ACTA ARITHMETICA
Online First version

Density problems related to the ideal counting functions of number fields

by

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Abstract. Let K_1 and K_2 be two number fields of degree k, and let $a_{K_i}(p)$ denote the ideal counting function in the Dedekind zeta function of K_i. The fields K_1 and K_2 are called *arithmetically equivalent* if they share the same Dedekind zeta function. We investigate the analytic density of the set S of primes p for which $a_{K_1}(p) \neq a_{K_2}(p)$ when K_1 and K_2 are not arithmetically equivalent. We confirm a conjecture of Mantilla-Soler (2023) by showing that this analytic density is at least $2/k^2$. We also study density problems associated with $a_K(p)$ for a single number field K.

1. Introduction. Let K be a number field of degree k with Dedekind zeta function

$$\zeta_K(s) = \sum_{\mathfrak{n}} \frac{1}{\mathfrak{N}(\mathfrak{n})^s} = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}.$$

The coefficient $a_K(n)$ counts the number of integral ideals of K with norm n and is of fundamental importance in number theory. For a prime p, we have $0 \leq a_K(p) \leq k$. In this paper, we focus on density problems related to the distribution of $a_K(p)$.

For a set S of prime numbers, its *analytic density* is defined as

$$\delta(S) := \lim_{s\to 1^+} \Big(\sum_{p\in S} \frac{1}{p^s}\Big) \Big/ \Big(\sum_{p} \frac{1}{p^s}\Big).$$

A central problem addressed here is how to distinguish arithmetically equivalent number fields via the statistical behavior of $a_K(p)$. Let K_1 and K_2 be two number fields with Dedekind zeta functions $\zeta_{K_i}(s) = \sum_{n=1}^{\infty} a_{K_i}(n) n^{-s}$. Two number fields with identical Dedekind zeta functions are called *arithmetically equivalent*. Determining when number fields are arithmetically equivalent

2020 *Mathematics Subject Classification*: Primary 11R42; Secondary 11R45.
Key words and phrases: arithmetically equivalent field, Dedekind zeta function, analytic density.
Received 14 October 2025; revised 16 March 2026.
Published online 3 September 2026.

DOI: 10.4064/aa251014-28-4

is a classical problem, to which extensive research has been devoted (see [KK22, K23, K24, P77, SP95, V23]). We know that arithmetically equivalent fields share the same degree, discriminant, Galois closure, and other arithmetic invariants; two number fields are arithmetically equivalent if and only if they have the same arithmetic type.

A natural question arises: which density of primes p satisfying $a_{K_1}(p) = a_{K_2}(p)$ suffices to guarantee that K_1 and K_2 are arithmetically equivalent? In [M23], Mantilla-Soler studied this problem and proved that if two degree-k number fields K_1 and K_2 are not arithmetically equivalent, then the set of primes p with $a_{K_1}(p) \neq a_{K_2}(p)$ has density at least $\frac{1}{4k^2}$. Moreover, Mantilla-Soler made the following conjecture.

Conjecture 1.1 ([M23]). *If K_1 and K_2 are not arithmetically equivalent, then the set $\{p \mid a_{K_1}(p) \neq a_{K_2}(p)\}$ has analytic density at least $2/k^2$.*

In this paper, we prove this conjecture.

Theorem 1.2. *Let K_1 and K_2 be two non-arithmetically-equivalent number fields of degree k, and let $S = \{p \mid a_{K_1}(p) \neq a_{K_2}(p)\}$. Then*

$$\delta(S) \geq 2/k^2.$$

An immediate corollary is the following.

Corollary 1.3. *If the set $\{p \mid a_{K_1}(p) = a_{K_2}(p)\}$ has density at least $1 - 2/k^2$, then K_1 and K_2 are arithmetically equivalent.*

Mantilla-Soler's proof relies on a counting theorem in algebraic geometry. In contrast, we adopt a Galois representation approach, which relates the coefficients $a_K(p)$ to the character of an Artin representation. Through a detailed analysis of these representations, we prove the theorem via the Chebotarev Density Theorem. Our method is analogous to that in [C17] and [CJ19], which studies density problems for Fourier coefficients of modular forms and automorphic representations.

In Section 5, we present explicit computations for number fields of small degree and show that our estimate is sharp for degrees $k \leq 4$.

We then turn to the density results for the ideal counting function of a single number field. Let K be a number field of degree k over $\mathbb{Q}$. For a prime p that decomposes in K as $p\mathcal{O}_K = \mathfrak{p}_1 \cdots \mathfrak{p}_g$ with each $\mathfrak{p}_i$ unramified of residue degree f_i, the coefficient $a_K(p)$ is the number of indices i for which $f_i = 1$.

Let $f(x)$ be an irreducible polynomial of degree k over $\mathbb{Q}$ with integral coefficients, α a root, and $K = \mathbb{Q}(\alpha)$. For an unramified prime p, the number of solutions to $f(x) \equiv 0 \pmod{p}$ equals $a_K(p)$ (see [K07] for a detailed discussion). Thus, studying $a_K(p)$ provides information about the behavior of roots of irreducible polynomials in finite fields.

For a number field K of degree k, define for each i the set

$$S_i = \{p \mid a_K(p) = i\}.$$

We aim to estimate $\delta(S_i)$. For Galois number fields, the density is simple: if $K/\mathbb{Q}$ is Galois, then for an unramified prime p,

$$a_K(p) = \begin{cases} k & \text{if } p \text{ splits completely in } K, \\ 0 & \text{if } p \text{ is inert in } K. \end{cases}$$

Since only finitely many primes ramify, for Galois K we have $\delta(S_k) = \frac{1}{k}$ and $\delta(S_0) = \frac{k-1}{k}$.

For non-Galois K, the density of S_i can still be computed using the Chebotarev Density Theorem. Let $\bar{K}$ denote the Galois closure of K and $G = \mathrm{Gal}(\bar{K}/\mathbb{Q})$. The induced representation $\rho_K = \mathrm{Ind}_K^{\mathbb{Q}} 1_K$ is an Artin representation of G with Artin L-function $L(\rho_K, s) = \zeta_K(s)$. For a prime p, let σ_p be a Frobenius element in G; then $\chi_{\rho_K}(\sigma_p) = a_K(p)$. By Chebotarev, $\delta(S_i)$ equals the density of the conjugacy classes in G on which χ_{ρ_K} takes the value i.

Our next theorem gives an upper bound for $\delta(S_i)$.

THEOREM 1.4. *Let K be a number field of degree k. Then*

$$\delta(S_0) \le \frac{k-1}{k}, \qquad \delta(S_i) \le \frac{k-1}{k(i-1)} \quad \textit{for } i = 2, \ldots, k.$$

The proof is given in Section 4.

2. Preliminaries. Let $K/\mathbb{Q}$ be a number field of degree k with Galois closure $\bar{K}$, and $G = \mathrm{Gal}(\bar{K}/\mathbb{Q})$. For a prime p, let σ_p be a Frobenius element in G and I_p the associated inertial group. Let V be a finite-dimensional complex vector space and let $\rho : G \to \mathrm{GL}(V)$ be an Artin representation. The *Artin L-function* of ρ is

$$L(\rho, s) := \prod_p L_p(\rho, s),$$

where

$$L_p(\rho, s) := \frac{1}{\det(1 - p^{-s}\rho(\sigma_p) \mid V^{I_p})}.$$

Let $H = \mathrm{Gal}(\bar{K}/K)$ and $\rho_K = \mathrm{Ind}_H^G 1_H$ be the induced Artin representation of dimension k. Then $L(\rho_K, s) = \zeta_K(s)$. Comparing coefficients gives $\chi_{\rho_K}(\sigma_p) = \mathrm{tr}(\rho_K(\sigma_p)) = a_K(p)$ for all primes p unramified in $\bar{K}$. Using the character formula for induced representations, we compute

$$\chi_{\rho_K}(\sigma_p) = \frac{1}{|H|} \sum_{g \in G,\, g\sigma_p g^{-1} \in H} \chi_{\rho_K}(g\sigma_p g^{-1}) = \frac{N_{\sigma_p}}{|H|}, \tag{2.1}$$

where $N_{\sigma_p} = |\{g \in G \mid g\sigma_p g^{-1} \in H\}|$.

Let ρ_1 and ρ_2 be two finite-dimensional complex representations of G. Denote by $\langle\rho_1,\rho_2\rangle_G$ the inner product of their characters. For any representation ρ, the multiplicity of the trivial representation in ρ is $\langle\rho,1_G\rangle_G$. The following lemma is critical to the proof of Theorem 1.2.

LEMMA 2.1. *Let G be a finite group, and let χ_1 and χ_2 be distinct characters of G having the same degree. Then*

$$\langle\chi_1-\chi_2,\chi_1-\chi_2\rangle_G \geq 2.$$

Proof. Write $\chi_1-\chi_2=\sum_{i=1}^t a_i\psi_i$, where ψ_i are distinct irreducible characters and a_i are integers. Then $\langle\chi_1-\chi_2,\chi_1-\chi_2\rangle_G=\sum_{i=1}^t a_i^2$. Since χ_1 and χ_2 are distinct and of the same degree, at least two a_i are non-zero, so the sum is at least 2. ∎

The foundational tool for studying the density of prime sets associated with number fields is the Chebotarev Density Theorem.

THEOREM 2.2 (Chebotarev Density Theorem). *Let $L/\mathbb{Q}$ be a Galois extension with Galois group G, C a conjugacy class in G, and T the set of primes p unramified in L for which $\sigma_p\in C$. Then $\delta(T)=|C|/|G|$.*

Applying the Chebotarev Density Theorem, Krishnamoorthy [K25] proved the following proposition.

PROPOSITION 2.3 ([K25]). *Let G be the Galois group of a Galois extension $L/\mathbb{Q}$. For any two Artin representations ρ_1 and ρ_2 of G,*

$$\sum_p \frac{\chi_{\rho_1}(\sigma_p)\overline{\chi_{\rho_2}(\sigma_p)}}{p^s}=\langle\rho_1,\rho_2\rangle_G\sum_p\frac{1}{p^s}+O(1)\quad (s\to 1^+).$$

In particular, with $\rho_1=\rho_K$ and $\rho_2=1_G$, we obtain

$$\sum_p\frac{a_K(p)}{p^s}=\sum_p\frac{1}{p^s}+O(1)\quad (s\to 1^+).\tag{2.2}$$

3. Density estimates for non-arithmetically-equivalent fields. Let K_1,K_2 be number fields of degree k over $\mathbb{Q}$. As before, K_1 and K_2 are arithmetically equivalent if and only if $\zeta_{K_1}(s)=\zeta_{K_2}(s)$. From [M19], we know that K_1 and K_2 are arithmetically equivalent if and only if $a_{K_1}(p)=a_{K_2}(p)$ for almost all primes p. Define

$$S:=\{p\mid a_{K_1}(p)\neq a_{K_2}(p)\}.$$

In this section, we establish a lower bound for $\delta(S)$.

We first consider the Galois case, where the density can be computed explicitly. Suppose K_1 and K_2 are Galois of degree k over $\mathbb{Q}$, and let $L=K_1\cap K_2$ be the intermediate field of degree l over $\mathbb{Q}$. For an unramified prime p, $a_{K_i}(p)=k$ if p splits completely in K_i and $a_{K_i}(p)=0$ otherwise.

The density of S is therefore the density of primes splitting completely in exactly one of the two fields, leading to the following proposition.

PROPOSITION 3.1. *If K_1 and K_2 are Galois number fields of degree k and $[K_1 \cap K_2 : \mathbb{Q}] = l$, then $\delta(S) = \frac{2}{k}\big(1 - \frac{l}{k}\big)$.*

For the general (non-Galois) case, let $\bar{K}_1, \bar{K}_2$ be the Galois closures of K_1, K_2 with Galois groups $G_i = \mathrm{Gal}(\bar{K}_i/\mathbb{Q})$. Let $\bar{K}$ be the Galois closure of K_1K_2 and $G = \mathrm{Gal}(\bar{K}/\mathbb{Q})$. Every representation of G_i extends to a representation of G via the quotient $G \to G_i$. Let $\rho_{K_i} = \mathrm{Ind}_{K_i}^{\mathbb{Q}} 1_{K_i}$, viewed as a representation of G. Define

$$m_{K_i} = \langle \rho_{K_i}, \rho_{K_i} \rangle_G, \qquad m_{1,2} = \langle \rho_{K_1}, \rho_{K_2} \rangle_G.$$

LEMMA 3.2. *We have the asymptotic estimate*

$$\sum_p \frac{a_{K_1}(p)a_{K_2}(p)}{p^s} = m_{1,2} \sum_p \frac{1}{p^s} + O(1) \qquad (s \to 1^+).$$

Proof. For primes p unramified in $\bar{K}$, we have $\chi_{\rho_{K_i}}(p) = a_{K_i}(p)$. Since only finitely many primes ramify, Proposition 2.3 gives

$$\begin{aligned} \sum_p \frac{a_{K_1}(p)a_{K_2}(p)}{p^s} &= \sum_p \frac{\chi_{\rho_{K_1}}(p)\chi_{\rho_{K_2}}(p)}{p^s} \\ &= \langle \rho_{K_1} \otimes \rho_{K_2}, 1_G \rangle_G \sum_p \frac{1}{p^s} + O(1) \\ &= m_{1,2} \sum_p \frac{1}{p^s} + O(1) \qquad (s \to 1^+). \ \blacksquare \end{aligned}$$

LEMMA 3.3. *If K_1 and K_2 are not arithmetically equivalent, then*

$$m_{K_1} + m_{K_2} - 2m_{1,2} \geq 2.$$

Proof. Since K_1 and K_2 are not arithmetically equivalent, the characters $\chi_{\rho_{K_1}}$ and $\chi_{\rho_{K_2}}$ are distinct and of the same degree. By definition,

$$\begin{aligned} m_{K_1} + m_{K_2} - 2m_{1,2} &= \langle \chi_{\rho_{K_1}}, \chi_{\rho_{K_1}} \rangle_G + \langle \chi_{\rho_{K_2}}, \chi_{\rho_{K_2}} \rangle_G - 2\langle \chi_{\rho_{K_1}}, \chi_{\rho_{K_2}} \rangle_G \\ &= \langle \chi_{\rho_{K_1}} - \chi_{\rho_{K_2}}, \chi_{\rho_{K_1}} - \chi_{\rho_{K_2}} \rangle_G. \end{aligned}$$

The lemma follows immediately from Lemma 2.1. $\blacksquare$

PROPOSITION 3.4. *If K_1 and K_2 are not arithmetically equivalent, then*

$$\sum_p \frac{(a_{K_1}(p) - a_{K_2}(p))^2}{p^s} \geq 2 \sum_p \frac{1}{p^s} + O(1) \qquad (s \to 1^+).$$

Proof. From Lemmas 3.2 and 3.3,

$$\begin{aligned}\sum_p \frac{(a_{K_1}(p)-a_{K_2}(p))^2}{p^s} &= \sum_p \frac{a_{K_1}(p)^2}{p^s} + \sum_p \frac{a_{K_2}(p)^2}{p^s} - 2\sum_p \frac{a_{K_1}(p)a_{K_2}(p)}{p^s} \\ &= (m_{K_1}+m_{K_2}-2m_{1,2})\sum_p \frac{1}{p^s} + O(1) \\ &\geq 2\sum_p \frac{1}{p^s} + O(1) \quad (s \to 1^+). \ \blacksquare\end{aligned}$$

We now prove Theorem 1.2.

Proof. Since $0 \leq a_{K_1}(p), a_{K_2}(p) \leq k$, we have $|a_{K_1}(p) - a_{K_2}(p)| \leq k$. Hence,

$$\sum_{p\in S} \frac{(a_{K_1}(p)-a_{K_2}(p))^2}{p^s} \leq k^2 \sum_{p\in S}\frac{1}{p^s}.$$

Combined with Proposition 3.4, this gives

$$2\sum_p \frac{1}{p^s} + O(1) \leq k^2 \sum_{p\in S}\frac{1}{p^s} + O(1) \quad (s \to 1^+).$$

Dividing both sides by $\sum_p p^{-s}$ and taking the limit as $s \to 1^+$ yields $\delta(S) \geq 2/k^2$. $\blacksquare$

4. Density results for a single number field K. Let $K/\mathbb{Q}$ be a number field of degree k. For $0 \leq i \leq k$, define

$$S_i := \{p \mid a_K(p) = i\},$$

and let $\delta(S_i)$ denote its analytic density. For an unramified prime p, we have $\chi_{\rho_K}(\sigma_p) = a_K(p)$. Since only finitely many primes ramify, the Chebotarev Density Theorem implies the following.

PROPOSITION 4.1. *Let C_i be the union of all conjugacy classes of $G = \mathrm{Gal}(\bar{K}/\mathbb{Q})$ on which χ_{ρ_K} takes the value i. Then $\delta(S_i) = |C_i|/|G|$.*

The densities $\delta(S_i)$ are well known for Galois extensions $K/\mathbb{Q}$. Proposition 4.1 allows us to compute $\delta(S_i)$ for many non-Galois number fields. We illustrate this with dihedral extensions, a key example for Section 5.

EXAMPLE 4.2 (Dihedral extensions). Let $K/\mathbb{Q}$ be a number field of degree k with Galois closure $\bar{K}$ and Galois group $G \simeq D_k$, the dihedral group of order $2k$. The generators are u, v with $u^k = v^2 = 1$, $uvu = v$. The character χ_{ρ_K} is given by

- $\chi_{\rho_K}(1) = k$,
- $\chi_{\rho_K}(u^i) = 0$ for $i \neq 0$,

- $\chi_{\rho_K}(u^i v) = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even and } i \text{ is even,} \\ 0 & \text{if } k \text{ is even and } i \text{ is odd.} \end{cases}$

See [YY26] for a proof of the character formula. Proposition 4.1 gives the densities by counting the elements in each conjugacy class of D_k:

- If k is odd: $\delta(S_0) = \frac{k-1}{2k}$, $\delta(S_1) = \frac{1}{2}$, $\delta(S_k) = \frac{1}{2k}$.
- If k is even: $\delta(S_0) = \frac{3k-2}{4k}$, $\delta(S_2) = \frac{1}{4}$, $\delta(S_k) = \frac{1}{2k}$.

We now derive an upper bound for $\delta(S_i)$. First, the densities satisfy

$$\sum_{i=0}^{k} \delta(S_i) = 1. \tag{4.1}$$

From Proposition 2.3 with ρ_K the trivial representation, we have

$$\sum_{i=0}^{k} \sum_{p \in S_i} \frac{i}{p^s} = \sum_{p} \frac{1}{p^s} + O(1) \quad (s \to 1^+).$$

Taking the limit as $s \to 1^+$ gives

$$\sum_{i=1}^{k} i\delta(S_i) = 1. \tag{4.2}$$

Subtracting (4.2) from (4.1) yields

$$\delta(S_0) = \sum_{i=1}^{k} (i-1)\delta(S_i). \tag{4.3}$$

For $2 \leq i \leq k$, we immediately obtain

$$0 \leq \delta(S_i) \leq \frac{\delta(S_0)}{i-1}. \tag{4.4}$$

Moreover, from (4.3) we have

$$\delta(S_0) \leq (k-1) \sum_{i=1}^{k} \delta(S_i) = (k-1)(1 - \delta(S_0)),$$

since $i - 1 \leq k - 1$ for all i. Solving for $\delta(S_0)$ gives $\delta(S_0) \leq \frac{k-1}{k}$. Combining these inequalities proves the following theorem.

THEOREM 4.3. *Let K be a number field of degree k over $\mathbb{Q}$. Then*

$$\delta(S_0) \leq \frac{k-1}{k}, \qquad \delta(S_i) \leq \frac{k-1}{k(i-1)} \quad \textit{for } i = 2, \ldots, k.$$

5. Examples for degree ≤ 4. In this section, we give explicit examples of number fields of degree at most 4 to show that the estimate $\delta(S) \geq 2/k^2$ from Theorem 1.2 is sharp for these degrees. Using Proposition 4.1, we compute the exact density of $S = \{p \mid a_{K_1}(p) \neq a_{K_2}(p)\}$ for pairs of non-arithmetically-equivalent number fields K_1, K_2 of degree $k = 2, 3, 4$.

5.1. Quadratic fields ($k = 2$). Let K_1 and K_2 be two quadratic number fields (all quadratic fields are Galois over $\mathbb{Q}$). By Proposition 3.1 with $l = [K_1 \cap K_2 : \mathbb{Q}] = 1$, we have

$$\delta(S) = \frac{2}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{2}.$$

Thus the bound $2/k^2 = 2/4 = 1/2$ is attained, so it is sharp for $k = 2$.

5.2. Cubic fields ($k = 3$). A cubic field can be either Galois (with Galois group C_3) or non-normal (with Galois group D_3). Let K_1 and K_2 be non-arithmetically-equivalent cubic fields. We consider the following cases.

(1) *Both K_1 and K_2 are Galois.* By Proposition 3.1, $\delta(S) = \frac{2}{3}\left(1 - \frac{1}{3}\right) = \frac{4}{9}$.

(2) *One field is Galois, the other is not.* Without loss of generality, we may assume that K_1 is Galois and K_2 is a non-Galois field with Galois group D_3. Let K be the Galois closure of K_1K_2. Then K is Galois of degree 18 with Galois group $G \simeq C_3 \times D_3$. The Frobenius elements for K_1 and K_2 are independent because G is a direct product. We know:

- For K_1: $\delta(S_0) = 2/3$, $\delta(S_3) = 1/3$.
- For K_2: $\delta(S_0) = 1/3$, $\delta(S_1) = 1/2$, $\delta(S_3) = 1/6$.

The density of primes with $a_{K_1}(p) = a_{K_2}(p)$ is

$$\frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{6} = \frac{5}{18}.$$

Hence $\delta(S) = 1 - \frac{5}{18} = \frac{13}{18}$.

(3) *Both K_1 and K_2 are non-Galois.* Let K_1, K_2 be non-Galois cubic fields with Galois closures $\bar{K}_1, \bar{K}_2$, so $\mathrm{Gal}(\bar{K}_i/\mathbb{Q}) \simeq D_3$. We consider two subcases.

First, suppose $\bar{K}_1 \cap \bar{K}_2 = \mathbb{Q}$. Then $\mathrm{Gal}(K/\mathbb{Q}) \simeq D_3 \times D_3$ and the Frobenius elements are independent. In this case,

$$\delta(S) = 1 - \frac{1}{9} - \frac{1}{4} - \frac{1}{36} = \frac{11}{18}.$$

Second, if $\bar{K}_1 \cap \bar{K}_2$ is a quadratic field over $\mathbb{Q}$, then $\mathrm{Gal}(K/\mathbb{Q})$ is a semi-direct product of C_3 and D_3. The two characters are given by the following table (where “1” denotes the identity element).

Conjugacy class	1	2	3A	3B	3C	3D
Size	1	9	2	2	2	2
$\chi_{\rho_{K_1}}$	3	1	3	0	0	0
$\chi_{\rho_{K_2}}$	3	1	0	3	0	0

The only primes with $a_{K_1}(p) = a_{K_2}(p)$ are those in classes 3A and 3B, so

$$\delta(S) = \frac{2+2}{18} = \frac{2}{9}.$$

Our estimate for $k = 3$ is $2/k^2 = 2/9$, which matches the minimal density in all cases. Hence, the bound is sharp for $k = 3$.

5.3. Quartic fields ($k = 4$)**.** Quartic fields are more complex; we focus on a case where the estimate is sharp. Let $\bar{K}$ be a Galois number field of degree 32 over $\mathbb{Q}$ with Galois group G. Assume G is the semidirect product of C_4 and D_4, generated by

$$G = \langle a, r, s \mid a^4 = r^4 = s^2 = 1,\ ar = ra,\ sas = a^{-1},\ srs = r^{-1} \rangle.$$

Let $\langle a \rangle$, $\langle ar^2 \rangle$ be normal subgroups of G of order 4, fixing Galois subfields $\bar{K}_1$ and $\bar{K}_2$, respectively. Let K_1 and K_2 be quartic subfields of $\bar{K}_1$ and $\bar{K}_2$, and ρ_{K_1}, ρ_{K_2} the corresponding Artin representations, viewed as representations of G. As calculated in Example 4.2, the characters are

$$\chi_{\rho_{K_1}}(a^i, r^j, s^k) = \begin{cases} 4 & \text{if } j = 1,\ k = 0, \\ 2 & \text{if } j = 2, 4 \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_{\rho_{K_2}}(a^i, r^j, s^k) = \begin{cases} 4 & \text{if } j - 2i \equiv 1 \pmod 4,\ k = 0, \\ 2 & \text{if } j - 2i \equiv 2, 4 \pmod 4 \text{ and } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The characters differ only for elements with $k = 0$, $j = 1, 3$, and $i = 1, 3$, and there are four such elements in G. By the Chebotarev Density Theorem,

$$\delta(S) = \frac{4}{|G|} = \frac{1}{8}.$$

Since $2/k^2 = 2/16 = 1/8$, the estimate is again sharp.

Acknowledgements. The author would like to express sincere gratitude to the anonymous referee for the valuable suggestions on the statements and proof of the main theorems, and in particular for the insightful comments on Lemma 2.1, which have greatly simplified and clarified the proofs in this paper.

Funding. This work was supported by the Natural Science Foundation of Shandong Province (Grant No. ZR2022QA097).

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