BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Online First version

HARMONIC ANALYSIS ON EUCLIDEAN SPACES

A note on the Muckenhoupt class $A_1(\mathbb{R})$

by

Eleftherios N. NIKOLIDAKIS and Andreas G. TOLIAS

Presented by Piotr BILER

Summary. We provide a direct proof of the best possible reverse Hölder inequality satisfied by every A_1-weight defined on $(0,1)$ with A_1-constant equal to $c > 1$.

1. Introduction. An integrable function $\varphi\colon (0,1) \to \mathbb{R}^+$ is called an A_1-*weight* if there exists $c > 1$ such that

$$\frac{1}{|I|}\int_I \varphi(x)\,dx \le c \operatorname*{ess\,inf}_{x\in I} \varphi(x) \tag{1.1}$$

for every subinterval I of $(0,1)$.

The minimum constant c for which (1.1) holds for all such intervals I is called the A_1-*constant* of φ and denoted by $[\varphi]_{A_1}$.

An equivalent definition of an A_1-weight is as follows: For every $\varphi \in L^1((0,1))$ the maximal function $\mathcal{M}\varphi$ is defined by

$$\mathcal{M}\varphi(x) = \sup\left\{\frac{1}{|I|}\int_I |\varphi| : x \in I,\ I \text{ is an open subinterval of } (0,1)\right\} \tag{1.2}$$

for every $x \in (0,1)$. It is easily seen that (1.1) is equivalent to the condition

$$\mathcal{M}\varphi(x) \le c\,\varphi(x) \quad \text{for almost every } x \in (0,1). \tag{1.3}$$

We denote by $A_1((0,1),c)$ the class of all positive functions on $(0,1)$ which satisfy (1.1) or equivalently (1.3).

In a similar manner one can define the class $A_1(J,c)$ for any interval $J \subset \mathbb{R}$ and any $c > 1$, as the collection of all functions $\varphi\colon J \to \mathbb{R}^+$ which

2020 *Mathematics Subject Classification*: Primary 42B25.
Key words and phrases: weight, maximal operator, reverse Hölder inequality.
Received 10 March 2026.
Published online 14 July 2026.

satisfy (1.1) for all subintervals I of J. For $\varphi \in L^1(J)$, $\mathcal{M}_J\varphi\colon J \to \mathbb{R}^+$ is defined by (1.2), where the supremum is now taken over all open subintervals of J that contain x. Then $\varphi \in A_1(J, c)$ if and only if $\mathcal{M}_J\varphi \le c\varphi$ almost everywhere on J.

For φ, J as above, the quantity $[\varphi]_{A_1}^J$, which is the A_1-constant of φ on the interval J, is defined analogously to the case $J = (0, 1)$. Consequently, $[\varphi]_{A_1}$ denotes $[\varphi]_{A_1}^{(0,1)}$ and $\mathcal{M}\varphi$ stands for $\mathcal{M}_{(0,1)}\varphi$, for any nonnegative $\varphi \in L^1((0, 1))$.

A notion that is connected to the class $A_1((0, 1), c)$ is that of the decreasing rearrangement of φ, which is denoted by φ^*, where $\varphi^*\colon (0, 1) \to \mathbb{R}$ is the unique decreasing, right continuous function on $(0, 1)$, equimeasurable to φ. The function φ^* can also be given by a specific formula (see for example [1, p. 39]).

It is proved in [2] that for any $\varphi \in A_1(((0, 1), c)$, the function φ^* also belongs to $A_1((0, 1), c)$, that is,

$$\frac{1}{t}\int_0^t \varphi^*(u)\,du \le c\varphi^*(t) \qquad \text{for every } t \in (0, 1). \tag{1.4}$$

Moreover, in [2] the following is proved.

Theorem 1.1. *If $\varphi\colon (0, 1) \to \mathbb{R}^+$ is in the class $A_1((0, 1), c)$, then the reverse Hölder inequality*

$$\frac{1}{|I|}\int_I \varphi^p(x)\,dx \le \frac{1}{c^{p-1}(c + p - cp)}\left(\frac{1}{|I|}\int_I \varphi(x)\,dx\right)^p \tag{1.5}$$

is satisfied for every interval $I \subset (0, 1)$ and every $p \in \left[1, \frac{c}{c-1}\right)$.

Moreover, the constant on the right-hand side of (1.5) is the best possible such that the inequality is satisfied for every weight φ with A_1-constant equal to c and all p with $1 \le p < \frac{c}{c-1}$.

The above theorem in fact implies that the best possible range of values of p for which an A_1-weight with A_1-constant $[\varphi]_{A_1}$ equal to c is L^p-integrable on $(0, 1)$ is the interval $\left[1, \frac{c}{c-1}\right)$.

For the above two results, alternative proofs have been given in [4]. The proofs in [2] and [4] use the notion of the decreasing rearrangement of φ.

In this short note we provide a direct proof of the above theorem without passing through the notion of decreasing rearrangement. Lastly, we should mention that the dyadic version of the above problem has been studied and completely solved in [3].

2. Proof of Theorem 1.1. Let $\varphi\colon (0, 1) \to \mathbb{R}^+$ be an integrable function such that $\mathcal{M}\varphi \le c\varphi$ almost everywhere on $(0, 1)$. We let $f := \int_0^1 \varphi(x)\,dx$

and for every $\lambda > 0$ we define

$$E_\lambda = [\mathcal{M}\varphi > \lambda] = \{x \in (0,1) : \mathcal{M}\varphi(x) > \lambda\}.$$

Note that for $\lambda \in (0, f)$ we have $E_\lambda = (0,1)$. Suppose now that $\lambda \geq f$. By definition the set E_λ is open, thus it can be written as an at most countable disjoint union $\bigcup_{j \in D_\lambda} I_j$ of open subintervals of $(0,1)$. We consider the following two cases:

Case 1: $(I_j)_{j \in D_\lambda} = \{(0,1)\}$, so that $E_\lambda = (0,1)$. In this case

$$|E_\lambda| = 1 \geq \frac{f}{\lambda} = \frac{\int_{E_\lambda} \varphi(x)\, dx}{\lambda}.$$

Case 2: The family $(I_j)_{j \in D_\lambda}$ differs from $\{(0,1)\}$. Then for each $j \in D_\lambda$ and each $\delta > 1$ close enough to 1, one can define an open subinterval $I_{j,\delta}$ of $(0,1)$ for which $I_j \subset I_{j,\delta}$ and $|I_{j,\delta}| = \delta |I_j|$.

Then, for each such j and δ, $I_{j,\delta}$ is not contained in E_λ, thus there exists a point $x_{j,\delta} \in I_{j,\delta}$ for which $\mathcal{M}\varphi(x_{j,\delta}) \leq \lambda$. Hence, by definition of $\mathcal{M}\varphi$, we get

$$\frac{1}{|I_{j,\delta}|} \int\limits_{I_{j,\delta}} \varphi(x)\, dx \leq \lambda$$

for every $\delta > 1$ sufficiently close to 1. Letting $\delta \to 1^+$ shows that

$$\frac{1}{|I_j|} \int\limits_{I_j} \varphi(x)\, dx \leq \lambda.$$

Since this holds for every $j \in D_\lambda$, and $(I_j)_{j \in D_\lambda}$ are pairwise disjoint with union E_λ, we obtain $\frac{1}{|E_\lambda|} \int_{E_\lambda} \varphi(x)\, dx \leq \lambda$, which implies that

$$|E_\lambda| \geq \frac{1}{\lambda} \int\limits_{E_\lambda} \varphi(x)\, dx. \tag{2.1}$$

We have just proved that inequality (2.1) holds for any $\lambda \geq f = \int_0^1 \varphi(x)\, dx$.

Consider now any $p \in \left[1, \frac{c}{c-1}\right)$. Multiplying (2.1) with $p\lambda^{p-1}$ and integrating on $[f, +\infty)$ we obtain

$$\int\limits_f^{+\infty} p\lambda^{p-1} |E_\lambda|\, d\lambda \geq \int\limits_f^{+\infty} p\lambda^{p-2} \Big(\int\limits_{E_\lambda} \varphi(x)\, dx \Big)\, d\lambda. \tag{2.2}$$

By Fubini's theorem and taking into account that $E_\lambda = [\mathcal{M}\varphi > \lambda]$ we get

$$\int\limits_0^1 (\mathcal{M}\varphi(x))^p\, dx = \int\limits_0^{+\infty} p\lambda^{p-1} |E_\lambda|\, d\lambda. \tag{2.3}$$

Using (2.3) and (2.2) and taking into account that $E_\lambda = (0,1)$ when $0 < \lambda < f$ we get

$$(2.4)\qquad \int_0^1 (\mathcal{M}\varphi(x))^p\,dx = \int_0^f p\lambda^{p-1}|E_\lambda|\,d\lambda + \int_f^{+\infty} p\lambda^{p-1}|E_\lambda|\,d\lambda$$
$$\geq \int_0^f p\lambda^{p-1}\,d\lambda + \int_f^{+\infty} p\lambda^{p-2}\Big(\int_{E_\lambda}\varphi(x)\,dx\Big)\,d\lambda$$
$$= f^p + \int_f^{+\infty}\int_{E_\lambda} p\,\lambda^{p-2}\varphi(x)\,dx\,d\lambda.$$

We now apply Fubini's theorem to the integral on the right-hand side of (2.4):

$$(2.5)\qquad \int_f^{+\infty}\int_{E_\lambda} p\,\lambda^{p-2}\varphi(x)\,dx\,d\lambda = \int_{E_f}\varphi(x)\int_f^{\mathcal{M}\varphi(x)} p\,\lambda^{p-2}\,d\lambda\,dx$$
$$= \int_{E_f}\varphi(x)\frac{p}{p-1}\big((\mathcal{M}\varphi(x))^{p-1} - f^{p-1}\big)\,dx$$
$$= \int_0^1 \frac{p}{p-1}\varphi(x)\big((\mathcal{M}\varphi(x))^{p-1} - f^{p-1}\big)\,dx$$
$$= -\frac{p}{p-1}f^p + \frac{p}{p-1}\int_0^1 \varphi(x)(\mathcal{M}\varphi(x))^{p-1}\,dx,$$

where we have taken into account that $\mathcal{M}\phi(x) \geq f$ for all $x \in (0,1)$.

From (2.4) and (2.5) we get

$$\int_0^1 (\mathcal{M}\varphi(x))^p\,dx \geq -\frac{f^p}{p-1} + \frac{p}{p-1}\int_0^1 \varphi(x)(\mathcal{M}\varphi(x))^{p-1}\,dx$$

and thus

$$(2.6)\qquad \int_0^1 \{p\varphi(x)(\mathcal{M}\varphi(x))^{p-1} - (p-1)(\mathcal{M}\varphi(x))^p\}\,dx \leq f^p.$$

Consider now, for each $y > 0$, the function $h_y\colon [y,+\infty) \to \mathbb{R}$ given by

$$h_y(t) = pyt^{p-1} - (p-1)t^p.$$

We notice that

$$h_y'(t) = p(p-1)yt^{p-2} - p(p-1)t^{p-1} = p(p-1)t^{p-2}(y-t) < 0$$

for all $t > y$, thus h_y is strictly decreasing on $[y,+\infty)$. Hence for every t with $y \leq t \leq cy$, we have

$$(2.7)\qquad h_y(t) \geq h_y(cy) = y^p c^{p-1}(c+p-cp).$$

We also notice that

$$\varphi(x) \le \mathcal{M}\varphi(x) \le c\varphi(x) \quad \text{for almost every } x \in (0,1),$$

where the left-hand inequality follows by Lebesgue's differentiation theorem for $L^1(0,1)$-functions, while the right-hand inequality follows from our assumption that $\varphi \in A_1((0,1),c)$. Applying (2.7) for $y = \varphi(x)$ and $t = \mathcal{M}\varphi(x)$, we find that for almost all $x \in (0,1)$,

$$p\varphi(x)(\mathcal{M}\varphi(x))^{p-1} - (p-1)(\mathcal{M}\varphi(x))^p \ge \varphi^p(x)c^{p-1}(c+p-cp). \tag{2.8}$$

Combining now (2.6) and (2.8), we deduce that

$$c^{p-1}(c+p-cp)\int_0^1 \varphi^p(x)\,dx \le f^p = \Big(\int_0^1 \varphi(x)\,dx\Big)^p. \tag{2.9}$$

Since $p \in \big[1, \frac{c}{c-1}\big)$, we see that $c+p-cp > 0$, and therefore from (2.9) we get

$$\int_0^1 \varphi^p(x)\,dx \le \frac{1}{c^{p-1}(c+p-cp)}\Big(\int_0^1 \varphi(x)\,dx\Big)^p,$$

which is (1.5) for the whole interval $I = (0,1)$. Now if one considers an arbitrary interval $I \subset (0,1)$, then, by defining $\varphi_I := \varphi|I \colon I \to \mathbb{R}^+$, we easily see that

$$\mathcal{M}_I\varphi_I \le c\varphi_I \quad \text{a.e. in } I.$$

Using then the same arguments as in the above proof, but now working on I, we get inequality (1.5) for this specific I, concluding the proof of the reverse Hölder inequality.

Finally, we prove the sharpness of (1.5). Let $c > 1$ and consider the function $\varphi\colon (0,1) \to \mathbb{R}^+$ given by

$$\varphi(t) = t^{-1+1/c}, \qquad t \in (0,1).$$

Then $\int_0^1 \varphi(t)\,dt = c$, and $\frac{1}{t}\int_0^t \varphi(u)\,du = c\varphi(t)$ for all $t \in (0,1)$, and since φ is decreasing on $(0,1)$, we find that $[\varphi]_{A_1} = c$.

Moreover, $\int_0^1 \varphi^p(t)\,dt = \frac{1}{1+(-1+1/c)p}$ for any $p \in \big[1, \frac{c}{c-1}\big)$. Thus for any p in that interval we have

$$\frac{\int_0^1 \varphi^p(t)\,dt}{(\int_0^1 \varphi(t)\,dt)^p} = \frac{1}{c^p(1+(-1+1/c)p)} = \frac{1}{c^{p-1}(c+p-cp)},$$

which gives us the sharpness of (1.5) for any $p \in \big[1, \frac{c}{c-1}\big)$. ∎

References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, 1988.

[2] B. Bojarski, C. Sbordone and I. Wik, *The Muckenhoupt class* $A_1(\mathbb{R})$, Studia Math. 101 (1992), 155–163.

[3] A. Melas, *A sharp L^p-inequality for dyadic A_1 weights on $\mathbb{R}^n$*, Bull. London Math. Soc. 37 (2005), 919–926.

[4] E. N. Nikolidakis, *A_1 weights on $\mathbb{R}$, an alternative approach*, Ann. Acad. Sci. Fenn. Math. 40 (2015), 343–355.

Eleftherios N. Nikolidakis, Andreas G. Tolias
Department of Mathematics
University of Ioannina
Ioannina 45110, Greece
E-mail: enikolid@uoi.gr
atolias@uoi.gr