

The crossing matrix and the extended first Johnson homomorphism of a braid group

by

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Abstract. We compare two crossed homomorphisms on a braid group, one defined diagrammatically and the other defined algebraically. We show that these crossed homomorphisms are essentially the same, and compute them in detail for half twists, namely elements conjugate to the standard generators of the braid group or to their inverses.

1. Introduction. Let $m \geq 2$ be a positive integer and B_m the braid group on m strands. In this paper, we study two crossed homomorphisms on B_m .

One crossed homomorphism is defined diagrammatically and called the *crossing matrix*, which was introduced by Burillo, Gutierrez, Krstić and Nitecki [4]. The crossing matrix $C(\beta)$ of an m -braid β is an $m \times m$ matrix whose (i, j) -entry counts the crossings of the i th strand over the j th strand, taking the signs into account. This yields a crossed homomorphism

$$C : B_m \rightarrow \text{Mat}_m^0,$$

where the target is the abelian group of $m \times m$ integer matrices with zero diagonal entries.

Burillo et al. [4] completely characterized the crossing matrices of braids, thereby determining the image $C(B_m)$. From various motivations, the problem of determining $C(S)$ for certain subsets S of B_m has been studied. For example, when S is the set of positive pure braids, a conjecture completely characterizing $C(S)$ was proposed in [4]. Several authors, including [6, 17, 15], have investigated this conjecture and obtained partial results, but a complete solution has not yet been achieved.

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The other crossed homomorphism is defined algebraically. Recall that the group B_m acts naturally on the free group F_m of rank m , and we have an embedding $B_m \hookrightarrow \text{Aut}(F_m)$ called the Artin representation. Let $H = F_m^{\text{abel}}$ be the abelianization of F_m , which is free abelian of rank m . In a cohomological study of $\text{Aut}(F_m)$, Kawazumi [11] introduced a crossed homomorphism $\tau_1^\theta : \text{Aut}(F_m) \rightarrow \text{Hom}(H, H^{\otimes 2})$ called the *extended first Johnson homomorphism* (associated with the standard Magnus expansion). By restriction, we obtain a map from B_m which actually takes values in a smaller space:

$$\tau_1^\theta : B_m \rightarrow \text{Hom}(H, \wedge^2 H).$$

In [12], Kawazumi used this map to study the cohomology of the braid group with twisted coefficients.

Since both maps C and τ_1^θ are crossed homomorphisms, it is natural to seek their relationship. Our first main result, which is Theorem 4.2, shows that they carry essentially the same information: there is an injective map $\delta : \text{Mat}_m^0 \rightarrow \text{Hom}(H, \wedge^2 H)$ such that $\tau_1^\theta = \delta \circ C$.

Our second main result concerns the computation of these invariants for half twists, that is, elements conjugate to the standard generators of B_m or to their inverses. Each half twist is determined by its sign and the isotopy class of its cord, a certain arc in an m -marked disk. In Theorem 5.2, we give a formula for the crossing matrix of a half twist in terms of a homological invariant of the corresponding cord. This implies that we can determine the crossing matrices of half twists.

Let Q be a *quandle*, that is, a set with a binary operation which axiomatizes the notion of conjugation in a group (see [5]). The *Hurwitz action* on the m -fold direct product of Q is a natural action of the braid group B_m on Q^m . The *Hurwitz equivalence problem* asks whether given two elements of Q^m are sent to each other by the Hurwitz action, or not. When Q is the set of half twists in B_n , solving the problem implies a complete invariant of *simple surface braids of degree n with m branch points*, which are, roughly speaking, a 2-dimensional version of classical braids on n strands (see [7]). However, no algorithm to solve the problem is known for general quandles, including the set of half twists. We expect to obtain invariants of simple surface braids using Theorem 5.2.

The map τ_1^θ extends to a family of maps $\{\tau_k^\theta\}_{k \geq 1}$, called the extended higher Johnson homomorphisms [11]. It would be interesting to extend the equality $\tau_1^\theta = \delta \circ C$ to a diagrammatic description of the higher invariants $\tau_2^\theta, \tau_3^\theta, \dots$ on the braid group.

This paper is organized as follows. In Section 2, we introduce terminology for braids and recall the definition of the crossing matrix. In Section 3, we recall the definition of the extended first Johnson homomorphism (to be precise, associated with the standard Magnus expansion). In Section 4, we

prove Theorem 4.2. In Section 5, we recall the definition of a cord of a half twist, introduce its homological invariant, and prove Theorem 5.2.

Convention. Groups act always on the left. All the homology groups that we consider will be over the integers. Let G be a group acting on a commutative group A . Then the *semi-direct product* $A \rtimes G$ is a group structure on the cartesian product $A \times G$ defined by

$$(a_1, g_1)(a_2, g_2) = (a_1 + g_1 \cdot a_2, g_1 g_2).$$

A map $c : G \rightarrow A$ is called a *crossed homomorphism* if it satisfies

$$c(g_1 g_2) = c(g_1) + g_1 \cdot c(g_2)$$

for any $g_1, g_2 \in G$.

In this paper, we often consider the following situation. Let $\varpi : G \rightarrow K$ be a group homomorphism, and suppose that K acts on a commutative group A . The group G acts on A through ϖ . Let $c : G \rightarrow A$ be a crossed homomorphism. Then we have the following group homomorphism:

$$\tilde{c} : G \rightarrow A \rtimes K, \quad \tilde{c}(g) = (c(g), \varpi(g)).$$

2. The crossing matrix of a braid group. First we review the definitions of geometric braids, braid diagrams, and the braid groups. Our convention for braid diagrams follows the standard one described, for example, in [3, Section 1.1.1] or [10, Section 1.2.2].

Let $m \geq 2$ be a positive integer. Let $I = [0, 1]$ be the unit interval in $\mathbb{R}$ and let $D = I \times I$ be the closed domain in $\mathbb{R}^2$. For $i \in \{1, \dots, m\}$, let $q_i = (i/(m+1), 1/2) \in D$, and let $Q_m = \{q_1, \dots, q_m\}$. A *geometric m -braid* β is a 1-manifold embedded in $D \times I$ with the boundary condition $\partial\beta = \beta \cap \partial(D \times I) = Q_m \times \{0, 1\}$ such that the restriction $\pi|_\beta : \beta \rightarrow I$ of the natural projection $\pi : D \times I \rightarrow I$ is an m -fold covering map. We call $Q_m \times \{0\}$ (resp. $Q_m \times \{1\}$) the set of *starting points* (resp. the set of *terminal points*) of geometric m -braids. Take a *string* α of a geometric m -braid β , that is a connected component of β . If $\alpha \cap (D \times \{0\}) = \{q_i\} \times \{0\}$ (resp. $\alpha \cap (D \times \{1\}) = \{q_j\} \times \{1\}$), then we call the point q_i (resp. the point q_j) the *starting point* (resp. the *terminal point*) of α . The string of β starting from q_i is called the *i th string of β* . Following [3, Section 1.1.1] and [10, Section 1.2.2], we draw geometric m -braids with the I -coordinate directed downward; thus the starting points $Q_m \times \{0\}$ appear at the top and the terminal points $Q_m \times \{1\}$ appear at the bottom.

Let B_m be the set of the equivalence classes of geometric m -braids under ambient isotopies of $D \times I$ that fix $\partial(D \times I)$ pointwise. The set B_m forms a group with the “stacking product” as follows: If β_1 and β_2 are (the equivalence

classes of) m -braids then $\beta_1\beta_2$ is the (equivalence class of) m -braid obtained by placing β_1 above β_2 so that the terminal points of β_1 coincide with the starting points of β_2 . To be more precise, by rescaling the vertical coordinates we modify β_1 and β_2 to proper submanifolds in $D \times [0, 1/2]$ and $D \times [1/2, 1]$, respectively, and take their union in $D \times I$. We call the group B_m the *braid group on m strands*. For simplicity, we denote the equivalence class of a geometric m -braid β by the same letter $\beta \in B_m$, and the strings of an element of B_m refer to those of its geometric representative.

A geometric m -braid β is called *generic* if its image under the projection $D \times I \rightarrow I \times I$, $(x, y, t) \mapsto (x, t)$, has transverse double points only. Endowing the image of β with the over/under information of strings around each transverse double point by a small break in the under arc, we obtain the *diagram* of a generic geometric m -braid β , which we denote by Γ_β . The double points in Γ_β are called *crossings*. There are two types of crossings, positive and negative: see Figure 1.

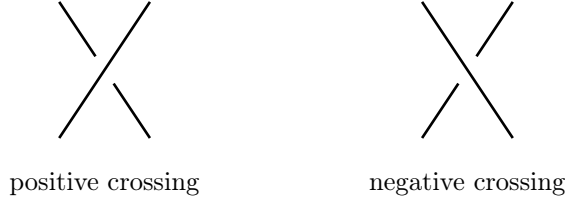


Fig. 1. Positive and negative crossings

Any element in B_m admits a generic representative. Abusing notation, for $\beta \in B_m$ we denote by Γ_β the diagram of its generic representative (which is not unique).

It is well-known that B_m has the following presentation [1]:

$$B_m = \left\langle \sigma_1, \dots, \sigma_{m-1} \left| \begin{array}{ll} \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & (|i - j| = 1), \\ \sigma_i \sigma_j = \sigma_j \sigma_i & (|i - j| > 1) \end{array} \right. \right\rangle,$$

where σ_i is called the i th standard generator of B_m , and is defined as the m -braid shown in Figure 2.

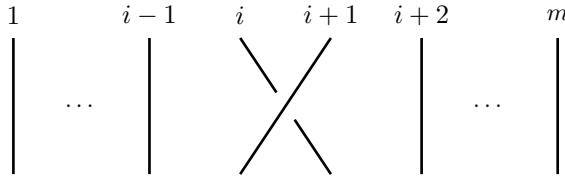


Fig. 2. The standard generator $\sigma_i \in B_m$

Let S_m be the symmetric group of degree m , namely the set of bijections from $\{1, \dots, m\}$ to itself. The group operation of S_m is the composition of functions, i.e. the product $\pi_1\pi_2$ is defined by $(\pi_1\pi_2)(j) = \pi_1(\pi_2(j))$ for each $\pi_1, \pi_2 \in S_m$ and each $j \in \{1, \dots, m\}$.

Take an element $\beta \in B_m$. For each $j \in \{1, \dots, m\}$, let $|\beta|(j)$ be the integer with $1 \leq |\beta|(j) \leq m$ such that the terminal point of the $|\beta|(j)$ th string of β is q_j . Then we have a permutation $|\beta| \in S_m$ and have a surjective group homomorphism $|\cdot| : B_m \rightarrow S_m$, $\beta \mapsto |\beta|$.

We recall the definition of the *crossing matrix* of m -braids [4]. Let $\beta \in B_m$ and take its diagram Γ_β . The crossing matrix $C(\beta)$ of β is defined to be the $m \times m$ matrix whose (i, j) entry is the algebraic number of crossings in Γ_β between the i th and j th strings, where the i th string is over the j th string. Here, we assign $+1$ to positive crossings and -1 to negative crossings. By definition, the diagonal entries of $C(\beta)$ are zero. Note that $C(\beta)$ does not depend on the choice of a diagram Γ_β . Figure 3 shows an example for $\beta = \sigma_2^{-1}\sigma_1^2\sigma_2^3\sigma_1^{-1}\sigma_2 \in B_3$. Let Mat_m^0 be the set of $m \times m$ integer matrices

$$\beta = \quad C(\beta) = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

Fig. 3. The crossing matrix for $\beta = \sigma_2^{-1}\sigma_1^2\sigma_2^3\sigma_1^{-1}\sigma_2$.

whose diagonal entries are all 0. We have a map

$$C : B_m \rightarrow \text{Mat}_m^0, \quad \beta \mapsto C(\beta).$$

Matrix addition endows Mat_m^0 with a commutative group structure. For $M \in \text{Mat}_m^0$ and a permutation $\pi \in S_m$, let $\pi(M) \in \text{Mat}_m^0$ be the matrix obtained by rearranging the rows and columns of M according to π , i.e. the (i, j) entry of $\pi(M)$ is the $(\pi^{-1}(i), \pi^{-1}(j))$ entry of M . Then we have an action of S_m on Mat_m^0 .

Note that for any $\beta_1, \beta_2 \in B_m$, the i th string of β_1 is connected to the $|\beta_1|^{-1}(i)$ th string of β_2 , and thus we have the following lemma.

LEMMA 2.1 ([4]). *The map C is a crossed homomorphism. Namely, for any $\beta_1, \beta_2 \in B_m$, $C(\beta_1\beta_2) = C(\beta_1) + |\beta_1|(C(\beta_2))$. Thus we have a group homomorphism $\tilde{C} : B_m \rightarrow \text{Mat}_m^0 \rtimes S_m$ defined by $\tilde{C}(\beta) = (C(\beta), |\beta|)$.*

Let $H = H_m$ be the m -fold direct sum of the infinite cyclic group $\mathbb{Z}$, which is generated by m letters $X_1, \dots, X_m$. The group S_m acts on H by $\pi(X_k) = X_{\pi(k)}$. For $M = (\alpha_{kl}) \in \text{Mat}_m^0$ and $i \in \{1, \dots, m\}$, let $f_i(M)$ be the element of H defined by

$$f_i(M) = \sum_{k=1}^m \alpha_{ki} X_k.$$

LEMMA 2.2. *For any $\pi \in S_m$, any $M \in \text{Mat}_m^0$ and any $i \in \{1, \dots, m\}$, we have $f_i(\pi(M)) = \pi(f_{\pi^{-1}(i)}(M))$.*

Proof. Put $(\alpha'_{kl}) = \pi(M) \in \text{Mat}_m^0$. Since $\alpha'_{kl} = \alpha_{\pi^{-1}(k)\pi^{-1}(l)}$, we have

$$\begin{aligned} f_i(\pi(M)) &= \sum_{k=1}^m \alpha'_{ki} X_k = \sum_{k=1}^m \alpha_{\pi^{-1}(k)\pi^{-1}(i)} X_k = \pi \left(\sum_{k=1}^m \alpha_{\pi^{-1}(k)\pi^{-1}(i)} X_{\pi^{-1}(k)} \right) \\ &= \pi \left(\sum_{k=1}^m \alpha_{k\pi^{-1}(i)} X_k \right) = \pi(f_{\pi^{-1}(i)}(M)). \end{aligned}$$

Thus, we obtain the result. ■

Take $\beta \in B_m$ and $i \in \{1, \dots, m\}$. Let $f_i(\beta) = f_i(C(\beta)) \in H$. Note that $f_i(\beta)$ gives the algebraic count of how many times the i th string of β has passed under each of the other strings. We call $f_i(\beta)$ the i th *diving combinational information* of β . Then we have the following corollary.

LEMMA 2.3. *For any $\beta_1, \beta_2 \in B_m$ and any $i \in \{1, \dots, m\}$, we have $f_i(\beta_1\beta_2) = f_i(\beta_1) + |\beta_1|f_{|\beta_1|^{-1}(i)}(\beta_2)$.*

Proof. By Lemma 2.1, $f_i(C(\beta_1\beta_2)) = f_i(C(\beta_1)) + f_i(|\beta_1|C(\beta_2))$. By Lemma 2.2, $f_i(|\beta_1|C(\beta_2)) = |\beta_1|f_{|\beta_1|^{-1}(i)}(C(\beta_2))$, and we obtain the result. ■

3. The extended first Johnson homomorphism of the automorphism group of a free group. In this section, we review the extended first Johnson homomorphism on the automorphism group of a free group associated with the standard Magnus expansion. For more details, see [11, 14]. See also [13, 16] for the description of the (higher) Johnson homomorphism in terms of the standard Magnus expansion.

Let F_m be the free group generated by m letters $x_1, \dots, x_m$. The abelianization $F_m^{\text{abel}} = F_m/[F_m, F_m]$ is a free abelian group of rank m . For $x \in F_m$, let $[x] \in F_m^{\text{abel}}$ be the image of x by the canonical projection $F_m \rightarrow F_m^{\text{abel}}$. Then, the elements $[x_i]$, $1 \leq i \leq m$, form a $\mathbb{Z}$ -basis of F_m^{abel} . We identify F_m^{abel} with H , the group introduced in Section 2, by setting $X_i := [x_i]$. For an integer $n > 0$, let $H^{\otimes n} = H \otimes \dots \otimes H$ (n times) be the n -fold tensor product

and let $H^{\otimes 0} = \mathbb{Z}$. The completed tensor algebra

$$\widehat{T} := \prod_{n=0}^{\infty} H^{\otimes n}$$

is naturally isomorphic to the ring of non-commutative formal power series in X_i .

The *standard Magnus expansion* of F_m is a unique map $\theta : F_m \rightarrow \widehat{T}$ characterized by the following properties:

- (M0) $\theta(e) = 1$, where e is the identity element of F_m ;
- (M1) $\theta(xy) = \theta(x)\theta(y)$ for any $x, y \in F_m$;
- (M2) $\theta(x_i) = 1 + X_i$ for any $1 \leq i \leq m$.

Note that $\theta(x_i^{-1}) = 1 - X_i + X_i^{\otimes 2} - X_i^{\otimes 3} + \dots$. In general, for any $x \in F_m$ one can write

$$\theta(x) = 1 + \theta_1(x) + \theta_2(x) + \dots,$$

where $\theta_n(x) \in H^{\otimes n}$. In this paper, we will use only the terms of θ up to degree two. For the degree one part, we have

$$\theta_1(x) = [x]$$

for any $x \in F_m$. For the degree two part, we will use the following property:

LEMMA 3.1. *For any $x, y \in F_m$, we have*

$$\theta_2(y^{-1}xy) = \theta_2(x) + ([x] \otimes [y] - [y] \otimes [x]).$$

Proof. Modulo terms of degree ≥ 3 , we have $\theta(y^{-1}) \equiv 1 - [y] + ([y]^{\otimes 2} - \theta_2(y))$. Hence

$$\begin{aligned} \theta(y^{-1}xy) &= \theta(y^{-1})\theta(x)\theta(y) \\ &\equiv (1 - [y] + ([y]^{\otimes 2} - \theta_2(y)))(1 + [x] + \theta_2(x))(1 + [y] + \theta_2(y)) \\ &\equiv 1 + [x] + (\theta_2(x) + [x] \otimes [y] - [y] \otimes [x]). \end{aligned}$$

This proves the lemma. ■

Let $\text{Aut}(F_m)$ and $\text{Aut}(H)$ be the automorphism group of the free group F_m and the free abelian group H , respectively. There is a natural map $\text{Aut}(F_m) \rightarrow \text{Aut}(H)$ (which is known to be surjective). For $\varphi \in \text{Aut}(F_m)$, we denote by $|\varphi| \in \text{Aut}(H)$ the image of φ by this map. The natural action of $\text{Aut}(H)$ on H induces an action on the tensor product $H^{\otimes 2}$ and hence on the space $\text{Hom}(H, H^{\otimes 2})$ of group homomorphisms from H to $H^{\otimes 2}$. Explicitly, this action, which we denote by $\odot$, is given as follows: for $g \in \text{Aut}(H)$ and $F \in \text{Hom}(H, H^{\otimes 2})$,

$$(g \odot F)(X) = g^{\otimes 2}(F(g^{-1}(X))), \quad X \in H.$$

Now let $\varphi \in \text{Aut}(F_m)$. Then the map

$$\tau_1^\theta(\varphi) : H \rightarrow H^{\otimes 2}, \quad [x] \mapsto \theta_2(x) - |\varphi|^{\otimes 2}(\theta_2(\varphi^{-1}(x))),$$

is a group homomorphism [11, Lemma 2.2]. The resulting map

$$\tau_1^\theta : \text{Aut}(F_m) \rightarrow \text{Hom}(H, H^{\otimes 2}), \quad \varphi \mapsto \tau_1^\theta(\varphi),$$

is called the *extended first Johnson homomorphism*. It is actually not a homomorphism; it is rather a crossed homomorphism:

LEMMA 3.2 ([11], Lemma 2.1). *For any $\varphi, \psi \in \text{Aut}(F_m)$,*

$$\tau_1^\theta(\varphi\psi) = \tau_1^\theta(\varphi) + |\varphi| \odot \tau_1^\theta(\psi).$$

Thus we obtain the following group homomorphism:

$$\tilde{\tau}_1^\theta : \text{Aut}(F_m) \rightarrow \text{Hom}(H, H^{\otimes 2}) \rtimes \text{Aut}(H), \quad \varphi \mapsto (\tau_1^\theta(\varphi), |\varphi|).$$

REMARK 3.3. The *IA-automorphism subgroup* of $\text{Aut}(F_m)$ is defined to be $\text{IA}(F_m) := \{\varphi \in \text{Aut}(F_m) \mid |\varphi| = \text{id}_H\}$. The restriction $\tau_1 = \tau_1^\theta|_{\text{IA}(F_m)}$ is a homomorphism, and it is called the *first Johnson homomorphism*. In fact, one can extend τ_1 to the whole group $\text{Aut}(F_m)$ using any *generalized Magnus expansion*, which is a map $\theta : F_m \rightarrow \hat{T}$ satisfying conditions (M0), (M1), (M2) and a condition milder than (M2):

(M2') $\theta(x_i) = 1 + X_i + (\text{terms of degree } \geq 2)$ for any $1 \leq i \leq m$.

See [11] for more details.

4. The extended first Johnson homomorphism of a braid group.

The Artin representation (see [1, 3]) is an injective group homomorphism

$$\Phi : B_m \rightarrow \text{Aut}(F_m)$$

such that

$$\sigma_i \mapsto \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_k \mapsto x_k \quad \text{for } k \neq i, i+1 \end{cases}$$

and

$$\sigma_i^{-1} \mapsto \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_k \mapsto x_k \quad \text{for } k \neq i, i+1. \end{cases}$$

Through Φ , we regard B_m as a subgroup of $\text{Aut}(F_m)$. For any $\beta \in B_m$, the automorphism $|\Phi(\beta)| \in \text{Aut}(H)$ is identical to the action of the permutation $|\beta| \in S_m$.

We also recall the geometric background of the Artin representation, since we will use it in Section 5. Let $q_0 = (1/2, 0) \in \partial D$ and let x_i be (the homotopy class of) a loop in $D \setminus Q_m$ with base point q_0 which goes along the segment $\overline{q_0 q_i}$, turns around q_i clockwise and comes back along the segment. The fundamental group $\pi_1(D \setminus Q_m, q_0)$ is a free group with m generators $x_1, \dots, x_m$, and we naturally identify it with the free group F_m .

Since the braid group B_m is isomorphic to the mapping class group of the m -marked disk (D, Q_m) relative to the boundary (see [1, 3]), the group B_m acts naturally on $\pi_1(D \setminus Q_m, q_0) = F_m$. This action amounts to the homomorphism Φ . For example, the standard generator σ_i corresponds to the isotopy class of a self-homeomorphism of (D, Q_m) which twists a sufficiently small disk neighborhood of the segment $\overline{q_i q_{i+1}}$ by 180° counterclockwise. From this description, we can deduce the formula for $\Phi(\sigma_i)$. See Figure 4.

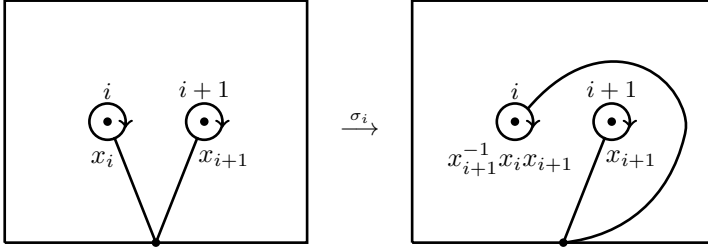


Fig. 4. The action of σ_i on generators $x_1, \dots, x_m$

We now consider the restriction of the extended first Johnson homomorphism to B_m . Let $\bigwedge^2 H$ be the second exterior power of H , which we regard as an $\text{Aut}(H)$ -submodule of $H^{\otimes 2}$ by $\bigwedge^2 H \rightarrow H^{\otimes 2}, X \wedge Y \mapsto X \otimes Y - Y \otimes X$.

The following proposition was proved by Kawazumi [12].

PROPOSITION 4.1 ([12, Lemma 2.1]). *For the i th standard generator σ_i of B_m , we have*

- (1) $\tau_1^\theta(\sigma_i)(X_i) = X_i \wedge X_{i+1}$ and $\tau_1^\theta(\sigma_i)(X_k) = 0$ if $k \neq i$;
- (2) $\tau_1^\theta(\sigma_i^{-1})(X_{i+1}) = X_i \wedge X_{i+1}$ and $\tau_1^\theta(\sigma_i^{-1})(X_k) = 0$ if $k \neq i+1$.

To demonstrate how the definition of τ_1^θ works, we show here the proof of the first formula in Proposition 4.1(1). We have $\theta_2(x_i) = 0$ and compute $\theta_2(\sigma_i^{-1}(x_i)) = \theta_2(x_i x_{i+1} x_i^{-1}) = \theta_2(x_i) + X_i \wedge X_{i+1} = X_i \wedge X_{i+1}$ using Lemma 3.1. Hence

$$\tau_1^\theta(\sigma_i)(X_i) = -|\sigma_i|^{\otimes 2}(\theta_2(\sigma_i^{-1}(x_i))) = -|\sigma_i|^{\otimes 2}(X_i \wedge X_{i+1}) = X_i \wedge X_{i+1}.$$

By Proposition 4.1 and the fact that τ_1^θ is a crossed homomorphism, we see that the restriction of τ_1^θ to B_m takes values in $\text{Hom}(H, \bigwedge^2 H)$.

Let $\delta : \text{Mat}_m^0 \rightarrow \text{Hom}(H, \bigwedge^2 H)$ be the map defined by

$$\delta(M)(X_i) := X_i \wedge f_i(M),$$

where $f_i(M) \in H$ is the element defined in Section 2. The map δ is injective

and S_m -equivariant. To see the S_m -equivariance, we compute

$$\begin{aligned}\delta(\pi(M))(X_i) &= X_i \wedge f_i(\pi(M)) \\ &= X_i \wedge \pi(f_{\pi^{-1}(i)}(M)) \\ &= \pi^{\otimes 2}(X_{\pi^{-1}(i)} \wedge f_{\pi^{-1}(i)}(M)) \\ &= (\pi \odot \delta(M))(X_i).\end{aligned}$$

Here, in the second line we have used Lemma 2.2.

THEOREM 4.2.

$$\tau_1^\theta = \delta \circ C : B_m \rightarrow \text{Hom}(H, \bigwedge^2 H).$$

In other words, for each $\beta \in B_m$ and for each element $i \in \{1, \dots, m\}$, we have $\tau_1^\theta(\beta)(X_i) = X_i \wedge f_i(\beta)$.

Proof. Since both the maps τ_1^θ and $\delta \circ C$ are crossed homomorphisms, it is sufficient to compare their values on the standard generators of B_m .

Let $i, j \in \{1, \dots, m\}$. By Proposition 4.1(1), $\tau_1^\theta(\sigma_j)(X_i) = X_i \wedge X_{i+1}$ if $j = i$, and $\tau_1^\theta(\sigma_j)(X_i) = 0$ otherwise. On the other hand, only the $(j+1, j)$ entry of the crossing matrix $C(\sigma_j)$ of σ_j is 1 and all the others are 0. Hence, $f_i(\sigma_j) = X_{i+1}$ if $j = i$, and $f_i(\sigma_j) = 0$ otherwise. Thus, $\tau_1^\theta(\sigma_j)(X_i) = X_i \wedge f_i(\sigma_j)$. This completes the proof. ■

We have the following commutative diagram:

$$\begin{array}{ccc} & B_m & \\ \tilde{C} \swarrow & & \searrow \tilde{\tau}_1^\theta \\ \text{Mat}_m^0 \rtimes S_m & \longrightarrow & \text{Hom}(H, \bigwedge^2 H) \rtimes \text{Aut}(H) \end{array}$$

Here, the horizontal arrow is induced from δ and the natural inclusion $S_m \rightarrow \text{Aut}(H)$. Thus we can rewrite the target of the restriction of the map $\tilde{\tau}_1^\theta$ to B_m as follows:

$$\tilde{\tau}_1^\theta : B_m \rightarrow \text{Hom}(H, \bigwedge^2 H) \rtimes S_m, \quad \beta \mapsto (\tau_1^\theta(\beta), |\beta|).$$

5. Half twists and cords in a disk. In this section, we describe the diving combinatorial information of half twists. Let us introduce a piece of notation: for two elements g and h in a group, we define the conjugate of g by h to be $g * h = h^{-1}gh$.

Recall that the standard generators σ_i of B_m are conjugate to each other since $\sigma_{i+1} = \sigma_i * (\sigma_{i+1}\sigma_i)$ for any $i \in \{1, \dots, m-2\}$. Any element of B_m conjugate to these generators (resp. the inverses of these generators) is called a *positive* (resp. *negative*) *half twist*. Let T_m^+ and T_m^- be the sets of positive (resp. negative) half twists in B_m . We also use the notation

$$T_m = T_m^+ \cup T_m^-$$

for the set of all half twists in B_m . Recall that $|\cdot| : B_m \rightarrow S_m$ is the natural projection. Then $|T_m| \subset S_m$ is the set of transpositions of S_m . We also remark that $|T_m| = |T_m^+| = |T_m^-|$. For $i, j \in \{1, \dots, m\}$ with $i < j$, let us denote by $(i\ j) \in S_m$ the transposition of i and j , and set

$$\begin{aligned} T(i\ j)_m^+ &= \{\beta \in T_m^+ \mid |\beta| = (i\ j)\}, \\ T(i\ j)_m^- &= \{\beta \in T_m^- \mid |\beta| = (i\ j)\}. \end{aligned}$$

We also set $T(i\ j)_m = T(i\ j)_m^+ \cup T(i\ j)_m^-$.

Half twists correspond to geometric objects in the m -marked disk (D, Q_m) , called *cords*. For $i, j \in \{1, \dots, m\}$ satisfying $i < j$, an (i, j) -cord on (D, Q_m) is an embedding $\gamma : [0, 1] \rightarrow (\text{Int } D \setminus Q_m) \cup \{q_i, q_j\}$ such that $\gamma(0) = q_i$ and $\gamma(1) = q_j$. Two cords γ_1 and γ_2 are *isotopic* if they are ambiently isotopic to each other by an isotopy of D that fixes Q_m and ∂D pointwise.

For an (i, j) -cord γ and a sign $\varepsilon \in \{\pm 1\}$, consider a self-homeomorphism of (D, Q_m) which twists a sufficiently small disk neighborhood of the simple arc $\gamma([0, 1])$ by 180° counterclockwise if $\varepsilon = +1$ and 180° clockwise if $\varepsilon = -1$. The corresponding braid is a half twist. This assignment gives rise to the following bijections:

$$\begin{aligned} T(i\ j)_m &\cong \{\text{isotopy classes of } (i, j)\text{-cords on } (D, Q_m)\} \times \{\pm 1\}, \\ T_m &\cong \{\text{isotopy classes of cords on } (D, Q_m)\} \times \{\pm 1\} \end{aligned}$$

which respect the signs. See [8, Lemma 4.1] or [2, Lemma 4.7]. For a half twist β we denote by γ_β the (isotopy class of the) corresponding cord, and we call γ_β a *cord of* β . For example, the cord of the standard generator σ_i is the path traversing the segment $\overline{q_i q_{i+1}}$ from q_i to q_{i+1} .

For each $k \in \{1, \dots, m\}$, take a small closed disk D_k centered at q_k , and let l_k be the loop which traverses the boundary of D_k clockwise. Also, let α_k be the path traversing the segment $\overline{q_0 q_k}$ from q_0 to q_k , where $q_0 = (1/2, 0) \in \partial D$. The first homology group $H_1(D \setminus Q_m)$ is naturally identified with $\pi_1(D \setminus Q_m, q_0)^{\text{abel}} = F_m^{\text{abel}} = H$ and the homology class $[l_k]$ with X_k , for $1 \leq k \leq m$.

Take $i, j \in \{1, \dots, m\}$ satisfying $i < j$. Let

$$D_{i,j} = D \setminus \bigsqcup_{k \neq i,j} \text{Int } D_k.$$

The first homology group $H_1(D_{i,j})$, which we may view as a subgroup of H , is generated by X_k for $k \in \{1, \dots, m\} \setminus \{i, j\}$. For an (i, j) -cord γ , let $\tilde{\gamma}$ be a loop with base point q_0 given by $\tilde{\gamma} = \alpha_i \cdot \gamma \cdot \alpha_j^{-1}$. We call its first homology class $[\tilde{\gamma}] \in H_1(D_{i,j})$ the *first homology class of* γ .

Let $\mathcal{H}_{i,j} = H_1(D_{i,j}) = \bigoplus_{k \neq i,j} \mathbb{Z}X_k$ and let $\tilde{\mathcal{H}} = \bigsqcup_{i < j} \mathcal{H}_{i,j}$. Let $v : T(i\ j)_m \rightarrow \mathcal{H}_{i,j} \times \{\pm 1\}$ be a map defined by $v(\beta) = ([\tilde{\gamma}_\beta], \varepsilon(\beta))$ for $\beta \in T(i\ j)_m$, where $\varepsilon(\beta) = +1$ if $\beta \in T(i\ j)_m^+$ or $\varepsilon(\beta) = -1$ if $\beta \in T(i\ j)_m^-$. This

map can be extended to a map

$$v : T_m \rightarrow \tilde{\mathcal{H}} \times \{\pm 1\}.$$

For example, let $\beta = \sigma_1 * (\sigma_2 \sigma_3^{-1} \sigma_4^{-2} \sigma_1^{-2}) \in T_5^+$ as on the left-hand side of Figure 5. Then a cord γ_β of β is shown on the right-hand side of the figure. We see that the first homology class $[\gamma_\beta]$ is $2X_2 - X_5$. Hence, $v(\beta) = (2X_2 - X_5, +1) \in \mathcal{H}_{1,4} \times \{\pm 1\}$.

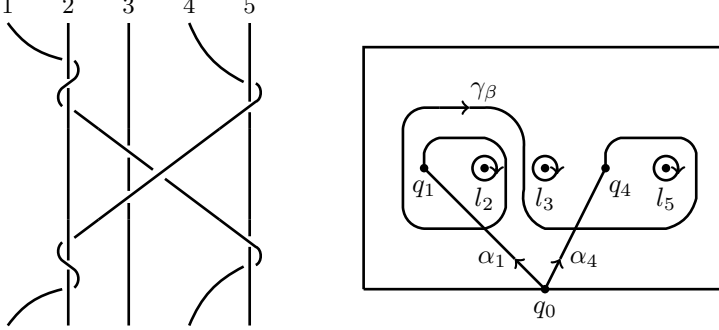


Fig. 5. The 5-braid $\beta = \sigma_1 * (\sigma_2 \sigma_3^{-1} \sigma_4^{-2} \sigma_1^{-2})$ and its cord γ_β .

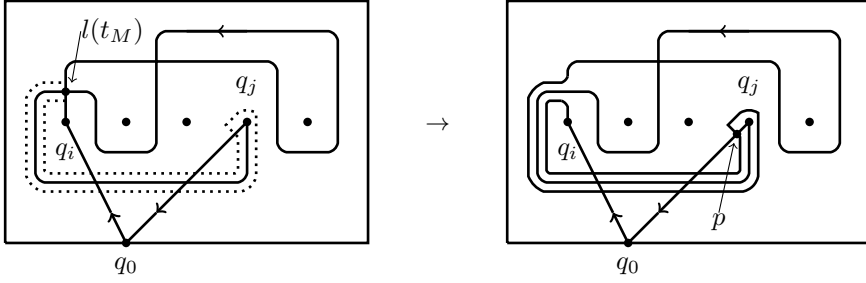
PROPOSITION 5.1. *For each $i, j \in \{1, \dots, m\}$ with $i < j$ and for each $\varepsilon \in \{\pm 1\}$, we have $v(T(i, j)_m^\varepsilon) = \mathcal{H}_{i,j} \times \{\varepsilon\}$. Therefore, the map $v : T_m \rightarrow \tilde{\mathcal{H}} \times \{\pm 1\}$ is surjective.*

Proof. We will show that any element in $\mathcal{H}_{i,j}$ is the homology class of some (i, j) -cord. Take an element $X \in \mathcal{H}_{i,j}$. Let $l : [0, 1] \rightarrow D_{i,j}$ be a loop based at q_0 whose homology class is X . By a suitable homotopy in $(D_{i,j}, q_0)$, we may assume that the following conditions are satisfied:

- l is an immersion with transverse double points only;
- $l(\frac{1}{3}) = q_i$, $l(\frac{2}{3}) = q_j$, and the curves $l([0, \frac{1}{3}])$ and $l([\frac{2}{3}, 1])$ are segments $\overline{q_0 q_i}$ and $\overline{q_0 q_j}$, respectively.

Let Σ be the set of self-intersections of the path $l([\frac{1}{3}, \frac{2}{3}])$. If $\Sigma = \emptyset$, we are done: a reparametrization γ of $l|_{[\frac{1}{3}, \frac{2}{3}]}$ is an (i, j) -cord with $l = \alpha_i \cdot \gamma \cdot \alpha_j^{-1}$, and hence the homology class of γ is $[l] = X$.

When $\Sigma \neq \emptyset$, we modify l as follows. Let t_M be the maximal number in the preimage $l^{-1}(\Sigma)$. There exists an ambient isotopy of $D_{i,j}$ fixing q_0 and $Q_m \setminus \{q_i, q_j\}$ pointwise which pushes out $l(t_M)$ through q_j , i.e. which slides the intersection $l(t_M)$ through the curve $l([t_M, \frac{2}{3}])$ towards q_j , passing through q_j and reaching the interior of $\overline{q_0 q_j}$. See Figure 6. Here, p stands for the result of pushing out $l(t_M)$ by the ambient isotopy. (See also [9].) This isotopy reduces the number of points in Σ . By repeating this process, we may assume that $\Sigma = \emptyset$. This completes the proof. ■


 Fig. 6. Pushing out $l(t_M)$ through q_j

For $k \in \{1, \dots, m\} \setminus \{i, j\}$, let ζ_k (resp. η_k) be the intersection of the vertical segment $\{k/(m+1)\} \times [1/2, 1]$ (resp. $\{k/(m+1)\} \times [0, 1/2]$) with $D_{i,j}$, oriented upwards. We use the same letters ζ_k and η_k for their homology classes in $H_1(D_{i,j}, \partial D_{i,j})$. Note that we have

$$\zeta_k + \eta_k = 0 \in H_1(D_{i,j}, \partial D_{i,j}).$$

Using the intersection pairing $(\cdot) : H_1(D_{i,j}) \times H_1(D_{i,j}, \partial D_{i,j}) \rightarrow \mathbb{Z}$, we can write

$$[\widetilde{\gamma}_\beta] = \sum_{k \neq i, j} ([\widetilde{\gamma}_\beta] \cdot \zeta_k) X_k.$$

Since the elements $f_i(\beta)$ recover the crossing matrix $C(\beta)$, the following theorem gives a formula for the crossing matrix of a half twist.

THEOREM 5.2. *Let $\beta \in T(i, j)_m$.*

- (1) *If β is positive, then $f_i(\beta) = [\widetilde{\gamma}_\beta] + X_j$ and $f_j(\beta) = -[\widetilde{\gamma}_\beta]$.*
- (2) *If β is negative, then $f_i(\beta) = [\widetilde{\gamma}_\beta]$ and $f_j(\beta) = -[\widetilde{\gamma}_\beta] - X_i$.*
- (3) *For $k \neq i, j$, we have*

$$f_k(\beta) = \begin{cases} ([\widetilde{\gamma}_\beta] \cdot \zeta_k - 1)(X_i - X_j) & \text{if } i < k < j, \\ ([\widetilde{\gamma}_\beta] \cdot \zeta_k)(X_i - X_j) & \text{if } k < i \text{ or } j < k. \end{cases}$$

Proof. For $k \in \{1, \dots, m\}$, D_k has been defined as a small disk centered at q_k . We may assume that all these disks have the same radius. For simplicity, we denote $\gamma = \gamma_\beta$. By a suitable isotopy, we can arrange that the image of γ lies in $D_{i,j}$ and that there exists a real number $\varepsilon > 0$ such that

- $\gamma(t) = q_i + t\overrightarrow{q_i q_j}$ for $t \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$,
- $\gamma(t) \notin D_i \cup D_j$ for $t \in [\varepsilon, 1 - \varepsilon]$,
- on the interval $[\varepsilon, 1 - \varepsilon]$, γ is transverse to δ_k and η_k for all $k \in \{1, \dots, m\} \setminus \{i, j\}$, as well as to the vertical segments $\delta_l^- := \{l/(m+1)\} \times [0, 1/2]$ and $\delta_l^+ := \{l/(m+1)\} \times (1/2, 1]$ for $l \in \{i, j\}$, both oriented upwards.

See Figure 7.

each contributing an amount equal to the negative of the sign of the intersection. By our arrangement of γ , the algebraic intersection number of γ with the union of δ_j^+ and δ_j^- is zero. Hence the contributions from I_1 and I_3 cancel, and the coefficient of X_j in $f_i(\beta)$ is 1.

In a similar manner, we see that the coefficient of X_i in $f_j(\beta)$ is zero. In this case, the contributions from I_1 and I_3 cancel as well, and we have no contribution from I_2 . This completes the proof of (1).

The case (2) is proved similarly, so we omit it.

(3) Similarly to the case (1), we have

$$f_k(\beta) = -(\gamma \cdot \eta_k)(X_i - X_j),$$

where $(\gamma \cdot \eta_k)$ is the number of positive intersections of γ and η_k minus the number of negative intersections of γ and η_k . Furthermore, since $\tilde{\gamma} = \alpha_i \cdot \gamma \cdot \alpha_j^{-1}$, we have

$$(\gamma \cdot \eta_k) = \begin{cases} ([\tilde{\gamma}] \cdot \eta_k) + 1 & \text{if } i < k < j, \\ ([\tilde{\gamma}] \cdot \eta_k) & \text{if } k < i \text{ or } j < k. \end{cases}$$

Since $([\tilde{\gamma}] \cdot \eta_k) = -([\tilde{\gamma}] \cdot \zeta_k)$, we obtain the desired formula. ■

COROLLARY 5.3. *Fix $i, j \in \{1, \dots, m\}$ with $i < j$ and $\varepsilon \in \{\pm 1\}$. For $\beta, \beta' \in T(i, j)_m$, we have $[\widetilde{\gamma_\beta}] = [\widetilde{\gamma_{\beta'}}]$ if and only if $\tau_1^\theta(\beta) = \tau_1^\theta(\beta')$.*

Proof. This follows from Theorems 4.2 and 5.2. ■

Recall the group homomorphism $\tilde{\tau}_1^\theta : B_m \rightarrow \text{Hom}(H, \wedge^2 H) \rtimes S_m$ defined by $\tilde{\tau}_1^\theta(\beta) = (\tau_1^\theta(\beta), |\beta|)$. For each $\varepsilon \in \{\pm 1\}$ and each element $(f, \varepsilon) \in \mathcal{H}_{i,j} \times \{\varepsilon\}$, there exists an element $\beta \in T_m^\varepsilon$ such that $(f, \varepsilon) = v(\beta) = ([\gamma_\beta], \varepsilon)$ by Proposition 5.1. By Corollary 5.3, the value $\tilde{\tau}_1^\theta(\beta)$ does not depend on the choice of β . Thus, the map

$$\mu : \mathcal{H}_{i,j} \times \{\varepsilon\} \rightarrow \tilde{\tau}_1^\theta(T(i, j)_m^\varepsilon), \quad \mu((f, \varepsilon)) = \tilde{\tau}_1^\theta(\beta),$$

where β is an element of T_m^ε such that $(f, \varepsilon) = v(\beta)$, is well-defined. This map is extended to a map $\mu : \tilde{\mathcal{H}} \times \{\pm 1\} \rightarrow \tilde{\tau}_1^\theta(T_m)$. By construction, the following diagram commutes:

$$\begin{array}{ccc} & T_m & \\ v \swarrow & & \searrow \tilde{\tau}_1^\theta \\ \tilde{\mathcal{H}} \times \{\pm 1\} & \xrightarrow{\mu} & \tilde{\tau}_1^\theta(T_m) \end{array}$$

By Corollary 5.3, we have the following proposition.

PROPOSITION 5.4. *The map $\mu : \mathcal{H}_{i,j} \times \{\varepsilon\} \rightarrow \tilde{\tau}_1^\theta(T(i, j)_m^\varepsilon)$ is bijective. Therefore, the map $\mu : \tilde{\mathcal{H}} \times \{\pm 1\} \rightarrow \tilde{\tau}_1^\theta(T_m)$ is bijective.*

In a future work, we plan to discuss an application of this proposition to the Hurwitz action of the braid group on the cartesian products of the set of half twists.

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