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WIDTHS OF REGULAR COMPONENTS FOR AN n-REGULAR TREE $T(n)$

BY

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Abstract. Let $(T(n), \Omega)$ be the covering of the generalized Kronecker quiver $K(n)$, where Ω is a bipartite orientation. Given a regular Auslander–Reiten component $\mathcal{D}$ of $\mathrm{mod}(T(n), \Omega)$, we introduce two invariants: the width $\mathcal{W}(\mathcal{D})$ and the number $b(\mathcal{D})$ of flow modules. We show that

$$\mathcal{W}(\mathcal{D}) \geq \frac{b(\mathcal{D})+1}{2}.$$

In particular, we get $\{\mathcal{W}(\mathcal{D}) \mid \mathcal{D} \text{ is a regular component}\} = \mathbb{N}$.

1. Introduction. Let k be an algebraically closed field. The generalized Kronecker quiver $K(n)$ is just the quiver with two vertices 1, 2, and n arrows $\gamma_1, \dots, \gamma_n : 1 \to 2$. It is popular to study the representations of $K(n)$ when studying finite groups, moduli spaces, quiver Grassmannians, and in many other areas of mathematics (see [14, 1, 9]). Let $\mathrm{rep}_k(K(n))$ denote the category of finite-dimensional representations of $K(n)$ and let $\mathcal{K}_n = kK(n)$ be the *path algebra* of $K(n)$ (see 2.2). If $\mathrm{mod}\,\mathcal{K}_n$ denotes the category of finite-dimensional modules over $\mathcal{K}_n$, then there exists an equivalence between the categories $\mathrm{rep}_k(K(n))$ and $\mathrm{mod}\,\mathcal{K}_n$. We will identify these two categories frequently.

When $n \leq 2$, it is known how to classify the indecomposable modules in $\mathrm{mod}\,\mathcal{K}_n$ [13, XIX.3]. When $n \geq 3$, however, it is hopeless to classify all the indecomposable modules since the representation type is wild (see [13, XVIII], [6, 1.3]). Hence it is desirable to find invariants in the category $\mathrm{mod}\,\mathcal{K}_n$. J. Worch introduced the *width* $\mathcal{W}(\mathcal{C})$ for a regular component $\mathcal{C}$ of $\mathrm{mod}\,\mathcal{K}_n$ in 2013, and she showed that $\mathcal{W}(\mathcal{C}) \geq 0$ [14, Section 3]. C. M. Ringel found two invariants: *the smallest radius* and *the number of flow modules* for the covering quiver of $K(n)$ in 2018 (see [10, 11, 4]). Then it is natural to ask what the width of a regular component $\mathcal{D}$ of the covering is.

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From now on, we let $n \geq 3$. Let $M \in \operatorname{mod} \mathcal{K}_n$. Then we have a linear operator $x_\alpha^M = \alpha_1 M(\gamma_1) + \alpha_2 M(\gamma_2) + \cdots + \alpha_n M(\gamma_n) : M_1 \to M_2$ for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n \setminus \{0\}$. Thus we can define two full subcategories of $\operatorname{mod} \mathcal{K}_n$ with the equal kernels property and with the equal images property (see [14, Definition 2.1]): $\mathrm{EKP}_n := \{M \in \operatorname{mod} \mathcal{K}_n \mid \forall \alpha \in k^n \setminus \{0\} : x_\alpha^M \text{ is injective}\}$ and $\mathrm{EIP}_n := \{M \in \operatorname{mod} \mathcal{K}_n \mid \forall \alpha \in k^n \setminus \{0\} : x_\alpha^M \text{ is surjective}\}$. One shows that $\mathrm{EIP}_n \cap \mathrm{EKP}_n = (0)$, and the regular Auslander–Reiten (AR) components of the category $\operatorname{mod} \mathcal{K}_n$ are of the form $\mathbb{Z}A_\infty$ (see [8]). Let $M \in \operatorname{mod} \mathcal{K}_n$ be an indecomposable module and let τ be the Auslander–Reiten translation [2, IV.2.3]. Then a module M is said to be *regular* if $\tau^m(M) \neq 0$ for any $m \in \mathbb{Z}$. Suppose that $M \in \mathrm{EKP}_n$ is regular. Then there exists some positive integer m such that $\tau^m M \in \mathrm{EIP}_n$ (see [14, Section 3]). This suggests looking for an invariant connecting $\mathrm{EIP}_n \cap \mathcal{C}$ and $\mathrm{EKP}_n \cap \mathcal{C}$ for every regular AR component $\mathcal{C}$ of $\operatorname{mod} \mathcal{K}_n$. Then we can define the *width* $\mathcal{W}(\mathcal{C})$ (≥ 0) of $\mathcal{C}$ [14, Section 3]. In this note, we investigate the width from another point of view using covering theory.

Let $(T(n), \Omega)$ be the universal covering of $K(n)$ [3, Section 7], i.e., $T(n)$ is an n-regular tree with a bipartite orientation Ω (see 2.2). Let $\operatorname{mod}(T(n), \Omega)$ be the category of finite-dimensional representations of $(T(n), \Omega)$. Then $\operatorname{mod}(T(n), \Omega)$ also admits AR-sequences (see [4, 2.2]). Similarly, we have widths for the regular components $\mathcal{D}$ of $\operatorname{mod}(T(n), \Omega)$. C. M. Ringel defined an invariant $b(\mathcal{D})$ (the number of flow modules) for the components $\mathcal{D}$ in 2018 (see [10]). We study the components $\mathcal{D}$ and show that $\mathcal{W}(\mathcal{D}) > 0$.

2. Preliminaries. Throughout, k will denote an algebraically closed field.

2.1. Trees. A *graph* $G = (G_0, G_1)$ consists of two sets: G_0 (the set of *vertices*) and G_1 (the set of *edges*). Two vertices $a, b \in G_0$ are called *neighbours* if $\{a, b\}$ is an edge in G_1. For a graph G, by an *orientation* we mean a map $\Omega : G_1 \to G_0 \times G_0$ such that $\Omega(\{a, a'\})$ is either (a, a') or (a', a). We call (G, Ω) an *oriented graph*. We write $a \to a'$ if $\Omega(\{a, a'\}) = (a, a')$ and call a the *start*, a' the *target*, and $a \to a'$ the *arrow*. Further we call a a *sink* (or a *source*) if it is not a start (or the target, respectively) of any arrow. If any vertex is a sink or a source, then the orientation Ω will be called *bipartite*. A *subgraph* of a graph $G = (G_0, G_1)$ is a graph $G' = (G'_0, G'_1)$ such that $G'_0 \subseteq G_0$ and $G'_1 \subseteq G_1$. A *walk* of length $t \geq 0$ with *start* a_0 and *target* a_t in a graph G is a finite sequence

$$l_t = (a_0 \mid \{a_0, a_1\}, \{a_1, a_2\}, \ldots, \{a_{t-1}, a_t\} \mid a_t)$$

such that a_{i-1} and a_i are neighbours for $1 \leq i \leq t$, and $a_{j-1} \neq a_{j+1}$ for $1 \leq j < t$. We sometimes use $(a_0, a_1, \ldots, a_t)$ to denote the walk l_t if there only exists one walk connecting a_0 and a_t. The graph G is called *finite* if G_0

and G_1 are finite sets, and *connected* if for any two vertices $a, b \in G_0$, there exists a walk that connects them. A graph G is said to be a *tree* if it is connected and there only exist walks of length 0 that connect a vertex with itself. Moreover, for any two different vertices a, b in a tree G we can only find one walk that connects them. Suppose that G is a finite tree. Then we call the walks of maximal possible length the *diameter walks* of G, and we use $d(G)$ to denote their length. Let $l_t = (a_0, a_1, \ldots, a_t)$ be a walk in G. If $t = 2r$, then we call a_r or $\{a_r\}$ the *centre* of l_t and r its *radius*. If $t = 2r+1$, then we call $\{a_r, a_{r+1}\}$ the *centre* of l_t and r its *radius*. In fact, all diameter walks of a finite tree G have the same centre [10, 2.1]. Hence we take the centre and the radius of diameter walks of a finite tree G as the centre and the radius of the G itself and we use $C(G)$ and $r(G)$ to denote them, respectively.

2.2. Quivers. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where $s, t : Q_1 \to Q_0$. The elements in Q_0 are called *vertices* or *points*, and the elements in Q_1 are called *arrows*, respectively. For every arrow $\alpha \in Q_1$, we call the vertex $s(\alpha)$ its *start*, and the vertex $t(\alpha)$ its *target*. Note that an oriented graph (G, Ω) can be seen as a quiver without loops and multiple arrows.

Let Q be a quiver, and let $x \in Q_0$. We define two sets

$$x^+ := \{y \in Q_0 \mid \exists \alpha \in Q_1 : t(\alpha) = y \text{ and } s(\alpha) = x\},$$
$$x^- := \{y \in Q_0 \mid \exists \alpha \in Q_1 : s(\alpha) = y \text{ and } t(\alpha) = x\}.$$

The *neighbourhood* $\mathcal{N}(x)$ of x is $x^+ \cup x^-$. We say that x is a *leaf* if x has at most one neighbour. A sequence $(a \mid \alpha_1, \ldots, \alpha_r \mid b)$ in Q is called a *path* with start a and target b if $s(\alpha_1) = a$, $t(\alpha_r) = b$, and $t(\alpha_k) = s(\alpha_{k+1})$ for each $1 \leq k < r$. Moreover, each vertex $x \in Q_0$ is a *trivial path* of length 0. The quiver Q is said to be *locally bounded* if for every $x \in Q_0$ the neighbourhood $\mathcal{N}(x)$ is finite and there exists $m_x \in \mathbb{N}$ such that each path in Q which starts or ends in x is of length $\leq m_x$. In this note, when we talk about a quiver Q, we always assume that it is locally bounded.

Let $\delta_{a,b}$ be the Kronecker delta, and let $l_1 = (a_0 \mid \alpha_1, \ldots, \alpha_p \mid a_p)$ and $l_2 = (b_0 \mid \beta_1, \ldots, \beta_q \mid b_q)$ be two paths in Q. We define the product of l_1 and l_2 by

$$l_1 l_2 = \delta_{a_p, b_0}(a_0 \mid \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \mid b_q).$$

Then we define the k-algebra kQ to be the k-linear span of all paths of length $l \geq 0$ (see [2, II.1]). We call kQ the *path algebra* of Q. When Q_0 is finite, kQ is an associative and finite-dimensional k-algebra. A *finite-dimensional representation* $M = ((M_x)_{x \in Q_0}, (M(\alpha))_{\alpha \in Q_1})$ of Q consists of vector spaces M_x and k-linear maps $M(\alpha) : M_{s(\alpha)} \to M_{t(\alpha)}$ such that $\dim_k M := \sum_{x \in Q_0} \dim_k M_x$ is finite. A *morphism* $f : M \to N$ between two representations is a collection $(f_z)_{z \in Q_0}$ of k-linear maps such that for each arrow $\alpha : x \to y$, we have $f_y \circ M(\alpha) = N(\alpha) \circ f_x$. We write $\operatorname{rep}_k(Q)$ for the category of finite-dimensional representations of Q over k. If $\operatorname{mod} kQ$ denotes the category of finite-dimensional left kQ-modules, then the categories $\operatorname{mod} kQ$ and $\operatorname{rep}_k(Q)$ are equivalent (see [2, III 1.6]).

We use $(T(n), \Omega)$ to denote the universal covering of $K(n)$ with a bipartite orientation Ω [3, Section 7]. Moreover, $(T(n), \Omega)_0$ and $(T(n), \Omega)_1$ are the sets of vertices and arrows in the quiver $(T(n), \Omega)$, respectively. Then there exists a push-down functor $\pi_\lambda : \operatorname{rep}_k(T(n), \Omega) \to \operatorname{rep}_k(K(n))$, and π_λ is exact (see [3]). We follow the notations of [3, Section 7] and give the following example.

Example 1. Let $M \in \operatorname{mod}(T(3), \Omega)$ be

$$\begin{array}{ccccccccccc}
0 \longrightarrow & k & \xleftarrow{\gamma_1^1} & k & \xrightarrow{\gamma_3^1} & k & \xleftarrow{\gamma_1^2} & k & \xrightarrow{\gamma_3^2} & k & \longleftarrow 0 \\
& \uparrow & & \downarrow{\scriptstyle \gamma_2^1} & & \uparrow & & \downarrow{\scriptstyle \gamma_2^2} & & \uparrow & \\
& 0 & & k & & 0 & & k & & 0 &
\end{array}$$

where every $M(\gamma_j^i)$ is the identity map and $\pi(\gamma_j^i) = \gamma_j$, $i, j \in \{1, 2, 3\}$. In fact, M is indecomposable [12, Proposition 1]. Moreover, $\pi_\lambda(M) = N$, where

$$N = \begin{tikzcd} k^2 \arrow[r, bend left, "\gamma_1"] \arrow[r, "\gamma_2"] \arrow[r, bend right, "\gamma_3"'] & k^5 \end{tikzcd}$$

with

$$N(\gamma_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad N(\gamma_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad N(\gamma_3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let σ_x be the reflection functor of a sink $x \in (T(n), \Omega)_0$ (see [2, VII.5]). For any two sinks $x, y \in (T(n), \Omega)_0$, we have $\sigma_x \sigma_y = \sigma_y \sigma_x$. Then we can define a reflection functor $\sigma : \operatorname{rep}_k(T(n), \Omega) \to \operatorname{rep}_k(T(n), \sigma\Omega)$ at all sinks, where $\sigma\Omega$ is the opposite orientation. The functor σ is independent of the order used in the composition and acts on the finite-dimensional representations, so that for any non-trivial representation $M \in \operatorname{rep}_k(T(n), \Omega)$ there exist sinks $x_1, \ldots, x_M$ such that $\sigma(M) = \sigma_{x_1} \ldots \sigma_{x_M}(M)$. Hence the functor σ is well-defined. We sometimes call the functor σ the *shift functor*. Similarly, we have a reflection functor σ^- at all sources of $(T(n), \Omega)_0$.

We call the representations in $\operatorname{rep}_k(T(n), \Omega)$ the *graded Kronecker modules* (or *graded modules*, or simply *modules*), and we use $\operatorname{mod}(T(n), \Omega)$ to denote the category of those modules.

Definition. Let $M \in \operatorname{mod}(T(n), \Omega)$ be an indecomposable module. We say that M is *regular* if $\sigma^t(M) \neq 0$ for any $t \in \mathbb{Z}$.

In fact, an indecomposable module $M \in \mathrm{mod}(T(n), \Omega)$ is regular if and only if $\pi_\lambda(M)$ is regular (see [2, VIII.2.1, VIII.2.14], [3, Corollary 7.2]). Moreover, this can be checked by looking at the dimension vector of $\pi_\lambda(M)$ (see [5, 15]). Hence one can show that M is regular in Example 1 since $\pi_\lambda(M) = N$ is regular.

The functors σ, σ^- send regular indecomposable modules to regular indecomposable modules [7, Lemma 3.1.2]. Furthermore, the functors σ and σ^- induce quasi-inverse equivalences of the k-linear full subcategories of $\mathrm{mod}(T(n), \Omega)$ and $\mathrm{mod}(T(n), \sigma\Omega)$ consisting of regular modules [7, Lemma 3.2.7]. Let $M \in \mathrm{mod}(T(n), \Omega)$ be a regular indecomposable module. Then $\sigma\sigma^-(M) \cong \sigma^-\sigma(M) \cong M$ and $\sigma^2 M \cong \tau M$, where τ is the Auslander–Reiten translation [10, 2.6]. Let $D_{T(n)} : \mathrm{mod}(T(n), \Omega) \to \mathrm{mod}(T(n), \sigma\Omega)$ be the duality as in [3, 7.2]. Then we have $D_{T(n)} \circ \sigma(M) \cong \sigma^- \circ D_{T(n)}(M)$, $D_{T(n)} \circ \sigma^-(M) \cong \sigma \circ D_{T(n)}(M)$ [7, Lemma 3.2.8].

2.3. Graded Kronecker modules. Let $\emptyset \neq S \subseteq (T(n), \Omega)_0$ be a set of vertices. We use $T(S)$ to denote the unique minimal tree in $(T(n), \Omega)$ containing S. Let $x \in (T(n), \Omega)_0$, and let $M \in \mathrm{mod}(T(n), \Omega)$. Define $\mathrm{supp}(M) := \{y \in (T(n), \Omega)_0 \mid M_y \neq (0)\}$, and call it the *support* of M. Let $T(M) = T(\mathrm{supp}(M))$. Then the vertex x is said to be a *leaf* of M if x is a leaf of $T(M)$. Suppose that M is indecomposable. Then $T(M)$ is a finite tree, and we can define $d(M) = d(T(M))$, $C(M) = C(T(M))$ and $r(M) = r(T(M))$, and call them the *diameter*, the *centre*, and the *radius* of M, respectively [10, 2.4]. The module M is said to be a *sink* (or *source*) *module* if the diameter walks of $T(M)$ start and end in sinks (or sources), and M is a *flow module* when $d(M)$ is odd. We can actually find some $i \in \mathbb{Z}$ such that $M_0 = \sigma^{-i} M$ is a sink module with the smallest possible radius [10, Theorem 1]. We define the *index* $\iota(M) := t$ when $M = \sigma^t M_0$ for some $t \in \mathbb{Z}$.

Let $M \in \mathrm{mod}(T(n), \Omega)$ be a sink (or source) module with centre c and radius $r \geq 1$. Then M is said to be *complete* if for any walk $(x(0), x(1), \ldots, x(r))$ in $(T(n), \Omega)$ with $x(r) = c$ such that $x(i) \in \mathrm{supp}(M)$ for all $i \in \{1, \ldots, r\}$, we have $\dim_k M_{x(0)} = \dim_k M_{x(1)}$. When M is a flow module with radius $r \geq 1$, we say that M is *complete* if for any walk $(x(0), x(1), \ldots, x(r))$ in $(T(n), \Omega)_0$ with $x(r) \in C(M)$, $x(r-1) \notin C(M)$ and $x(i) \in \mathrm{supp}(M)$ for all $i \in \{1, \ldots, r\}$, we have $\dim_k M_{x(0)} = \dim_k M_{x(1)}$. Otherwise, we call an indecomposable module M *incomplete* [10, 2.5].

2.4. Regular components. We give a short introduction to Auslander–Reiten quivers for finite-dimensional k-algebras. The readers can refer to [4] and [2, IV.4] for locally bounded categories.

Let $\mathcal{A}$ be a finite-dimensional k-algebra, and let $\mathrm{mod}\,\mathcal{A}$ be the category of all finite-dimensional left modules of $\mathcal{A}$. Let $X, Y \in \mathrm{mod}\,\mathcal{A}$ be indecomposable. We say that a homomorphism $f : X \to Y$ is *irreducible* if it is neither a split monomorphism nor a split epimorphism, and if $f = f_1 \circ f_2$, then either f_1 is a split epimorphism or f_2 is a split monomorphism. Let $\mathrm{rad}_{\mathcal{A}}$ be the radical of $\mathrm{mod}\,\mathcal{A}$. Then we call the set

$$\mathrm{Irr}(X,Y) := \mathrm{rad}_{\mathcal{A}}(X,Y)/\mathrm{rad}^2_{\mathcal{A}}(X,Y)$$

the *space of irreducible morphisms* [2, IV.1.6]. In fact, a homomorphism $f : X \to Y$ is irreducible if and only if $f \in \mathrm{rad}_{\mathcal{A}}(X,Y) \setminus \mathrm{rad}^2_{\mathcal{A}}(X,Y)$.

Definition. The *Auslander–Reiten quiver* (*translation quiver*) $\Gamma(\mathcal{A})$ of $\mathcal{A}$ is defined as follows:

(a) The points $\Gamma(\mathcal{A})_0$ correspond to the isomorphism classes $[X]$ of indecomposable modules X in $\mathrm{mod}\,\mathcal{A}$.
(b) Let $[X], [Y]$ be two points in $\Gamma(\mathcal{A})_0$. Then there are $\dim_k \mathrm{Irr}(X,Y)$ arrows from $[X]$ to $[Y]$ in $\Gamma(\mathcal{A})_1$.

Let $X, Y \in \mathrm{mod}\,\mathcal{A}$ be indecomposable. We say that X is a *predecessor* of Y if there exists a path from $[X]$ to $[Y]$ in $\Gamma(\mathcal{A})$, i.e. a chain of irreducible maps from X to Y. We say that X is a *successor* of Y if there exists a path from $[Y]$ to $[X]$ in $\Gamma(\mathcal{A})$, i.e. a chain of irreducible maps from Y to X. We define the *cone* $(X \to)$ to be the set of all successors of X, and $(\to Y)$ to be the set of all predecessors of Y.

Let X be a non-projective indecomposable module (or let Y be a non-injective indecomposable module). Then there exists a unique short exact sequence, called the *Auslander–Reiten sequence*, or simply *AR-sequence* (or *almost split sequence*)

$$0 \to Y \xrightarrow{f} \bigoplus_{i=1}^{t} M_i^{n_i} \xrightarrow{g} X \to 0,$$

where Y (X, respectively) is indecomposable, and the modules M_i are pairwise non-isomorphic and indecomposable. Suppose that

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_t \end{bmatrix},\ g = [g_1, \dots, g_t], \quad \text{where} \quad f_i = \begin{bmatrix} f_{i1} \\ \vdots \\ f_{in_i} \end{bmatrix},\ g_i = [g_{i1}, \dots, g_{in_i}].$$

Then the maps $f_{i1}, f_{i2}, \dots, f_{in_i} : Y \to M_i$ and $g_{i1}, \dots, g_{in_i} : M_i \to X$ correspond to bases of $\mathrm{Irr}(Y, M_i)$ and $\mathrm{Irr}(M_i, X)$, respectively. We write $Y = \tau X$ (or $X = \tau^- Y$) and we denote this in $\Gamma(\mathcal{A})$ by $[Y] \dashleftarrow [X]$. Note that we sometimes do not distinguish between X (or Y) and its isomorphism class $[X]$ (or $[Y]$) in $\Gamma(\mathcal{A})$. Moreover, we say that an indecomposable module $M \in \mathrm{mod}\,\mathcal{A}$ *preprojective* (or *preinjective*) if there exists $m \in \mathbb{N}_0$ such that $\tau^m M$ ($\tau^{-m} M$, respectively) is projective (injective, respectively); otherwise, M is said to be *regular*.

We call a connected component $\mathcal{C}$ of $\Gamma(\mathcal{A})$ *regular* if it consists of regular modules. It is well-known that the regular components of $\operatorname{mod}\mathcal{K}_n$ are of the type $\mathbb{Z}A_\infty$ (see [8]), and they are of the form in Figure 1. A module M is

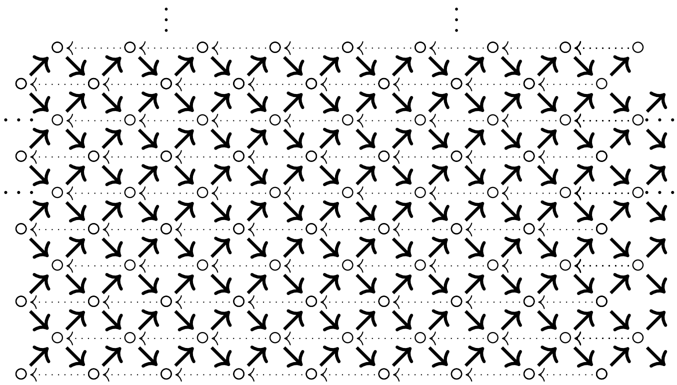

Fig. 1. Regular component $\mathbb{Z}A_\infty$

called *quasi-simple* if it is located in the bottom row of a $\mathbb{Z}A_\infty$-component. Let M be a module in a regular $\mathbb{Z}A_\infty$-component $\mathcal{C}$. Then there exists a quasi-simple module $X \in \mathcal{C}$ (or $Y \in \mathcal{C}$) such that we can find a chain of irreducible monomorphisms $X = X_1 \to \cdots \to X_{s-1} \to X_s = M$ (or irreducible epimorphisms $M = Y_s \to Y_{s-1} \to \cdots \to Y_1 = Y$). We call s the *quasi-length* of M and the module X (or Y) *quasi-socle* (or *quasi-top*) of M. It can be shown that M is uniquely determined by its quasi-length and quasi-socle (or quasi-top), whence we can define $\mathrm{ql}(M) := s =$ the quasi-length of M.

Given a regular $\mathbb{Z}A_\infty$-component $\mathcal{C}$, there are uniquely determined quasi-simple modules $M_\mathcal{C}$ and $W_\mathcal{C}$ in $\mathcal{C}$ such that the cone $(M_\mathcal{C} \to)$ of all successors of $M_\mathcal{C}$ satisfies $(M_\mathcal{C} \to) = \mathrm{EKP}_n \cap \mathcal{C}$ and the cone $(\to W_\mathcal{C})$ of all predecessors of $W_\mathcal{C}$ satisfies $(\to W_\mathcal{C}) = \mathrm{EIP}_n \cap \mathcal{C}$ [14, Theorem 3.3]. Hence we can measure the distance between $M_\mathcal{C}$ and $W_\mathcal{C}$.

Definition. Let $\mathcal{C}$ be a regular AR component of $\operatorname{mod}\mathcal{K}_n$. Define $\mathcal{W}(\mathcal{C})$ to be an integer $\mathcal{W}(\mathcal{C})$ satisfying $\tau^{\mathcal{W}(\mathcal{C})+1} M_\mathcal{C} = W_\mathcal{C}$. We call the *width* of $\mathcal{C}$.

We now define

$$\mathrm{Inj}_{n,\Omega} := \{M \in \operatorname{mod}(T(n),\Omega) \mid \forall \delta \in (T(n),\Omega)_1 : M(\delta) \text{ is injective}\},$$
$$\mathrm{Surj}_{n,\Omega} := \{M \in \operatorname{mod}(T(n),\Omega) \mid \forall \delta \in (T(n),\Omega)_1 : M(\delta) \text{ is surjective}\}.$$

Let $M \in \mathrm{mod}(T(n), \Omega)$ be an indecomposable module. By [3, Theorem 8.1], we know that $M \in \mathrm{Inj}_{n,\Omega}$ if and only if $\pi_\lambda(M) \in \mathrm{EKP}_n$.

A component of category $\mathrm{mod}(T(n), \Omega)$ is said to be *regular* if it contains a regular module. Let $\mathcal{D}$ be a regular component of $\mathrm{mod}(T(n), \Omega)$. According to [3, Corollary 7.2], the map $\mathcal{D} \to \pi_\lambda(\mathcal{D})$ is an isomorphism of translation quivers. Consequently, the regular component $\mathcal{D}$ is also of the type $\mathbb{Z}A_\infty$. Hence it is similar to the category $\mathrm{mod}\,\mathcal{K}_n$, and there exist unique quasi-simple modules $M_\mathcal{D}, W_\mathcal{D}$ in $\mathcal{D}$ such that $(M_\mathcal{D} \to) = \mathrm{Inj}_{n,\Omega} \cap \mathcal{D}$ and $(\to W_\mathcal{D}) = \mathrm{Surj}_{n,\Omega} \cap \mathcal{D}$. Then we have the following.

DEFINITION. Let $\mathcal{D}$ be a regular AR component of $\mathrm{mod}(T(n), \Omega)$. We define the *width* of $\mathcal{D}$ to be an integer $\mathcal{W}(\mathcal{D})$ satisfying $\tau^{\mathcal{W}(\mathcal{D})+1} M_\mathcal{D} = W_\mathcal{D}$.

3. Graded modules in regular components. We use $\mathcal{R}$ to denote the set of regular AR components of $\mathrm{mod}(T(n), \Omega)$, and we now focus on a component $\mathcal{D} \in \mathcal{R}$ in this section. Let $\sigma\mathcal{D} = \sigma^-\mathcal{D}$ denote the corresponding component under the action of reflection functors σ and σ^-. Let $M \in \mathrm{mod}(T(n), \Omega)$ be a regular indecomposable module. We define $\mathcal{N}_M(b) := \mathcal{N}(b) \cap T(M)_0$, for $b \in T(M)_0$.

LEMMA 3.1. *Suppose that $M \in \mathrm{Inj}_{n,\Omega}$ is a regular indecomposable module. Then*

(a) *M is a sink module and $d(M) \geq 4$.*
(b) *Any leaf of $T(\sigma M)$ is a sink. In particular, σM is a sink module.*

Proof. According to the definition, every map is injective in M, that is, every leaf of $T(M)$ is a sink, whence M is a sink module. Note that σM is also indecomposable. Suppose that there exists a source b_0 which is a leaf in $T(\sigma M)$. Let $\alpha : b_0 \to b_1$ be an arrow of $T(\sigma M)_1$ and the k-linear map $\varphi = (\sigma M)(\alpha) : (\sigma M)_{b_0} \to (\sigma M)_{b_1}$ be non-trivial. Since $\sigma^-(\sigma M) \cong M$, there exists a map $\varphi' = M(\alpha') : M_{b_1} = (\sigma M)_{b_1} \to M_{b_0} = \mathrm{coker}\,\varphi$ that is not injective in M, this is a contradiction. Hence all leaves of $T(\sigma M)_0$ are sinks and σM is a sink module. Moreover, M is complete and $d(\sigma M) = d(M) - 2$, $C(\sigma M) = C(M)$ [10, 3.1].

Since M is a sink and regular module, it follows that $d(M) \geq 4$: if $d(M) = 2$, then we have $d(\sigma M) = 0$, that is, $|T(\sigma M)_0| = 1$, so that σM is not regular. ■

COROLLARY 3.2. *Suppose that $N \in \mathrm{Surj}_{n,\Omega}$ is a regular indecomposable module. Then*

(a) *N is a source module and $d(N) \geq 4$.*
(b) *Any leaf of $T(\sigma^- N)$ is a source. In particular, $\sigma^- N$ is a source module.*

Proof. Since $N \in \mathrm{Surj}_{n,\Omega}$, we obtain $D_{T(n)}(N) \in \mathrm{Inj}_{n,\sigma\Omega}$. By Lemma 3.1, we know that N is a source module and $d(N) \geq 4$. Similarly, we get (b). ∎

We give an example to show how the modules change in their σ-orbits.

EXAMPLE 2. Let $N = k \xleftarrow{\lambda_1} k \xrightarrow{\lambda_2} k \in \mathrm{mod}(T(3), \Omega)$, where $\lambda_1, \lambda_2 \in k \setminus \{0\}$. Then N is indecomposable [12, Proposition 1]. Actually, N is regular since $\pi_\lambda(N)$ is regular. Let $\mathcal{D}$ be a regular component. Without loss of generality, we assume that $N \in \mathcal{D}$. Let $M = \sigma^{-2}N$. It is easy to check that $M \in \mathrm{Inj}_{3,\Omega} \cap \mathcal{D}$. Note that $N \notin \mathrm{Inj}_{3,\Omega} \cap \mathcal{D}$ and N is quasi-simple [5, Proposition 3.4]. Then M is quasi-simple and $M = M_{\mathcal{D}}$. One can check that $\sigma^- M$ is a sink module.

LEMMA 3.3. *Suppose that $M \in \mathrm{Inj}_{n,\Omega} \cap \mathcal{D}$ is a regular indecomposable module. Then $\sigma^- M \in \mathrm{Inj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$.*

Proof. Let $M' = \sigma^- M$. We want to prove that $M' \in \mathrm{Inj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$. Since M is a sink module, all neighbours of any source $x \in T(M)_0$ still belong to $T(M)_0$. We have an injective map $f_i : M_x \to M_{y_i}$ for every $i \in \{1, \dots, n\}$ and $y_i \in \mathcal{N}_M(x)$. Put

$$M'_x := \Big(\bigoplus_{y_i \in \mathcal{N}_M(x)} M_{y_i}\Big)/\operatorname{im} f, \quad \text{where} \quad f(m) = \begin{bmatrix} f_1(m) \\ \vdots \\ f_n(m) \end{bmatrix} \text{ for } m \in M_x.$$

For each i, we have to show that the map

$$g_i : M'_{y_i} = M_{y_i} \to M'_x, \quad m_i \mapsto (0, \dots, 0, \overline{m}_i, 0, \dots, 0),$$

is injective, where g_i maps m_i to the ith position in M'_x with the image $\overline{m}_i$. Now let $m_i \in \ker g_i$. Then $(0, \dots, 0, \overline{m}_i, 0, \dots, 0) \in \operatorname{im} f$. Hence there exists $m \in M_x$ such that $m_i = f_i(m)$, while $f_j(m) = 0$ for all $j \neq i$. Since each f_j is injective (actually, $\bigcap_{j \neq i} \ker f_j = (0)$ suffices), it follows that $m = 0$, whence $m_i = 0$. Consequently, g_i is injective. ∎

LEMMA 3.4. *Suppose that $N \in \mathrm{Surj}_{n,\Omega} \cap \mathcal{D}$ is a regular indecomposable module. Then $\sigma N \in \mathrm{Surj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$.*

Proof. Since $D_{T(n)}(N) \in \mathrm{Inj}_{n,\sigma\Omega}$, it follows from Lemma 3.3 that $\sigma^- \circ D_{T(n)}(N) \cong D_{T(n)} \circ \sigma(N) \in \mathrm{Inj}_{n,\sigma\Omega}$, that is, $\sigma N \in \mathrm{Surj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$. ∎

Let $M \in \mathcal{D}$ be a regular indecomposable module. Recall that $\mathrm{ql}(M)$ is the quasi-length of M in $\mathcal{D}$. Then we have the following.

LEMMA 3.5. $\iota(M_{\sigma\mathcal{D}}) \in \{\iota(M_{\mathcal{D}}) - 1, \iota(M_{\mathcal{D}}) + 1\}$.

Proof. Let $M = M_{\mathcal{D}}$, and let $N \in \mathrm{Inj}_{n,\Omega} \cap \mathcal{D}$ with $ql(N) = 1$. Since M is regular with $ql(M) = 1$, we get $N = \tau^{-t}M = \sigma^{-2t}M$ and $\iota(N) = \iota(M) - 2t$ for some $t \in \mathbb{N}_0$ in the regular component $\mathcal{D}$. Hence M is the one with the biggest index $\iota(M)$ in its σ-orbit located in $\mathrm{Inj}_{n,\Omega} \cap \mathcal{D}$. On the other hand, we have $\sigma^- M \in \mathrm{Inj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$ by Lemma 3.3. Moreover, we have $ql(\sigma^- M) = ql(M_{\sigma\mathcal{D}}) = ql(M)$, and $\iota(\sigma^- M) \leq \iota(M_{\sigma\mathcal{D}})$. Then $\iota(M_{\sigma\mathcal{D}}) \geq \iota(M_{\mathcal{D}}) - 1$ and $\iota(M_{\sigma\mathcal{D}}) \neq \iota(M_{\mathcal{D}})$. Otherwise, suppose that $\iota(M_{\sigma\mathcal{D}}) \leq \iota(M) - 3$. Then $\iota(\sigma^- M) = \iota(M) - 1 > \iota(M_{\sigma\mathcal{D}})$, a contradiction. Similarly, suppose that $\iota(M_{\sigma\mathcal{D}}) \geq \iota(M) + 3$. We have $\iota(\sigma^- M_{\sigma\mathcal{D}}) = \iota(M_{\sigma\mathcal{D}}) - 1 > \iota(M)$ and $\sigma^- M_{\sigma\mathcal{D}} \in \mathrm{Inj}_{n,\Omega} \cap \mathcal{D}$ by Lemma 3.3, a contradiction. Hence $\iota(M_{\sigma\mathcal{D}}) \in \{\iota(M) - 1, \iota(M) + 1\}$. ∎

LEMMA 3.6. $\iota(W_{\sigma\mathcal{D}}) \in \{\iota(W_{\mathcal{D}}) - 1, \iota(W_{\mathcal{D}}) + 1\}$.

Proof. Let $U \in \mathrm{Surj}_{n,\Omega} \cap \mathcal{D}$ with $ql(U) = 1$. Similarly, we can see that $U = \tau^s W_{\mathcal{D}} = \sigma^{2s} W_{\mathcal{D}}$ and $\iota(U) = \iota(W_{\mathcal{D}}) + 2s$ for some $s \in \mathbb{N}_0$. Hence $\iota(U) \geq \iota(W_{\mathcal{D}})$. By Lemma 3.4, we have $\sigma W_{\sigma\mathcal{D}} \in \mathrm{Surj}_{n,\Omega} \cap \mathcal{D}$. Suppose that $\iota(W_{\sigma\mathcal{D}}) \leq \iota(W_{\mathcal{D}}) - 3$. Then $\iota(\sigma W_{\sigma\mathcal{D}}) = \iota(W_{\sigma\mathcal{D}}) + 1 < \iota(W_{\mathcal{D}})$, a contradiction. Suppose that $\iota(W_{\sigma\mathcal{D}}) \geq \iota(W_{\mathcal{D}}) + 3$. Then $\iota(\sigma W_{\mathcal{D}}) = \iota(W_{\mathcal{D}}) + 1 < \iota(W_{\sigma\mathcal{D}})$ and $\sigma W_{\mathcal{D}} \in \mathrm{Surj}_{n,\sigma\Omega} \cap \sigma\mathcal{D}$ by Lemma 3.4, a contradiction. Hence $\iota(W_{\sigma\mathcal{D}}) < \iota(W_{\mathcal{D}}) + 3$. Finally, we have $\iota(W_{\sigma\mathcal{D}}) = \iota(W_{\mathcal{D}}) + 2s' + 1$ for some $s' \in \mathbb{Z}$. ∎

Let $M \in \mathcal{D}$ be a regular indecomposable module, and let $b(M)$ be the number of flow modules in the σ-orbit of M. Then $b(M) = b(N)$ for any $N \in \mathcal{D}$ [10, Section 7]. Hence we can define $b(\mathcal{D}) := b(M)$.

THEOREM 3.7. *Let $\mathcal{D}$ be a regular component of* $\mathrm{mod}(T(n), \Omega)$. *Then*

$$|\mathcal{W}(\mathcal{D}) - \mathcal{W}(\sigma\mathcal{D})| \leq 1 \quad \text{and} \quad \frac{b(\mathcal{D}) + 1}{2} \leq \mathcal{W}(\mathcal{D}).$$

Proof. Since $\tau = \sigma^2$, we have $(\sigma^2)^{(\mathcal{W}(\mathcal{D})+1)} M_{\mathcal{D}} = W_{\mathcal{D}}$. Then $2(\mathcal{W}(\mathcal{D})+1) + \iota(M_{\mathcal{D}}) = \iota(W_{\mathcal{D}})$, that is, $\mathcal{W}(\mathcal{D}) = \frac{\iota(W_{\mathcal{D}}) - \iota(M_{\mathcal{D}})}{2} - 1$. According to Lemmas 3.5 and 3.6, we have

$$\frac{\iota(W_{\mathcal{D}}) - 1 - (\iota(M_{\mathcal{D}}) + 1)}{2} - 1 \leq \mathcal{W}(\sigma D) = \frac{\iota(W_{\sigma\mathcal{D}}) - \iota(M_{\sigma\mathcal{D}})}{2} - 1 \leq \frac{\iota(W_{\mathcal{D}}) + 1 - (\iota(M_{\mathcal{D}}) - 1)}{2} - 1. \tag{1}$$

Thus

$$\mathcal{W}(\mathcal{D}) - 1 \leq \mathcal{W}(\sigma\mathcal{D}) \leq \mathcal{W}(\mathcal{D}) + 1. \tag{2}$$

According to Lemma 3.1 and Corollary 3.2, there are only sink modules contained in $\mathrm{Inj}_{n,\Omega}$ and source modules contained in $\mathrm{Surj}_{n,\Omega}$. Let N be a regular indecomposable module in the σ-orbit of $M_{\mathcal{D}}$. Then $N = \sigma^q M_{\mathcal{D}}$ for some $q \in \mathbb{N}_0$. Suppose that N is a flow module. Since we already know that $\sigma M_{\mathcal{D}}$ is a sink module and $\sigma^- W_{\mathcal{D}}$ is a source module, we have

$$\iota(M_{\mathcal{D}}) + 1 = \iota(\sigma M_{\mathcal{D}}) < \iota(N) = q + \iota(M_{\mathcal{D}}) < \iota(\sigma^- W_{\mathcal{D}}) = \iota(W_{\mathcal{D}}) - 1.$$

Hence $1 < q < \iota(W_{\mathcal{D}}) - 1 - \iota(M_{\mathcal{D}}) = 2\mathcal{W}(\mathcal{D}) + 1$, that is, $q \in \{2, \ldots, 2\mathcal{W}(\mathcal{D})\}$. Then the number of flow modules is $b(N) \leq 2\mathcal{W}(\mathcal{D}) - 1$. However, $b(\mathcal{D}) = b(N)$. Hence $\frac{b(\mathcal{D})+1}{2} \leq \mathcal{W}(\mathcal{D})$. ■

Corollary 3.8. *Let $\mathcal{D}$ be a regular component of* $\operatorname{mod}(T(n), \Omega)$. *Suppose that $\mathcal{W}(\mathcal{D}) = \mathcal{W}(\sigma\mathcal{D})$. Then $\mathcal{W}(\mathcal{D}) \geq \frac{b(\mathcal{D})+2}{2}$.*

Proof. We define two sets

$$\mathcal{M} := \mathcal{D} \setminus (\mathrm{Inj}_{n,\Omega} \cup \mathrm{Surj}_{n,\Omega}) \quad \text{and} \quad \mathcal{M}(\sigma) := \sigma\mathcal{D} \setminus (\mathrm{Inj}_{n,\sigma\Omega} \cup \mathrm{Surj}_{n,\sigma\Omega}).$$

If $\mathcal{W}(\mathcal{D}) \leq \frac{b(\mathcal{D})+1}{2}$, then we have $\mathcal{W}(\mathcal{D}) = \frac{b(\mathcal{D})+1}{2}$ by Theorem 3.7. Hence we get $\mathcal{W}(\mathcal{D}) = \mathcal{W}(\sigma\mathcal{D}) = \frac{b(\mathcal{D})+1}{2}$. That is, there are $b(\mathcal{D}) + 1$ quasi-simple modules contained in $\mathcal{M} \cup \mathcal{M}(\sigma)$. However, there exist at least one sink module and one source module that are quasi-simple in $\mathcal{M} \cup \mathcal{M}(\sigma)$ by Lemma 3.1 and Corollary 3.2. Hence there are at least $b(\mathcal{D}) + 2$ quasi-simple modules contained in $\mathcal{M} \cup \mathcal{M}(\sigma)$, and this is a contradiction. ■

We now give an example with $b(\mathcal{D}) = 0$.

Example 3. Let $M \in \operatorname{mod}(T(3), \Omega)$ be the module

$$\begin{tikzcd}
0 \arrow[r] & k & k \arrow[l, "\lambda_1"'] \arrow[r, "\lambda_2"] \arrow[d, "\lambda_5"] & k & k \arrow[l, "\lambda_3"'] \arrow[r, "\lambda_4"] \arrow[d, "\lambda_7"] & k & 0 \arrow[l] \\
 & 0 \arrow[u] & k & k \arrow[u, "\lambda_6"] \arrow[dl, "\lambda_8"'] \arrow[dr, "\lambda_9"] & k & 0 \arrow[u] & \\
 & & k & & k & &
\end{tikzcd},$$

where $\lambda_i \in k \setminus \{0\}$, $i \in \{1, \ldots, 9\}$. Then M is indecomposable and $M \in \mathrm{Inj}_{3,\Omega}$ [12, Proposition 1]. Actually, M is quasi-simple [5, Proposition 3.4]. We can see that M is regular since $\pi_\lambda(M)$ is regular. Furthermore, we have $b(M) = 0$ and $M = M_{\mathcal{D}}$ for some regular component $\mathcal{D}$ of $\operatorname{mod}(T(3), \Omega)$. It is easy to see that $\sigma^3 M$ and $\sigma^4 M$ are source modules and maps in them are all surjective. Since $\sigma^2 M \notin \mathrm{Surj}_{3,\Omega}$, we obtain $\sigma^4 M = \tau^2 M = W_{\mathcal{D}}$ and $\mathcal{W}(\mathcal{D}) = 1$. Moreover, $\sigma M \notin \mathrm{Inj}_{3,\sigma\Omega} \cup \mathrm{Surj}_{3,\sigma\Omega}$, whence $\mathcal{W}(\sigma\mathcal{D}) = 1$ by Lemma 3.3.

Proposition 3.9. $\{\mathcal{W}(\mathcal{D}) \mid \mathcal{D} \in \mathcal{R}\} = \mathbb{N}$.

Proof. According to Theorem 3.7, we get $\mathcal{W}(\mathcal{D}) \geq \frac{b(\mathcal{D})+1}{2} > 0$ for any $\mathcal{D} \in \mathcal{R}$. Then $\{\mathcal{W}(\mathcal{D}) \mid \mathcal{D} \in \mathcal{R}\} \subseteq \mathbb{N}$. However, we already know $\mathbb{N} \subseteq \{\mathcal{W}(\mathcal{D}) \mid \mathcal{D} \in \mathcal{R}\}$ by [3, Theorem 10.3(b)]. ■

Remark. If there existed a regular component $\mathcal{D}$ of $\operatorname{mod}(T(n), \Omega)$ such that $\mathcal{W}(\mathcal{D}) = 0$, then $\tau M_{\mathcal{D}} = W_{\mathcal{D}} = \sigma^2 M_{\mathcal{D}}$. Hence the module $\sigma M_{\mathcal{D}}$ would be a sink module and $\sigma^- W_{\mathcal{D}} \cong \sigma M_{\mathcal{D}}$ would be a source module by Lemma 3.1(b) and Corollary 3.2(b). Apparently, this is a contradiction.

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