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Fractional parts of powers of negative rationals

by

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Abstract. We prove that for any real number $\xi \neq 0$ and any coprime integers $p > q \geq 1$, if ξ is irrational or $q > 1$, then the image in $\mathbb{R}/\mathbb{Z}$ of the sequence $(\xi(-p/q)^n)_{n\geq 0}$ is not contained in any interval of length less than $(1 + q/p - q^2/p^2)/p$.

1. Introduction. Distribution modulo 1 of geometric progressions has been studied extensively. One intriguing open question considered by Mahler [M] is the following: Does there exist a real number $\xi > 0$ such that $\{\xi(3/2)^n\} < 1/2$ for all $n \geq 0$? Here $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of a real number x. Flatto, Lagarias, and Pollington [FLP, Theorem 1.4] proved that for any real number $\xi > 0$ and coprime integers $p > q > 1$, we have

$$\limsup_{n\to\infty}\{\xi(p/q)^n\} - \liminf_{n\to\infty}\{\xi(p/q)^n\} \geq 1/p.$$

Dubickas ([D1, Theorem 1], [D2, Theorem 2]) proved a more general result which implies that for all real numbers $\xi \neq 0$ and η and for any rational number $\alpha = p/q$ with integers $p > q \geq 1$, if $\xi \notin \mathbb{Q}$ or $\alpha \notin \mathbb{Z}$, then

$$\limsup_{n\to\infty}\{\xi\alpha^n + \eta\} - \liminf_{n\to\infty}\{\xi\alpha^n + \eta\} \geq 1/p. \tag{1.1}$$

In fact, Dubickas' proof of (1.1) also holds for $\alpha = -p/q$. In this paper, we improve the bound (1.1) in the case where α is a negative rational number.

THEOREM 1.1. *Let $\xi \neq 0$ and η be real numbers and let $p > q \geq 1$ be integers. Assume that $\xi \notin \mathbb{Q}$ or $p/q \notin \mathbb{Z}$. Then*

$$\limsup_{n\to\infty}\{\xi(-p/q)^n + \eta\} - \liminf_{n\to\infty}\{\xi(-p/q)^n + \eta\} \geq (1 + r - r^2)/p, \tag{1.2}$$

where $r = q/p$.

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Dubickas proved various bounds for the limit values of $(\{\xi(-p/q)^n\})_{n\geq 0}$ [D3, Theorems 2, 4] and $(\|\xi(-p/q)^n\|)_{n\geq 0}$ [D2, Theorem 3] (where $\|x\| = \min(\{x\}, 1-\{x\})$ denotes the distance to the nearest integer), but no bound of the form (1.2) better than $1/p$ has been established to the best of our knowledge. In the case $(p,q) = (3,2)$, which is the negative analogue of Mahler's question, our bound is $(1+2/3-4/9)/3 = 11/27$.

For a sequence $S = (s_n)_{n\geq 0}$ we let $S_n(x) = \sum_{i=0}^{\infty} s_{n+i}x^i$ denote the (shifted) generating series. We deduce Theorem 1.1 from the following optimal bound.

THEOREM 1.2. *For any real number* $0 < r < 1$ *and any bounded sequence* $S = (s_n)_{n\geq 0}$ *of integers that is not ultimately periodic, we have*

$$\limsup_{n\to\infty} S_n(-r) - \liminf_{n\to\infty} S_n(-r) \geq 1 + r - r^2. \tag{1.3}$$

Moreover, there exists a bounded sequence S *of integers that is not ultimately periodic such that equality holds in* (1.3) *for all* $0 < r < 1$.

Our proofs use techniques developed by Dubickas [D1, D2]. One key fact that allows one to deduce Theorem 1.1 from (1.3) is that the sequence $(p\lfloor\xi(-p/q)^n+\eta\rfloor + q\lfloor\xi(-p/q)^{n+1}+\eta\rfloor)_{n\geq 0}$ of integers is not ultimately periodic. This was observed in various settings by Dubickas and Novikas ([DN, Lemma 2], [D1, Theorem 3], [D2, Lemma 1], [D3, Lemma 9]). The inequality (1.3) is deduced from the Morse–Hedlund theorem on subword complexity. One difference from Dubickas' proof of (1.1) is that we exploit the alternating sign in $S_n(-r)$, which leads to the optimal bound in Theorem 1.2.

It follows immediately from Theorem 1.1 that the image in $\mathbb{R}/\mathbb{Z}$ of the sequence $(\xi(-p/q)^n)_{n\geq 0}$ is not contained in any interval of length less than $(1+r-r^2)/p$. In subsequent work [LZ], we will examine the question whether the image can lie in an interval of length equal to $(1+r-r^2)/p$. In particular, we will show that the bound in Theorem 1.1 is the best possible in the case $q = 1$. The analogous question for powers of positive rational numbers was studied by several authors [FLP, B1, BD, D4, D5].

2. Proof of Theorem 1.2. Our proof of (1.3) uses the following consequence of the Morse–Hedlund theorem on subword complexity [MH1, Theorem 7.3].

LEMMA 2.1. *Let* W *be an infinite word over a finite alphabet that is not ultimately periodic and let* $n \geq 0$. *Then there exist a word* U *of length* n *and letters* $s \neq t$ *such that* sU *and* tU *are subwords of* W.

Proof. This is a special case of [D2, Lemma 2] or [B2, Corollary A.4]. We include a proof for the sake of completeness. Let $p_W(n)$ denote the number of subwords of W of length n. By the Morse–Hedlund theorem, $p_W(n+1) > p_W(n)$. The assertion then follows from the pigeonhole principle. ∎

We use Sturmian words in the proof of the last assertion of Theorem 1.2. Recall that an infinite word $s_0 s_1 \dots$ over the alphabet $\{0,1\}$ or a sequence $(s_0, s_1, \dots)$ with values in $\{0,1\}$ is *Sturmian* if there exists an irrational $\theta \in (0,1)$ and a $\rho \in \mathbb{R}$ such that

$$s_n = \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor \tag{2.1}$$

for all $n \geq 0$ or

$$s_n = \lceil (n+1)\theta + \rho \rceil - \lceil n\theta + \rho \rceil \tag{2.2}$$

for all $n \geq 0$.

Below, for a letter a, a^l denotes the word $a \dots a$ of length l.

Lemma 2.2. *Let $S = s_0 s_1 \dots$ be a Sturmian word over the alphabet $\{0,1\}$. Then S is not ultimately periodic. Moreover, for all $m, n \geq 0$, the word $(s_m - s_n)(s_{m+1} - s_{n+1}) \dots$ has no subword of the form $10^l 1$ or $(-1)0^l(-1)$ for $l \geq 0$. Furthermore, if $\sigma_n = \sum_{i=0}^{n-1} s_i$ denotes the partial sum, then, for all $m, n, k \geq 0$, we have $(\sigma_{m+k} - \sigma_m) - (\sigma_{n+k} - \sigma_n) \in \{-1, 0, 1\}$.*

Proof. The first assertion is standard and the second and third assertions are equivalent forms of the fact that Sturmian words are balanced (see [MH2] or [L, Theorem 2.1.13]). We include a proof for the sake of completeness. The first assertion follows from the identity

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} s_i}{n} = \lim_{n\to\infty} \frac{\lfloor n\theta + \rho \rfloor - \lfloor \rho \rfloor}{n} = \theta$$

and the observation that the limit is rational if $(s_n)_{n\geq 0}$ is ultimately periodic. The second and third assertions follow from the observation that, in the case (2.1), we have

$$\begin{aligned}
(\sigma_{m+k} - \sigma_m) - (\sigma_{n+k} - \sigma_n) &= \sum_{i=0}^{k-1} (s_{m+i} - s_{n+i}) \\
&= (\lfloor (m+k)\theta + \rho \rfloor - \lfloor m\theta + \rho \rfloor) - (\lfloor (n+k)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor) \\
&= -(\{(m+k)\theta + \rho\} - \{m\theta + \rho\}) + (\{(n+k)\theta + \rho\} - \{n\theta + \rho\}) \\
&\in (-2, 2) \cap \mathbb{Z} = \{-1, 0, 1\}
\end{aligned}$$

and the same holds in the case (2.2) after replacing $\lfloor \cdot \rfloor$ by $\lceil \cdot \rceil$ and $\{\cdot\}$ by $-\{-(\cdot)\}$. ∎

Let $S = (s_n)_{n\geq 0}$ be a sequence of integers. For $n, k \geq 0$, we have

$$S_n(x) - x^k S_{n+k}(x) = \sum_{i=0}^{k-1} s_{n+i} x^i.$$

Thus, for $n, m, k \geq 0$, we have

$$(S_n(x) - S_m(x)) - x^k(S_{n+k}(x) - S_{m+k}(x)) = \sum_{i=0}^{k-1} (s_{n+i} - s_{m+i})x^i. \tag{2.3}$$

We will say that we apply formula (2.3) to the subwords $(s_n, \ldots, s_{n+k-1})$ and $(s_m, \ldots, s_{m+k-1})$.

We now state and prove an analogue of Theorem 1.2 for $S_n(r)$. Although not logically required, we hope that it makes the proof of Theorem 1.2 easier to understand. The inequality (2.4) is a special case of [D2, Lemma 3].

PROPOSITION 2.3. *For any real number $0 < r < 1$ and any bounded sequence $S = (s_n)_{n\geq 0}$ of integers that is not ultimately periodic, we have*

$$\limsup_{n\to\infty} S_n(r) - \liminf_{n\to\infty} S_n(r) \geq 1. \tag{2.4}$$

Moreover, there exists a bounded sequence S of integers that is not ultimately periodic such that equality holds in (2.4) *for all* $0 < r < 1$.

Proof. We include a proof of (2.4) for the sake of completeness. Assume that the set of limit values of $(S_n(r))_{n\geq 0}$ is contained in an interval (A, B) of length < 1. Then, up to shifting (and truncating) S, we may assume that $S_n(r) \in (A, B)$ for all $n \geq 0$. Let $\mu = \max_{n\geq 0} s_n$ and $\nu = \min_{n\geq 0} s_n$. Since S is not ultimately periodic, we have $\mu > \nu$. By (2.3),

$$2 > (1 + r)(B - A) > \mu - \nu.$$

Thus $\mu - \nu = 1$ and consequently $s_n \in \{\nu, \mu\}$ for all $n \geq 0$. We regard S as a word $s_0 s_1 \ldots$ over the alphabet $\{\nu, \mu\}$. Let $n \geq 0$. By Lemma 2.1, there exists a word $U = u_1 \ldots u_n$ of length n such that μU and νU are subwords of S. By (2.3) applied to these subwords,

$$(1 + r^{n+1})(B - A) > \mu - \nu = 1.$$

Since $n \geq 0$ is arbitrary, we get $B - A \geq 1$, a contradiction. This finishes the proof of (2.4).

Now let S be a Sturmian sequence with values in $\{0, 1\}$. Then for all $m, n \geq 0$,

$$S_m(r) - S_n(r) = \sum_{i=0}^{\infty} (s_{m+i} - s_{n+i})r^i \leq 1.$$

Here in the inequality we used the fact that the sequence $(s_{m+i} - s_{n+i})_{i\geq 0}$ takes values in $\{-1, 0, 1\}$ and alternates in sign if all zeroes are removed (Lemma 2.2). ∎

Proof of Theorem 1.2. Assume that the set of limit values of $(S_n(-r))_{n\geq 0}$ is contained in an interval (A, B) of length $< 1 + r - r^2$. Then, up to shifting (and truncating) S, we may assume that $S_n(-r) \in (A, B)$ for all $n \geq 0$.

Let $\mu = \max_{n\geq 0} s_n$ and $\nu = \min_{n\geq 0} s_n$. Since S is not ultimately periodic, we have $\mu > \nu$. Moreover, there exists a subword (μ, μ') of S with $\mu' < \mu$. Similarly, there exists a subword (ν, ν') of S with $\nu' > \nu$. If $\mu - \nu \geq 2$, then, by (2.3) applied to the subwords (μ, μ') and (ν, ν'),

$$\begin{aligned}2 - (1-r)(1-r^3) &= (1+r^2)(1+r-r^2)\\ &> (\mu-\nu) - r(\mu'-\nu') \geq (\mu-\nu) - r(\mu-\nu-2) \geq 2,\end{aligned}$$

which is a contradiction. Thus $\mu - \nu = 1$. Up to replacing s_n by $s_n - \nu$, we may assume that $s_n \in \{0,1\}$ for all $n \geq 0$. For notational convenience, we now regard S as a word $s_0 s_1 \ldots$ over the alphabet $\{0,1\}$. By the above, 01 and 10 are subwords of S.

Note that S has a subword of the form $10a$, where $a \in \{0,1\}$. If 010 is a subword of S, then, by (2.3) applied to these subwords, we have

$$1 + r - r^2(1-r)(1-r^2) = (1+r^3)(1+r-r^2) > 1 + r + ar^2 \geq 1 + r,$$

which is a contradiction. Thus 010 is not a subword of S. Similarly, 101 is not a subword of S.

Next note that 0111 and 1000 are not both subwords of S. Indeed, otherwise, by (2.3) applied to these subwords, we would have

$$\begin{aligned}1 + r - r^2 + r^3 - r^3(1-r)(1-r^2) &= (1+r^4)(1+r-r^2)\\ &> 1 + r - r^2 + r^3.\end{aligned}$$

Up to replacing s_n by $1 - s_n$ for all $n \geq 0$, we may assume that 0111 is not a subword of S. Then all blocks of zeroes in S are separated by 11. Thus $S = 1^y 0^{z_0} 110^{z_1} 110^{z_2} \ldots$, where $y \geq 0$, $z_0 \geq 1$, and $z_n \geq 2$ for all $n \geq 1$.

Assume that z_n is odd for some $n \geq 1$. Then, by (2.3) applied to the subwords $10^{z_n}1$ and $0110^{z_n - 1}$ of S, we have

$$\begin{aligned}1 + r - r^2 + r^{z_n+1} - r^{z_n+1}(1-r)(1-r^2) &= (1+r^{z_n+2})(1+r-r^2)\\ &> 1 + r - r^2 + r^{z_n+1},\end{aligned}$$

a contradiction.

Thus z_n is even for all $n \geq 1$. Assume that $z_m - z_n > 2$ for some $m, n \geq 1$. Then both 10^{z_n+3} and $0110^{z_n}1$ are subwords of S. Thus, by (2.3) applied to these subwords, we have

$$\begin{aligned}1 + r - r^2 + r^{z_n+3} - r^{z_n+3}(1-r)(1-r^2) &= (1+r^{z_n+4})(1+r-r^2)\\ &> 1 + r - r^2 + r^{z_n+3},\end{aligned}$$

a contradiction.

Thus there exists an even integer $k \geq 2$ such that $z_n \in \{k, k+2\}$ for all $n \geq 1$. Since S is not ultimately periodic, neither is the word $Z = z_1 z_2 \ldots$. Let $n \geq 0$ be an integer. By Lemma 2.1 applied to Z, there exists a word $U = u_1 \ldots u_n$ of length n over $\{k, k+2\}$ such that $(k+2)U$ and kU are subwords of Z. Then $10^{k+2}110^{u_1}\dots110^{u_n}$ and $0110^k110^{u_1}\dots110^{u_n}$ are subwords of S. Thus, by (2.3) applied to these subwords, we have

$$(1+r^m)(B-A) > 1+r-r^2,$$

where $m = k+3+\sum_{i=1}^{n}(u_i+2) > 4n$. Since $n \geq 0$ is arbitrary, we get $B-A \geq 1+r-r^2$, a contradiction. This finishes the proof of (1.3).

Let $k \geq 2$ be an even integer and let $W = w_0w_1\dots$ be a Sturmian word over the alphabet $\{0,1\}$. Let $z_n = k+2w_n$ for $n \geq 0$. Consider the word $S = 0^{z_0}110^{z_1}11\dots$. We will show that

$$S_j(-r) - S_l(-r) \leq 1+r-r^2 \tag{2.5}$$

for all $j,l \geq 0$.

For $n \geq 0$, let $\sigma_n = -1+\sum_{i=0}^{n-1}(z_i+2) = -1+(k+2)n+2\tau_n$, where $\tau_n = \sum_{i=0}^{n-1} w_i$. Then $s_j = 1$ if and only if $j = \sigma_n$ or $j = \sigma_n - 1$ for some $n \geq 1$. For $n \geq 1$, we have

$$\begin{aligned} S_{\sigma_n}(-r) &= 1-\sum_{i=n+1}^{\infty} r^{\sigma_i-\sigma_n-1}(1-r) \\ &> 1-\sum_{i=n+1}^{\infty} r^{2(i-n)-1}(1-r) > 1-\frac{r(1-r)}{1-r^2} = \frac{1}{1+r} \end{aligned} \tag{2.6}$$

and

$$S_{\sigma_n-1}(-r) = \sum_{i=n}^{\infty} r^{\sigma_i-\sigma_n}(1-r) > 0.$$

In (2.6) we have used the estimate $\sigma_i-\sigma_n \geq (k+2)(i-n) > 2(i-n)$ for $i \geq n+1$. We observe that $S_{\sigma_n-1}(-r) = 1-rS_{\sigma_n}(-r) < S_{\sigma_n}(-r)$ by (2.6). Moreover, for $\sigma_{n-1} < j < \sigma_n$, we have $S_j(-r) = (-r)^{\sigma_n-j-1}S_{\sigma_n-1}(-r)$. Thus

$$\max_{\sigma_{n-1}<j\leq\sigma_n} S_j(-r) = S_{\sigma_n}(-r), \qquad \min_{\sigma_{n-1}<j\leq\sigma_n} S_j(-r) = S_{\sigma_n-2}(-r).$$

Therefore, in order to show (2.5), we may assume that $j = \sigma_m$ and $l = \sigma_n - 2$ for some $m,n \geq 1$. In this case,

$$\begin{aligned} S_{\sigma_m}(-r) - S_{\sigma_n-2}(-r) &= 1+r-r^2-\sum_{i=1}^{\infty}(r^{\sigma_{m+i}-\sigma_m-1}-r^{\sigma_{n+i}-\sigma_n+1})(1-r) \\ &\leq 1+r-r^2. \end{aligned}$$

Here in the inequality we have used the inequality

$$(\sigma_{n+i}-\sigma_n)-(\sigma_{m+i}-\sigma_m) = 2(\tau_{n+i}-\tau_n)-2(\tau_{m+i}-\tau_m) \geq -2,$$

which follows from Lemma 2.2. ∎

3. Proof of Theorem 1.1. For $n \geq 0$, let

$$\begin{aligned} x_n &= \lfloor \xi(-p/q)^n + \eta \rfloor, \\ s_n &= -px_n - qx_{n+1}. \end{aligned} \tag{3.1}$$

We deduce Theorem 1.1 from (1.3) using the following lemma.

LEMMA 3.1. *The sequence $(s_n)_{n\geq 0}$ is not ultimately periodic.*

Proof. For $n \geq 0$, let

$$\begin{aligned} t_n = ps_{2n} - qs_{2n+1} &= -p^2 x_{2n} + q^2 x_{2n+2} \\ &= -p^2 \lfloor \xi(p^2/q^2)^n + \eta \rfloor + q^2 \lfloor \xi(p^2/q^2)^{n+1} + \eta \rfloor. \end{aligned}$$

By [D2, Lemma 1] applied to $\alpha = p^2/q^2$, $(t_n)_{n\geq 0}$ is not ultimately periodic. Assume that there exist $N \geq 0$ and $l \geq 1$ such that $s_{n+l} = s_n$ for all $n \geq N$. Then $t_{n+l} = t_n$ for all $n \geq N/2$, a contradiction. ∎

Proof of Theorem 1.1. For $n \geq 0$, let

$$y_n = \{\xi(-p/q)^n + \eta\} - \eta.$$

Since $x_n + y_n = \xi(-p/q)^n$, we have

$$s_n = py_n + qy_{n+1}. \tag{3.2}$$

By (3.1) and (3.2), $(s_n)_{n\geq 0}$ is a bounded sequence of integers. Thus, by Lemma 3.1 and Theorem 1.2, (1.3) holds. Since

$$S_n(-r) = \sum_{i=0}^{\infty} s_{n+i}(-r)^i = \sum_{i=0}^{\infty} p(y_{n+i} + ry_{n+i+1})(-r)^i = py_n,$$

we see that (1.3) is equivalent to

$$\limsup_{n\to\infty} y_n - \liminf_{n\to\infty} y_n \geq (1 + r - r^2)/p,$$

and hence to (1.2). ∎

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References

[B1] Y. Bugeaud, *Linear mod one transformations and the distribution of fractional parts $\{\xi(p/q)^n\}$*, Acta Arith. 114 (2004), 301–311.

[B2] Y. Bugeaud, *Distribution Modulo One and Diophantine Approximation*, Cambridge Tracts in Math. 193, Cambridge Univ. Press, Cambridge, 2012.

[BD] Y. Bugeaud and A. Dubickas, *Fractional parts of powers and Sturmian words*, C. R. Math. Acad. Sci. Paris 341 (2005), 69–74.

[D1] A. Dubickas, *Arithmetical properties of powers of algebraic numbers*, Bull. London Math. Soc. 38 (2006), 70–80.

[D2] A. Dubickas, *On the distance from a rational power to the nearest integer*, J. Number Theory 117 (2006), 222–239.

[D3] A. Dubickas, *On a sequence related to that of Thue–Morse and its applications*, Discrete Math. 307 (2007), 1082–1093.

[D4] A. Dubickas, *Powers of a rational number modulo 1 cannot lie in a small interval*, Acta Arith. 137 (2009), 233–239.

[D5] A. Dubickas, *Fractional parts of powers of large rational numbers*, Discrete Math. 342 (2019), 1949–1955.

[DN] A. Dubickas and A. Novikas, *Integer parts of powers of rational numbers*, Math. Z. 251 (2005), 635–648.

[FLP] L. Flatto, J. C. Lagarias and A. D. Pollington, *On the range of fractional parts* $\{\xi(p/q)^n\}$, Acta Arith. 70 (1995), 125–147.

[L] M. Lothaire, *Algebraic Combinatorics on Words*, Encyclopedia Math. Appl. 90, Cambridge Univ. Press, Cambridge, 2002.

[LZ] Q. Lu and W. Zheng, *Alternating geometric progressions modulo one and Sturmian words*, Combin. Number Theory, to appear; arXiv:2603.22045v3 (2026).

[M] K. Mahler, *An unsolved problem on the powers of* $3/2$, J. Austral. Math. Soc. 8 (1968), 313–321.

[MH1] M. Morse and G. A. Hedlund, *Symbolic dynamics*, Amer. J. Math. 60 (1938), 815–866.

[MH2] M. Morse and G. A. Hedlund, *Symbolic dynamics II. Sturmian trajectories*, Amer. J. Math. 62 (1940), 1–42.

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