphi_n(q)+\varepsilon} \max(1, 2^{\pm(\varphi_n(q)+\varepsilon)})$. Fix $s \in \mathbb{N}$. It is evident that $|x_{ij} - x_{tj}| > \rho_{ij} + \rho_{tj} \geq 2\rho_{s+1} = \rho_s$ for $ij \neq tj$ with $\rho_{s+1} \leq \rho_{ij} < \rho_s$ and $\rho_{s+1} \leq \rho_{tj} < \rho_s$ By utilizing Besicovitch's Covering Theorem for the collection of balls $\{\mathscr{B}(x_{ij}, \rho_s) \mid \rho_{s+1} \leq \rho_{ij} < \rho_s\}$, we can establish the existence of ξ subfamilies $I_{s1}, \ldots, I_{s\xi}$ of $\{ij \mid \rho_{s+1} \leq \rho_{ij} < \rho_s\}$ such that $\{ij \mid \rho_{s+1} \leq \rho_{ij} < \rho_s\} = \bigcup_l I_{sl}$. Furthermore, for any $ij, tj \in I_{sl}$ with $ij \neq tj$, we have $\mathscr{B}(x_{ij}, \rho_s) \cap \mathscr{B}(x_{i_1 j}, \rho_s) = \emptyset$. Then

$$\begin{aligned}
\sum_{\rho_{s+1} \leq \rho_{ij} < \rho_s} \sum_{j=1}^{\theta} \nu(\mathscr{B}(x_{ij}, \rho_s))^q &\leq \sum_{l=1}^{\xi} \sum_{ij \in I_{sl}} \sum_{j=1}^{\theta} \nu(\mathscr{B}(x_{ij}, \rho_s))^q \\
&\leq \sum_{l=1}^{\xi} \sum_{j=1}^{\theta} \mathscr{M}^{q,p}_{\nu,\rho_s}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho_s)) \\
&= \xi\theta M^{q,p}_{\nu,\rho_s}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho_s)) \qquad (6.10)
\end{aligned}$$

for all $s \in \mathbb{N}$ and $q \in \mathbb{R}$.

It now follows from (6.7), (6.9) and (6.10) that

$$\begin{aligned}
\sum_i \nu(\mathscr{B}(x_i, \rho_i))^q (2\rho_i)^{\varphi_n(q)+\varepsilon} &= \sum_s \sum_{\rho_{s+1} \leq \rho_i < \rho_s} \nu(\mathscr{B}(x_i, \rho_i))^q (2\rho_i)^{\varphi_n(q)+\varepsilon} \\
&\leq \sum_s \sum_{\rho_{s+1} \leq \rho_{ij} < \rho_s} \sum_{j=1}^{\theta} \nu(\mathscr{B}(x_{ij}, \rho_{ij}))^q (2\rho_{ij})^{\varphi_n(q)+\varepsilon} \\
&\leq \begin{cases} c \displaystyle\sum_s \sum_{\rho_{s+1} \leq \rho_{ij} < \rho_s} \sum_{j=1}^{\theta} \nu(\mathscr{B}(x_{ij}, \rho_{s+1}))^q \rho_{s+1}^{\varphi_n(q)+\varepsilon} & \text{if } q \leq 0, \\ c \displaystyle\sum_s \sum_{\rho_{s+1} \leq \rho_{ij} < \rho_s} \sum_{j=1}^{\theta} \nu(\mathscr{B}(x_{ij}, \rho_s))^q \rho_s^{\varphi_n(q)+\varepsilon} & \text{if } q \geq 0, \end{cases} \\
&\leq \begin{cases} c\theta \displaystyle\sum_s \mathscr{M}^{q,p}_{\nu,\rho_{s+1}}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho_{s+1})) \rho_{s+1}^{\varphi_n(q)+\varepsilon} & \text{if } q \leq 0, \\ e\theta \displaystyle\sum_s \xi \mathscr{M}^{q,p}_{\nu,\rho_s}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \rho_s)) \rho_s^{\varphi_n(q)+\varepsilon} & \text{if } q \geq 0, \end{cases} \\
&\leq c\theta\xi \sum_s \rho_s^{\varphi_n(q)+\varepsilon} Q^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho_s) \leq c\theta\xi M \sum_s \rho_s^{\varphi_n(q)+\varepsilon} \rho_s^{-\varphi_n(q)} \\
&= c\theta\xi M \sum_s \frac{\delta^\varepsilon}{2^{\varepsilon(s-1)}} = c\theta\xi M \frac{2^\varepsilon}{2^\varepsilon - 1} \delta^\varepsilon = \mathscr{A}^* \delta^\varepsilon,
\end{aligned}$$

where $\mathscr{A}^* = c\theta\xi M \frac{2^\varepsilon}{2^\varepsilon - 1} < \infty$. We therefore deduce that

$$\overline{\mathscr{H}}^{q,\varphi_n(q)+\varepsilon}_{\nu,\delta}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) \leq \mathscr{A}^* \delta^\varepsilon,$$

and

$$\mathscr{H}^{q,\varphi_n(q)+\varepsilon}_{\nu}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) = 0.$$

This completes the proof of assertion (2)(a).

(b) The proof is identical to that of (2)(a), so it is omitted. ∎

6.2. Measure of overlaps and Jacobians. The objective of this section is to demonstrate that the overlaps $f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})$ have negligible $\mathscr{P}_\nu^{q,\overline{\beta}(q)}$ and $\mathscr{H}_\nu^{q,\underline{\beta}(q)}$ measures for all $\boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T}$ with $|\boldsymbol{\alpha}| = |\boldsymbol{\sigma}|$ and $\boldsymbol{\alpha} \neq \boldsymbol{\sigma}$, and to establish the almost sure existence of the Jacobian of $f_{\boldsymbol{\alpha}}(\mathscr{K})$ and determine its almost sure value (Propositions 6.10 and 6.11). Let $q \in \mathbb{R}$ and $E \subseteq \mathbb{R}^d$. Given a locally finite measure ν on $\mathbb{R}^d$ and a bi-measurable mapping $f : \mathbb{R}^d \to \mathbb{R}^d$, the *lower* and *upper qth order Jacobians* of f on E with respect to ν are defined as follows:

$$\underline{J}_\nu^q(f, E) = \liminf_{\rho \to 0} \inf_{x \in E} \left(\frac{\nu f(\mathscr{B}(x,\rho))}{\nu \mathscr{B}(x,\rho)} \right)^q,$$

$$\overline{J}_\nu^q(f, E) = \limsup_{\rho \to 0} \sup_{x \in E} \left(\frac{\nu f(\mathscr{B}(x,\rho))}{\nu \mathscr{B}(x,\rho)} \right)^q.$$

If $\underline{J}_\nu^q(f, E) = \overline{J}_\nu^q(f, E)$ are identical, we denote this common value by $J_\nu^q(f, E)$ and refer to it as the *qth order Jacobian* of f on E with respect to ν. The significance of these Jacobians lies in their role in determining the scaling characteristics of $\mathscr{H}_\nu^{q,t}$ and $\mathscr{P}_\nu^{q,t}$. This is formally stated in the following lemma.

LEMMA 6.9. *Let ν denote a probability measure on $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a similarity map, meaning there exists a constant $C \in (0, \infty)$ such that $|f(x) - f(y)| = C|x - y|$ for all $x, y \in \mathbb{R}^d$. Suppose that $f(\operatorname{supp} \nu) \subseteq \operatorname{supp} \nu$. Let $q, t \in \mathbb{R}$ and $E \subseteq \operatorname{supp} \nu$. Then*

(1) $\underline{J}_\nu^q(f, E) C^t \mathscr{P}_\nu^{q,t}(E) \le \mathscr{P}_\nu^{q,t}(fE) \le \overline{J}_\nu^q(f, E) C^t \mathscr{P}_\nu^{q,t}(E)$,
(2) $\underline{J}_\nu^q(f, E) C^t \mathscr{H}_\nu^{q,t}(E) \le \mathscr{H}_\nu^{q,t}(fE) \le \overline{J}_\nu^q(f, E) C^t \mathscr{H}_\nu^{q,t}(E)$.

Proof. This follows straightforwardly from the definitions (see also [23]). ∎

PROPOSITION 6.10. *Assume that $\{f_{n,j} \mid n \ge 1,\ 1 \le j \le a_n\}$ satisfies the* (SOSC) *and let $q \in \mathbb{R}$. Denote $\dim_\nu^q(\mathscr{K}) = \underline{\beta}(q) = t$ and $\operatorname{Dim}_\nu^q(\mathscr{K}) = \overline{\beta}(q) = s$.*

(1) *For $\boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T}$ with $|\boldsymbol{\alpha}| = |\boldsymbol{\sigma}|$ and $\boldsymbol{\alpha} \neq \boldsymbol{\sigma}$, we have*

$$\mathscr{H}_\nu^{q,t}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) = \mathscr{P}_\nu^{q,s}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) = 0.$$

(2) *For $q \ge 0$ and all $\boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T}$ with $|\boldsymbol{\alpha}| = |\boldsymbol{\sigma}|$ and $\boldsymbol{\alpha} \neq \boldsymbol{\sigma}$, we have*

$$\mathscr{H}_\nu^{q,t}(f_{\boldsymbol{\alpha}}^{-1}(f_{\boldsymbol{\sigma}}(\mathscr{K}))) = \mathscr{P}_\nu^{q,s}(f_{\boldsymbol{\alpha}}^{-1}(f_{\boldsymbol{\sigma}}(\mathscr{K}))) = 0.$$

Proof. (1) Use the same technique as in Theorem 6.2.

(2) By making use of Lemma 6.9, we obtain

$$\begin{aligned}\mathscr{P}_\nu^{q,s}(f_{\boldsymbol{\alpha}}^{-1}(f_{\boldsymbol{\sigma}}(\mathscr{K}))) &= \mathscr{P}_\nu^{q,s}(f_{\boldsymbol{\alpha}}^{-1}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K}))) \\ &\le |I_\alpha|^{-s} \overline{J}_\nu^q(f_{\boldsymbol{\alpha}}^{-1}, \mathscr{K}) \mathscr{P}_\nu^{q,s}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})).\end{aligned}$$

It now suffices to show that $\overline{J}_\nu^q(f_{\boldsymbol{\alpha}}^{-1}, \mathscr{K}) < \infty$ (because $\mathscr{P}_\nu^{q,s}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})) = 0$). So

$$\begin{aligned}\overline{J}_\nu^q(f_{\boldsymbol{\alpha}}^{-1}, \mathscr{K}) &= \limsup_{\rho \to 0} \sup_{x \in \mathscr{K}} \left(\frac{\nu(f_{\boldsymbol{\alpha}}^{-1} \mathscr{B}(x,\rho))}{\nu \mathscr{B}(x,\rho)} \right)^q \\ &= \limsup_{\rho \to 0} \sup_{x \in \mathscr{K}} \left(\frac{p_{\boldsymbol{\alpha}} \nu(f_{\boldsymbol{\alpha}}^{-1} \mathscr{B}(x,\rho))}{\sum_{|\boldsymbol{\sigma}| = |\boldsymbol{\alpha}|} p_{\boldsymbol{\sigma}} \nu(f_{\boldsymbol{\sigma}}^{-1} \mathscr{B}(x,\rho))} \right)^q p_{\boldsymbol{\alpha}}^{-q} \le p_{\boldsymbol{\alpha}}^{-q} < \infty. \ ∎\end{aligned}$$

PROPOSITION 6.11. *Assume that* $\{f_{n,j} \mid n \geq 1,\, 1 \leq j \leq a_n\}$ *satisfies the* (SOSC). *Let* $q \in \mathbb{R}$ *and* $\boldsymbol{\alpha} \in \mathscr{T}$. *Denote* $\dim_\nu^q(\mathscr{K}) = \underline{\beta}(q) = t$ *and* $\mathrm{Dim}_\nu^q(\mathscr{K}) = \overline{\beta}(q) = s$. *For* $\delta > 0$ *write*

$$\Delta_{\boldsymbol{\alpha},\delta} = \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\delta).$$

(1) *If* $\mathscr{H}_\nu^{q,t}(\mathscr{K}) \leq \mathscr{P}_\nu^{q,s}(\mathscr{K}) < \infty$, *then*

$$\lim_{\delta\to 0} \mathscr{H}_\nu^{q,t}(\Delta_{\boldsymbol{\alpha},\delta}) = 0, \qquad \lim_{\delta\to 0} \mathscr{P}_\nu^{q,s}(\Delta_{\boldsymbol{\alpha},\delta}) = 0.$$

(2) *If* $\mathscr{H}_\nu^{q,t}(\mathscr{K}) \leq \mathscr{P}_\nu^{q,s}(\mathscr{K}) < \infty$. *Then*

$$\lim_{\delta\to 0} \mathscr{H}_\nu^{q,t}(f_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\alpha},\delta}) = 0, \qquad \lim_{\delta\to 0} \mathscr{P}_\nu^{q,s}(f_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\alpha},\delta}) = 0.$$

(3) *For each* $\delta > 0$,

$$\underline{J}_\nu^q(f_{\boldsymbol{\alpha}}, \mathscr{K} \setminus \Delta_{\boldsymbol{\alpha},\delta}) = \overline{J}_\nu^q(f_{\boldsymbol{\alpha}}, \mathscr{K} \setminus \Delta_{\boldsymbol{\alpha},\delta}) = p_{\boldsymbol{\alpha}}^q.$$

Proof. (1) Since

$$\lim_{\delta\to 0} \Delta_{\boldsymbol{\alpha},\delta} = \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} f_{\boldsymbol{\alpha}}^{-1} f_{\boldsymbol{\sigma}}(\mathscr{K}),$$

one can use Proposition 6.10(2).

(2) Since

$$\lim_{\delta\to 0} f_{\boldsymbol{\alpha}}(\Delta_{\boldsymbol{\alpha},\delta}) = \lim_{\delta\to 0} \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} (f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\delta)) = \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} (f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})),$$

one can use Proposition 6.10(1).

(3) Let $x \in \mathscr{K} \setminus \Delta_{\boldsymbol{\alpha},\delta}$. Hence

$$f_{\boldsymbol{\alpha}}(x) \notin \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\delta),$$

then

$$f_{\boldsymbol{\alpha}}\mathscr{B}(x,\rho) \cap \mathscr{K} \subseteq f_{\boldsymbol{\alpha}}(\mathscr{K}) \setminus \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} f_{\boldsymbol{\sigma}}(\mathscr{K})$$

for $0 < \rho < \delta$, and consequently

$$\frac{\nu f_{\boldsymbol{\alpha}}\mathscr{B}(x,\rho)}{\nu\mathscr{B}(x,\rho)} = \frac{\sum_{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|} \nu(f_{\boldsymbol{\sigma}}^{-1} f_{\boldsymbol{\alpha}}\mathscr{B}(x,\rho))}{\nu\mathscr{B}(x,\rho)} = \frac{p_{\boldsymbol{\alpha}}\nu\mathscr{B}(x,\rho)}{\nu\mathscr{B}(x,\rho)} = p_{\boldsymbol{\alpha}}$$

for all $0 < \rho < \delta$. ∎

6.3. Proof of Theorem 3.1 in the case of (SOSC) and $q \geq 0$. This proof mirrors that of Theorem 3.1 under the (SSC) assumption, with the only difference being the use of the following lemma in place of Lemma 3.7.

Lemma 6.12. *Suppose that* $\{f_{n,j} \mid n \geq 1,\, 1 \leq j \leq a_n\}$ *satisfies the* (SOSC). *Fix* $q \geq 0$. *Consider the functions* $h_{1,n}, h_{2,n}$ *defined as follows:*

$$h_{1,n}(Y) = e^{-Y\beta_n(q)}\Big(\mathscr{M}^{q,c}_{\nu,e^{-Y}}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,c}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}e^{-Y}}(\mathscr{K})\Big),$$
$$h_{2,n}(Y) = e^{-Y\beta_n(q)}\Big(\mathscr{M}^{q,p}_{\nu,e^{-Y}}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,p}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}e^{-Y}}(\mathscr{K})\Big).$$

Then there exists $C > 0$ *such that*

$$|h_{1,n}(Y)| \leq Ce^{-T(\beta_n(q)-\varphi_n(q))}, \qquad |h_{2,n}(Y)| \leq Ce^{-T(\beta_n(q)-\varphi_n(q))}$$

for all $Y > 0$. *Specifically, if* $\beta_n(q) - \varphi_n(q) > 0$, *we can deduce from the preceding remark on the Renewal Theorem that both* h_1 *and* h_2 *are directly Riemann integrable.*

Proof. This can be immediately derived from the following lemma, along with Propositions 6.7 and 6.8. ■

Lemma 6.13. *Suppose that* $\{f_{n,j} \mid n \geq 1,\, 1 \leq j \leq a_n\}$ *satisfies the* (SOSC). *Let* $q \geq 0$ *and* $n \geq 1$.

(1) *For all* $\rho > 0$ *we have*

$$\Big|\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,p}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}\rho}(\mathscr{K})\Big| \leq \sum_{|\boldsymbol{\alpha}|=n}\sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho).$$

(2) *For all* $\rho > 0$ *we have*

$$\begin{aligned}\Big|\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,c}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}\rho}(\mathscr{K})\Big| &\leq \sum_{|\boldsymbol{\alpha}|=n}\sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{P}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho)\\ &\leq 5^d\sum_{|\boldsymbol{\alpha}|=n}\sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} \mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho).\end{aligned}$$

Proof. (1) can be readily derived from Remark 3.6. The proof of the first inequality in (2) is analogous to the proofs of Lemmas 3.4 and 3.5, and is therefore omitted. To establish the second inequality of (2), it is sufficient to demonstrate that $\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K}) \leq 5^d\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})$ for all sets $\mathscr{K}$ and all $\rho > 0$. It can be observed that there exists a family $B = (\mathscr{B}(x_i,\rho))_i$ of balls with $x_i \in \mathscr{K}$ and $\mathscr{K}$ contained in $\bigcup_i \mathscr{B}(x_i,\rho)$ such that B can be divided into 5^d subfamilies $B_1 = (\mathscr{B}(x_{1i},\rho))_i, \dots, B_{5^d} = (\mathscr{B}(x_{5^d i},\rho))_i$ where each B_s consists of mutually disjoint balls. Consequently,

$$\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K}) \leq \sum_i \nu(\mathscr{B}(x_i,\rho))^q = \sum_{s=1}^{5^d}\sum_i \nu(\mathscr{B}(x_{si},\rho))^q \leq \sum_{s=1}^{5^d}\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K}) = 5^d\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K}). \ ■$$

6.4. Proof of Theorem 3.1 in the case of (SOSC) and $q < 0$. Put first

$$\eta = \min_{n\geq 1,\, 1\leq j\leq a_n}\frac{\log p_{n,j}}{\log b_{n,j}}, \qquad \tau = \max_{n\geq 1,\, 1\leq j\leq a_n}\frac{\log p_{n,j}}{\log b_{n,j}}.$$

THEOREM 6.14. *Suppose a list* $\{f_{n,j} \mid n \geq 1,\, 1 \leq j \leq a_n\}$ *of contracting similarities satisfies the* (SOSC)*. Assume* $q < 0$ *is such that* $\beta_n(q) > \varphi_n(q) + (\eta - \tau)q$ *for all* $n \in \mathbb{N}$*. Let* $\boldsymbol{\alpha} \in \mathscr{T}$*.*

(1) *The arithmetic case: If* $\big\{\log\big(\frac{1}{b_{1,\alpha_1}}\big), \ldots, \log\big(\frac{1}{b_{n,\alpha_n}}\big)\big\}$ *is contained in a discrete* (*additive*) *subgroup of* $\mathbb{R}$*, then there exist multiplicatively periodic functions* $\overline{\pi}_q, \underline{\pi}_q, \overline{\Pi}_q, \underline{\Pi}_q : (0,\infty) \to \mathbb{R}$ *with period* e^θ *satisfying*

$$\overline{\pi}_q(e^{\pm\theta}\rho) = \overline{\pi}_q(\rho), \quad \underline{\pi}_q(e^{\pm\theta}\rho) = \underline{\pi}_q(\rho), \quad \overline{\Pi}_q(e^{\pm\theta}\rho) = \overline{\Pi}_q(\rho), \underline{\Pi}_q(e^{\pm\theta}\rho) = \underline{\Pi}_q(\rho)$$

for all $\rho > 0$*, such that*

$$\frac{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}} = \overline{\pi}_q(\rho) + \epsilon(\rho), \qquad \frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}} = \overline{\Pi}_q(\rho) + \epsilon(\rho),$$
$$\frac{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}{\rho^{-\underline{\beta}(q)}} = \underline{\pi}_q(\rho) + \epsilon(\rho), \qquad \frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\underline{\beta}(q)}} = \underline{\Pi}_q(\rho) + \epsilon(\rho),$$

where $\epsilon(\rho) \to 0$ *as* $\rho \to 0$*.*

(2) *The non-arithmetic case: If* $\big\{\log\big(\frac{1}{b_{1,\alpha_1}}\big), \ldots, \log\big(\frac{1}{b_{n,\alpha_n}}\big)\big\}$ *is not contained in a discrete* (*additive*) *subgroup of* $\mathbb{R}$*, then there exist constants* $\overline{c}, \underline{c}, \overline{s}, \underline{s} \in \mathbb{R}$ *such that*

$$\frac{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}} = \overline{c} + \epsilon(\rho), \qquad \frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}} = \overline{s} + \epsilon(\rho),$$
$$\frac{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}} = \underline{c} + \epsilon(\rho), \qquad \frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\overline{\beta}(q)}} = \underline{s} + \epsilon(\rho),$$

where $\epsilon(\rho) \to 0$ *as* $\rho \to 0$*.*

In order to prove Theorem 6.14, we first derive the following auxiliary results.

PROPOSITION 6.15. *Suppose that* $\{f_{n,j} \mid n \geq 1,\, 1 \leq j \leq a_n\}$ *satisfies the* (SOSC)*. For each* $\varepsilon > 0$ *there exist positive constants* m_ε *and* ρ_ε *such that*

$$\frac{1}{m_\varepsilon}\rho^{\tau+\epsilon} \leq \nu(\mathscr{B}(x,\rho)) \leq m_\varepsilon \rho^{\eta-\varepsilon}$$

for all $x \in \mathscr{K}$ *and all* $0 < \rho < \rho_\varepsilon$*.*

Proof. First, note that

$$\eta \leq \frac{\log p_{\boldsymbol{\alpha}}}{|I_{\boldsymbol{\alpha}}|} \leq \tau \tag{6.11}$$

for all $\boldsymbol{\sigma} \in \mathscr{T}$.

We can safely assume that $|\mathscr{K}| = 1$. Select $\rho_\varepsilon > 0$ such that for $0 < \rho < \rho_\varepsilon$, the condition

$$1 - \frac{\log b_{\min}}{\log \rho} \geq \frac{1}{1 + \frac{\varepsilon}{\tau}}$$

holds. Fix $x \in \mathscr{K}$ and choose $\boldsymbol{\alpha} \in \mathscr{T}_n$ such that $\boldsymbol{\pi}(\boldsymbol{\alpha}) = x$ (where $\boldsymbol{\pi} : T_n \to \mathbb{R}^d$, $\{\boldsymbol{\pi}(\boldsymbol{\alpha})\} = \bigcap_n f_{\boldsymbol{\alpha}|n}(\mathscr{K})$). Then, fix $0 < \rho < \rho_\varepsilon$. Finally, denote by n the unique positive integer such that $|I_{\boldsymbol{\alpha}|n}| \leq \rho < |I_{\boldsymbol{\alpha}|(n-1)}|$. Notice that since $x = \boldsymbol{\pi}(\boldsymbol{\alpha}) \in f_{\boldsymbol{\alpha}|n}(\mathscr{K})$ and $|f_{\boldsymbol{\alpha}|n}(\mathscr{K})| = |I_{\boldsymbol{\alpha}|n}|\,|\mathscr{K}| = |I_{\boldsymbol{\alpha}|n}| \leq \rho$ (given our assumption $|\mathscr{K}| = 1$), it follows that

$f_{\boldsymbol{\alpha}|n}(\mathscr{K}) \subseteq \mathscr{B}(x,\rho)$. Consequently (by Lemma 6.4),

$$p_{\boldsymbol{\alpha}|n} = \nu(f_{\boldsymbol{\alpha}|n}(\mathscr{K})) \le \nu(\mathscr{B}(x,\rho)).$$

We conclude from this and (6.11) that

$$\tau \ge \frac{p_{\boldsymbol{\alpha}|n}}{\log|I_{\boldsymbol{\alpha}|n}|} \ge \frac{\log\nu(\mathscr{B}(x,\rho))}{\log|I_{\boldsymbol{\alpha}|n}|} = \frac{\log\rho}{\log|I_{\boldsymbol{\alpha}|n}|}\,\frac{\log\nu(\mathscr{B}(x,\rho))}{\log\rho}. \tag{6.12}$$

Rearranging the inequality

$$\rho \le |I_{\boldsymbol{\alpha}|(n-1)}| \le \frac{|I_{\boldsymbol{\alpha}|n}|}{b_{\min}}$$

yields

$$\frac{\log\rho}{\log|I_{\boldsymbol{\alpha}|n}|} \ge 1 - \frac{\log b_{\min}}{\log|I_{\boldsymbol{\alpha}|n}|}.$$

This, in conjunction with (6.12), yields

$$\tau \ge \left(1 - \frac{\log b_{\min}}{\log|I_{\boldsymbol{\alpha}|n}|}\right)\frac{\log\nu(\mathscr{B}(x,\rho))}{\log\rho}. \tag{6.13}$$

Furthermore, rearranging the inequality $|I_{\boldsymbol{\alpha}|n}| \le \rho$ reveals that

$$1 - \frac{\log b_{\min}}{\log|I_{\boldsymbol{\alpha}|n}|} \ge 1 - \frac{\log b_{\min}}{\log\rho} \ge \frac{1}{1+\frac{\varepsilon}{\tau}}.$$

Therefore, we deduce from (6.13) that

$$\tau \ge \frac{1}{1+\frac{\varepsilon}{\tau}}\,\frac{\log\nu(\mathscr{B}(x,\rho))}{\log\rho},$$

Then $\rho^{\tau+\varepsilon} \le \nu(\mathscr{B}(x,\rho))$ for all $x \in \mathscr{K}$ and all $0 < \rho < \rho_\varepsilon$.

For $x \in \mathscr{K}$ and $\rho > 0$ write

$$\mathscr{A}(x,\rho) = \{\boldsymbol{\alpha} \in \mathscr{T} \mid |I_{\boldsymbol{\alpha}}| \le \rho < |I_{\boldsymbol{\alpha}|(|\boldsymbol{\alpha}|-1)}|,\ f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(x,\rho) \ne \emptyset\}.$$

So there is a constant $c > 0$ such that

$$|\mathscr{A}(x,\rho)| \le c \tag{6.14}$$

for all $x \in \mathscr{K}$ and $\rho > 0$. Indeed, let O be the open set in the (SOSC). Put now

$$\mathscr{B}(x,\rho) = \{\boldsymbol{\alpha} \in \mathscr{T} \mid |I_{\boldsymbol{\alpha}}| \le \rho < |I_{\boldsymbol{\alpha}|(|\boldsymbol{\alpha}|-1)}|,\ f_{\boldsymbol{\alpha}}(\overline{O}) \cap \mathscr{B}(x,\rho) \ne \emptyset\}.$$

For $x \in \mathbb{K}$ and $\rho > 0$, it is well known that $f_{\boldsymbol{\alpha}}(\mathscr{K}) \subseteq f_{\boldsymbol{\alpha}}(\overline{O})$ for all $\boldsymbol{\alpha} \in \mathscr{T}$, implying $\mathscr{A}(x,\rho) \subseteq \mathscr{B}(x,\rho)$. Given that O is non-empty, bounded and open, there exist constants $m_1, m_2 > 0$ such that O contains a ball of radius m_1 and is contained in a ball of radius m_2. Therefore, if $\boldsymbol{\alpha} \in \mathscr{B}(x,\rho)$, then $f_{\boldsymbol{\alpha}}(O)$ contains a ball of radius $|I_{\boldsymbol{\alpha}}|m_1$. Since $|I_{\boldsymbol{\alpha}}|m_1 \ge |I_{\boldsymbol{\alpha}|(|\boldsymbol{\alpha}|-1)}|b_{\min}m_1 \ge (b_{\min}m_1)\rho$, we infer that $f_{\boldsymbol{\alpha}}(O)$ contains a ball of radius $(b_{\min}m_1)\rho$. Similarly, if $\boldsymbol{\alpha} \in \mathscr{B}(x,\rho)$, then $f_{\boldsymbol{\alpha}}(O)$ is contained in a ball of radius $|I_{\boldsymbol{\alpha}}|m_2$. Given $|I_{\boldsymbol{\alpha}}|m_2 \le m_2\rho$, it follows that $f_{\boldsymbol{\alpha}}(O)$ is contained in a ball of radius $m_2\rho$. Furthermore, since the sets $(f_{\boldsymbol{\alpha}}(O))_{\boldsymbol{\alpha}\in\mathscr{B}(x,\rho)}$ are pairwise disjoint (due to $f_i(O) \cap f_j(O) = \emptyset$ for $i \ne j$), Lemma 6.5 implies

$$|\mathscr{A}(x,\rho)| \le |\mathscr{B}(x,\rho)| \le \left(\frac{1+2m_2}{b_{\min}m_1}\right)^d.$$

This completes the proof of (6.14).

Notice that for every $x \in \mathscr{K}$ and $\rho > 0$, we have

$$\mathscr{B}(x,\rho) \subseteq \bigcup_{\boldsymbol{\alpha}\in\mathscr{B}(x,\rho)} f_{\boldsymbol{\alpha}}(\mathscr{K}).$$

Consequently, from (6.14) and Lemma 6.4, we derive

$$\begin{aligned}\nu(\mathscr{B}(x,\rho)) &\le \sum_{\boldsymbol{\alpha}\in\mathscr{B}(x,\rho)} \nu(f_{\boldsymbol{\alpha}}(\mathscr{K})) = \sum_{\boldsymbol{\alpha}\in\mathscr{B}(x,\rho)} p_{\boldsymbol{\alpha}} \\ &\le |\mathscr{B}(x,\rho)| \sup_{\boldsymbol{\alpha}\in\mathscr{B}(x,\rho)} p_{\boldsymbol{\alpha}} \le c \sup_{\boldsymbol{\alpha}\in\mathscr{B}(x,\rho)} p_{\boldsymbol{\alpha}}.\end{aligned}$$

From this and (6.11) we obtain

$$\begin{aligned}\frac{\log\nu(\mathscr{B}(x,\rho))}{\log\rho} &\ge \frac{\log c}{\log\rho} + \frac{\sup_{\boldsymbol{\alpha}\in\mathscr{B}(x,\rho)}\log p_{\boldsymbol{\alpha}}}{\log\rho} = \frac{\log c}{\log\rho} + \inf_{\boldsymbol{\alpha}\in\mathscr{B}(x,\rho)} \frac{\log p_{\boldsymbol{\alpha}}}{\log\rho} \\ &= \frac{\log c}{\log\rho} + \inf_{\boldsymbol{\alpha}\in\mathscr{B}(x,\rho)} \frac{\log p_{\boldsymbol{\alpha}}}{\log|I_{\boldsymbol{\alpha}}|}\frac{\log|I_{\boldsymbol{\alpha}}|}{\log\rho} \ge \frac{\log c}{\log\rho} + \eta \inf_{\boldsymbol{\alpha}\in\mathscr{B}(x,\rho)} \delta\frac{\log|I_{\boldsymbol{\alpha}}|}{\log\rho}\end{aligned}$$

for all $x \in \mathscr{K}$ and all $0 < \rho < 1$. However, if $\boldsymbol{\alpha} \in \mathscr{B}(x,\rho)$, then $|I_{\boldsymbol{\alpha}}| \le \rho$, and consequently $\frac{\log|I_{\boldsymbol{\alpha}}|}{\log\rho} \ge 1$. Therefore, we deduce from the last inequality that

$$\frac{\log\nu(\mathscr{B}(x,\rho))}{\log\rho} \ge \frac{\log c}{\log\rho} + \eta.$$

Hence $\nu(\mathscr{B}(x,\rho)) \le c\rho^{\eta} \le c\rho^{\eta-\varepsilon}$ for all $x \in \mathscr{K}$ and all $0 < \rho < 1$. This completes the proof. ∎

LEMMA 6.16. *Suppose that* $\{f_{n,j} \mid n \ge 1,\ 1 \le j \le a_n\}$ *satisfies the* (SOSC). *Fix* $q < 0$ *and* $n \in \mathbb{N}$. *Suppose there exist positive constants* b, B, c_0 *and* ρ_0 *where* $b \le B$ *such that*

$$\frac{1}{c_0}\rho^B \le \nu(\mathscr{B}(x,\rho)) \le c_0\rho^b$$

for all $x \in \mathscr{K}$ *and all* $0 < \rho < \rho_0$.

(1) *There is a positive constant* c_n *such that for* $\boldsymbol{\alpha} \in \mathscr{T}$ *with* $|\alpha| = n$ *and* $0 < \rho < \rho_0$, *we have*

$$-c_n\rho^{(B-b)q} \sum_{\substack{|\boldsymbol{\sigma}|=n \\ \boldsymbol{\alpha}\ne\boldsymbol{\sigma}}} \mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho) + p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,p}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}) \le \mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})).$$

(2) *There is a positive constant* c_n *such that for* $\boldsymbol{\alpha} \in \mathscr{T}$ *with* $|\alpha| = n$ *and* $0 < \rho < \rho_0$, *we have*

$$-c_n\rho^{(B-b)q} \sum_{\substack{|\boldsymbol{\sigma}|=n \\ \boldsymbol{\alpha}\ne\boldsymbol{\sigma}}} \mathscr{P}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho) + p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,c}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}) \le \mathscr{M}^{q,c}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})).$$

Proof. (1) Let $\rho > 0$ and let F be an $|I_{\boldsymbol{\alpha}}|^{-1}\rho$-separated subset of $\mathscr{K}$. Write

$$R = f_{\boldsymbol{\alpha}}F \setminus \bigcup_{\substack{|\boldsymbol{\sigma}|=n \\ \boldsymbol{\alpha}\ne\boldsymbol{\sigma}}} \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho), \qquad H = f_{\boldsymbol{\alpha}}F \cap \bigcup_{\substack{|\boldsymbol{\sigma}|=n \\ \boldsymbol{\alpha}\ne\boldsymbol{\sigma}}} \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho).$$

Note that $f^{-1}_{\boldsymbol{\sigma}}\mathscr{B}(x,\rho) = \emptyset$ for all $x \in R$ and for all $\boldsymbol{\sigma} \in \mathscr{T}$ with $|\boldsymbol{\sigma}| = n$ and $\boldsymbol{\alpha} \ne \boldsymbol{\sigma}$. We obtain

$$\nu(\mathscr{B}(x,\rho)) = \sum_{\boldsymbol{\sigma}} p_{\boldsymbol{\sigma}}\nu(f^{-1}_{\boldsymbol{\sigma}}\mathscr{B}(x,\rho)) = p_{\boldsymbol{\alpha}}\nu(f^{-1}_{\boldsymbol{\alpha}}\mathscr{B}(x,\rho))$$

for all $x \in R$. This implies that

$$\begin{aligned}
\mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})) &\geq \sum_{x\in R} \nu(\mathscr{B}(x,\rho))^q + \sum_{x\in H} \nu(\mathscr{B}(x,\rho))^q \\
&= \sum_{x\in R} (p_{\boldsymbol{\alpha}} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho)))^q + \sum_{x\in H} \nu(\mathscr{B}(x,\rho))^q \\
&= \sum_{x\in R\cup H} p_{\boldsymbol{\alpha}}^q \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q - \sum_{x\in H} p_{\boldsymbol{\alpha}}^q \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q + \sum_{x\in H} \nu(\mathscr{B}(x,\rho))^q \\
&\geq p_{\boldsymbol{\alpha}}^q \sum_{x\in f_{\boldsymbol{\alpha}} F} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q - p_{\boldsymbol{\alpha}}^q \sum_{x\in H} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q) \\
&= p_{\boldsymbol{\alpha}}^q \sum_{x\in F} \nu(\mathscr{B}(x,|I_{\boldsymbol{\alpha}}|^{-1}\rho))^q - p_{\boldsymbol{\alpha}}^q \sum_{x\in H} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q. \qquad (6.15)
\end{aligned}$$

Next, we evaluate $p_{\boldsymbol{\alpha}}^q \sum_{x\in H} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q$. Specifically, we will demonstrate the existence of a constant c_n satisfying

$$p_{\boldsymbol{\alpha}}^q \sum_{x\in H} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q \leq c_n \rho^{(B-b)q} \sum_{\substack{|\boldsymbol{\sigma}|=n \\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}} \mathscr{Q}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho). \qquad (6.16)$$

To establish (6.16), our initial step involves recognizing H as a ρ-separated set satisfying

$$H \subseteq \bigcup_{\substack{|\boldsymbol{\sigma}|=n \\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}} f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho). \qquad (6.17)$$

Consider x and y within H, which is contained in $f_{\boldsymbol{\alpha}}(F)$. Consequently, we can find x_0 and y_0 from F such that $f_{\boldsymbol{\alpha}}(x_0) = x$ and $f_{\boldsymbol{\alpha}}(y_0) = y$. Thus,

$$|x-y| = |I_{\boldsymbol{\alpha}}|\,|x_0-y_0| > 2|I_{\boldsymbol{\alpha}}|\,|I_{\boldsymbol{\alpha}}|^{-1}\rho = 2\rho$$

(since F constitutes an $|I_{\boldsymbol{\alpha}}|^{-1}\rho$-separated set). Hence, H is a ρ-separated set. Now (6.17) naturally ensues from

$$H = f_{\boldsymbol{\alpha}}(F) \cap \bigcup_{\substack{|\boldsymbol{\sigma}|=n \\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}} \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho) \subseteq \bigcup_{\substack{|\boldsymbol{\sigma}|=n \\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}} f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho).$$

Subsequently, we observe that for all x and all $\boldsymbol{\sigma} \in \mathscr{T}$ with $|\boldsymbol{\sigma}| = n$,

$$\frac{\nu(f_{\boldsymbol{\sigma}}^{-1}\mathscr{B}(x,\rho))}{\nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))} = \frac{\nu\mathscr{B}(f_{\boldsymbol{\sigma}}^{-1}x,|I_{\boldsymbol{\sigma}}|^{-1}\rho)}{\nu\mathscr{B}(f_{\boldsymbol{\alpha}}^{-1}x,|I_{\boldsymbol{\alpha}}|^{-1}\rho)} \leq c_0^2 \frac{|I_{\boldsymbol{\sigma}}|^{-b}\rho^b}{|I_{\boldsymbol{\alpha}}|^{-B}\rho^B} \leq c_0^2 \left(\frac{b_{\max}^B}{b_{\min}^b}\right)^n \rho^{-(B-b)}.$$

This implies that for all x we have

$$\begin{aligned}
\nu(\mathscr{B}(x,\rho)) &= \sum_{|\boldsymbol{\sigma}|=n} p_{\boldsymbol{\sigma}} \nu(f_{\boldsymbol{\sigma}}^{-1}\mathscr{B}(x,\rho)) \\
&\leq c_0^2 \left(\frac{b_{\max}^B}{b_{\min}^b}\right)^n \sum_{|\boldsymbol{\sigma}|=n} p_{\boldsymbol{\sigma}} \rho^{-(B-b)} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho)) \\
&= c_0^2 \left(\frac{b_{\max}^B}{b_{\min}^b}\right)^n \rho^{-(B-b)} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho)).
\end{aligned}$$

Since $q < 0$, we deduce from the last inequality that for all x,

$$\nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q \leq c_0^{-2q}\left(\frac{b_{\max}^B}{b_{\min}^b}\right)^n \rho^{(B-b)q}\nu(\mathscr{B}(x,\rho))^q. \tag{6.18}$$

Combining (6.17) and (6.18) we obtain

$$\begin{aligned} p_{\boldsymbol{\alpha}}^q \sum_{x\in H} \nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q &\leq c_n\rho^{(B-b)q}\sum_{x\in H}\nu(\mathscr{B}(x,\rho))^q \\ &\leq c_n\rho^{(B-b)q}\mathscr{M}_{\nu,\rho}^{q,p}\Big(\bigcup_{\substack{|\boldsymbol{\sigma}|=n\\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}} f_{\boldsymbol{\alpha}}(\mathscr{K})\cap\mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho)\Big) \\ &\leq c_n\rho^{(B-b)q}\sum_{\substack{|\boldsymbol{\sigma}|=n\\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}}\mathscr{M}_{\nu,\rho}^{q,p}(f_{\boldsymbol{\alpha}}(\mathscr{K})\cap\mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}),\rho)) \\ &= c_n\rho^{(B-b)q}\sum_{\substack{|\boldsymbol{\sigma}|=n\\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}}\mathscr{Q}_{\boldsymbol{\alpha},\boldsymbol{\sigma}}^q(\rho), \end{aligned}$$

where $c_n = c_0^{-2q}(b_{\max}^B/b_{\min}^b)^{-nq}(\min_i p_i)^{nq}$. This proves (6.16).

Finally, by combining (6.15) and (6.16) we get

$$\begin{aligned} \mathscr{M}_{\nu,\rho}^{q,p}(f_{\boldsymbol{\alpha}}(\mathscr{K})) &\geq p_{\boldsymbol{\alpha}}^q\sum_{x\in F}\nu(\mathscr{B}(x,|I_{\boldsymbol{\alpha}}|^{-1}\rho))^q - p_{\boldsymbol{\alpha}}^q\sum_{x\in H}\nu(f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(x,\rho))^q \\ &\geq p_{\boldsymbol{\alpha}}^q\sum_{x\in F}\nu(\mathscr{B}(x,|I_{\boldsymbol{\alpha}}|^{-1}\rho))^q - c_n\rho^{(B-b)q}\sum_{\substack{|\boldsymbol{\sigma}|=n\\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}}\mathscr{Q}_{\boldsymbol{\alpha},\boldsymbol{\sigma}}^q(\rho). \end{aligned}$$

Taking the supremum over all $|I_{\boldsymbol{\alpha}}|^{-1}\rho$-separated subsets F of $\mathscr{K}$ the desired result yields.

(2) The proof is identical to the proof of (1). ■

LEMMA 6.17. *Suppose that* $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ *satisfies the* (SOSC). *Let* $q < 0$ *and* $n \geq 1$. *Suppose there exist positive constants* b, B, c_0 *and* ρ_0 *where* $b < B$ *such that*

$$\frac{1}{c_0}\rho^B \leq \nu(\mathscr{B}(x,\rho)) \leq c_0\rho^b$$

for all $x \in \mathscr{K}$ *and all* $0 < \rho < \rho_0$.

(1) *There is a positive constant* c_n *such that for* $0 < \rho < \rho_0$,

$$\Big|\mathscr{M}_{\nu,\rho}^{q,p}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p_{\boldsymbol{\alpha}}^q\mathscr{M}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}\rho}^{q,p}(\mathscr{K})\Big| \leq (1 + c_n\rho^{(B-b)q})\sum_{\substack{|\boldsymbol{\alpha}|=|\boldsymbol{\sigma}|=n\\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}}\mathscr{Q}_{\boldsymbol{\alpha},\boldsymbol{\sigma}}^q(\rho).$$

(2) *There is a positive constant* c_n *such that for* $0 < \rho < \rho_0$,

$$\begin{aligned} \Big|\mathscr{M}_{\nu,\rho}^{q,c}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p_{\boldsymbol{\alpha}}^q\mathscr{M}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}\rho}^{q,c}(\mathscr{K})\Big| &\leq (1 + c_n\rho^{(B-b)q})\sum_{\substack{|\boldsymbol{\alpha}|=|\boldsymbol{\sigma}|=n\\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}}\mathscr{P}_{\boldsymbol{\alpha},\boldsymbol{\sigma}}^q(\rho) \\ &\leq 5^d(1 + c_n\rho^{(B-b)q})\sum_{\substack{|\boldsymbol{\alpha}|=|\boldsymbol{\sigma}|=n\\ \boldsymbol{\alpha}\neq\boldsymbol{\sigma}}}\mathscr{Q}_{\boldsymbol{\alpha},\boldsymbol{\sigma}}^q(\rho). \end{aligned}$$

Proof. (1) can be readily derived from Lemmas 3.4 and 6.16. The proof of the first inequality in (2) is analogous to the proofs of Lemmas 3.4 and 6.16, and is therefore omitted. To establish the second ineqality of (2), it is sufficient to demonstrate that $\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K}) \le 5^d \mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})$ for all sets $\mathscr{K}$ and all $\rho > 0$. It can be observed that there exists a family $B = (\mathscr{B}(x_i,\rho))_i$ of balls with $x_i \in \mathscr{K}$ and $\mathscr{K}$ contained in $\bigcup_i \mathscr{B}(x_i,\rho)$ such that B can be divided into 5^d subfamilies $B_1 = (\mathscr{B}(x_{1i},\rho))_i, \dots, B_{5^d} = (\mathscr{B}(x_{5^d i},\rho))_i$ where each B_s consists of mutually disjoint balls. Consequently,

$$\begin{aligned}\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K}) &\le \sum_i \nu(\mathscr{B}(x_i,\rho))^q = \sum_{s=1}^{5^d}\sum_i \nu(\mathscr{B}(x_{si},\rho))^q \\ &\le \sum_{s=1}^{5^d} \mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K}) = 5^d \mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K}). \ \blacksquare\end{aligned}$$

LEMMA 6.18. *Suppose that* $\{f_{n,j} \mid n \ge 1,\ 1 \le j \le a_n\}$ *satisfies the* (SOSC). *Fix* $q < 0$ *and* $n \ge 1$. *Suppose there exist positive constants* b, B, c_0 *and* ρ_0 *where* $b \le B$ *such that*

$$\frac{1}{c_0}\rho^B \le \nu(\mathscr{B}(x,\rho)) \le c_0 \rho^b$$

for all $x \in \mathscr{K}$ *and all* $0 < \rho < \rho_0$. *Consider the functions* $h_{1,n}, h_{2,n}$ *defined by*

$$\begin{aligned} h_{1,n}(Y) &= e^{-Y\beta_n(q)}\Big(\mathscr{M}^{q,c}_{\nu,e^{-Y}}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}} \mathscr{M}^{q,c}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}e^{-Y}}(\mathscr{K})\Big), \\ h_{2,n}(Y) &= e^{-Y\beta_n(q)}\Big(\mathscr{M}^{q,p}_{\nu,e^{-Y}}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}} \mathscr{M}^{q,p}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}e^{-Y}}(\mathscr{K})\Big). \end{aligned}$$

Then there exists $C > 0$ *such that*

$$\begin{aligned} |h_{1,n}(Y)| &\le Ce^{-T(\beta_n(q)-(\varphi_n(q)-(B-b)q))}, \\ |h_{2,n}(Y)| &\le Ce^{-T(\beta_n(q)-(\varphi_n(q)-(B-b)q))}, \end{aligned}$$

for all $Y > 0$.

Proof. This can be immediately derived from Lemma 6.17, along with Propositions 6.7 and 6.8. ■

Proof of Theorem 6.14. (1) Define $H_n : \mathbb{R} \to \mathbb{R}$ and $h_n : [0,\infty) \to \mathbb{R}$ by

$$\begin{aligned} H_n(Y) &= e^{-Y\beta_n(q)}\mathscr{M}^{q,p}_{\nu,e^{-Y}}(\mathscr{K}), \\ h_n(Y) &= e^{-Y\beta_n(q)}\Big(\mathscr{M}^{q,p}_{\nu,e^{-Y}}(\mathscr{K}) - \sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}} \mathscr{M}^{q,p}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}e^{-Y}}(\mathscr{K})\Big). \end{aligned}$$

Consider the probability measure

$$P = \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} |I_{\boldsymbol{\alpha}}|^{\beta_n(q)} \delta_{\log \frac{1}{|I_{\boldsymbol{\alpha}}|}}.$$

It is evident that $H_n \in L^1(\mathbb{R}, P)$, and by Lemma 6.18, it follows that h_n is directly Riemann integrable on $[0,\infty)$. Next, for all $Y > 0$,

$$\begin{aligned} H_n(Y) &= e^{-Y\beta_n(q)}\mathscr{M}^{q,p}_{\nu,e^{-Y}}(\mathscr{K}) \\ &= e^{-Y\beta_n(q)}\Big(\sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,p}_{\nu,|I_{\boldsymbol{\alpha}}|^{-1}e^{-Y}}(\mathscr{K}) + e^{Y\beta_n(q)}h(Y)\Big) \\ &= \sum_{|\boldsymbol{\alpha}|=n} p^q_{\boldsymbol{\alpha}}|I_{\boldsymbol{\alpha}}|^{\beta_n(q)}H_n\Big(Y-\log\frac{1}{|I_{\boldsymbol{\alpha}}|}\Big) + h_n(Y) \\ &= \int_0^\infty H_n(Y-y)\,dP(y) + h_n(Y). \end{aligned}$$

We now make use of the Renewal Theorem. In the arithmetic case we get

$$\frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\beta_n(q)}} = \Pi_{q,n}(\rho) + \epsilon(\rho),$$

where $\Pi_{q,n}$ is a multiplicatively periodic function.

In the non-arithmetic case,

$$\frac{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}{\rho^{-\beta_n(q)}} = s_n + \epsilon(\rho).$$

The desired result is now reached by taking $\overline{\Pi}_q = \limsup_{n\to\infty}\Pi_{q,n}$, $\underline{\Pi}_q = \liminf_{n\to\infty}\Pi_{q,n}$, $\overline{s} = \limsup_{n\to\infty} s_n$ and $\underline{s} = \liminf_{n\to\infty} s_n$.

(2) The analogous result for $\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})$ is demonstrated in a similar manner. ■

6.5. Proof of Theorem 4.1 in the case of (SOSC) and $q \geq 0$. This proof is identical to that of Theorem 4.1 under the (SSC) assumption, except that the following lemma and Proposition 6.11 are used in place of Lemmas 4.3 and 4.4.

LEMMA 6.19. *Suppose that* $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ *satisfies the* (SOSC). *Let* $q \in \mathbb{R}$ *and* $\boldsymbol{\alpha} \in \mathscr{T}$.

(1) *For* $(F,\rho) \in \Gamma^q$ *we have*

$$\mu^q_{(F,\rho)}(f_{\boldsymbol{\alpha}}(\mathscr{K})) \leq \frac{1}{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}\Big(p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,c}_{\nu,\rho/|I_{\boldsymbol{\alpha}}|}(\mathscr{K}) + 2\cdot 5^d \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\theta}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\theta}}} \mathscr{Q}^q_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\rho)\Big).$$

(2) *For* $(F,\rho) \in \Lambda^q$ *we have*

$$\nu^q_{(F,\rho)}(f_{\boldsymbol{\alpha}}(\mathscr{K})) \leq \frac{1}{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})}\Big(p^q_{\boldsymbol{\alpha}}\mathscr{M}^{q,p}_{\nu,\rho/|I_{\boldsymbol{\alpha}}|}(\mathscr{K}) + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\theta}|=|\boldsymbol{\alpha}|\\ \boldsymbol{\sigma}\neq\boldsymbol{\theta}}} \mathscr{Q}^q_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\rho)\Big).$$

Proof. (1) We have

$$\mu^q_{(F,\rho)} = \frac{1}{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})}\sum_{x\in F\cap f_{\boldsymbol{\alpha}}(\mathscr{K})} \nu(\mathscr{B}(x,\rho))^q.$$

Since $(F,\rho) \in \Gamma^q$, we get

$$\sum_{x\in F\cap f_{\boldsymbol{\alpha}}(\mathscr{K})} \nu(\mathscr{B}(x,\rho))^q \leq \mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K}\cap\mathscr{B}(f_{\boldsymbol{\alpha}}(\mathscr{K}),\rho)). \tag{6.19}$$

Indeed, otherwise, there exists a ρ-spanning subset R of $\mathscr{K} \cap \mathscr{B}(f_{\boldsymbol{\alpha}}(\mathscr{K}), \rho)$ such that

$$\sum_{x \in R} \nu(\mathscr{B}(x,\rho))^q < \sum_{x \in F \cap f_{\boldsymbol{\alpha}}(\mathscr{K})} \nu(\mathscr{B}(x,\rho))^q.$$

Since $H = R \cup (F \setminus f_{\boldsymbol{\alpha}}(\mathscr{K}))$ is clearly a ρ-spanning subset of $\mathscr{K}$, we conclude that

$$\begin{aligned}
\mathscr{N}^q_{\nu,\rho}(\mathscr{K}) &\le \sum_{x \in H} \nu(\mathscr{B}(x,\rho))^q \\
&\le \sum_{x \in R} \nu(\mathscr{B}(x,\rho))^q + \sum_{x \in F \setminus f_{\boldsymbol{\alpha}}(\mathscr{K})} \nu(\mathscr{B}(x,\rho))^q \\
&< \sum_{x \in F \cap f_{\boldsymbol{\alpha}}(\mathscr{K})} \nu(\mathscr{B}(x,\rho))^q + \sum_{x \in F \setminus f_{\boldsymbol{\alpha}}(\mathscr{K})} \nu(\mathscr{B}(x,\rho))^q \\
&= \sum_{x \in F} \nu(\mathscr{B}(x,\rho))^q.
\end{aligned}$$

However, this contradicts the fact that $(F,\rho) \in \Gamma^q$, meaning

$$\sum_{x \in F} \nu(\mathscr{B}(x,\rho))^q = \mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K}).$$

From (6.19) and the inequality $\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{A} \cup B) \le \mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{A}) + \mathscr{M}^{q,c}_{\nu,\rho}(B)$ for all $\mathscr{A}, B \subseteq \mathbb{R}^d$, we get

$$\begin{aligned}
\mu^q_{(F,\rho)}(f_{\boldsymbol{\alpha}}(\mathscr{K}) &\le \frac{1}{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})} \mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K} \cap \mathscr{B}(f_{\boldsymbol{\alpha}}(\mathscr{K}), \rho)) \\
&= \frac{1}{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})} \mathscr{M}^{q,c}_{\nu,\rho}\Big(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cup \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma} \ne \boldsymbol{\alpha}}} (f_{\boldsymbol{\sigma}}(\mathscr{K}) \cap \mathscr{B}(f_{\boldsymbol{\alpha}}(\mathscr{K}), \rho))\Big) \\
&\le \frac{1}{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})} \Big(\mathscr{M}^{q,c}_{\nu,\rho}(f_{\boldsymbol{\alpha}}\mathscr{K}) + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma} \ne \boldsymbol{\alpha}}} \mathscr{P}^q_{\boldsymbol{\sigma},\boldsymbol{\alpha}}(\rho)\Big).
\end{aligned}$$

By using Lemma 3.5, we obtain

$$\begin{aligned}
\mu^q_{(F,\rho)}(f_{\boldsymbol{\alpha}}(\mathscr{K})) &\le \frac{1}{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})} \Big(p^q_{\boldsymbol{\alpha}} \mathscr{M}^{q,c}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}) + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma} \ne \boldsymbol{\alpha}}} \mathscr{P}^q_{\boldsymbol{\alpha},\boldsymbol{\sigma}}(\rho) + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma} \ne \boldsymbol{\alpha}}} \mathscr{P}^q_{\boldsymbol{\sigma},\boldsymbol{\alpha}}(\rho)\Big) \\
&\le \frac{1}{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})} \Big(p^q_{\boldsymbol{\alpha}} \mathscr{M}^{q,c}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}) + 2 \cdot 5^d \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\theta}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma} \ne \boldsymbol{\theta}}} \mathscr{Q}^q_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\rho)\Big).
\end{aligned}$$

(2) By using Lemma 3.5 we obtain

$$\begin{aligned}
\nu^q_{(F,\rho)}(f_{\boldsymbol{\alpha}}(\mathscr{K})) &= \frac{1}{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})} \sum_{x \in F \cap f_{\boldsymbol{\alpha}}(\mathscr{K})} \nu(\mathscr{B}(x,\rho))^q \le \frac{1}{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})} \mathscr{M}^{q,p}_{\nu,\rho}(f_{\boldsymbol{\alpha}}(\mathscr{K})) \\
&\le \frac{1}{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})} \Big(p^q_{\boldsymbol{\alpha}} \mathscr{M}^{q,p}_{\nu,\rho|I_{\boldsymbol{\alpha}}|^{-1}}(\mathscr{K}) + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma} \ne \boldsymbol{\alpha}}} \mathscr{Q}^q_{\boldsymbol{\sigma},\boldsymbol{\alpha}}(\rho)\Big).
\end{aligned}$$

This completes the proof. ∎

6.6. Proof of Theorem 4.1 in the case of (SOSC) and $q < 0$

Theorem 6.20. *Given a list* $\{f_{n,j} \mid n \geq 1, 1 \leq j \leq a_n\}$ *of contracting similarities satisfying the* (SOSC), *assume that* $q < 0$ *with* $\overline{\beta}(q) > \varphi(q) - (\tau - \eta)q$, *and if there exists* n_0 *such that* $\beta_{n_0}(q) \leq \underline{\beta}(q)$ *assume that* $\underline{\beta}(q) > \varphi(q) - (\tau - \eta)q$. *Denote* $\dim_\nu^q(\mathscr{K}) = \underline{\beta}(q) = t$ *and* $\mathrm{Dim}_\nu^q(\mathscr{K}) = \overline{\beta}(q) = s$. *Define probability measures* $\mu^q_{(F,\rho)}$ *and* $\nu^q_{(F,\rho)}$ *on* $\mathbb{R}^d$ *by*

$$\mu^q_{(F,\rho)} = \frac{1}{\mathscr{M}^{q,c}_{\nu,\rho}(\mathscr{K})} \sum_{x \in F} \nu(\mathscr{B}(x,\rho))^q \delta_x \quad \text{for } (F,\rho) \in \Gamma_q,$$

$$\nu^q_{(F,\rho)} = \frac{1}{\mathscr{M}^{q,p}_{\nu,\rho}(\mathscr{K})} \sum_{x \in F} \nu(\mathscr{B}(x,\rho))^q \delta_x \quad \text{for } (F,\rho) \in \Lambda_q,$$

where δ_x *is the Dirac measure at* $x \in \mathbb{R}^d$.

(1) *We have*

$$\mu^q_{(F,\rho)} \to \frac{\mathscr{P}^{q,s}_\nu \llcorner \mathscr{K}}{\mathscr{P}^{q,s}_\nu(\mathscr{K})} \quad \text{weakly},$$

$$\nu^q_{(F,\rho)} \to \frac{\mathscr{P}^{q,s}_\nu \llcorner \mathscr{K}}{\mathscr{P}^{q,s}_\nu(\mathscr{K})} \quad \text{weakly}.$$

(2) *If there exists* n_0 *such that* $\beta_{n_0}(q) \leq \underline{\beta}(q)$, *then*

$$\mu^q_{(F,\rho)} \to \frac{\mathscr{H}^{q,t}_\nu \llcorner \mathscr{K}}{\mathscr{H}^{q,t}_\nu(\mathscr{K})} \quad \text{weakly},$$

$$\nu^q_{(F,\rho)} \to \frac{\mathscr{H}^{q,t}_\nu \llcorner \mathscr{K}}{\mathscr{H}^{q,t}_\nu(\mathscr{K})} \quad \text{weakly}.$$

Before proving Theorem 6.20, we establish the following intermediate propositions.

Proposition 6.21. *Given a list* $\{f_{n,j} \mid n \geq 1, 1 \leq j \leq a_n\}$ *of contracting similarities satisfying the* (SOSC), *denote* $\dim_\nu^q(\mathscr{K}) = \underline{\beta}(q) = t$ *and* $\mathrm{Dim}_\nu^q(\mathscr{K}) = \overline{\beta}(q) = s$. *Let* $q < 0$.

(1) *If* $\overline{\beta}(q) > \varphi(q) - (\tau - \eta)q$, *then for all distinct* $\boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T}$ *with* $|\boldsymbol{\alpha}| = |\boldsymbol{\sigma}|$,

$$\mathscr{P}^{q,s}_\nu(f_{\boldsymbol{\alpha}}^{-1}(f_{\boldsymbol{\sigma}}(\mathscr{K}))) = 0.$$

(2) *If there exists* n_0 *such that* $\beta_{n_0}(q) \leq \underline{\beta}(q)$ *and if* $\underline{\beta}(q) > \varphi(q) - (\tau - \eta)q$, *then for all distinct* $\boldsymbol{\alpha}, \boldsymbol{\sigma} \in \mathscr{T}$,

$$\mathscr{H}^{q,t}_\nu(f_{\boldsymbol{\alpha}}^{-1}(f_{\boldsymbol{\sigma}}(\mathscr{K}))) = 0.$$

Proof. (1) Let $\{\mathscr{B}(x_i,\rho_i)\}_i$ be a centered δ-packing of $f_{\boldsymbol{\alpha}}^{-1}(f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K}))$. Notice that $\{\mathscr{B}(f_{\boldsymbol{\alpha}} x_i, \rho_i |I_{\boldsymbol{\alpha}}|)\}_i$ forms a centered $|I_{\boldsymbol{\alpha}}|\delta$-packing of $f_{\boldsymbol{\alpha}}(\mathscr{K}) \cap f_{\boldsymbol{\sigma}}(\mathscr{K})$. By applying Proposition 6.15, we obtain

$$\begin{aligned} \frac{\nu(f_{\boldsymbol{\alpha}}^{-1} f_{\boldsymbol{\alpha}} \mathscr{B}(x_i,\rho_i))}{\nu(f_{\boldsymbol{\sigma}}^{-1} f_{\boldsymbol{\alpha}} \mathscr{B}(x_i,\rho_i))} &= \frac{\nu \mathscr{B}(x_i,\rho_i)}{\nu(\mathscr{B}(f_{\boldsymbol{\sigma}}^{-1} f_{\boldsymbol{\alpha}} x_i, |I_{\boldsymbol{\sigma}}|^{-1} |I_{\boldsymbol{\alpha}}| \rho_i))} \\ &\geq m_\varepsilon^{-2} \frac{\rho_i^{\tau+\varepsilon}}{|I_{\boldsymbol{\sigma}}|^{-\eta+\varepsilon} |I_{\boldsymbol{\alpha}}|^{\eta-\varepsilon} \rho_i^{\eta-\varepsilon}} \\ &\geq m_\varepsilon^{-2} \left(\frac{b_{\min}}{b_{\max}}\right)^{(\eta-\varepsilon)n} \rho_i^{\tau-\eta+2\varepsilon} \end{aligned}$$

for each $\varepsilon > 0$. Consequently,

$$\begin{aligned}\nu(\mathscr{B}(f_{\boldsymbol\alpha}x_i,|I_{\boldsymbol\alpha}|\rho_i)) = \nu(f_{\boldsymbol\alpha}\mathscr{B}(x_i,\rho_i)) &= \sum_{|\boldsymbol\sigma|=n} p_{\boldsymbol\sigma}\nu(f_{\boldsymbol\sigma}^{-1}f_{\boldsymbol\alpha}\mathscr{B}(x_i,\rho_i))\\ &\le m_\varepsilon^2\bigg(\frac{b_{\max}}{b_{\min}}\bigg)^{(\eta-\varepsilon)n}\rho_i^{-\tau+\eta-2\varepsilon}\sum_{|\boldsymbol\sigma|=n} p_{\boldsymbol\sigma}\nu(f_{\boldsymbol\alpha}^{-1}f_{\boldsymbol\alpha}\mathscr{B}(x_i,\rho_i))\\ &= m_\varepsilon^2\bigg(\frac{b_{\max}}{b_{\min}}\bigg)^{(\eta-\varepsilon)n}\rho_i^{-\tau+\eta-2\varepsilon}\nu(\mathscr{B}(x_i,\rho_i)).\end{aligned}$$

Thus

$$\begin{aligned}\nu(\mathscr{B}(x_i,\rho_i)) &\ge m_\varepsilon^{-2}\bigg(\frac{b_{\min}}{b_{\max}}\bigg)^{(\eta-\varepsilon)n}\rho_i^{\tau-\eta+2\varepsilon}\nu(\mathscr{B}(f_{\boldsymbol\alpha}x_i,|I_{\boldsymbol\alpha}|\rho_i))\\ &= m_0\rho_i^{\tau-\eta+2\varepsilon}\nu(\mathscr{B}(f_{\boldsymbol\alpha}x_i,|I_{\boldsymbol\alpha}|\rho_i)),\end{aligned}$$

where $m_0 = m_\varepsilon^{-2}(b_{\min}/b_{\max})^{(\eta-\varepsilon)n}$.

Write $w_k = |I_{\boldsymbol\alpha}|\delta/2^{k-1}$ for $k \in \mathbb{N}$. As $q < 0$, we have

$$\begin{aligned}&\sum_i \nu(\mathscr{B}(x_i,\rho_i))^q(2\rho_i)^{\overline{\beta}(q)}\\ &\quad\le m_0^q 2^{\overline{\beta}(q)}\sum_i \nu(\mathscr{B}(f_{\boldsymbol\alpha}x_i,|I_{\boldsymbol\alpha}|\rho_i))^q(\rho_i)^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)}\\ &\quad\le m_0^q 2^{\overline{\beta}(q)}\sum_k\sum_{w_{k+1}\le|I_{\boldsymbol\alpha}|\rho_i<w_k}\nu(\mathscr{B}(f_{\boldsymbol\alpha}x_i,|I_{\boldsymbol\alpha}|\rho_i))^q(\rho_i)^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)}\\ &\quad\le m_0^q 4^{\overline{\beta}(q)}\bigg(\frac{1}{|I_{\boldsymbol\alpha}|}\bigg)^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)}\\ &\qquad\times\sum_k\sum_{w_{k+1}\le|I_{\boldsymbol\alpha}|\rho_i<w_k}\nu(\mathscr{B}(f_{\boldsymbol\alpha}x_i,w_{k+1}))^q(w_{k+1})^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)}\\ &\quad\le m_0^q 4^{\overline{\beta}(q)}\bigg(\frac{1}{|I_{\boldsymbol\alpha}|}\bigg)^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)}\\ &\qquad\times\sum_k\mathscr{M}^{q,p}_{\nu,w_{k+1}}(f_{\boldsymbol\alpha}(\mathscr{K})\cap\mathscr{B}(f_{\boldsymbol\sigma}(\mathscr{K}),w_{k+1}))(w_{k+1})^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)}\\ &\quad= m_0^q 4^{\overline{\beta}(q)}\bigg(\frac{1}{|I_{\boldsymbol\alpha}|}\bigg)^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)}\sum_k\mathscr{Q}^q_{\boldsymbol\alpha,\boldsymbol\sigma}(w_{k+1})(w_{k+1})^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)}\\ &\quad\le m_0^q 4^{\overline{\beta}(q)}M\bigg(\frac{1}{|I_{\boldsymbol\alpha}|}\bigg)^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)}\sum_k(w_{k+1})^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)-\varphi(q)}\\ &\quad= m_0^q 4^{\overline{\beta}(q)}M\bigg(\frac{1}{|I_{\boldsymbol\alpha}|}\bigg)^{\varphi(q)}\frac{1}{2^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)-\varphi(q)}-1}\delta^{q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)-\varphi(q)}.\end{aligned}$$

Since $q(\tau-\eta+2\varepsilon)+\overline{\beta}(q)-\varphi(q) > 0$, it follows that

$$\mathscr{P}^{q,s}_\nu(f_{\boldsymbol\alpha}^{-1}(f_{\boldsymbol\sigma}(\mathscr{K}))) = \mathscr{P}^{q,\overline{\beta}(q)}_\nu(f_{\boldsymbol\alpha}^{-1}(f_{\boldsymbol\sigma}(\mathscr{K}))) = 0.$$

(2) The proof is the same as for (1). ■

Proposition 6.22. *Assume that* $\{f_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ *satisfies the* (SOSC). *Let* $q < 0$ *and* $\boldsymbol{\alpha} \in \mathscr{T}$. *Suppose that*

$$\overline{\beta}(q) > \varphi_n(q) - (\tau - \eta)q.$$

For $\delta > 0$, *write*

$$\Delta_{\boldsymbol{\alpha},\delta} = \bigcup_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\alpha}}} f_{\boldsymbol{\alpha}}^{-1}\mathscr{B}(f_{\boldsymbol{\sigma}}(\mathscr{K}), \delta).$$

(1) *If* $\mathscr{H}_\nu^{q,t}(\mathscr{K}) \leq \mathscr{P}_\nu^{q,s}(\mathscr{K}) < \infty$, *then*

$$\lim_{\delta\to 0} \mathscr{H}_\nu^{q,t}(\Delta_{\boldsymbol{\alpha},\delta}) = 0, \qquad \lim_{\delta\to 0} \mathscr{P}_\nu^{q,s}(\Delta_{\boldsymbol{\alpha},\delta}) = 0.$$

(2) *If* $\mathscr{H}_\nu^{q,t}(\mathscr{K}) \leq \mathscr{P}_\nu^{q,s}(\mathscr{K}) < \infty$, *then*

$$\lim_{\delta\to 0} \mathscr{H}_\nu^{q,t}(f_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\alpha},\delta}) = 0, \qquad \lim_{\delta\to 0} \mathscr{P}_\nu^{q,s}(f_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\alpha},\delta}) = 0.$$

(3) *For each* $\delta > 0$,

$$\underline{J}_\nu^q(f_{\boldsymbol{\alpha}}, \mathscr{K} \setminus \Delta_{\boldsymbol{\alpha},\delta}) = \overline{J}_\nu^q(f_{\boldsymbol{\alpha}}, \mathscr{K} \setminus \Delta_{\boldsymbol{\alpha},\delta}) = p_{\boldsymbol{\alpha}}^q.$$

Proof. Argue as in the proof of Proposition 6.11. ■

Proof of Theorem 6.20. (1) We can find a positive integer n' satisfying

$$\overline{\beta}(q) > \varphi_n(q) - (\tau - \eta)q$$

for every $n \geq n'$.

Let $\{S_{n,j} \mid n \geq 1,\ 1 \leq j \leq a_n\}$ satisfy the (SOSC) and $q < 0$. For $\boldsymbol{\alpha} \in T$ with $n \geq n'$ define $h_{\boldsymbol{\alpha}} : (0, \infty) \to \mathbb{R}$ by

$$h_{\boldsymbol{\alpha}}(\rho) = \frac{\mathcal{M}^{q,p}_{\mu,\rho/|I_{\boldsymbol{\alpha}}|}(\mathcal{K})}{(\rho/|I_{\boldsymbol{\alpha}}|)^{-\overline{\beta}(q)}} \Big/ \frac{\mathcal{M}^{q,p}_{\mu,\rho}(\mathcal{K})}{\rho^{-\overline{\beta}(q)}}.$$

It is clear from Theorem 3.1 that

$$h_{\boldsymbol{\alpha}}(\rho) \to 1 \qquad \text{as } \rho \to 0. \tag{6.20}$$

Moreover, Propositions 6.7 and 6.8 demonstrate that for distinct $\boldsymbol{\sigma}, \boldsymbol{\theta} \in T$ with $|\boldsymbol{\sigma}| = |\boldsymbol{\theta}| = |\boldsymbol{\alpha}|$ there exists a positive constant $C_{\boldsymbol{\sigma},\boldsymbol{\theta}}$ such that

$$\mathcal{Q}^q_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\rho) \leq C_{\boldsymbol{\sigma},\boldsymbol{\theta}}\rho^{-\gamma_n(q)} \qquad \text{for all } \rho > 0.$$

Then for $(F, \rho) \in \Lambda^q(\mathcal{K})$ from Lemma 6.19 we get

$$\begin{aligned}
\mu^q_{(F,\rho)}(S_{\boldsymbol{\alpha}}(\mathcal{K})) &\leq \frac{1}{\mathcal{M}^{q,p}_{\mu,\rho}(\mathcal{K})}\Big(p^q_{\boldsymbol{\alpha}}\mathcal{M}^{q,p}_{\mu,\rho/|I_{\boldsymbol{\alpha}}|}(\mathcal{K}) + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\theta}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\theta}}} \mathcal{Q}^q_{\boldsymbol{\sigma},\boldsymbol{\theta}}(\rho)\Big) \\
&\leq p^q_{\boldsymbol{\alpha}}|I_{\boldsymbol{\alpha}}|^{\overline{\beta}(q)}h_{\boldsymbol{\alpha}}(\rho) + \sum_{\substack{|\boldsymbol{\sigma}|=|\boldsymbol{\theta}|=|\boldsymbol{\alpha}| \\ \boldsymbol{\sigma}\neq\boldsymbol{\theta}}} C_{\boldsymbol{\sigma},\boldsymbol{\theta}}\frac{\rho^{-\gamma(q)}}{\mathcal{M}^{q,p}_{\mu,\rho}(\mathcal{K})} \\
&= p^q_{\boldsymbol{\alpha}}|I_{\boldsymbol{\alpha}}|^s h_{\boldsymbol{\alpha}}(r) + C\rho^{\overline{\beta}(q)-\gamma(q)}\left(\frac{\mathcal{M}^{q,p}_{\mu,\rho}(\mathcal{K})}{\rho^{-\overline{\beta}(q)}}\right)^{-1} \\
&= g_{\boldsymbol{\alpha}}(\rho),
\end{aligned} \tag{6.21}$$

where $C = \sum_{|\boldsymbol{\sigma}|=|\boldsymbol{\theta}|=|\boldsymbol{\alpha}|,\, \boldsymbol{\sigma}\neq\boldsymbol{\theta}} C_{\boldsymbol{\sigma},\boldsymbol{\theta}}$. Since $\rho^{\overline{\beta}(q)-\gamma(q)} \to 0$ as $\rho \to 0$ (because $\overline{\beta}(q) - \gamma(q) > 0$) and $\mathcal{M}^{q,p}_{\mu,\rho}(\mathcal{K})/\rho^{\overline{\beta}(q)}$ remains bounded away from 0 for sufficiently small ρ by Theorem 3.1, it follows from (6.20) and (6.21) that

$$g_{\boldsymbol{\alpha}}(\rho) \to p^q_{\boldsymbol{\alpha}} |I_{\boldsymbol{\alpha}}|^s \quad \text{as } \rho \to 0 \text{ for all } \boldsymbol{\alpha} \in T. \tag{6.22}$$

Consequently, for $m \in \mathbb{N}$ we have

$$\begin{aligned} 1 = \mu^q_{(F,\rho)}(\mathcal{K}) &\le \sum_{|\boldsymbol{\alpha}|=m} \mu^q_{(F,\rho)}(S_{\boldsymbol{\alpha}}(\mathcal{K})) \\ &\le \sum_{|\boldsymbol{\alpha}|=m} g_{\boldsymbol{\alpha}}(\rho) \to \sum_{|\boldsymbol{\alpha}|=m} p^q_{\boldsymbol{\alpha}} |I_{\boldsymbol{\alpha}}|^s \le 1 \quad \text{as } \rho \to 0. \end{aligned} \tag{6.23}$$

We conclude now that

$$\mu^q_{(F,\rho)}(S_{\boldsymbol{\alpha}}(\mathcal{K})) \to p^q_{\boldsymbol{\alpha}} |I_{\boldsymbol{\alpha}}|^s \quad \text{as } \rho \to 0 \text{ for all } \boldsymbol{\alpha} \in T. \tag{6.24}$$

For brevity, define $H = \mathcal{P}^{q,s}_{\mu} \llcorner \mathcal{K}/\mathcal{P}^{q,s}_{\mu}(\mathcal{K})$. Next, for $\delta > 0$ and $\boldsymbol{\alpha} \in T$, let $\Delta_{\boldsymbol{\alpha},\delta}$ be as defined in Proposition 6.22. It now follows straightforwardly from Lemma 6.9 and Proposition 6.22 that

$$\begin{aligned} H(S_{\boldsymbol{\alpha}}(\mathcal{K})) &= \lim_{\delta\to 0} \frac{\mathcal{P}^{q,s}_{\mu}(S_{\boldsymbol{\alpha}}(\mathcal{K} \setminus \Delta_{\alpha,\delta}))}{\mathcal{P}^{q,s}_{\mu}(\mathcal{K})} \\ &= \lim_{\delta\to 0} |I_{\boldsymbol{\alpha}}|^s J^q_{\mu}(S_{\boldsymbol{\alpha}}(\mathcal{K} \setminus \Delta_{\alpha,\delta})) \frac{\mathcal{P}^{q,s}_{\mu}(\mathcal{K} \setminus \Delta_{\boldsymbol{\alpha},\delta})}{\mathcal{P}^{q,s}_{\mu}(\mathcal{K})} \\ &= \lim_{\delta\to 0} |I_{\boldsymbol{\alpha}}|^s p^q_{\boldsymbol{\alpha}} \frac{\mathcal{P}^{q,s}_{\mu}(\mathcal{K} \setminus \Delta_{\boldsymbol{\alpha},\delta})}{\mathcal{P}^{q,s}_{\mu}(\mathcal{K})} \\ &= |I_{\boldsymbol{\alpha}}|^s p^q_{\boldsymbol{\alpha}}. \end{aligned} \tag{6.25}$$

By combining (6.24) and (6.25) we have

$$\mu^q_{(F,\rho)}(S_{\boldsymbol{\alpha}}(\mathcal{K})) \to H(S_{\boldsymbol{\alpha}}(\mathcal{K})) \quad \text{as } \rho \to 0 \text{ for all } \boldsymbol{\alpha} \in T \text{ with } n \ge n'. \tag{6.26}$$

Next, let $f : \mathcal{K} \to \mathbb{R}$ be a continuous function, and let $\varepsilon > 0$. Since f is uniformly continuous and $\max_{|\boldsymbol{\alpha}|=m} |S_{\boldsymbol{\alpha}}(\mathcal{K})| \le (b_{\max})^m |\mathcal{K}| \to 0$ as $m \to \infty$, there exists $M \in \mathbb{N}$ such that $|f(x) - f(y)| \le \varepsilon/2$ for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = M$ and for $x, y \in S_{\boldsymbol{\alpha}}(\mathcal{K})$. Furthermore, it follows from (6.26) that for each $\boldsymbol{\alpha} \in T$ with $|\boldsymbol{\alpha}| = M$ there exists $\rho_{\boldsymbol{\alpha}}$ such that

$$|\mu^q_{(F,\rho)}(S_{\boldsymbol{\alpha}}(\mathcal{K})) - H(S_{\boldsymbol{\alpha}}(\mathcal{K}))| \le \varepsilon/(2n^M \|f\|_\infty)$$

for all $(F,\rho) \in \Lambda^q$ with $0 < \rho < \rho_{\boldsymbol{\alpha}}$.

Then, for each $(F,\rho) \in \Lambda^q$ with $0 < \rho < \min_{\boldsymbol{\alpha}=M} \rho_{\boldsymbol{\alpha}}$ we obtain

$$\begin{aligned} \left| \int f \, d\mu^q_{(F,\rho)} - \int f \, dH \right| &\le \sum_{|\boldsymbol{\alpha}|=M} \left| \int_{S_{\boldsymbol{\alpha}}(\mathcal{K})} f \, d\mu^q_{(F,\rho)} - \int_{S_{\boldsymbol{\alpha}}(\mathcal{K})} f \, dH \right| \\ &\le \sum_{|\boldsymbol{\alpha}|=M} \left(H(S_{\boldsymbol{\alpha}}(\mathcal{K})) \sup_{x,y \in S_{\boldsymbol{\alpha}}(\mathcal{K})} |f(x) - f(y)| + \|f\|_\infty \frac{\varepsilon}{2n^M \|f\|_\infty} \right) \\ &\le \frac{\varepsilon}{2} \sum_{|\boldsymbol{\alpha}|=M} H(S_{\boldsymbol{\alpha}}(\mathcal{K})) + \frac{\varepsilon}{2}. \end{aligned}$$

Finally, since $\mathcal{P}^s(S_{\boldsymbol{\alpha}}(\mathcal{K}) \cap S_{\boldsymbol{\sigma}}(\mathcal{K})) = 0$ for all distinct $\boldsymbol{\alpha}, \boldsymbol{\sigma} \in T$ with $|\boldsymbol{\alpha}| = |\boldsymbol{\sigma}|$ (refer to Proposition 6.21), it ensues that

$$\sum_{|\boldsymbol{\alpha}|=M} H(S_{\boldsymbol{\alpha}}(\mathcal{K})) = H\Big(\bigcup_{|\boldsymbol{\alpha}|=M} S_{\boldsymbol{\alpha}}(\mathcal{K})\Big) = H(\mathcal{K}) = 1,$$

and consequently

$$\left|\int f d\mu^q_{(F,\rho)} - \int f\, dH\right| \leq \varepsilon.$$

This concludes the proof (the analogous result for $\nu^q_{(F,\rho)}$ is established in a similar manner).

(2) We just add the condition that there exists n_0 satisfying $\beta_{n_0}(q) \leq \underline{\beta}(q)$. Then we use the same arguments as in the last statement. ∎

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