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Almost sure upper bound for random multiplicative functions

by

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Abstract. Let $\varepsilon > 0$. Let f be a Steinhaus or Rademacher random multiplicative function. We prove that almost surely, as $x \to +\infty$,

$$\sum_{n \leqslant x} f(n) \ll \sqrt{x}(\log_2 x)^{3/4+\varepsilon}.$$

1. Introduction. The aim of this article is to study large fluctuations of random multiplicative functions. Random multiplicative functions have been a very attractive topic in recent years. There are at least two models of random multiplicative functions that have been frequently studied in number theory and probability (see, for example, Harper [11, 10, 12], Lau–Tenenbaum–Wu [13], Chatterjee–Soundararajan [4], Benatar–Nishry–Rodgers [2]). Let $\mathcal{P}$ be the set of all prime numbers. A *Steinhaus random multiplicative function* is obtained by letting $(f(p))_{p\in\mathcal{P}}$ be a sequence of independent Steinhaus random variables (i.e. distributed uniformly on the complex unit circle $\{|z| = 1\}$), and then setting

$$f(n) := \prod_{p^a || n} f(p)^a \quad \text{for all } n \in \mathbb{N}$$

where $p^a \parallel n$ means that p^a is the highest power of p that divides n. A *Rademacher multiplicative function* is obtained by letting $(f(p))_{p\in\mathcal{P}}$ be independent Rademacher random variables (i.e. taking values ± 1 with probability $1/2$ each), and setting

$$f(n) = \begin{cases} \prod_{p|n} f(n) & \text{if } n \text{ is squarefree,} \\ 0 & \text{otherwise.} \end{cases}$$

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The Rademacher model was introduced by Wintner [18] in 1944 as a heuristic model of the Möbius function μ (see the introduction in [13]). With a little change, one can obtain the probabilistic model for a real primitive Dirichlet character (see Granville–Soundararajan [7]). Steinhaus random multiplicative functions model the randomly chosen Dirichlet character χ or continuous characters $n \mapsto n^{it}$; see for example Granville–Soundararajan [6, Section 2].

A classical result in the study of sums of independent random variables is the Law of the Iterated Logarithm, which predicts the almost sure size of the largest fluctuation of those sums. Let $(\xi_k)_{k\in\mathbb{N}}$ be an independent sequence of real random variables taking value ± 1 with probability $1/2$ each. Khinchin's Law of the Iterated Logarithm says that almost surely,

$$\limsup_{N\to+\infty} \frac{|\sum_{k\leqslant N} \xi_k|}{\sqrt{2N\log_2 N}} = 1.$$

Here and in the sequel, $\log_k$ denotes the k-fold iterated logarithm. See for instance Gut [8, Theorem 8.8.1]. Note that the largest fluctuations as N varies (of the size $\sqrt{N\log_2 N}$) are somewhat larger than the random fluctuations one expects at a fixed point ($\mathbb{E}[|\sum_{k\leqslant N}\xi_k|] \asymp \sqrt{N}$).

Khinchin's theorem cannot be applied for random multiplicative functions, because their values are not independent. However, following Harper (see [12, end of Introduction]), we believe that a suitable version of the Law of the Iterated Logarithm might hold.

For f a multiplicative function, we denote $M_f(x) := \sum_{n\leqslant x} f(n)$. Wintner [18] studied the case where f is a Rademacher random multiplicative function, and he was able to prove that, for any fixed $\varepsilon > 0$,

$$M_f(x) = O(x^{1/2+\varepsilon}) \quad \text{and} \quad M_f(x) = \Omega(x^{1/2-\varepsilon}) \quad \text{almost surely.}$$

This has been further improved by Erdős [5] who proved that almost surely one has the bound $O(\sqrt{x}\log^A x)$ and almost surely one does not have the bound $O(\sqrt{x}\log^{-B} x)$, for some nonnegative real numbers A and B. In the 1980s, Halász [9] introduced several novel ideas which made a further progress. By conditioning coupled with hypercontractive inequalities, he proved in the Rademacher case that

$$M_f(x) = O\big(\sqrt{x}\exp(A\sqrt{\log_2 x\log_3 x})\big)$$

for some nonnegative A. Recently, Lau–Tenenbaum–Wu [13] (see also Basquin [1]) improved the analysis of hypercontractive inequalities in Halász's argument, establishing, for the Rademacher case, an almost sure upper bound $O(\sqrt{x}(\log_2 x)^{2+\varepsilon})$.

For the lower bound, Harper [12] proved that for any function $V(x)$ tending to infinity as $x \to +\infty$, there almost surely exist arbitrarily large values of x for which

$$|M_f(x)| \gg \frac{\sqrt{x}(\log_2 x)^{1/4}}{V(x)}.$$

Moreover, Mastrostefano [14] recently proved an upper bound for the sum restricted to integers with a large prime factor. We denote by $P(n)$ the largest prime factor of n, with the convention $P(1) = 1$. In [14], it was proved that almost surely, as $x \to +\infty$,

$$\sum_{\substack{n \leqslant x \\ P(n) > \sqrt{x}}} f(n) \ll \sqrt{x}(\log_2 x)^{1/4+\varepsilon}.$$

This bound should be compared with the first moment: Harper [11] proved that as $x \to +\infty$,

$$\mathbb{E}[|M_f(x)|] \asymp \frac{\sqrt{x}}{(\log_2 x)^{1/4}}.$$

This discrepancy of a factor $\sqrt{\log_2 x}$ between the first moment and the almost sure behaviour is similar to the Law of the Iterated Logarithm for independent random variables. For this reason Harper conjectured that for any fixed $\varepsilon > 0$, we might have almost surely, as $x \to +\infty$,

$$M_f(x) \ll \sqrt{x}(\log_2 x)^{1/4+\varepsilon}$$

for both Steinhaus and Rademacher cases (see [12, Introduction] for more details).

The main objective of this work is to strengthen the result of Lau–Tenenbaum–Wu to achieve a $3/4$ improvement.

THEOREM 1.1. *Let $\varepsilon > 0$. Let f be a Steinhaus or Rademacher random multiplicative function. Then almost surely, as $x \to +\infty$,*

$$M_f(x) \ll \sqrt{x}(\log_2 x)^{3/4+\varepsilon}. \tag{1}$$

2. Sketch of the proof. Let f be a Steinhaus or Rademacher multiplicative function. As is typical when aiming to establish almost sure bounds for a sum of random variables, we will focus on a sequence of "test points" (say x_i). These test points are chosen to be sparse, yet sufficiently close to allow for manageable control over the increments of f between consecutive elements. We will then consolidate the information obtained from each test point using the first Borel–Cantelli lemma. The second step is to split $M_f(x_i)$ according to the largest prime factor $P(n)$ of the summation variable n. Recall that $v_p(n)$ denotes the p-adic valuation of an integer n (i.e. the exponent of p in the prime factorization of n). Let $(y_j)_{0 \leqslant j \leqslant J}$ be a nondecreasing sequence such that $J \asymp \log_2 x_i$. The choice of y_j is such that $\log y_{j-1} \sim \log y_j$. One could choose $\log y_j \sim \ell^c \log y_{j-1}$ for some constant c, but this will not change the result. The reason behind this choice is to approximate the factor $\frac{1}{\log(x_i/z)}$ that will arise later for $x_i/y_j < z \leqslant x_i/y_{j-1}$ with $\frac{1}{\log y_j}$. We have

$$M_f(x_i) = \sum_{\substack{n\leqslant x_i\\ P(n)\leqslant y_0}} f(n) + M_f^{(1)}(x_i) + M_f^{(2)}(x_i)$$

where

$$M_f^{(1)}(x_i) := \sum_{j=1}^{J} \sum_{y_{j-1}<p\leqslant y_j} f(p) \sum_{\substack{n\leqslant x_i/p\\ P(n)<p}} f(n), \tag{2}$$

$$M_f^{(2)}(x_i) := \sum_{j=1}^{J} \sideset{}{^{y_{j-1},y_j}}\sum_{\substack{y_{j-1}^2<d\leqslant x_i\\ v_{P(d)}(d)\geqslant 2}} f(d) \sum_{\substack{n\leqslant x_i/d\\ P(n)\leqslant y_{j-1}}} f(n) \tag{3}$$

with $\sum^{y,z}$ indicating summation restricted to integers all of whose prime factors belong to the interval $]y,z]$. Note that $M_f^{(2)}(x_i)$ is 0 in the Rademacher case. By choosing y_0 small enough, the sum $\sum_{n\leqslant x_i,\, P(n)\leqslant y_0} f(n)$ is "small", thus it can be neglected. We show first that $M_f^{(2)}(x_i)$ is "small"; this can be done in few steps. Moreover, the expectation, conditional on the value $f(q)$ for prime numbers $q<p$, of the quantity $f(p)\sum_{n\leqslant x_i/p,\, P(n)<p} f(n)$ is equal to 0. Therefore, $M_f^{(1)}(x_i)$ is a sum of martingale differences with some kind of conditional variance

$$V(x_i) = \sum_{y_0<p\leqslant y_J} \Big| \sum_{\substack{n\leqslant x_i/p\\ P(n)<p}} f(n)\Big|^2. \tag{4}$$

By smoothing the above quantity, we get

$$\widetilde{V}(y_j, x_i) \approx \int_{x_i/y_{j-1}}^{x_i/y_j} \frac{1}{\log(x_i/z)} \Big| \sum_{\substack{n\leqslant z\\ P(n)\leqslant x_i/z}} f(n)\Big|^2 \frac{\mathrm{d}z}{z^2}. \tag{5}$$

One could compare the right-hand expression above to equation (7.1) in [14]. In our case, the friable condition $P(n)\leqslant x_i/z$ varies with z, making it impossible to transform the relevant quantity into an Euler product. To eliminate this dependence, we proceed by bounding (5) by

$$\frac{1}{\log y_j} \int_0^{+\infty} \sup_{y_{j-1}<p\leqslant y_j} \Big| \sum_{\substack{n\leqslant z\\ P(n)\leqslant p}} f(n)\Big|^2 \frac{\mathrm{d}z}{z^2}.$$

By conditioning on $\mathcal{F}_{j-1}$, the σ-algebra generated by $\{f(p);\, p \leqslant y_{j-1}\}$, one can bound the above quantity by

$$\mathbb{E}\left[\frac{1}{\log y_j} \int_{-\infty}^{+\infty} \frac{|F_j(1/2+it)|^2}{|1/2+it|^2}\,\mathrm{d}t \;\middle|\; \mathcal{F}_{j-1}\right].$$

This will give rise to a submartingale sequence. A new version of Hoeffding's inequality (see Lemma 3.10) allows us to reduce the problem to studying the expression (4). The use of this inequality is exactly what gives a strong exponential upper bound. More specifically, the goal becomes to prove that almost surely, as $x_i \to +\infty$, we have $V(x_i) \leqslant x_i(\log_2 x_i)^{1/2}$. The key point here is to notice that among the terms in the sum $V(x_i)$ that contribute significantly are $\sum_{y_{j-1}<p\leqslant y_j} |\sum_{n\leqslant x_i/p,\, P(n)<p} f(n)|^2$ such that y_j is very "close" to x_i (see Section 7.5). In fact, the number of y_j close to x_i, that contribute much to the sum, is less than $(\log_2 x_i)^{\varepsilon/50}$. Thus we are reduced again to studying

$$\widetilde{V}(y_j, x_i) := \sum_{y_{j-1}<p\leqslant y_j} \Big| \sum_{\substack{n\leqslant x_i/p \\ P(n)<p}} f(n)\Big|^2. \tag{6}$$

By smoothing and adjusting $\widetilde{V}(y_j, x_i)$ a little, this will give rise to a supermartingale sequence $(U_{j,i})_{j\geqslant 0}$ where $U_{0,i} = I_0$ does not depend on x_i. At the end, we use Harper's low moment result. Roughly speaking, we get $V(x_i) \leqslant x_i(\log_2 x_i)^{1/2}$ almost surely as $x_i \to +\infty$.

3. Preliminary results

3.1. Notation. Let us start with some definitions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We call every sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of increasing sub-σ-algebras of $\mathcal{F}$ a *filtration*. We say that a sequence $(Z_n)_{n\in\mathbb{N}}$ of real random variables is a *submartingale* (resp. *supermartingale*) *sequence* with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if the following properties are satisfied:

- Z_n is $\mathcal{F}_n$-measurable,
- $\mathbb{E}[|Z_n|] < +\infty$,
- $\mathbb{E}[Z_{n+1} \,|\, \mathcal{F}_n] \geqslant Z_n$ (resp. $\mathbb{E}[Z_{n+1} \,|\, \mathcal{F}_n] \leqslant Z_n$) almost surely.

We say that $(Z_n)_{n\in\mathbb{N}}$ is a *martingale difference sequence* with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if

- Z_n is $\mathcal{F}_n$ measurable,
- $\mathbb{E}[|Z_n|] < +\infty$,
- $\mathbb{E}[Z_{n+1} \,|\, \mathcal{F}_n] = 0$ almost surely.

An event $E \in \mathcal{F}$ happens *almost surely* if $\mathbb{P}[E] = 1$.

If Z is a random variable and $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{F}$ are some sub-σ-algebras, we have

$$\mathbb{E}\big[\mathbb{E}[Z \,|\, \mathcal{H}_2] \,\big|\, \mathcal{H}_1\big] = \mathbb{E}[Z \,|\, \mathcal{H}_1].$$

3.2. Known tools

LEMMA 3.1. *Let f be a Steinhaus or Rademacher random multiplicative function. For every sequence $(a_n)_{n\geqslant 1}$ of complex numbers and every positive integer $m \geqslant 1$, we have*

$$\mathbb{E}\Big[\Big|\sum_{n\geqslant 1} a_n f(n)\Big|^{2m}\Big] \leqslant \Big(\sum_{n\geqslant 1} |a_n|^2 \tau_{2m-1}(n)\Big)^m$$

where $\tau_m(n)$ is the m-fold divisor function.

Proof. See Bonami [3, Chapter III], and also Halász [9, Lemma 2] for another proof. This result was already used by Lau–Tenenbaum–Wu [13], Harper [10] and recently by Mastrostefano [14]. ∎

LEMMA 3.2 ([2, Lemma 3.1]). *Let $m \geqslant 2$ an integer. Then, uniformly for $x \geqslant 3$, we have*

$$\sum_{n\leqslant x} \tau_m(n) \leqslant x(2\log x)^{m-1}.$$

LEMMA 3.3 (Parseval's identity, [15, (5.26)]). *Let $(a_n)_{n\in\mathbb{N}^*}$ be a sequence of complex numbers, let $A(s) := \sum_{n=1}^{+\infty} \frac{a_n}{n^s}$ denote the corresponding Dirichlet series and let σ_c be its abscissa of convergence. Then for any $\sigma > \max(0, \sigma_c)$, we have*

$$\int_0^{+\infty} \frac{|\sum_{n\leqslant x} a_n|^2}{x^{1+2\sigma}}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Big|\frac{A(\sigma+it)}{\sigma+it}\Big|^2\,\mathrm{d}t.$$

We define the following parameter:

$$a_f = \begin{cases} 1 & \text{if } f \text{ is a Rademacher multiplicative function,} \\ -1 & \text{if } f \text{ is a Steinhaus multiplicative function.} \end{cases} \tag{7}$$

LEMMA 3.4 (Euler product result, Mastrostefano [14, Lemma 2.4]). *Let f be a Rademacher or Steinhaus random multiplicative function. Let t be a real number and $2 \leqslant x \leqslant y$. Then*

$$\mathbb{E}\Big[\prod_{x<p\leqslant y} \Big|1 + a_f \frac{f(p)}{p^{1/2+it}}\Big|^{2a_f}\Big] = \prod_{x<p\leqslant y} \Big(1 + \frac{a_f}{p}\Big)^{a_f}.$$

Let $(A_n)_{n\geqslant 1}$ be sequence of events. Recall that

$$\limsup_{n\to+\infty} A_n = \bigcap_{n\geqslant 1} \bigcup_{k\geqslant n} A_k.$$

Lemma 3.5 (Borel–Cantelli's First Lemma, [8, Theorem 2.18.1]). *Let $(A_n)_{n\geqslant 1}$ be a sequence of events. If $\sum_{n=1}^{+\infty}\mathbb{P}[A_n]<+\infty$ then*

$$\mathbb{P}\Big[\limsup_{n\to+\infty} A_n\Big]=0.$$

Lemma 3.6. *Let $b_0,b_1,\dots,b_n$ be any complex numbers. Then*

$$\int_0^1\Big|b_0+\sum_{k=1}^{n}\mathrm{e}^{i2\pi k\vartheta}b_k\Big|\,\mathrm{d}\vartheta\geqslant|b_0|. \tag{8}$$

Proof. Directly, we have

$$\int_0^1\Big|b_0+\sum_{k=1}^{n}\mathrm{e}^{i2\pi k\vartheta}b_k\Big|\,\mathrm{d}\vartheta\geqslant\Big|\int_0^1\Big(b_0+\sum_{k=1}^{n}\mathrm{e}^{i2\pi k\vartheta}b_k\Big)\,\mathrm{d}\vartheta\Big|=|b_0|. \ \blacksquare$$

Lemma 3.7 ([8, Theorem 10.2.1]). *Suppose that a sequence $\{(Z_n,\mathcal{F}_n)\}_{n\geqslant 0}$ of real random variables and σ-algebras is a supermartingale. Then, for any integers n,m such that $n\geqslant m$, we have*

$$\mathbb{E}[Z_n\,|\,\mathcal{F}_m]\leqslant Z_m.$$

Lemma 3.8 (Doob's inequality, [8, Theorem 10.9.1]). *Let $a>0$. Suppose that a sequence $\{(Z_n,\mathcal{F}_n)\}_{n\geqslant 0}$ of real random variables and σ-algebras is a nonnegative submartingale (resp. supermartingale). Then*

$$\mathbb{P}\Big[\max_{0\leqslant j\leqslant n} Z_j>a\Big]\leqslant\frac{\mathbb{E}[Z_n]}{a}\qquad\Big(\text{resp. }\mathbb{P}\Big[\max_{0\leqslant j\leqslant n} Z_j>a\Big]\leqslant\frac{\mathbb{E}[Z_0]}{a}\Big).$$

Lemma 3.9 (Doob's L^r-inequality, [8, Theorem 10.9.4]). *Let $r>1$. Suppose that a sequence $\{(Z_n,\mathcal{F}_n)\}_{0\leqslant n\leqslant N}$ of real random variables and σ-algebras is a nonnegative submartingale bounded in L^r. Then*

$$\mathbb{E}\Big[\Big(\max_{0\leqslant n\leqslant N} Z_n\Big)^r\Big]\leqslant\Big(\frac{r}{r-1}\Big)^r\max_{0\leqslant n\leqslant N}\mathbb{E}[Z_n^r].$$

3.3. New tools. We have the following version of Azuma/Hoeffding's inequality.

Lemma 3.10. *Let $Z=(Z_n)_{1\leqslant n\leqslant N}$ be a complex martingale difference sequence with respect to a filtration $(\mathcal{F}_n)_{1\leqslant n\leqslant N}$. Assume that for each n, Z_n is bounded almost surely. Furthermore, assume that $|Z_n|\leqslant S_n$ almost surely where $(S_n)_{1\leqslant n\leqslant N}$ is a real predictable process with respect to the same filtration (i.e. for each n, S_n is $\mathcal{F}_{n-1}$-measurable). Assume that $\sum_{1\leqslant n\leqslant N}S_n^2\leqslant T$ almost surely where T is a deterministic constant. Then, for any $\varepsilon>0$, we have*

$$\mathbb{P}\Big[\Big|\sum_{1\leqslant n\leqslant N} Z_n\Big|\geqslant\varepsilon\Big]\leqslant 2\exp\Big(\frac{-\varepsilon^2}{10T}\Big).$$

Proof. We define the conditional expectation $\mathbb{E}_n[\cdot] := \mathbb{E}[\cdot \mid \mathcal{F}_n]$. Following [16, proof of Theorem 3], we define $g_n := \sum_{k=1}^n Z_k$ with $g_0 := 0$. We set $Z_0 := 0$ and $S_0 := 0$. We now define, for each $n \geqslant 1$, $\lambda > 0$ and $t \geqslant 0$,

$$H_n(t) := \mathbb{E}_{n-1}[\cosh(\lambda|g_{n-1} + tZ_n|)]$$

where $\cosh x := \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$. Note that $H_n \geqslant 0$. For any positive differentiable function u, we have

$$(\cosh u)'' = u'^2 \cosh u + u'' \sinh u \leqslant (\cosh u)(u'^2 + |u''|u).$$

Here, we have used the inequality

$$\sinh u \leqslant u \cosh u.$$

Thus, by taking u to be $\lambda|g_{n-1} + tZ_n|$ we find that $u'^2 + |u''|u$ is less than

(9)
$$\lambda^2\biggl(2\biggl(\frac{\Re(Z_n)(\Re(g_{n-1}) + t\Re(Z_n)) + \Im(Z_n)(\Im(g_{n-1}) + t\Im(Z_n))}{|g_{n-1} + tZ_n|}\biggr)^2 + |Z_n|^2\biggr).$$

By using

$$\frac{|\Re(g_{n-1}) + t\Re(Z_n)|}{|g_{n-1} + tZ_n|}, \frac{|\Im(g_{n-1}) + t\Im(Z_n)|}{|g_{n-1} + tZ_n|} \leqslant 1,$$

we deduce that (9) is less than

$$\lambda^2\bigl(2(|\Re(Z_n)| + |\Im(Z_n)|)^2 + |Z_n|^2\bigr), \tag{10}$$

and since $|\Re(Z_n)| + |\Im(Z_n)| \leqslant \sqrt{2}|Z_n|$, we conclude that

$$\begin{aligned} H_n''(t) &\leqslant \mathbb{E}_{n-1}[5\lambda^2|Z_n|^2 \cosh(\lambda|g_{n-1} + tZ_n|)] \\ &\leqslant 5\lambda^2 S_n^2 \mathbb{E}_{n-1}[\cosh(\lambda|g_{n-1} + tZ_n|)] = 5\lambda^2 S_n^2 H_n(t). \end{aligned} \tag{11}$$

Since $\mathbb{E}_{n-1}[Z_n] = 0$ and g_{n-1} is $\mathcal{F}_{n-1}$-measurable, we have

$$\begin{aligned} H_n'(0) &= \lambda\mathbb{E}_{n-1}\biggl[(\Re(Z_n)\Re(g_{n-1}) + \Im(Z_n)\Im(g_{n-1}))\frac{\sinh(\lambda|g_{n-1}|)}{|g_{n-1}|}\biggr] \\ &= \lambda\bigl(\mathbb{E}_{n-1}[\Re(Z_n)]\Re(g_{n-1}) + \mathbb{E}_{n-1}[\Im(Z_n)]\Im(g_{n-1})\bigr)\frac{\sinh(\lambda|g_{n-1}|)}{|g_{n-1}|} \\ &= 0. \end{aligned}$$

Now by [16, Lemma 3], we get

$$H_n(t) \leqslant H_n(0)\exp\bigl(\tfrac{5}{2}t^2\lambda^2 S_n^2\bigr).$$

In particular,

$$\begin{aligned} H_n(1) = \mathbb{E}_{n-1}[\cosh(\lambda|g_n|)] &\leqslant \mathbb{E}_{n-1}[\cosh(\lambda|g_{n-1}|)]\exp\bigl(\tfrac{5}{2}\lambda^2 S_n^2\bigr) \\ &= \cosh(\lambda|g_{n-1}|)\exp\bigl(\tfrac{5}{2}\lambda^2 S_n^2\bigr). \end{aligned}$$

We now define, for each $n \geqslant 0$,

$$G_n := \exp\left(\frac{-5\lambda^2}{2}\sum_{k=0}^{n} S_k^2\right)\cosh(\lambda|g_n|).$$

Since by assumption, S_n is $\mathcal{F}_{n-1}$-measurable, we get

$$\begin{aligned}\mathbb{E}_{n-1}[G_n] &= \mathbb{E}_{n-1}\left[\exp\left(\frac{-5\lambda^2}{2}\sum_{k=0}^{n} S_k^2\right)\cosh(\lambda|g_n|)\right] \\ &= \exp\left(\frac{-5\lambda^2}{2}\sum_{k=0}^{n} S_k^2\right)\mathbb{E}_{n-1}[\cosh(\lambda|g_n|)] \\ &\leqslant \exp\left(\frac{-5\lambda^2}{2}\sum_{k=0}^{n} S_k^2\right)\exp\left(\frac{5\lambda^2}{2}S_n^2\right)\cosh(\lambda|g_{n-1}|) = G_{n-1}.\end{aligned}$$

We deduce that G_n is a supermartingale with $\mathbb{E}G_0 = 1$. Thus by Doob's inequality, we have

$$\begin{aligned}\mathbb{P}[|g_N| \geqslant \varepsilon] &\leqslant \mathbb{P}\Big[\sup_{0\leqslant n\leqslant N} |g_n| \geqslant \varepsilon\Big] \\ &\leqslant \mathbb{P}\left[\sup_{0\leqslant n\leqslant N} G_n \geqslant \exp\left(\frac{-5\lambda^2 T}{2}\right)\cosh\lambda\varepsilon\right] \\ &\leqslant \frac{\exp\left(\frac{5\lambda^2 T}{2}\right)}{\cosh\lambda\varepsilon}\mathbb{E}G_0 \leqslant 2\exp\left(\frac{5\lambda^2 T}{2} - \lambda\varepsilon\right).\end{aligned}$$

By choosing λ to be $\varepsilon/(5T)$, we get the result. ∎

LEMMA 3.11. *Let $Z = (Z_n)_{1\leqslant n\leqslant N}$ be a complex martingale difference sequence with respect to a filtration $\mathcal{F} = (\mathcal{F}_n)_{1\leqslant n\leqslant N}$. Assume that for each n, Z_n is bounded almost surely (say $|Z_n| \leqslant b_n$ almost surely where b_n is some real number). Furthermore, assume that $|Z_n| \leqslant S_n$ almost surely where $(S_n)_{1\leqslant n\leqslant N}$ is a real predictable process with respect to the same filtration. Consider the event $\Sigma := \{\sum_{1\leqslant n\leqslant N} S_n^2 \leqslant T\}$ where T is a deterministic constant. Then, for any $\varepsilon > 0$,*

$$\mathbb{P}\left[\left\{\Big|\sum_{1\leqslant n\leqslant N} Z_n\Big| \geqslant \varepsilon\right\} \cap \Sigma\right] \leqslant 2\exp\left(\frac{-\varepsilon^2}{10T}\right).$$

Proof. We define

$$\widetilde{S}_n := S_n \mathbb{1}_{\sum_{k=1}^{n} S_k^2 \leqslant T} \quad \text{and} \quad \widetilde{Z}_n := Z_n \mathbb{1}_{\sum_{k=1}^{n} S_k^2 \leqslant T}.$$

It is clear that $(\widetilde{Z}_n)_{1\leqslant n\leqslant N}$ is a martingale difference sequence which is almost surely bounded. Note that $(\widetilde{S}_n)_{1\leqslant n\leqslant N}$ is a predictable process with respect to the filtration $\mathcal{F}$ and $\sum_{1\leqslant n\leqslant N} \widetilde{S}_n^2 \leqslant T$. Thus, all assumptions of Lemma 3.10

are satisfied for $(\widetilde{Z}_n)_{1\leqslant n\leqslant N}$ and $(\widetilde{S}_n)_{1\leqslant n\leqslant N}$. Note that under the condition Σ, we have

$$\sum_{1\leqslant n\leqslant N} Z_n = \sum_{1\leqslant n\leqslant N} \widetilde{Z}_n,$$

and thus, by Lemma 3.10,

$$\begin{aligned}\mathbb{P}\Big[\Big\{\Big|\sum_{1\leqslant n\leqslant N} Z_n\Big| \geqslant \varepsilon\Big\} \cap \Sigma\Big] &\leqslant \mathbb{P}\Big[\Big\{\Big|\sum_{1\leqslant n\leqslant N} \widetilde{Z}_n\Big| \geqslant \varepsilon\Big\} \cap \Sigma\Big] \\ &\leqslant \mathbb{P}\Big[\Big|\sum_{1\leqslant n\leqslant N} \widetilde{Z}_n\Big| \geqslant \varepsilon\Big] \leqslant 2\exp\Big(\frac{-\varepsilon^2}{10T}\Big). \quad \blacksquare\end{aligned}$$

4. Reduction of the problem. From now on, f is a Steinhaus or Rademacher random multiplicative function. The goal of this section is to reduce the problem to a simpler one. We want to prove that the event

$$\mathcal{A} := \{|M_f(x)| > 4\sqrt{x}(\log_2 x)^{3/4+\varepsilon} \text{ for infinitely many } x\}$$

holds with null probability. As in Lau–Tenenbaum–Wu [13], Basquin [1], Mastrostefano [14] and, more generally, in some proofs of the Law of the Iterated Logarithm ([8, Theorem 8.1] for example), the idea is to assess the event $\mathcal{A}$ on a suitable sequence of test points. We keep the same test points as in Lau–Tenenbaum–Wu [13, Lemma 2.3]. We take $x_i := \lfloor \mathrm{e}^{i^{c_0}} \rfloor$ where c_0 is a "small constant" in $]0, 1[$ that will be chosen in Lemma 4.1 below. As in [14, end of Section 2] we choose $X_\ell := \mathrm{e}^{2^{\ell^K}}$ where $K := 25/\varepsilon$. Set

$$\mathcal{A}_\ell := \Big\{ \sup_{X_{\ell-1} < x_{i-1} \leqslant X_\ell} \ \sup_{x_{i-1} < x \leqslant x_i} \frac{|M_f(x)|}{\sqrt{x}R(x)} > 4 \Big\}$$

where $R(x) := (\log_2 x)^{3/4+\varepsilon}$. One can easily see that $\mathcal{A} \subset \bigcup_{\ell\geqslant 1} \mathcal{A}_\ell$. We have the following upper bound. Recall that $x_i = \lfloor \mathrm{e}^{i^{c_0}} \rfloor$.

Lemma 4.1. *For any $A > 0$, there exists $c_0 = c_0(A)$ small enough such that almost surely, as $x_i \to +\infty$,*

$$\max_{x_{i-1} < x \leqslant x_i} |M_f(x) - M_f(x_{i-1})| \ll_A \frac{\sqrt{x_i}}{(\log x_i)^A}. \tag{12}$$

Remark. For $A = 1$ we can take $c_0 = \frac{1}{350}$. From now on, we take $c_0 \leqslant \frac{1}{10^3}$.

Proof of Lemma 4.1. See [13, Lemma 2.3]. Lau–Tenenbaum–Wu state the result for the Rademacher case, but it can be easily extended to the Steinhaus case by following the same argument. $\blacksquare$

Thus it suffices to prove that $\sum_{\ell\geqslant 1}\mathbb{P}[\mathcal{B}_\ell]<+\infty$ where

$$\mathcal{B}_\ell := \Big\{\sup_{X_{\ell-1}<x_i\leqslant X_\ell}\frac{|M_f(x_i)|}{\sqrt{x_i}R(x_i)}>3\Big\}.$$

5. Upper bound of $\mathbb{P}[\mathcal{B}_\ell]$

5.1. Setting up the model. In this subsection, we give the basic idea of our approach. Arguing as in [13, proof of Lemma 3.1], with a change of variables, let $x_i\in\,]X_{\ell-1},X_\ell]$ and take

$$y_0=\exp(2^{\ell^K(1-K/\ell)})=\exp\{(\log X_\ell)^{1-K/\ell}\},\qquad y_j=\exp\{\mathrm{e}^{j/\ell}(\log X_\ell)^{1-K/\ell}\}.$$

Let J be minimal with $y_J\geqslant X_\ell$, which means $J_\ell=J:=\lceil K\ell^K\log 2\rceil\ll\ell^K$. Note that $\ell^K=\frac{1}{\log 2}\log_2 X_\ell\asymp\log_2 x_i\asymp\log_2 y_j$ for any $x_i\in\,]X_{\ell-1},X_\ell]$ and $1\leqslant j\leqslant J$.

We start by splitting $M_f(x_i)$ according to the size of the largest prime factor $P(n)$ of n. Let

$$\Psi_f(x,y):=\sum_{\substack{n\leqslant x\\ P(n)\leqslant y}}f(n)\quad\text{and}\quad \Psi_f'(x,y):=\sum_{\substack{n\leqslant x\\ P(n)<y}}f(n).$$

We have

$$M_f(x_i)=\Psi_f(x_i,y_0)+M_f^{(1)}(x_i)+M_f^{(2)}(x_i)$$

with

$$M_f^{(1)}(x_i):=\sum_{y_0<p\leqslant y_J}Y_p\quad\text{where}\quad Y_p:=f(p)\Psi_f'(x_i/p,p),$$

$$M_f^{(2)}(x_i):=\sum_{j=1}^{J}\ \sum_{\substack{y_{j-1}^2<d\leqslant x_i\\ v_{P(d)}(d)\geqslant 2}}^{y_{j-1},y_j} f(d)\Psi_f(x_i/d,y_{j-1}).$$

One can see that $\mathcal{B}_\ell\subset\mathcal{B}_\ell^{(0)}\cup\mathcal{B}_\ell^{(1)}\cup\mathcal{B}_\ell^{(2)}$ where

$$\mathcal{B}_\ell^{(0)}:=\bigcup_{X_{\ell-1}<x_i\leqslant X_\ell}\{|\Psi_f(x_i,y_0)|>\sqrt{x_i}R(x_i)\},$$

$$\mathcal{B}_\ell^{(1)}:=\bigcup_{X_{\ell-1}<x_i\leqslant X_\ell}\{|M_f^{(1)}(x_i)|>\sqrt{x_i}R(x_i)\},$$

$$\mathcal{B}_\ell^{(2)}:=\bigcup_{X_{\ell-1}<x_i\leqslant X_\ell}\{|M_f^{(2)}(x_i)|>\sqrt{x_i}R(x_i)\}.$$

Proof of Theorem 1.1 assuming that $\sum_{\ell\geqslant 1}\mathbb{P}[\mathcal{B}_\ell^{(0)}]$, $\sum_{\ell\geqslant 1}\mathbb{P}[\mathcal{B}_\ell^{(1)}]$ and $\sum_{\ell\geqslant 1}\mathbb{P}[\mathcal{B}_\ell^{(2)}]$ converge. If the three series converge then $\sum_{\ell\geqslant 1}\mathbb{P}[\mathcal{B}_\ell]$ converges. By Borel–Cantelli's First Lemma 3.5, we get the conclusion. ∎

Lemma 5.1. *The series* $\sum_{\ell\geqslant 1}\mathbb{P}[\mathcal{B}_\ell^{(0)}]$ *converges.*

Proof. Note that $\Psi_1(x,y)=\#\{n\leqslant x;\ P(n)\leqslant y\}$. By Markov's inequality we have

$$\mathbb{P}[\mathcal{B}_\ell^{(0)}]\leqslant\sum_{X_{\ell-1}<x_i\leqslant X_\ell}\mathbb{P}[|\Psi_f(x_i,y_0)|>\sqrt{x_i}R(x_i)]\leqslant\sum_{X_{\ell-1}<x_i\leqslant X_\ell}\frac{\mathbb{E}[|\Psi_f(x_i,y_0)|^2]}{x_iR(x_i)^2}.$$

However, we know that $y_0\leqslant x_i^{1/\log_2 x_i}$ for ℓ large enough. Using [15, Theorem 7.6], we then have

$$\mathbb{E}[\Psi_f(x_i,y_0)^2]\leqslant\Psi_1(x_i,y_0)\leqslant\Psi_1(x_i,x_i^{1/\log_2 x_i})\ll x_i(\log x_i)^{-c\log_3 x_i}$$

where c is an absolute constant. Thus, the series

$$\sum_{\ell\geqslant 1}\mathbb{P}[\mathcal{B}_\ell^{(0)}]\leqslant\sum_{\ell\geqslant 1}\sum_{X_{\ell-1}<x_i\leqslant X_\ell}(\log x_i)^{-c\log_3 x_i}$$

converges. ∎

6. Bounding $\mathbb{P}[\mathcal{B}_\ell^{(2)}]$. We set

$$N_{ij}(f):=\sideset{}{^{y_{j-1},y_j}}\sum_{\substack{y_{j-1}^2<d\leqslant x_i\\ v_{P(d)}(d)\geqslant 2}}f(d)\sum_{\substack{n\leqslant x_i/d\\ P(n)\leqslant y_{j-1}}}f(n)$$

with

$$M_f^{(2)}(x_i)=\sum_{j=1}^{J}N_{ij}(f).$$

Recall that $\mathcal{F}_{j-1}$ is the σ-algebra generated by $\{f(p);\ p\leqslant y_{j-1}\}$. Let $m>1$. Following [13, Section 3], we have

$$\mathbb{E}(|N_{ij}(f)|^{2m}\mid\mathcal{F}_{j-1})\leqslant\Big(\sideset{}{^{y_{j-1},y_j}}\sum_{\substack{d\leqslant x_i\\ v_{P(d)}(d)\geqslant 2}}\tau_{2m-1}(d)|\Psi_f(x_i/d,y_{j-1})|^2\Big)^m.\tag{13}$$

We set

$$X_{i,j}:=\sideset{}{^{y_{j-1},y_j}}\sum_{\substack{d\leqslant x_i\\ v_{P(d)}(d)\geqslant 2}}\tau_{2m-1}(d)|\Psi_f(x_i/d,y_{j-1})|^2.$$

Note that

$$\begin{aligned}\mathbb{E}[X_{i,j}]&=\sideset{}{^{y_{j-1},y_j}}\sum_{\substack{d\leqslant x_i\\ v_{P(d)}(d)\geqslant 2}}\tau_{2m-1}(d)\mathbb{E}[|\Psi_f(x_i/d,y_{j-1})|^2]\\ &\leqslant x_i\sideset{}{^{y_{j-1},y_j}}\sum_{\substack{d\leqslant x_i\\ v_{P(d)}(d)\geqslant 2}}\frac{\tau_{2m-1}(d)}{d}.\end{aligned}$$

By writing $d = rp^2$ where $p = P(d)$, we have

$$\begin{aligned}(14)\quad \sideset{}{^{y_{j-1},y_j}}\sum_{\substack{d\leqslant x_i\\ v_{P(d)}(d)\geqslant 2}} \frac{\tau_{2m-1}(d)}{d} &\leqslant \Bigg(\sideset{}{^{y_{j-1},y_j}}\sum_{r\leqslant x_i/y_{j-1}^2} \frac{\tau_{2m-1}(r)}{r} + 1\Bigg) \sum_{y_{j-1}<p\leqslant y_j} \frac{(2m-1)^2}{p^2}\\ &\ll \frac{m^2}{y_{j-1}}\Bigg(\sideset{}{^{y_{j-1},y_j}}\sum_{r\leqslant x_i/y_{j-1}^2} \frac{\tau_{2m-1}(r)}{r} + 1\Bigg) \sum_{y_{j-1}<p\leqslant y_j} \frac{1}{p}.\end{aligned}$$

Note that

$$\sum_{y_{j-1}<p\leqslant y_j} \frac{1}{p} \ll \frac{1}{\ell} \leqslant 1.$$

Recall that $\tau_{2m-1}(r) \leqslant (2m-1)^{\Omega(r)}$. It is clear that

$$\begin{aligned}\sideset{}{^{y_{j-1},y_j}}\sum_{\frac{x_i}{y_j^2 z(1+1/X)}\leqslant r\leqslant \frac{x_i}{y_{j-1}^2 z}} \frac{\tau_{2m-1}(r)}{r} &\leqslant \sideset{}{^{y_{j-1},y_j}}\sum_{r\geqslant 1} \frac{(2m-1)^{\Omega(r)}}{r}\\ &= \prod_{y_{j-1}<p\leqslant y_j} \left(1-\frac{2m-1}{p}\right)^{-1} \leqslant \mathrm{e}^{cm/\ell}\end{aligned}$$

where c is a positive constant. Finally, we get

$$(15)\qquad \mathbb{E}[X_{i,j}] \ll \frac{x_i m^2 \mathrm{e}^{cm/\ell}}{y_{j-1}}.$$

We define the following events:

$$\mathcal{X}_\ell = \Bigg\{\sup_{\substack{X_{\ell-1}<x_i\leqslant X_\ell\\ 1\leqslant j\leqslant J}} \frac{X_{i,j}}{x_i} \leqslant \frac{1}{\ell^{10K}}\Bigg\} \quad \text{and} \quad \mathcal{X}_{\ell,i,j} = \Bigg\{\frac{X_{i,j}}{x_i} \leqslant \frac{1}{\ell^{10K}}\Bigg\}.$$

Lemma 6.1. *For $m \ll \ell^K$, $\sum_{\ell\geqslant 1} \mathbb{P}[\overline{\mathcal{X}_\ell}]$ converges.*

Proof. By using the bound (15), we get

$$\begin{aligned}\mathbb{P}[\overline{\mathcal{X}_\ell}] &\leqslant \sum_{j=1}^{J} \sum_{X_{\ell-1}<x_i\leqslant X_\ell} \frac{\ell^{10K}}{x_i}\mathbb{E}[X_{ij}] \ll \sum_{j=1}^{J} \sum_{X_{\ell-1}<x_i\leqslant X_\ell} \ell^{10K}\frac{m^2\mathrm{e}^{cm/\ell}}{y_{j-1}}\\ &\ll \ell^{11K}\frac{\mathrm{e}^{c_1\ell^K}}{2^{c_2\mathrm{e}^{\ell^K}}}\end{aligned}$$

where $c_1, c_2 > 0$ are absolute constants. Thus, $\sum_{\ell\geqslant 1} \mathbb{P}[\overline{\mathcal{X}_\ell}]$ converges. ■

Proposition 6.2. *The series $\sum_{\ell\geqslant 1} \mathbb{P}[\mathcal{B}_\ell^{(2)}]$ converges.*

Proof. By Cauchy–Schwarz's inequality, we have

$$\Big|\sum_{1\leqslant j\leqslant J} N_{ij}(f)\Big|^{2m} \leqslant J^{2m-1}\sum_{1\leqslant j\leqslant J}|N_{ij}(f)|^{2m}.$$

On the other hand,

$$\mathbb{P}[\mathcal{B}_\ell^{(2)}] \leqslant \mathbb{P}[\mathcal{B}_\ell^{(2)}\cap\mathcal{X}_\ell]+\mathbb{P}[\,\overline{\mathcal{X}_\ell}\,] \tag{16}$$

and

$$\begin{aligned}\mathbb{P}[\mathcal{B}_\ell^{(2)}\cap\mathcal{X}_\ell] &\leqslant \sum_{X_{\ell-1}<x_i\leqslant x_\ell}\sum_{j=1}^{J}\frac{\mathbb{E}[|N_{ij}(f)|^{2m}\mid \mathcal{X}_{\ell,i,j}]J^{2m-1}}{(x_iR(x_i)^2)^m} \\ &\ll \sum_{X_{\ell-1}<x_i\leqslant X_\ell}\sum_{j=1}^{J}\frac{1}{J}\Big(\frac{c_5J^2}{R(x_i)^2\ell^{10K}}\Big)^m \\ &\ll 2^{\ell^K/c_0}\Big(\frac{c_5J^2}{R(X_\ell)^2\ell^{10K}}\Big)^m\end{aligned} \tag{17}$$

where c_5 is an absolute constant. By taking $m=\ell^K$ and recalling that $J\ll\ell^K$, we get

$$\mathbb{P}[\mathcal{B}_\ell^{(2)}\cap\mathcal{X}_\ell]\ll 2^{\ell^K/c_0}\Big(\frac{c_6}{\ell^{19K/2}}\Big)^{\ell^K}$$

where c_6 is an absolute constant. Thus, $\sum_{\ell\geqslant1}\mathbb{P}[\mathcal{B}_\ell^{(2)}\cap\mathcal{X}_\ell]$ converges. By Lemma 6.1 and (16), so does $\sum_{\ell\geqslant1}\mathbb{P}[\mathcal{B}_\ell^{(2)}]$. ∎

7. Bounding $\mathbb{P}[\mathcal{B}_\ell^{(1)}]$. From now on, we denote by p,q two prime numbers. We consider the filtration $\{\mathcal{F}_p\}_{p\in\mathcal{P}}$ where $\mathcal{F}_p$ denotes the σ-algebra generated by the random variables $f(q)$ for $q<p$. One can see that $\mathbb{E}[Y_p\mid\mathcal{F}_p]=0$. Thus $(Y_p)_{p\in\mathcal{P}}$ is a martingale difference.

Set

$$V_\ell(x_i;f):=\sum_{y_0<p\leqslant y_J}|\Psi_f'(x_i/p,p)|^2. \tag{18}$$

7.1. Simplifying and smoothing $V_\ell(x_i;f)$. Let X be a large real such that $\log X\asymp\ell^K$. Let p be a prime and $p<t\leqslant p(1+1/X)$. Let

$$\Psi_f'(x,z,y):=\sum_{\substack{z<n\leqslant x\\ P(n)<y}}f(n).$$

Using the bound

$$|\Psi_f'(x_i/p,p)|^2\leqslant 2|\Psi_f'(x_i/t,p)|^2+2|\Psi_f'(x_i/p,x_i/t,p)|^2,$$

we have $V_\ell(x_i;f)\leqslant 2L_\ell(x_i;f)+2W_\ell(x_i;f)$ with

$$L_\ell(x_i;f) := \sum_{y_0<p\leqslant y_J} \frac{X}{p} \int\limits_{p}^{p(1+1/X)} |\Psi_f'(x_i/t,p)|^2 \,\mathrm{d}t, \tag{19}$$

$$W_\ell(x_i;f) := \sum_{y_0<p\leqslant y_J} \frac{X}{p} \int\limits_{p}^{p(1+1/X)} |\Psi_f'(x_i/p,x_i/t,p)|^2 \,\mathrm{d}t. \tag{20}$$

Let us start by cleaning up $L_\ell(x_i;f)$. We have

$$\begin{aligned} L_\ell(x_i;f) &= \sum_{y_0<p\leqslant y_J} \frac{X}{p} \int\limits_{p}^{p(1+1/X)} |\Psi_f'(x_i/t,p)|^2 \,\mathrm{d}t \\ &= \sum_{\substack{j=1\\ x_i\geqslant y_{j-1}}}^{J} \sum_{y_{j-1}<p\leqslant y_j} \frac{X}{p} \int\limits_{p}^{p(1+1/X)} |\Psi_f'(x_i/t,p)|^2 \,\mathrm{d}t. \end{aligned}$$

By swapping integration and summation, we have

$$\begin{aligned} L_\ell(x_i;f) &= \sum_{\substack{j=1\\ x_i\geqslant y_{j-1}}}^{J} \int\limits_{y_{j-1}}^{y_j(1+1/X)} X \sum_{y_{j-1}<p\leqslant y_j} \frac{1}{p} \mathbb{1}_{p<t<p(1+1/X)} |\Psi_f'(x_i/t,p)|^2 \,\mathrm{d}t \\ &\leqslant \sum_{\substack{j=1\\ x_i\geqslant y_{j-1}}}^{J} \int\limits_{y_{j-1}}^{y_j(1+1/X)} X \sum_{\max\{t/(1+1/X),y_{j-1}\}<p\leqslant \min\{t,y_j\}} \frac{1}{p} |\Psi_f'(x_i/t,p)|^2 \,\mathrm{d}t \\ &\leqslant x_i L_\ell^{(1)}(x_i;f) + x_i L_\ell^{(2)}(x_i;f) \end{aligned}$$

where

$$L_\ell^{(1)}(x_i;f) := \frac{1}{x_i} \sum_{\substack{j=1\\ x_i\geqslant y_{j-1}}}^{J} \int\limits_{y_{j-1}}^{y_j} X \sum_{t/(1+1/X)<p\leqslant t} \frac{1}{p} |\Psi_f'(x_i/t,p)|^2 \,\mathrm{d}t \tag{21}$$

and

$$\begin{aligned} &L_\ell^{(2)}(x_i;f) \\ &\quad := \frac{1}{x_i} \sum_{\substack{j=1\\ x_i\geqslant y_{j-1}}}^{J} \int\limits_{y_j}^{y_j(1+1/X)} X \sum_{\max\{t/(1+1/X),y_{j-1}\}<p\leqslant y_j} \frac{1}{p} |\Psi_f'(x_i/t,p)|^2 \,\mathrm{d}t. \end{aligned} \tag{22}$$

By changing the variable $z := x_i/t$, we find

$$L_\ell^{(1)}(x_i;f) = \sum_{\substack{j=1\\ x_i\geqslant y_{j-1}}}^{J} \int\limits_{x_i/y_j}^{x_i/y_{j-1}} X \sum_{\frac{x_i}{z(1+1/X)}<p\leqslant \frac{x_i}{z}} \frac{1}{p} |\Psi_f'(z,p)|^2 \,\frac{\mathrm{d}z}{z^2},$$

$$L_\ell^{(2)}(x_i;f)=\sum_{\substack{j=1\\ x_i\geqslant y_{j-1}}}^{J}\int\limits_{\frac{x_i}{y_j(1+1/X)}}^{x_i/y_j} X\sum_{\max\{\frac{x_i}{z(1+1/X)},y_{j-1}\}<p\leqslant y_j}\frac{1}{p}|\Psi_f'(z,p)|^2\,\frac{\mathrm{d}z}{z^2}.$$

Let us first focus on j such that $1\leqslant\frac{\log x_i}{\log y_{j-1}}\leqslant\ell^{100K}$ in the $L_\ell^{(1)}(x_i;f)$'s sum. By the strong form of Mertens' theorem (with error term given by the Prime Number Theorem) we have

$$\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant x_i/z}\frac{1}{p}=\log\Bigg(\frac{\log(x_i/z)}{\log\big(\frac{x_i}{z(1+1/X)}\big)}\Bigg)+O\Bigg(\exp\Bigg(-C\sqrt{\log\Big(\frac{x_i/z}{1+1/X}\Big)}\Bigg)\Bigg)$$

where C is an absolute constant. Since $x_i/y_j<z\leqslant x_i/y_{j-1}$ and $\log y_j=\mathrm{e}^{1/\ell}\log y_{j-1}$, we get $\log(x_i/z)\asymp\log y_{j-1}$. On the other hand, by assumption, $\log X\asymp\log_2 y_{j-1}\asymp\ell^K$. Hence

$$\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant\frac{x_i}{z}}\frac{1}{p}\ll\frac{1}{X\log y_{j-1}}.\tag{23}$$

Thus, we have

$$\begin{aligned}
&\sum_{\substack{j=1\\ 1\leqslant\frac{\log x_i}{\log y_{j-1}}\leqslant\ell^{100K}}}^{J}\int\limits_{x_i/y_j}^{x_i/y_{j-1}} X\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant\frac{x_i}{z}}\frac{1}{p}|\Psi_f'(z,p)|^2\,\frac{\mathrm{d}z}{z^2}\\
&\leqslant\sum_{\substack{j=1\\ 1\leqslant\frac{\log x_i}{\log y_{j-1}}\leqslant\ell^{100K}}}^{J}\int\limits_{x_i/y_j}^{x_i/y_{j-1}} X\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant\frac{x_i}{z}}\frac{1}{p}\sup_{\frac{x_i}{z(1+1/X)}<q\leqslant\frac{x_i}{z}}|\Psi_f'(z,q)|^2\,\frac{\mathrm{d}z}{z^2}\\
&\ll\sum_{\substack{j=1\\ 1\leqslant\frac{\log x_i}{\log y_{j-1}}\leqslant\ell^{100K}}}^{J}\frac{1}{\log y_j}\int\limits_{x_i/y_j}^{x_i/y_{j-1}}\sup_{\frac{x_i}{z(1+1/X)}<q\leqslant\frac{x_i}{z}}|\Psi_f'(z,q)|^2\,\frac{\mathrm{d}z}{z^2}.
\end{aligned}$$

Further,

$$\begin{aligned}
|\Psi_f'(z,q)|^2&=\Bigg|\Psi_f(z,x_i/z)-\sum_{\substack{n\leqslant z\\ \frac{x_i}{z(1+1/X)}<P(n)\leqslant x_i/z}}f(n)+\sum_{\substack{n\leqslant z\\ \frac{x_i}{z(1+1/X)}<P(n)<q}}f(n)\Bigg|^2\\
&\leqslant 4|\Psi_f(z,x_i/z)|^2+4\Bigg|\sum_{\substack{n\leqslant z\\ \frac{x_i}{z(1+1/X)}<P(n)\leqslant x_i/z}}f(n)\Bigg|^2+4\Bigg|\sum_{\substack{n\leqslant z\\ \frac{x_i}{z(1+1/X)}<P(n)<q}}f(n)\Bigg|^2.
\end{aligned}$$

We define

$$M_\ell(x_i,y_j;f) := \frac{1}{\log y_j}\int_{x_i/y_j}^{x_i/y_{j-1}} |\Psi_f(z,x_i/z)|^2\,\frac{\mathrm{d}z}{z^2},$$

$$\lambda_\ell^{(2)}(x_i,y_j;f) := \frac{1}{\log y_j}\int_{x_i/y_j}^{x_i/y_{j-1}} \sup_{\frac{x_i}{z(1+1/X)}\leqslant q\leqslant \frac{x_i}{z}}\Big|\sum_{\substack{n\leqslant z\\ \frac{x_i}{z(1+1/X)}<P(n)<q}} f(n)\Big|^2\,\frac{\mathrm{d}z}{z^2},$$

$$\lambda_\ell^{(3)}(x_i,y_j;f) := \frac{1}{\log y_j}\int_{x_i/y_j}^{x_i/y_{j-1}} \Big|\sum_{\substack{n\leqslant z\\ \frac{x_i}{z(1+1/X)}<P(n)\leqslant \frac{x_i}{z}}} f(n)\Big|^2\,\frac{\mathrm{d}z}{z^2},$$

$$L_\ell^{(12)}(x_i;f) := \sum_{\substack{j=1\\ \frac{\log x_i}{\log y_{j-1}}>\ell^{100K}}}^{J}\int_{x_i/y_j}^{x_i/y_{j-1}} X\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant\frac{x_i}{z}}\frac{1}{p}|\Psi_f'(z,p)|^2\,\frac{\mathrm{d}z}{z^2}.$$

Since the number of y_j such that $1\leqslant \frac{\log x_i}{\log y_{j-1}}\leqslant \ell^{100K}$ is less than $100K\ell\log\ell$, it follows that

$$\begin{aligned}\frac{L_\ell(x_i;f)}{x_i} &\ll \ell\log\ell\cdot \sup_{\substack{y_j\\ 1\leqslant\frac{\log x_i}{\log y_{j-1}}\leqslant\ell^{100K}}} M_\ell(x_i,y_j;f)\\ &\quad+\ell\log\ell\cdot\sum_{k=2}^{3}\sup_{\substack{y_j\\ 1\leqslant\frac{\log x_i}{\log y_{j-1}}\leqslant\ell^{100K}}}\lambda_\ell^{(k)}(x_i,y_j;f)\\ &\quad+L_\ell^{(12)}(x_i;f)+L_\ell^{(2)}(x_i;f)\\ &\leqslant \ell\log\ell\cdot\sup_{y_j}M_\ell(x_i,y_j;f)+\ell\log\ell\cdot\sum_{k=2}^{3}\sup_{y_j}\lambda_\ell^{(k)}(x_i,y_j;f)\\ &\quad+L_\ell^{(12)}(x_i;f)+L_\ell^{(2)}(x_i;f).\end{aligned}$$

Thus

$$\begin{aligned}\frac{V_\ell(x_i;f)}{x_i} &\ll \ell\log\ell\cdot\sup_{0\leqslant j\leqslant J}M_\ell(x_i,y_j;f)\\ &\quad+\ell\log\ell\cdot\sum_{k=2}^{3}\sup_{0\leqslant j\leqslant J}\lambda_\ell^{(k)}(x_i,y_j;f)\\ &\quad+L_\ell^{(12)}(x_i;f)+L_\ell^{(2)}(x_i;f)+\frac{W_\ell(x_i;f)}{x_i}.\end{aligned}\tag{24}$$

It turns out that $M_\ell(x_i,y_j;f)$ makes the largest contribution to the above right-hand sum. The other terms of the sum will be bounded straightforwardly. Let $T(\ell)=\ell^{10}$. We define, for each $k\in\{2,3\}$, the following probabilities:

$$\mathbb{P}_\ell^{\lambda,k} := \mathbb{P}\bigg[\sup_{X_{\ell-1}<x_i\leqslant X_\ell}\ \sup_{0\leqslant j\leqslant J} \lambda_\ell^{(k)}(x_i,y_j,f) > \frac{T(\ell)}{\ell^{K/2}\ell\log\ell}\bigg], \tag{25}$$

$$\mathbb{P}_\ell^{\lambda,1} := \mathbb{P}\bigg[\sup_{X_{\ell-1}<x_i\leqslant X_\ell}\ \sup_{0\leqslant j\leqslant J} M_\ell(x_i,y_j,f) > \frac{T(\ell)\ell^{K/2}}{\ell\log\ell}\bigg], \tag{26}$$

and we set

$$\begin{aligned}
\mathbb{P}_\ell^{(12)} &:= \mathbb{P}\bigg[\sup_{X_{\ell-1}<x_i\leqslant X_\ell} L_\ell^{(12)}(x_i;f) > \frac{T(\ell)}{\ell^{K/2}}\bigg],\\
\mathbb{P}_\ell^{(2)} &:= \mathbb{P}\bigg[\sup_{X_{\ell-1}<x_i\leqslant X_\ell} L_\ell^{(2)}(x_i;f) > \frac{T(\ell)}{\ell^{K/2}}\bigg],\\
\mathbb{P}_\ell^{W} &:= \mathbb{P}\bigg[\sup_{X_{\ell-1}<x_i\leqslant X_\ell} \frac{\ell^{K/2}W_\ell(x_i;f)}{x_i} > 1\bigg].
\end{aligned}$$

7.2. Bounding $\mathbb{P}_\ell^W$. The goal of this subsection is to prove the following.

PROPOSITION 7.1. *$\sum_{\ell\geqslant 1}\mathbb{P}_\ell^W$ converges.*

We need the following lemma.

LEMMA 7.2. *Let $r>1$ be an integer. Then*

$$\mathbb{P}_\ell^W \ll_r \sum_{X_{\ell-1}<x_i\leqslant X_\ell}\bigg(\frac{\ell^{K/2}}{\log x_i}\bigg)^r. \tag{27}$$

Proof. Using Markov's inequality for the power $r>1$, we have

$$\begin{aligned}
\mathbb{P}_\ell^W &\leqslant \frac{1}{T(\ell)^r}\sum_{X_{\ell-1}<x_i\leqslant X_\ell}\bigg(\frac{\ell^{K/2}}{x_i}\bigg)^r\mathbb{E}[W_\ell(x_i;f)^r]\\
&\leqslant \sum_{X_{\ell-1}<x_i\leqslant X_\ell}\bigg(\frac{\ell^{K/2}}{x_i}\bigg)^r\mathbb{E}[W_\ell(x_i;f)^r].
\end{aligned}$$

Following the steps of Harper's work [10] and more recently Mastrostefano [14, Section 6], we begin by applying Minkowski's inequality, and then bound the above expectation by

$$\mathbb{E}[W_\ell(x_i;f)^r] \leqslant \Bigg(\sum_{\substack{y_0<p\leqslant y_J\\ p\leqslant x_i}}\Bigg(\mathbb{E}\Bigg[\Bigg(\frac{X}{p}\int_p^{p(1+1/X)} |\Psi_f'(x_i/p,x_i/t,p)|^2\,\mathrm{d}t\Bigg)^r\Bigg]\Bigg)^{1/r}\Bigg)^r.$$

Then by applying Hölder's inequality to the normalized integral $\frac{X}{p}\int_p^{p(1+1/X)}\mathrm{d}t$ with parameters $1/r$ and $(r-1)/r$ we can bound the above by

$$\leqslant \Bigg(\sum_{\substack{y_0<p\leqslant y_J\\ p\leqslant x_i}}\Bigg(\frac{X}{p}\int\limits_{p}^{p(1+1/X)}\mathbb{E}[|\Psi_f'(x_i/p,x_i/t,p)|^{2r}]\,\mathrm{d}t\Bigg)^{1/r}\Bigg)^{r}. \tag{28}$$

Now let us focus on bounding the $2r$-th moment of the partial sum of f over short intervals. By arguing as in [10, proof of Proposition 2], we observe that when $\frac{x_i}{p}-\frac{x_i}{p(1+1/X)}<1$, the interval $]x_i/(p(1+1/X)),x_i/p]$ contains at most one integer. Hence

$$\frac{X}{p}\int\limits_{p}^{p(1+1/X)}\mathbb{E}[|\Psi_f'(x_i/p,x_i/t,p)|^{2r}]\,\mathrm{d}t\leqslant 1.$$

Otherwise we have $p\leqslant\frac{x_i}{1+X}$, and by applying Cauchy–Schwarz's inequality, we get

$$\begin{aligned}&\mathbb{E}[|\Psi_f'(x_i/p,x_i/t,p)|^{2r}]\\ &\qquad\leqslant\sqrt{\mathbb{E}[|\Psi_f'(x_i/p,x_i/t,p)|^{2}]\mathbb{E}[|\Psi_f'(x_i/p,x_i/t,p)|^{2(2r-1)}]}.\end{aligned}$$

Since $t\leqslant p(1+1/X)$, we have $\frac{x_i}{p}-\frac{x_i}{t}<\frac{x_i}{p(1+X)}$. Hence

$$\mathbb{E}[|\Psi_f'(x_i/p,x_i/t,p)|^{2}]\leqslant\sum_{\substack{x_i/t<n\leqslant x_i/p\\ P(n)<p}}1\ll\frac{x_i}{pX}.$$

For the second expectation, we apply Lemma 3.1:

$$\mathbb{E}[|\Psi_f'(x_i/p,x_i/t,p)|^{2(2r-1)}]\ll\Bigg(\sum_{\substack{x_i/t<n\leqslant x_i/p\\ P(n)<p}}\tau_{4r-3}(n)\Bigg)^{2r-1}.$$

By applying Lemma 3.2, we deduce that the above is

$$\ll\left(\frac{x_i}{p}(\log x_i)^{4r-4}\right)^{2r-1}.$$

Finally, for $p\leqslant\frac{x_i}{1+X}$ we get

$$\mathbb{E}[|\Psi_f'(x_i/p,x_i/t,p)|^{2r}]\ll_q\left(\frac{x_i}{p}\right)^{r}\frac{(\log x_i)^{(2r-2)(2r-1)}}{\sqrt{X}}. \tag{29}$$

By choosing $X=(\log x_i)^{8r^2-8r+4}$, we conclude that

$$\begin{aligned}\mathbb{E}[W_\ell(x_i;f)^r]&\ll_r\Bigg(\sum_{\frac{x_i}{1+X}<p\leqslant x_i}1+x_i\frac{(\log x_i)^{\frac{(2r-2)(2r-1)}{r}}}{X^{\frac{1}{2r}}}\sum_{p\leqslant\frac{x_i}{1+X}}\frac{1}{p}\Bigg)^{r}\\ &\ll_r\left(\frac{x_i}{\log x_i}\right)^{r}.\end{aligned} \tag{30}$$

Then

$$\mathbb{P}_\ell^W \ll_r \sum_{X_{\ell-1}<x_i\leqslant X_\ell} \left(\frac{\ell^{K/2}}{\log x_i}\right)^r,$$

which ends the proof. ∎

Proof of Proposition 7.1. By choosing $r > 1/c_0$ where c_0 is the constant of Lemma 4.1, we get the convergence of $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^W$. ∎

7.3. Bounding $\mathbb{P}_\ell^{\lambda,1}$. The goal of this subsection is to prove the convergence of $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{\lambda,1}$. For all $1\leqslant j\leqslant J$ we have

$$M_\ell(x_i,y_j;f) \leqslant U_j := \frac{1}{\log y_j}\int_0^{+\infty} \max_{y_{j-1}<p\leqslant y_j} |\Psi_f(z,p)|^2 \frac{\mathrm{d}z}{z^2},$$

and

$$M_\ell(x_i,y_0;f) \leqslant U_0 := \frac{1}{\log y_0}\int_0^{+\infty} |\Psi_f(z,y_0)|^2 \frac{\mathrm{d}z}{z^2}.$$

We set

$$I_j := \left(\frac{\log y_j}{\log y_0}\right)^{-1/\ell^K} \frac{1}{\log y_j}\int_{-\infty}^{+\infty} \left|\frac{F_j(1/2+it)}{1/2+it}\right|^2 \mathrm{d}t \tag{31}$$

where $F_j(s) = \prod_{p\leqslant y_j}\left(1 + a_f\frac{f(p)}{p^s}\right)^{a_f}$. The reason why we added the factor $\left(\frac{\log y_j}{\log y_0}\right)^{-1/\ell^K}$ is to make sure that $(I_j)_{0\leqslant j\leqslant J}$ is a supermartingale sequence with respect to the filtration $(\mathcal{F}_{y_j})_{0\leqslant j\leqslant J}$.

Recalling $T(\ell) = \ell^{10}$, we denote

$$T_1(\ell) = \frac{T(\ell)}{\ell\log\ell}. \tag{32}$$

Define

$$\mathcal{S} := \left\{I_j \leqslant \frac{T_1(\ell)^{1/2}}{\ell^{K/2}} \text{ for all } 0\leqslant j\leqslant J\right\} \quad\text{and}\quad \mathcal{S}_j := \left\{I_j \leqslant \frac{T_1(\ell)^{1/2}}{\ell^{K/2}}\right\}.$$

Now we have

$$\begin{aligned}\mathbb{P}_\ell^{\lambda,1} &\leqslant \mathbb{P}\Bigg[\bigcup_{0\leqslant j\leqslant J}\left\{U_j \geqslant \frac{T(\ell)\ell^{K/2}}{\ell\log\ell}\right\}\cap\{\mathcal{S}\}\Bigg] + \mathbb{P}[\overline{\mathcal{S}}]\\ &\leqslant \sum_{j=0}^{J}\mathbb{P}\Bigg[\left\{U_j \geqslant \frac{T(\ell)\ell^{K/2}}{\ell\log\ell}\right\}\cap\{\mathcal{S}_{j-1}\}\Bigg] + \mathbb{P}[\overline{\mathcal{S}}].\end{aligned}$$

Let us start by treating

$$\widetilde{\mathbb{P}}_j := \mathbb{P}\Bigg[\left\{U_j \geqslant \frac{T(\ell)\ell^{K/2}}{\ell\log\ell}\right\}\cap\{\mathcal{S}_{j-1}\}\Bigg].$$

By Markov's inequality, we have

$$\begin{aligned}\widetilde{\mathbb{P}}_j &\leqslant \mathbb{P}\Big[\Big\{U_j \geqslant \frac{T(\ell)\ell^{K/2}}{\ell\log\ell}\Big\}\,\Big|\,\{\mathcal{S}_{j-1}\}\Big]\\ &\leqslant \frac{\ell\log\ell}{T(\ell)\ell^{K/2}}\mathbb{E}[U_j \mid \mathcal{S}_{j-1}].\end{aligned}$$

Before going further, we need the following lemmas.

Lemma 7.3. *Let $z \geqslant 1$ and define*

$$X_q(z,q_0) := \sum_{\substack{n\leqslant z\\ q_0<P(n)\leqslant q}} f(n).$$

Then $(|X_q(z,q_0)|)_{q\in\mathcal{P}}$ is a submartingale under the filtration $(\mathcal{F}_q)$.

Proof. Indeed, let $q<p$ be two consecutive prime numbers. For the Rademacher case, we have

$$\begin{aligned}\mathbb{E}[|X_p(z,q_0)|\mid\mathcal{F}_p] &= \mathbb{E}\big[|X_q(z,q_0)+f(p)X_q(z/p,q_0)|\,\big|\,\mathcal{F}_p\big]\\ &= \tfrac12|X_q(z,q_0)+X_q(z/p,q_0)| + \tfrac12|X_q(z,q_0)-X_q(z/p,q_0)|\\ &\geqslant |X_q(z,q_0)|.\end{aligned}$$

For the Steinhaus case, let n be the smallest integer such that $z/p^n<1$. By applying Lemma 3.6, we have

$$\begin{aligned}\mathbb{E}\big[|X_p(z,q_0)|\,\big|\,\mathcal{F}_p\big] &= \mathbb{E}\Big[\Big|X_q(z,q_0)+\sum_{k=1}^{n} f(p)^k X_q(z/p^k,q_0)\Big|\,\Big|\,\mathcal{F}_p\Big]\\ &= \int_0^1\Big|X_q(z,q_0)+\sum_{k=1}^{n}\mathrm{e}^{i2\pi k\vartheta}X_q(z/p^k,q_0)\Big|\,\mathrm{d}\vartheta\\ &\geqslant |X_q(z,q_0)|.\end{aligned}$$

Then $\mathbb{E}\big[|X_p(z,q_0)|\,\big|\,\mathcal{F}_p\big] \geqslant |X_q(z,q_0)|$ in both cases. ∎

Lemma 7.4. *For ℓ large enough, the sequence $(I_j)_{j\geqslant 0}$ is a supermartingale with respect to the filtration $(\mathcal{F}_{y_j})_{j\geqslant 0}$.*

Proof. We have

$$\begin{aligned}\mathbb{E}[I_j\mid\mathcal{F}_{y_{j-1}}] &= \frac{1}{\log y_j}\Big(\frac{\log y_j}{\log y_0}\Big)^{-1/\ell^K}\int_{-\infty}^{+\infty}\mathbb{E}\Big[\Big|\frac{F_j(1/2+it)}{1/2+it}\Big|^2\,\Big|\,\mathcal{F}_{y_{j-1}}\Big]\,\mathrm{d}t\\ &= \frac{1}{\log y_j}\Big(\frac{\log y_j}{\log y_0}\Big)^{-1/\ell^K}\\ &\quad\times\int_{-\infty}^{+\infty}\mathbb{E}\Big[\prod_{y_{j-1}<p\leqslant y_j}\Big|1+a_f\frac{f(p)}{p^{1/2+it}}\Big|^{2a_f}\Big]\Big|\frac{F_{j-1}(1/2+it)}{1/2+it}\Big|^2\,\mathrm{d}t.\end{aligned}$$

Using Lemma 3.4 we have

$$\mathbb{E}\Big[\prod_{y_{j-1}<p\leqslant y_j}\Big|1+a_f\frac{f(p)}{p^{1/2+it}}\Big|^{2a_f}\Big]=\prod_{y_{j-1}<p\leqslant y_j}\Big(1+\frac{a_f}{p}\Big)^{a_f}.$$

For readability, we set

$$b(a_f,j):=\prod_{y_{j-1}<p\leqslant y_j}\Big(1+\frac{a_f}{p}\Big)^{a_f}\mathrm{e}^{-1/\ell}\Big(\frac{\log y_j}{\log y_{j-1}}\Big)^{-1/\ell^K}.$$

By collecting the previous computations together, we find

$$\mathbb{E}[I_j\,|\,\mathcal{F}_{y_{j-1}}]\leqslant b(a_f,j)I_{j-1}.$$

To end the proof it suffices to prove that $b(a_f,j)\leqslant 1$. Recall that $\frac{\log y_j}{\log y_{j-1}} = \mathrm{e}^{1/\ell}$. For ℓ large enough, we have

$$\begin{aligned}
b(a_f,j)&=\mathrm{e}^{-1/\ell}(\mathrm{e}^{1/\ell})^{-1/\ell^K}\prod_{y_{j-1}<p\leqslant y_j}\Big(1+a_f\frac{1}{p}\Big)^{a_f}\\
&=\exp\Big(-\frac{1}{\ell}+\frac{-1}{\ell^{K+1}}+a_f\sum_{y_{j-1}<p\leqslant y_j}\log\Big(1+a_f\frac{1}{p}\Big)\Big)\\
&=\exp\Big(\frac{-1}{\ell^{K+1}}+O\Big(\sum_{y_{j-1}\leqslant p<y_j}\frac{1}{p^2}\Big)+O\big(\mathrm{e}^{-C\sqrt{\log y_{j-1}}}\big)\Big)
\end{aligned}$$

for some constant C. We have $\sum_{y_{j-1}\leqslant p<y_j}\frac{1}{p^2}\ll\frac{1}{y_{j-1}}\ll\frac{1}{y_0}$. We get

$$b(a_f,j)=\exp\Big(\frac{-1}{\ell^{K+1}}+O\big(\mathrm{e}^{-C\sqrt{\log y_0}}\big)\Big)=\exp\Big(\frac{-1}{\ell^{K+1}}+O\big(\mathrm{e}^{-2^{\ell^K/4}}\big)\Big).$$

Thus, for large ℓ, we have $b(a_f,j)\leqslant 1$, and so $\mathbb{E}[I_j\,|\,\mathcal{F}_{y_{j-1}}]\leqslant I_{j-1}$. ∎

Using Lemma 7.3 for $q_0=1$, followed by Lemma 3.9 for $r=2$, we get

$$\begin{aligned}
\mathbb{E}\Big[\max_{y_{j-1}<p\leqslant y_j}|\Psi_f(z,p)|^2\,\Big|\,\mathcal{S}_{j-1}\Big]&\leqslant 4\max_{y_{j-1}<p\leqslant y_j}\mathbb{E}\big[|\Psi_f(z,p)|^2\,\big|\,\mathcal{S}_{j-1}\big]\\
&=4\mathbb{E}\big[|\Psi_f(z,y_j)|^2\,\big|\mathcal{S}_{j-1}\big].
\end{aligned}$$

Recall that $\big(\frac{\log y_j}{\log y_0}\big)^{-1/\ell^K}\asymp 1$. By using Lemma 3.3, we have

$$\begin{aligned}
\mathbb{E}[U_j\,|\,\mathcal{S}_{j-1}]&=\frac{1}{\log y_j}\int_0^{+\infty}\mathbb{E}\Big[\max_{y_{j-1}<p\leqslant y_j}|\Psi_f(z,p)|^2\,\Big|\,\mathcal{S}_{j-1}\Big]\frac{\mathrm{d}z}{z^2}\\
&\leqslant\frac{4}{\log y_j}\int_0^{+\infty}\mathbb{E}\big[|\Psi_f(z,y_j)|^2\,\big|\,\mathcal{S}_{j-1}\big]\frac{\mathrm{d}z}{z^2}
\end{aligned}$$

$$\ll \mathbb{E}\bigg[\Big(\frac{\log y_j}{\log y_0}\Big)^{-1/\ell^K} \frac{1}{\log y_j} \int_0^{+\infty} |\Psi_f(z,y_j)|^2 \frac{\mathrm{d}z}{z^2} \,\bigg|\, \mathcal{S}_{j-1}\bigg]$$
$$= \mathbb{E}[I_j \,|\, \mathcal{S}_{j-1}].$$

Now by using Lemma 7.4, we have $\mathbb{E}[I_j \,|\, \mathcal{F}_{j-1}] \leqslant I_{j-1}$. Thus

$$\mathbb{E}[U_j \,|\, \mathcal{S}_{j-1}] \ll \mathbb{E}[I_j \,|\, \mathcal{S}_{j-1}] = \mathbb{E}\big[\mathbb{E}[I_j \,|\, \mathcal{F}_{j-1}] \,\big|\, \mathcal{S}_{j-1}\big] \leqslant \mathbb{E}[I_{j-1} \,|\, \mathcal{S}_{j-1}]$$
$$\leqslant \frac{T_1(\ell)^{1/2}}{\ell^{K/2}}.$$

Thus, we finally get

$$\sum_{j=1}^{J} \widetilde{\mathbb{P}}_j \ll \sum_{j=1}^{J} \frac{T_1(\ell)^{1/2}}{\ell^{K/2}} \frac{\ell \log \ell}{T(\ell)\ell^{K/2}} \ll \frac{T_1(\ell)^{1/2} \ell \log \ell}{T(\ell)} = \frac{\sqrt{\ell \log \ell}}{T(\ell)^{1/2}}. \tag{33}$$

Since $T(\ell) = \ell^{10}$, we see that $\sum_{\ell \geqslant 1} \frac{\sqrt{\ell \log \ell}}{T(\ell)^{1/2}}$ converges.

Let us now consider $\mathbb{P}[\mathcal{S}]$. We define $\mathcal{A} := \{I_0 \leqslant T_1(\ell)^{1/4}/\ell^{K/2}\}$.

Lemma 7.5. *For ℓ large enough, $\mathbb{P}[\,\overline{\mathcal{S}}\,] \ll \frac{1}{T_1(\ell)^{1/6}}$.*

Proof. Indeed, we have

$$\mathbb{P}[\,\overline{\mathcal{S}}\,] \leqslant \mathbb{P}\bigg[\max_{0 \leqslant j \leqslant J} I_j > \frac{T_1(\ell)^{1/2}}{\ell^{K/2}} \,\bigg|\, \mathcal{A}\bigg] + \mathbb{P}[\overline{\mathcal{A}}].$$

Recall that, by Lemma 7.4, the sequence (I_j) is a supermartingale. Thus by Lemma 3.8, we get

$$\mathbb{P}\bigg[\max_{0 \leqslant j \leqslant J} I_j > \frac{T_1(\ell)^{1/2}}{\ell^{K/2}} \,\bigg|\, \mathcal{A}\bigg] \leqslant \frac{\ell^{K/2}}{T_1(\ell)^{1/2}} \mathbb{E}[I_0 \,|\, \mathcal{A}] \leqslant \frac{1}{T_1(\ell)^{1/4}}.$$

On the other hand, as in [11], in the paragraph entitled "Proof of the upper bound in Theorem 1, assuming Key Propositions 1 and 2", for the Steinhaus case we have, by taking $y_0 = x^{1/\mathrm{e}}$, $q = 2/3$ and $k = 0$,

$$\mathbb{E}[I_0^{2/3}] \ll \mathbb{E}\bigg[\bigg(\frac{1}{\log y_0} \int_{-\infty}^{+\infty} \bigg|\frac{F_0(1/2+it)}{1/2+it}\bigg|^2 \mathrm{d}t\bigg)^{2/3}\bigg] \ll \frac{1}{\ell^{K/3}}. \tag{34}$$

In the Rademacher case, (34) follows directly from [12, Key Propositions 3 and 4], using the same proof as in the Steinhaus case and by taking the same values $y_0 = x^{1/\mathrm{e}}$, $q = 2/3$ and $k = 0$. The conclusion follows easily from Markov's inequality:

$$\mathbb{P}\bigg[I_0 > \frac{T_1(\ell)^{1/4}}{\ell^{K/2}}\bigg] \leqslant \frac{\ell^{K/3}\mathbb{E}[I_0^{2/3}]}{T_1(\ell)^{1/6}} \ll \frac{1}{T_1(\ell)^{1/6}}. \blacksquare$$

Proposition 7.6. *$\sum_{\ell \geqslant 1} \mathbb{P}_\ell^{\lambda,1}$ converges.*

Proof. By combining Lemma 7.5 and inequality (33), we get

$$\mathbb{P}_\ell^{\lambda,1} \ll \frac{1}{T_1(\ell)^{1/6}}.$$

Since $T_1(\ell) = \frac{T(\ell)}{\ell \log \ell} \gg \ell^8$, $\sum_{\ell \geqslant 1} \mathbb{P}_\ell^{\lambda,1}$ converges. ∎

7.4. Bounding $\mathbb{P}_\ell^{\lambda,2}$ and $\mathbb{P}_\ell^{\lambda,3}$

LEMMA 7.7. *For $k = 2, 3$, the series $\sum_{\ell \geqslant 1} \mathbb{P}_\ell^{\lambda,k}$ converges.*

Proof. We have

$$\mathbb{P}_\ell^{\lambda,k} \leqslant \frac{\ell^K}{T_1(\ell)^2} \sum_{X_{\ell-1} < x_i \leqslant X_\ell} \sum_{0 \leqslant j \leqslant J} \mathbb{E}[(\lambda_\ell^{(k)}(x_i, y_j; f))^2].$$

For fixed z, we consider

$$X_q(z) := \sum_{\substack{n \leqslant z \\ \frac{x_i}{z(1+1/X)} < P(n) \leqslant q}} f(n).$$

By Lemma 7.3 for $q_0 = \frac{x_i}{z(1+1/X)}$, $(|X_q(z)|)_{q \in \mathcal{P}}$ is a submartingale under the filtration $(\mathcal{F}_p)$. Thus, by Cauchy–Schwarz's inequality and Doob's L^4-inequality (Lemma 3.9), we get

$$\begin{aligned}
\mathbb{E}[(\lambda_\ell^{(2)}(x_i, y_j; f))^2] &\leqslant \frac{1}{(\log y_j)^2} \Bigg(\int_{x_i/y_j}^{x_i/y_{j-1}} \frac{\mathrm{d}z}{z} \Bigg) \\
&\quad \times \int_{x_i/y_j}^{x_i/y_{j-1}} \mathbb{E}\Bigg[\Bigg(\sup_{\frac{x_i}{z(1+1/X)} \leqslant q \leqslant \frac{x_i}{z}} \Bigg| \sum_{\substack{n \leqslant z \\ \frac{x_i}{z(1+1/X)} < P(n) < q}} f(n) \Bigg|^4 \Bigg)\Bigg] \frac{\mathrm{d}z}{z^3} \\
&\ll \frac{1}{\log y_j} \int_{x_i/y_j}^{x_i/y_{j-1}} \mathbb{E}\Bigg[\Bigg| \sum_{\substack{n \leqslant z \\ \frac{x_i}{z(1+1/X)} < P(n) \leqslant \frac{x_i}{z}}} f(n) \Bigg|^4\Bigg] \frac{\mathrm{d}z}{z^3}.
\end{aligned}$$

By Cauchy–Schwarz's inequality, in the case of $\lambda_\ell^{(3)}(x_i, y_j; f)$ we get

$$\mathbb{E}[(\lambda_\ell^{(3)}(x_i, y_j; f))^2] \ll \frac{1}{\log y_j} \int_{x_i/y_j}^{x_i/y_{j-1}} \mathbb{E}\Bigg[\Bigg| \sum_{\substack{n \leqslant z \\ \frac{x_i}{z(1+1/X)} < P(n) \leqslant \frac{x_i}{z}}} f(n) \Bigg|^4\Bigg] \frac{\mathrm{d}z}{z^3}.$$

To give an upper bound of the fourth moment, we use Lemmas 3.1 and 3.2 to get

$$\mathbb{E}\Big[\Big|\sum_{\substack{n\leqslant z\\ \frac{x_i}{z(1+1/X)}<P(n)\leqslant\frac{x_i}{z}}} f(n)\Big|^4\Big] \leqslant \Big(\sum_{\substack{n\leqslant z\\ \frac{x_i}{z(1+1/X)}<P(n)\leqslant\frac{x_i}{z}}} \tau_3(n)\Big)^2$$
$$\leqslant \Big(3\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant\frac{x_i}{z}}\ \sum_{k\leqslant z/p}\tau_3(k)\Big)^2$$
$$\ll z^2(\log z)^4\Big(\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant\frac{x_i}{z}}\frac{1}{p}\Big)^2.$$

Since $x_i/y_j < z \leqslant x_i/y_{j-1}$, from (23) we have

$$\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant\frac{x_i}{z}}\frac{1}{p} \ll \frac{1}{X\log y_j}.$$

We thus get, in both cases ($k=2$ or 3),

$$\mathbb{E}[(\lambda_\ell^{(k)}(x_i,y_j;f))^2] \ll \frac{(\log x_i)^4}{X^2(\log y_j)^3}\int_{x_i/y_j}^{x_i/y_{j-1}}\frac{\mathrm{d}z}{z} \ll \frac{(\log x_i)^4}{X^2(\log y_0)^2}.$$

Consequently,

$$\mathbb{P}_\ell^{\lambda,k} \ll \frac{\ell^{2K}}{T_1(\ell)^2(\log y_0)^2}\sum_{X_{\ell-1}<x_i\leqslant X_\ell}\frac{(\log x_i)^4}{X^2}.$$

Now recall from the proof of Lemma 7.2 that $X=(\log x_i)^{8r^2-8r+4}$ where $r>1/c_0>1$ (c_0 is defined in Lemma 4.1). Since $2(8r^2-8r+4)-4>r$ for $r>1$, we have $c_0(2(8r^2-8r+4)-4)>1$. Thus

$$\mathbb{P}_\ell^{\lambda,k} \ll \frac{\ell^{2K}}{T_1(\ell)^2(\log y_0)^2}\sum_{i\leqslant 2^{\ell^K/c_0}}\frac{1}{i^{c_0(2(8r^2-8r+4)-4)}}$$
$$\ll \frac{\ell^{2K}}{T_1(\ell)^2(\log y_0)^2}.$$

Since $\log y_0 = \frac{2^{\ell^K}}{2^K\ell^{K-1}}$ and $T_1(\ell)\geqslant \ell^4$, $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{\lambda,k}$ converges. ∎

7.5. Bounding $\mathbb{P}_\ell^{(12)}$

LEMMA 7.8. *For ℓ large enough and $x_i\in\,]X_{\ell-1},X_\ell]$, we have*

$$\mathbb{E}[L_\ell^{(12)}(x_i;f)] \ll \ell^K \mathrm{e}^{\ell^{-99K}/2}. \tag{35}$$

Proof. Note first that for $p\leqslant x_i/z\leqslant y_j$, from [17, Theorem] we have

$$\mathbb{E}[|\Psi_f'(z,p)|^2] \ll z\mathrm{e}^{\frac{-\log z}{2\log p}} \leqslant z\mathrm{e}^{\frac{-\log z}{2\log y_j}}.$$

By using again the bound

$$\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant\frac{x_i}{z}}\frac{1}{p}\ll\frac{1}{X\log y_{j-1}}$$

we get

$$\begin{aligned}\mathbb{E}[L_\ell^{(12)}(x_i;f)] &\ll \sum_{\substack{j=1\\ \frac{\log x_i}{\log y_{j-1}}>\ell^{100K}}}^{J}\int_{x_i/y_j}^{x_i/y_{j-1}} X\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant\frac{x_i}{z}}\frac{1}{p}z\mathrm{e}^{\frac{-\log z}{2\log y_j}}\frac{\mathrm{d}z}{z^2}\\ &\ll \sum_{\substack{j=1\\ \frac{\log x_i}{\log y_{j-1}}>\ell^{100K}}}^{J}\frac{1}{\log y_j}\int_{x_i/y_j}^{x_i/y_{j-1}}\frac{1}{z^{1+\frac{1}{2\log y_j}}}\,\mathrm{d}z\ll\ell^K\mathrm{e}^{-\ell^{99K}/2}. \quad\blacksquare\end{aligned}$$

Lemma 7.9. *For $T\geqslant 1$, $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{(12)}$ converges.*

Proof. For ℓ large, by applying Lemma 7.8 we have

$$\begin{aligned}\mathbb{P}_\ell^{(12)} &\leqslant \sum_{X_{\ell-1}<x_i\leqslant X_\ell}\frac{\ell^{K/2}}{T(\ell)}\mathbb{E}[L_\ell^{(12)}(x_i;f)]\\ &\ll \sum_{X_{\ell-1}<x_i\leqslant X_\ell}\ell^{K/2}\ell^K\mathrm{e}^{-\ell^{99K}/2}\\ &\ll 2^{\ell^K/c_0}\ell^{3K/2}\mathrm{e}^{-\ell^{99K}/2}\ll\mathrm{e}^{-\ell^{98K}}.\end{aligned}$$

Thus $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{(12)}$ converges. ∎

7.6. Bounding $\mathbb{P}_\ell^{(2)}$

Lemma 7.10. *For $T\geqslant 1$, $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{(2)}$ converges.*

Proof. We get

$$\begin{aligned}\mathbb{E}[L_\ell^{(2)}(x_i;f)] &\ll \sum_{j=1}^{J}\int_{\frac{x_i}{y_j(1+1/X)}}^{x_i/y_j} X\sum_{\max\{\frac{x_i}{z(1+1/X)},y_{j-1}\}<p\leqslant y_j}\frac{1}{p}\mathrm{e}^{-\frac{\log z}{2\log p}}\frac{\mathrm{d}z}{z}\\ &\leqslant \sum_{j=1}^{J}\int_{\frac{x_i}{y_j(1+1/X)}}^{x_i/y_j} X\sum_{\max\{\frac{x_i}{z(1+1/X)},y_{j-1}\}<p\leqslant y_j}\frac{1}{p}\mathrm{e}^{-\frac{\log z}{2\log y_j}}\frac{\mathrm{d}z}{z}.\end{aligned}$$

Noting that $y_j\leqslant x_i/z$, by applying again (23) we have

$$\sum_{\max\{\frac{x_i}{z(1+1/X)},y_{j-1}\}<p\leqslant y_j}\frac{1}{p}\leqslant\sum_{\frac{x_i}{z(1+1/X)}<p\leqslant\frac{x_i}{z}}\frac{1}{p}\ll\frac{1}{X\log y_j}.$$

Returning to the expectation, we get

$$\mathbb{E}[L_\ell^{(2)}(x_i;f)] \ll \sum_{j=1}^{J} \frac{1}{\log y_j} \int\limits_{\frac{x_i}{y_j(1+1/X)}}^{x_i/y_j} \frac{1}{z^{1+\frac{1}{2\log y_j}}}\,\mathrm{d}z$$

$$\ll \sum_{j=1}^{J} \frac{1}{\mathrm{e}^{\frac{\log x_i}{2\log y_j}}}\big(\mathrm{e}^{\frac{\log(1+1/X)}{2\log y_j}}-1\big) \ll \sum_{j=1}^{J} \frac{1}{X\log y_j}.$$

Recall that X is chosen to be $(\log x_i)^{8r^2-8r+4}$ at the end of the proof of Lemma 7.2. We denote $r' := 8r^2-8r+4$. Note that $r'>1$. We then deduce

$$\mathbb{P}_\ell^{(2)} \ll \frac{\ell^{K/2}}{T(\ell)} \sum_{X_{\ell-1}<x_i\leqslant X_\ell} \sum_{j=1}^{J} \frac{1}{X\log y_j} \ll \frac{\ell^{K/2}}{T(\ell)} 2^{\ell^K/c_0} \frac{2^{K\ell^{K-1}}}{2^{r'(\ell-1)^K+\ell^K}}.$$

Since $r'>r>1/c_0$, we get the convergence of

$$\sum_{\ell\geqslant 1} 2^{\frac{K}{2}\frac{\log\ell}{\log 2}+\frac{1}{c_0}\ell^K+K\ell^{K-1}-r'(\ell-1)^K-\ell^K}.$$

Thus, $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{(2)}$ converges. ∎

7.7. Convergence of $\sum_{\ell\geqslant 1}\mathbb{P}[\mathcal{B}_\ell^{(1)}]$. From (24), there exists a constant C such that

$$C\frac{V_\ell(x_i;f)}{\ell^{K/2}x_i} \leqslant \frac{\ell\log\ell}{\ell^{K/2}} \sup_{1\leqslant y_j\leqslant J} M_\ell(x_i,y_j;f) + \log\ell\cdot\sum_{k=2}^{3}\sup_{1\leqslant y_j\leqslant J}\lambda_\ell^{(k)}(x_i,y_j;f)$$

$$+\frac{W_\ell(x_i;f)}{x_i} + L_\ell^{(12)}(x_i;f) + L_\ell^{(2)}(x_i;f).$$

We set

$$\mathbb{P}_\ell^V := P\bigg[\sup_{X_{\ell-1}<x_i\leqslant X_\ell} \frac{\ell^{K/2}V_\ell(x_i;f)}{x_i} > \frac{6T(\ell)}{C}\bigg].$$

Lemma 7.11. *The series $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^V$ converges.*

Proof. It suffices to observe that

$$\mathbb{P}_\ell^V \leqslant \mathbb{P}_\ell^{\lambda,1}+\mathbb{P}_\ell^{\lambda,2}+\mathbb{P}_\ell^{\lambda,3}+\mathbb{P}_\ell^{(12)}+\mathbb{P}_\ell^{(2)}+\mathbb{P}_\ell^{W}.$$

Since $T_1(\ell)\geqslant \ell^4$,

- by Proposition 7.6, $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{\lambda,1}$ converges,
- by Lemma 7.7, $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{\lambda,2}$ and $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{\lambda,3}$ converge,
- by Lemma 7.9, $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{(12)}$ converges,
- by Lemma 7.10, $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{(2)}$ converges,
- by Proposition 7.1, $\sum_{\ell\geqslant 1}\mathbb{P}_\ell^{W}$ converges. ∎

Now we define the following events:

$$\Sigma := \{\forall x_i \in\,]X_{\ell-1}, X_\ell],\ V_\ell(x_i; f) \leqslant 6\ell^{K/2} T(\ell) x_i / C\},$$

and for each $x_i \in\,]X_{\ell-1}, X_\ell]$,

$$\Sigma_i := \{V_\ell(x_i; f) \leqslant 6\ell^{K/2} T(\ell) x_i / C\}.$$

Note that $\mathbb{P}[\,\overline{\Sigma}\,] = \mathbb{P}_\ell^V$.

PROPOSITION 7.12. *The series* $\sum_{\ell \geqslant 1} \mathbb{P}[\mathcal{B}_\ell^{(1)}]$ *converges.*

Proof. Recall that $T(\ell) = \ell^{10}$; this guarantees the convergence of $\sum_{\ell \geqslant 1} \mathbb{P}_\ell^V$ by Lemma 7.11. We have

$$\begin{aligned}
\mathbb{P}[\mathcal{B}_\ell^{(1)}] &= \mathbb{P}\Big[\sup_{X_{\ell-1} < x_i \leqslant X_\ell} \frac{|M_f^{(1)}(x_i)|}{\sqrt{x_i} R(x_i)} > 1\Big] \\
&\leqslant \mathbb{P}\Big[\bigcup_{X_{\ell-1} < x_i \leqslant X_\ell} \Big\{\frac{|M_f^{(1)}(x_i)|}{\sqrt{x_i} R(x_i)} \geqslant 1\Big\} \cap \Sigma_i\Big] + \mathbb{P}_\ell^V \\
&\leqslant \sum_{X_{\ell-1} < x_i \leqslant X_\ell} \mathbb{P}[\{|M_f^{(1)}(x_i)| \geqslant \sqrt{x_i} R(x_i)\} \cap \Sigma_i] + \mathbb{P}_\ell^V.
\end{aligned}$$

At this stage, we cannot apply Lemma 3.10. In fact, under the event Σ_i, it is hard to say if $M_f^{(1)}(x_i)$ is still the sum of a martingale difference sequence. However, Lemma 3.11 allows us to give a strong upper bound under the event Σ_i:

$$\begin{aligned}
\mathbb{P}[\{|M_f^{(1)}(x_i)| \geqslant \sqrt{x_i} R(x_i)\} \cap \Sigma_i] &\leqslant \exp\Big(\frac{-2C x_i R(x_i)^2}{T(\ell)\ell^{K/2} x_i}\Big) \\
&\leqslant 2\exp(-C\ell^{K+2K\varepsilon-6})
\end{aligned}$$

where C is a positive constant. Since by assumption $K\varepsilon = 25$, we have

$$\sum_{X_{\ell-1} < x_i \leqslant X_\ell} \exp(-C_1 \ell^{K+2K\varepsilon-6}) \leqslant \exp\Big(\frac{\log 2}{c_0}\ell^K - C_1\, \ell^K \ell^{44}\Big).$$

Finally, we get

$$\mathbb{P}[\mathcal{B}_\ell^{(1)}] \ll \exp\Big(\frac{\log 2}{c_0}\ell^K - C_1\, \ell^K \ell^{44}\Big) + \mathbb{P}_\ell^V.$$

Thus $\sum_{\ell \geqslant 1} \mathbb{P}[\mathcal{B}_\ell^{(1)}]$ converges. ∎

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