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Rational points in Cantor sets in the complex plane

by

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Abstract. Let K be an imaginary quadratic field and let $\mathcal{O}_K$ be the ring of algebraic integers of K. For $\alpha \in \mathcal{O}_K$ with $|\alpha| > 1$, define

$$\mathcal{D}_\alpha = \bigcup_{n=0}^{\infty} \frac{\mathcal{O}_K}{\alpha^n}.$$

For $\beta \in \mathcal{O}_K$ with $|\beta| > 1$ and a finite subset $A \subset \mathcal{O}_K$, define

$$S_{\beta,A} = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{\beta^k} : a_k \in A \ \forall k \in \mathbb{N} \right\}.$$

Suppose that α and β are relatively prime. In this paper, we show that if $\dim_{\mathrm{H}} S_{\beta,A} < 1$, then the intersection $\mathcal{D}_\alpha \cap S_{\beta,A}$ is a finite set. In general, the threshold for the Hausdorff dimension of $S_{\beta,A}$ is sharp. If we further assume that $\mathcal{O}_K$ is a unique factorization domain and that $\overline{\alpha}$ and α are relatively prime, then we establish the finiteness of the intersection under the weaker condition $\dim_{\mathrm{H}} S_{\beta,A} < 2$. This extends the previously known results on the real line.

1. Introduction

1.1. Rational points in Cantor sets. Let $\mathbb{N}$ be the set of all positive integers, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}$, let $\mathbb{N}_{\geq k} := \mathbb{N} \cap [k, +\infty)$. Let $\#A$ denote the cardinality of a set A. The greatest common divisor of $m, n \in \mathbb{N}$ is denoted by $\gcd(m, n)$.

Given $q \in \mathbb{N}_{\geq 3}$ and a finite set $A \subset \mathbb{Q}$ with $\#A \geq 2$, we consider the *iterated function system* (IFS) $\mathcal{F}_{q,A} = \{\phi_a(x) = (x + a)/q : a \in A\}$. According to Hutchinson [4], there exists a unique non-empty compact subset $S_{q,A}$ of $\mathbb{R}$ such that

$$S_{q,A} = \bigcup_{a \in A} \phi_a(S_{q,A}),$$

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which is called a *self-similar set*. The set $S_{q,A}$ can be written algebraically as

$$S_{q,A} = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{q^k} : a_k \in A \ \forall k \in \mathbb{N} \right\}.$$

For $p \in \mathbb{N}_{\geq 2}$, let D_p be the set of all rational numbers in $\mathbb{R}$ having a finite p-ary expansion. That is,

$$D_p = \bigcup_{n=1}^{\infty} \frac{\mathbb{Z}}{p^n},$$

which is a proper subset of $\mathbb{Q}$ and is dense in $\mathbb{R}$.

The finiteness of the intersection of D_p and $S_{q,A}$ under the assumption that $\gcd(p,q) = 1$, has been extensively studied [13, 10, 1, 11, 12, 9, 5, 8]. In the solution to Problem H-339 in [15], Wall first showed that

$$D_2 \cap S_{3,\{0,2\}} = \left\{ 0, \frac{1}{4}, \frac{3}{4}, 1 \right\}.$$

And Wall [13] also proved that

$$D_{10} \cap S_{3,\{0,2\}} = \left\{ 0, \frac{1}{4}, \frac{3}{4}, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, \frac{1}{40}, \frac{3}{40}, \frac{9}{40}, \frac{13}{40}, \frac{27}{40}, \frac{31}{40}, \frac{37}{40}, \frac{39}{40}, 1 \right\}.$$

Note that $S_{3,\{0,2\}}$ is the classical middle third Cantor set. Later, Nagy [10] showed that the intersection $D_p \cap S_{3,\{0,2\}}$ is finite for each prime $p > 3$. Bloshchitsyn [1] generalized this result and proved that if $p > q^2$ is a prime and $A \subset \{0, 1, \ldots, q-1\}$ with $\#A < q$, then $D_p \cap S_{q,A}$ is a finite set. The general result was proved by Schleischitz [11, Corollary 4.4]: if $\gcd(p,q) = 1$, and $A \subset \{0, 1, \ldots, q-1\}$ with $\#A < q$, then the intersection $D_p \cap S_{q,A}$ is finite, a stronger version of which was obtained by Shparlinski [12] using a different method. Recently, Kong, Li and the second named author [8] established the same conclusion for $A \subset \mathbb{Q}$.

THEOREM 1.1 ([11, 12, 8]). *Let $p \in \mathbb{N}_{\geq 2}$, $q \in \mathbb{N}_{\geq 3}$ and let $A \subset \mathbb{Q}$ be a finite set. Suppose that $\gcd(p,q) = 1$. Then the intersection $D_p \cap S_{q,A}$ is finite if and only if $\dim_{\mathrm{H}} S_{q,A} < 1$.*

Here, $\dim_{\mathrm{H}} E$ denotes the Hausdorff dimension of a set E, and we refer the reader to [3] for the definition and properties of Hausdorff dimension. It is worth pointing out that Theorem 1.1 has been applied to study the spectral eigenvalue problems for self-similar spectral measures [8, 14].

The main purpose of this paper is to extend the sufficiency part of Theorem 1.1 to the complex plane.

1.2. Algebraic number theory. Here, we recall some basic notions in algebraic number theory, and we refer the reader to [7] for further details.

Let K be an algebraic number field, and let $\mathcal{O}_K$ denote the ring of algebraic integers of K. Note that $\mathcal{O}_K$ is a *Dedekind domain*, which means that every non-zero proper ideal of $\mathcal{O}_K$ can be written uniquely as a product of prime ideals of $\mathcal{O}_K$ except for the order of factors (cf. [7, Section 3.2]).

Let R be an integral domain and let I be a non-zero ideal of R. For $a, b \in R$, we say that a is *congruent to* b *module* I, denoted by $a \equiv b \pmod{I}$, if $a - b \in I$. Let $(R/I)^\times$ denote the unit group of the quotient ring R/I, which consists of all invertible elements for the multiplication in R/I. For two ideals I_1, I_2 of R, we say that I_1 *divides* I_2, denoted by $I_1 \mid I_2$, if there exists an ideal I_3 of R such that $I_2 = I_1 I_3$. For two ideals I_1, I_2 of $\mathcal{O}_K$, we have $I_1 \mid I_2$ if and only if $I_2 \subset I_1$ (cf. [7, Theorem 3.19(i)]). Two ideals I_1, I_2 of $\mathcal{O}_K$ are said to be *relatively prime* if $I_1 + I_2 = \mathcal{O}_K$. Two elements $\alpha, \beta \in \mathcal{O}_K$ are said to be relatively prime if $\alpha\mathcal{O}_K$ and $\beta\mathcal{O}_K$ are relatively prime, i.e., $(\alpha\mathcal{O}_K) + (\beta\mathcal{O}_K) = \mathcal{O}_K$.

Let I be a non-zero ideal of $\mathcal{O}_K$. The quotient ring $\mathcal{O}_K/I$ is always finite (cf. [7, Example 3.25]), and the number of elements of $\mathcal{O}_K/I$ is called the *norm* of I, denoted by $N(I)$. So $(\mathcal{O}_K/I)^\times$ is a finite group. For $\beta \in \mathcal{O}_K$, the notation $\beta \in (\mathcal{O}_K/I)^\times$ means that the equation $\beta x \equiv 1 \pmod{I}$ is solvable in $\mathcal{O}_K$. Then *the order of* β *modulo* I, denoted by $\mathrm{Ord}_I(\beta)$, is defined to be the smallest integer $n \in \mathbb{N}$ such that $\beta^n \equiv 1 \pmod{I}$.

Let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}_K$. Then $\mathcal{O}_K/\mathfrak{p}$ is a finite field. So $\mathfrak{p}$ contains a unique prime $p \in \mathbb{N}$ which is the characteristic of the finite field $\mathcal{O}_K/\mathfrak{p}$. It follows that $\mathfrak{p}$ contains $p\mathcal{O}_K$ and hence $\mathfrak{p} \mid (p\mathcal{O}_K)$. The largest integer $e \in \mathbb{N}$ such that $\mathfrak{p}^e \mid (p\mathcal{O}_K)$ is called the *index of ramification of* $\mathfrak{p}$. Note that $N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p}) = p^f$ for some $f \in \mathbb{N}$. The positive integer f is called the *residual degree of* $\mathfrak{p}$.

Let $K/\mathbb{Q}$ be an extension of degree n and let $p \in \mathbb{N}$ be a prime. We write $p\mathcal{O}_K = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\dots\mathfrak{p}_\ell^{e_\ell}$ as a product of powers of distinct prime ideals of $\mathcal{O}_K$, and let f_j denote the residual degree of $\mathfrak{p}_j$ for $1 \le j \le \ell$. Then we have $\sum_{j=1}^{\ell} e_j f_j = n$ (cf. [7, Proposition 4.4]). If K is a quadratic number field, then by [7, Theorem 4.11] the factorization of $p\mathcal{O}_K$ has one of the following forms:

(i) $p\mathcal{O}_K = \mathfrak{p}^2$, where $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$ and $\overline{\mathfrak{p}} = \mathfrak{p}$;
(ii) $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$ and $\overline{\mathfrak{p}} \neq \mathfrak{p}$;
(iii) $p\mathcal{O}_K$ is a prime ideal of $\mathcal{O}_K$.

Here, $\overline{\mathfrak{p}}$ is the conjugate prime ideal of $\mathfrak{p}$. Thus, in a quadratic number field, the index of ramification of a non-zero prime ideal is 1 or 2, and so is the residual degree.

1.3. Main results. Let $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field, where d is a negative square free integer. By [7, Theorem 2.9], we have

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2 \text{ or } 3 \pmod 4, \\ \mathbb{Z}\big[\frac{1+\sqrt{d}}{2}\big] & \text{if } d \equiv 1 \pmod 4. \end{cases} \tag{1.1}$$

It is easy to check that for each $z \in \mathcal{O}_K$, we have $|z|^2 \in \mathbb{Z}$.

For $\beta \in \mathcal{O}_K$ with $|\beta| > 1$ and a finite subset $A \subset \mathcal{O}_K$ with $\#A \geq 2$, let $S_{\beta,A}$ be the self-similar set in the complex plane generated by the IFS

$$\mathcal{F}_{\beta,A} = \Big\{\varphi_a(z) = \frac{z+a}{\beta} : a \in A\Big\}.$$

Similarly, the set $S_{\beta,A}$ can be written as

$$S_{\beta,A} = \Big\{\sum_{k=1}^{\infty} \frac{a_k}{\beta^k} : a_k \in A \ \forall k \in \mathbb{N}\Big\}. \tag{1.2}$$

For $\alpha \in \mathcal{O}_K$ with $|\alpha| > 1$, define

$$\mathcal{D}_\alpha = \bigcup_{n=1}^{\infty} \frac{\mathcal{O}_K}{\alpha^n}.$$

Let $\overline{z}$ denote the complex conjugate of $z \in \mathbb{C}$. Our main theorem is the following.

THEOREM 1.2. *Let K be an imaginary quadratic field. Let $\alpha, \beta \in \mathcal{O}_K$ with $|\alpha| > 1$ and $|\beta| > 1$. Let $A \subset \mathcal{O}_K$ be a finite set with $\#A \geq 2$. Suppose that α and β are relatively prime.*

(i) *If $\dim_{\mathrm{H}} S_{\beta,A} < 1$, then $\#(\mathcal{D}_\alpha \cap S_{\beta,A}) < +\infty$.*
(ii) *Suppose further that $\mathcal{O}_K$ is a unique factorization domain, and that $\overline{\alpha}$ and α are relatively prime. If $\dim_{\mathrm{H}} S_{\beta,A} < 2$, then $\#(\mathcal{D}_\alpha \cap S_{\beta,A}) < +\infty$.*

REMARK 1.3. (a) Note that K is the quotient field of $\mathcal{O}_K$ (cf. [7, Theorem 1.6(iii)]). For a finite subset $A \subset K$, we can find $0 \neq \theta \in \mathcal{O}_K$ such that $\theta A \subset \mathcal{O}_K$. Note that

$$\mathcal{D}_\alpha \cap S_{\beta,A} = \frac{1}{\theta}((\theta\mathcal{D}_\alpha) \cap S_{\beta,\theta A}) \subset \frac{1}{\theta}(\mathcal{D}_\alpha \cap S_{\beta,\theta A}).$$

Thus, Theorem 1.2 also holds for any finite subset $A \subset K$.

(b) For $\alpha, \beta \in \mathcal{O}_K$ with $|\alpha|, |\beta| > 1$, if $\gcd(|\alpha|^2, |\beta|^2) = 1$, then we have $(\alpha\overline{\alpha}\mathcal{O}_K) + (\beta\overline{\beta}\mathcal{O}_K) = \mathcal{O}_K$. It follows that $(\alpha\mathcal{O}_K) + (\beta\mathcal{O}_K) = \mathcal{O}_K$. Thus, the elements α and β are relatively prime in $\mathcal{O}_K$.

(c) For an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$, where d is a negative square free integer, $\mathcal{O}_K$ is a unique factorization domain only for $d = -1, -2, -3, -7, -11, -19, -43, -67, -163$ (cf. [7, Remark 3.6]).

(d) We write $\alpha\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_\ell^{e_\ell}$ as a product of powers of distinct prime ideals of $\mathcal{O}_K$. The condition that $\overline{\alpha}$ and α are relatively prime means that for any $1 \leq i, j \leq \ell$ we have $\mathfrak{p}_i \neq \overline{\mathfrak{p}_j}$ where $\overline{\mathfrak{p}_j}$ is the conjugate prime ideal of $\mathfrak{p}_j$.

Let $p_j \in \mathbb{N}$ be the unique prime contained in $\mathfrak{p}_j$. Since K is an imaginary quadratic field, we further see that $p_1, \dots, p_\ell$ are distinct.

(e) Let $K = \mathbb{Q}(\mathrm{i})$, where $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit. Then $\mathcal{O}_K = \mathbb{Z}[\mathrm{i}] = \{n + m\mathrm{i} : n, m \in \mathbb{Z}\}$ is the ring of Gaussian integers. Let $\alpha = p \in \mathbb{N}_{\geq 2}$, $\beta = q \in \mathbb{N}_{\geq 3}$, and $A \subset \mathbb{Q}$ a finite set. Note that $S_{q,A} \subset \mathbb{R}$. So we have $\mathcal{D}_p \cap S_{q,A} = D_p \cap S_{q,A}$. If p and q are coprime in $\mathbb{N}$, then p and q are also relatively prime in $\mathcal{O}_K$. If, moreover, $\dim_{\mathrm{H}} S_{q,A} < 1$, then by Theorem 1.2(i) and remark (a), we conclude that

$$\#(D_p \cap S_{q,A}) < +\infty.$$

Thus, the sufficiency part of Theorem 1.1 is a corollary of Theorem 1.2(i). By Theorem 1.1 this argument also shows that the threshold for the Hausdorff dimension of $S_{\beta,A}$ in Theorem 1.2(i) is sharp. Although $\mathbb{Z}[\mathrm{i}]$ is a unique factorization domain, we cannot apply Theorem 1.2(ii) for $\alpha = p \in \mathbb{N}_{\geq 2}$ because of $\overline{\alpha} = \alpha$.

(f) The proof of Theorem 1.2 follows the spirit of [11, 8]. Here, $\mathcal{O}_K$ is not a unique factorization domain in general. We need to deal with the prime ideal factorization.

(g) While preparing this manuscript, we became aware that Wu [16] independently proved Theorem 1.2(i) in the setting $K = \mathbb{Q}(\mathrm{i})$ and $\mathcal{O}_K = \mathbb{Z}[\mathrm{i}]$.

Finally, we give a concrete example in the Gaussian integer ring $\mathbb{Z}[\mathrm{i}]$. For $\theta = -n + \mathrm{i}$ with $n \in \mathbb{N}$, we have a canonical number system $(\theta, \{0, 1, \dots, n^2\})$ in $\mathbb{Z}[\mathrm{i}]$ [6]. This means that every Gaussian integer $\gamma \in \mathbb{Z}[\mathrm{i}]$ can be written uniquely as

$$\gamma = \sum_{k=0}^{\ell} \xi_k \theta^k \quad \text{with } \xi_0, \xi_1, \dots, \xi_\ell \in \{0, 1, \dots, n^2\}.$$

Example 1.4. Let $\alpha = -n + \mathrm{i}$ with $n \in \mathbb{N}$. Then

$$\mathcal{D}_\alpha = \Big\{ \sum_{k=-\ell}^{\ell} \xi_k \alpha^k : \ell \in \mathbb{N},\, \xi_k \in \{0, 1, \dots, n^2\}\ \forall\, -\ell \leq k \leq \ell \Big\}.$$

Suppose that n is an even integer. This implies that α and $\overline{\alpha}$ are relatively prime in $\mathbb{Z}[\mathrm{i}]$. Let $\beta = -m + \mathrm{i}$ with $m \in \mathbb{N}$, and let A be a proper subset of $\{0, 1, \dots, m^2\}$. Then

$$\dim_{\mathrm{H}} S_{\beta,A} = \frac{\log \#A}{\log |\beta|} < 2.$$

Suppose that $\gcd(n^2 + 1, m^2 + 1) = 1$, which implies that α and β are relatively prime in $\mathbb{Z}[\mathrm{i}]$. For instance, we may take $\alpha = -4 + \mathrm{i}$, $\beta = -2 + \mathrm{i}$, and $A = \{0, 1, 2, 3\}$. Then by Theorem 1.2(ii), the intersection $\mathcal{D}_\alpha \cap S_{\beta,A}$ is a finite set.

The rest of the paper is organized as follows. The proof of Theorem 1.2 relies on two key propositions, which will be proved in Sections 2 and 3, respectively. We finish the proof of Theorem 1.2 in Section 4.

2. Upper bound of period length for eventually periodic coding. In this section, let K be an imaginary quadratic field and let $\mathcal{O}_K$ be the ring of algebraic integers of K. By (1.1), for any non-zero element $z \in \mathcal{O}_K$ we have $|z| \geq 1$. The uniform discreteness of $\mathcal{O}_K$ is essential in the proof of Proposition 2.1 below. This is why we need to work in imaginary quadratic fields, rather than in more general algebraic number fields.

By (1.2), each element $z \in S_{\beta,A}$ has at least one expression of the form

$$z = \sum_{k=1}^{\infty} \frac{a_k}{\beta^k} \quad \text{with each } a_k \in A. \tag{2.1}$$

The infinite sequence $(a_k)_{k=1}^{\infty}$ in (2.1) is called a *coding* of z. The following proposition is inspired by [11, Theorem 4.3] and [8, Lemma 2.3].

PROPOSITION 2.1. *Let $\beta \in \mathcal{O}_K$ with $|\beta| > 1$ and let $A \subset \mathcal{O}_K$ be a finite subset with $\#A \geq 2$. Write $s = \dim_{\mathrm{H}} S_{\beta,A}$. Then for $\varepsilon > 0$ there exists a constant $c_1 = c_1(\varepsilon) > 0$ such that any complex number $\xi = v/u \in S_{\beta,A}$ with $v, u \in \mathcal{O}_K$ admits an eventually periodic coding with period length $\leq c_1|u|^{s+\varepsilon}$.*

Proof. Let $(a_k)_{k=1}^{\infty} \in A^{\mathbb{N}}$ be a coding of ξ, i.e.,

$$\xi = \sum_{k=1}^{\infty} \frac{a_k}{\beta^k}.$$

Consider the sequence $\{\xi_n\}_{n=0}^{\infty}$ defined by

$$\xi_0 := \xi, \quad \text{and} \quad \xi_n := \varphi_{a_n}^{-1}(\xi_{n-1}) = \beta\xi_{n-1} - a_n \quad \text{for } n \in \mathbb{N}. \tag{2.2}$$

Then

$$\xi_n = \sum_{k=1}^{\infty} \frac{a_{n+k}}{\beta^k} \in S_{\beta,A} \quad \forall n \in \mathbb{N}_0.$$

Note that $\beta \in \mathcal{O}_K$ and $A \subset \mathcal{O}_K$. By (2.2) we can recursively show that

$$\xi_n = \frac{v_n}{u} \quad \text{for some } v_n \in \mathcal{O}_K.$$

Let $\mathcal{C} := \{\xi_n : n \in \mathbb{N}_0\}$. Note that for any non-zero element $z \in \mathcal{O}_K$ we have $|z| \geq 1$. Thus for any distinct $z_1, z_2 \in \mathcal{C}$,

$$|z_1 - z_2| \geq \frac{1}{|u|}. \tag{2.3}$$

For $\delta > 0$ and a bounded set F, let $N_\delta(F)$ denote the smallest number of closed balls of radius δ that cover F. The *box-counting dimension* of F is

defined by

$$\dim_{\mathrm{B}} E = \lim_{\delta\to 0^+} \frac{\log N_\delta(E)}{-\log \delta}$$

if this limit exists (see [3] for more details). Since $S_{\beta,A}$ is a self-similar set, by [2, Corollary 3.3] we have

$$\dim_{\mathrm{B}} S_{\beta,A} = \dim_{\mathrm{H}} S_{\beta,A} = s.$$

For $\varepsilon > 0$, by the definition of box-counting dimension, there exists $\delta_0 > 0$ such that

$$N_\delta(S_{\beta,A}) \le \frac{1}{\delta^{s+\varepsilon}} \quad \forall 0 < \delta < \delta_0.$$

Thus, there is a constant $c = c(\varepsilon)$ such that

$$N_\delta(S_{\beta,A}) \le \frac{c}{\delta^{s+\varepsilon}} \quad \forall 0 < \delta < 1. \tag{2.4}$$

Note that $\mathcal{C} \subset S_{\beta,A}$. By (2.3) we have

$$N_{\frac{1}{3|u|}}(S_{\beta,A}) \ge N_{\frac{1}{3|u|}}(\mathcal{C}) \ge \#\mathcal{C}.$$

This together with (2.4) implies that $\#\mathcal{C} \le c(3|u|)^{s+\varepsilon}$. That is, there exists a constant $c_1 > 0$ depending on ε such that

$$\#\mathcal{C} \le c_1|u|^{s+\varepsilon}.$$

Thus, there are $0 \le k < n \le c_1|u|^{s+\varepsilon}$ such that $\xi_k = \xi_n$. By (2.2) this yields

$$\xi_k = \varphi_{a_{k+1}} \circ \varphi_{a_{k+2}} \circ \cdots \circ \varphi_{a_n}(\xi_n) = \varphi_{a_{k+1}} \circ \varphi_{a_{k+2}} \circ \cdots \circ \varphi_{a_n}(\xi_k).$$

This means that ξ_k has a periodic coding $(a_{k+1} \dots a_n)^\infty$. It follows that ξ has a eventually periodic coding $a_1 \dots a_k(a_{k+1} \dots a_n)^\infty$ with period length $n - k \le c_1|u|^{s+\varepsilon}$. ∎

3. Lower bound of the order. In this section, we work in general algebraic number fields. Let K be an algebraic number field and let $\mathcal{O}_K$ be the ring of algebraic integers of K.

Let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}_K$ and $\beta \in \mathcal{O}_K \setminus \mathfrak{p}$. For any $n \in \mathbb{N}$, suppose that $\beta\mathcal{O}_K + \mathfrak{p}^n \ne \mathcal{O}_K$. Then we can find a non-zero prime ideal $\mathfrak{p}'$ of $\mathcal{O}_K$ such that $\beta\mathcal{O}_K + \mathfrak{p}^n \subset \mathfrak{p}'$. Note that $\mathfrak{p}^n \subset \beta\mathcal{O}_K + \mathfrak{p}^n \subset \mathfrak{p}'$. This implies that $\mathfrak{p}' = \mathfrak{p}$. So we have $\beta\mathcal{O}_K + \mathfrak{p}^n \subset \mathfrak{p}$. It follows that $\beta \in \mathfrak{p}$, a contradiction. Thus, we conclude that $\beta\mathcal{O}_K + \mathfrak{p}^n = \mathcal{O}_K$ for any $n \in \mathbb{N}$. This means that the equation $\beta x \equiv 1 \pmod{\mathfrak{p}^n}$ is solvable in $\mathcal{O}_K$. That is, $\beta \in (\mathcal{O}_K/\mathfrak{p}^n)^\times$ and the order $\mathrm{Ord}_{\mathfrak{p}^n}(\beta)$ is well defined for any $n \in \mathbb{N}$.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be distinct non-zero prime ideals of $\mathcal{O}_K$ and $\beta \in \mathcal{O}_K \setminus \bigcup_{j=1}^k \mathfrak{p}_j$. For any $n_j \in \mathbb{N}$, we have $\beta \in (\mathcal{O}_K/\mathfrak{p}_j^{n_j})^\times$. Note that $\mathcal{O}_K$ is a Dedekind domain. By the Chinese remainder theorem (cf. [7, p. 60]), we have

$$\mathcal{O}_K/\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k} \cong \mathcal{O}_K/\mathfrak{p}_1^{n_1} \oplus \cdots \oplus \mathcal{O}_K/\mathfrak{p}_k^{n_k}.$$

Thus, we conclude that $\beta \in (\mathcal{O}_K/\mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_k^{n_k})^\times$ and the order $\mathrm{Ord}_{\mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_k^{n_k}}(\beta)$ is well defined for any tuple $(n_1,\ldots,n_k) \in \mathbb{N}_0^k \setminus \{\mathbf{0}\}$.

LEMMA 3.1. *Let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}_K$. Let $p \in \mathbb{N}$ be the unique prime contained in $\mathfrak{p}$, and let e be the index of ramification of $\mathfrak{p}$. For $\beta \in \mathcal{O}_K \setminus \mathfrak{p}$ with $|\beta| > 1$, there exists $n_0 \in \mathbb{N}$ such that*

$$\mathrm{Ord}_{\mathfrak{p}^{n_0+n}}(\beta) = p^{\lceil n/e \rceil} \cdot \mathrm{Ord}_{\mathfrak{p}^{n_0}}(\beta) \quad \forall n \in \mathbb{N}_0.$$

Proof. Write $m = \mathrm{Ord}_{\mathfrak{p}^{e+1}}(\beta)$. Then we have $(\beta^m - 1) \in \mathfrak{p}^{e+1}$, that is, $\mathfrak{p}^{e+1} \,|\, (\beta^m - 1)\mathcal{O}_K$. Since $|\beta| > 1$, we have $\beta^m - 1 \neq 0$. Let n_0 be the largest positive integer n such that $\mathfrak{p}^n \,|\, (\beta^m - 1)\mathcal{O}_K$. Then $n_0 \geq e+1$ and $\mathrm{Ord}_{\mathfrak{p}^{n_0}}(\beta) = m$.

We claim that

$$\beta^{mp^k} - 1 \in \mathfrak{p}^{n_0+ke} \setminus \mathfrak{p}^{n_0+ke+1} \quad \forall k \in \mathbb{N}_0. \tag{3.1}$$

We will prove (3.1) by induction on k. First, for $k = 0$ it holds because of the maximality of n_0. Next, we assume that

$$\beta^{mp^k} - 1 \in \mathfrak{p}^{n_0+ke} \setminus \mathfrak{p}^{n_0+ke+1}$$

for some $k \in \mathbb{N}_0$. So we can write

$$\beta^{mp^k} = 1 + \gamma \quad \text{with } \gamma \in \mathfrak{p}^{n_0+ke} \setminus \mathfrak{p}^{n_0+ke+1}.$$

Then

$$\beta^{mp^{k+1}} = (\beta^{mp^k})^p = (1+\gamma)^p = 1 + p\gamma + \sum_{\ell=2}^{p} \binom{p}{\ell} \gamma^\ell. \tag{3.2}$$

For $\ell \geq 2$, we have $\gamma^\ell \in \mathfrak{p}^{(n_0+ke)\ell} \subset \mathfrak{p}^{2n_0+ke} \subset \mathfrak{p}^{n_0+(k+1)e+1}$. It follows that

$$\sum_{\ell=2}^{p} \binom{p}{\ell} \gamma^\ell \in \mathfrak{p}^{n_0+(k+1)e+1}. \tag{3.3}$$

Note that $p \in \mathfrak{p}^e \setminus \mathfrak{p}^{e+1}$. Thus, $p\gamma \in \mathfrak{p}^{n_0+(k+1)e} \setminus \mathfrak{p}^{n_0+(k+1)e+1}$. Together with (3.3) and (3.2), we conclude that

$$\beta^{mp^{k+1}} - 1 \in \mathfrak{p}^{n_0+(k+1)e} \setminus \mathfrak{p}^{n_0+(k+1)e+1}.$$

That is, (3.1) follows by induction.

For any $k \in \mathbb{N}$, we first have $m = \mathrm{Ord}_{\mathfrak{p}^{n_0}}(\beta) \,|\, \mathrm{Ord}_{\mathfrak{p}^{n_0+ke}}(\beta)$. By (3.1),

$$\beta^{mp^k} \equiv 1 \pmod{\mathfrak{p}^{n_0+ke}} \quad \text{and} \quad \beta^{mp^{k-1}} \not\equiv 1 \pmod{\mathfrak{p}^{n_0+ke}}.$$

Note that p is a prime. Thus, we conclude that

$$\mathrm{Ord}_{\mathfrak{p}^{n_0+ke}}(\beta) = mp^k \quad \forall k \in \mathbb{N}. \tag{3.4}$$

For any $n \in \mathbb{N}$ with $e \nmid n$, write $n = ke + r$ where $k \in \mathbb{N}_0$ and $r \in \{1, \dots, e-1\}$. Then

$$\mathrm{Ord}_{\mathfrak{p}^{n_0+ke}}(\beta) \,|\, \mathrm{Ord}_{\mathfrak{p}^{n_0+n}}(\beta) \quad \text{and} \quad \mathrm{Ord}_{\mathfrak{p}^{n_0+n}}(\beta) \,|\, \mathrm{Ord}_{\mathfrak{p}^{n_0+(k+1)e}}(\beta).$$

Note that p is a prime. By (3.4) we obtain $\mathrm{Ord}_{\mathfrak{p}^{n_0+n}}(\beta) = mp^k$ or mp^{k+1}. By (3.1), we have $\beta^{mp^k} \not\equiv 1 \pmod{\mathfrak{p}^{n_0+n}}$. Thus,

$$\mathrm{Ord}_{\mathfrak{p}^{n_0+n}}(\beta) = mp^{k+1} = mp^{\lceil n/e \rceil}.$$

Together with (3.4), this completes the proof. ∎

PROPOSITION 3.2. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be distinct non-zero prime ideals of $\mathcal{O}_K$. For $1 \le j \le k$, let $p_j \in \mathbb{N}$ be the unique prime contained in $\mathfrak{p}_j$, and let e_j be the index of ramification of $\mathfrak{p}_j$. Let $\mathcal{P} = \{p_j : 1 \le j \le k\}$ and for $p \in \mathcal{P}$, let $\mathcal{I}_p = \{1 \le j \le k : p_j = p\}$. Then for $\beta \in \mathcal{O}_K \setminus \bigcup_{j=1}^k \mathfrak{p}_j$ with $|\beta| > 1$, there exists a constant $c_2 > 0$ such that*

$$\mathrm{Ord}_{\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}}(\beta) \ge c_2 \prod_{p \in \mathcal{P}} p^{\max\{\lceil n_j/e_j \rceil : j \in \mathcal{I}_p\}}$$

for any tuple $(n_1, \dots, n_k) \in \mathbb{N}_0^k \setminus \{\mathbf{0}\}$.

Proof. By Lemma 3.1, for any $1 \le j \le k$, there exists $m_j \in \mathbb{N}$ such that

$$p_j^{\lceil n/e_j \rceil} \,|\, p_j^{m_j} \cdot \mathrm{Ord}_{\mathfrak{p}_j^n}(\beta) \quad \forall n \in \mathbb{N}. \tag{3.5}$$

Fix $(n_1, \dots, n_k) \in \mathbb{N}_0^k \setminus \{\mathbf{0}\}$. Note that for $1 \le j \le k$,

$$\mathrm{Ord}_{\mathfrak{p}_j^{n_j}}(\beta) \,|\, \mathrm{Ord}_{\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}}(\beta).$$

That is,

$$p_j^{m_j} \cdot \mathrm{Ord}_{\mathfrak{p}_j^{n_j}}(\beta) \,|\, p_j^{m_j} \cdot \mathrm{Ord}_{\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}}(\beta).$$

Together with (3.5), we obtain

$$p_j^{\lceil n_j/e_j \rceil} \,|\, p_j^{m_j} \cdot \mathrm{Ord}_{\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}}(\beta) \quad \forall 1 \le j \le k.$$

Note that all primes in $\mathcal{P}$ are distinct. So we have

$$\prod_{p \in \mathcal{P}} p^{\max\{\lceil n_j/e_j \rceil : j \in \mathcal{I}_p\}} \,|\, p_1^{m_1} \cdots p_k^{m_k} \cdot \mathrm{Ord}_{\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}}(\beta).$$

It follows that

$$\mathrm{Ord}_{\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}}(\beta) \ge \frac{1}{p_1^{m_1} \cdots p_k^{m_k}} \prod_{p \in \mathcal{P}} p^{\max\{\lceil n_j/e_j \rceil : j \in \mathcal{I}_p\}},$$

as desired. ∎

4. Proof of Theorem 1.2. In this section, we will prove Theorem 1.2. Let K be an imaginary quadratic field and let $\mathcal{O}_K$ be the ring of algebraic integers of K.

For $\alpha \in \mathcal{O}_K$ with $|\alpha| > 1$, recall that

$$\mathcal{D}_\alpha = \bigcup_{n=1}^{\infty} \frac{\mathcal{O}_K}{\alpha^n}.$$

We write the ideal $\alpha\mathcal{O}_K$ as

$$\alpha\mathcal{O}_K = \mathfrak{p}_1^{b_1} \cdots \mathfrak{p}_\ell^{b_\ell},$$

a product of powers of distinct prime ideals. For $z \in \mathcal{D}_\alpha$, there exists $n \in \mathbb{N}$ such that $\alpha^n z \in \mathcal{O}_K$. It follows that $z\mathfrak{p}_1^{nb_1} \cdots \mathfrak{p}_\ell^{nb_\ell} \subset \mathcal{O}_K$. We define $\boldsymbol{n}_z$ to be a tuple $(n_1, \ldots, n_\ell) \in \mathbb{N}_0^\ell$ with smallest $n_1 + \cdots + n_\ell$ such that

$$z\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_\ell^{n_\ell} \subset \mathcal{O}_K.$$

For any $\boldsymbol{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}_0^\ell$, define

$$\mathcal{D}_\alpha^{\boldsymbol{n}} := \{z \in \mathcal{D}_\alpha : \boldsymbol{n}_z = (n_1, \ldots, n_\ell)\}.$$

Then the set $\mathcal{D}_\alpha$ can be written as the disjoint union of $\mathcal{D}_\alpha^{\boldsymbol{n}}$, i.e.,

$$\mathcal{D}_\alpha = \bigcup_{\boldsymbol{n} \in \mathbb{N}_0^\ell} \mathcal{D}_\alpha^{\boldsymbol{n}}. \tag{4.1}$$

For $1 \le j \le \ell$, let $p_j \in \mathbb{N}$ be the unique prime contained in $\mathfrak{p}_j$, let e_j be the index of ramification of $\mathfrak{p}_j$, and let f_j be the residual degree of $\mathfrak{p}_j$. Note that the degree of $K/\mathbb{Q}$ is 2. By [7, Proposition 4.4], we have $e_j \in \{1, 2\}$ and $f_j \in \{1, 2\}$ for all $1 \le j \le \ell$. Let $\mathcal{P} = \{p_j : 1 \le j \le \ell\}$ and for $p \in \mathcal{P}$, let $\mathcal{I}_p = \{1 \le j \le \ell : p_j = p\}$.

Proof of Theorem 1.2(i). For any tuple $\boldsymbol{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}_0^\ell$ and $z \in \mathcal{D}_\alpha^{\boldsymbol{n}}$, we have $z\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_\ell^{n_\ell} \subset \mathcal{O}_K$. Note that $p_j \in \mathfrak{p}_j$ for all $1 \le j \le \ell$. So, $zp_1^{n_1} \cdots p_\ell^{n_\ell} \in \mathcal{O}_K$. It follows that

$$\mathcal{D}_\alpha^{\boldsymbol{n}} \subset \frac{\mathcal{O}_K}{p_1^{n_1} \cdots p_\ell^{n_\ell}}.$$

This implies that $\mathcal{D}_\alpha^{\boldsymbol{n}}$ is uniformly discrete. Note that $S_{\beta,A}$ is compact. Thus, we conclude that

$$\#(\mathcal{D}_\alpha^{\boldsymbol{n}} \cap S_{\beta,A}) < +\infty \quad \forall \boldsymbol{n} \in \mathbb{N}_0^\ell.$$

In order to show $\#(\mathcal{D}_\alpha \cap S_{\beta,A}) < +\infty$, by (4.1) it suffices to find $n_0 \in \mathbb{N}$ such that for any $\boldsymbol{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}_0^\ell$ with $n_1 + \cdots + n_\ell \ge n_0$, we have

$$\mathcal{D}_\alpha^{\boldsymbol{n}} \cap S_{\beta,A} = \emptyset.$$

Assume that $s = \dim_{\mathrm{H}} S_{\beta,A} < 1$. Choose $\varepsilon > 0$ such that $s + \varepsilon < 1$. Let c_1 and c_2 be the constants appearing in Propositions 2.1 and 3.2, respectively.

Take a large integer $n_0 \in \mathbb{N}$ such that

$$(\sqrt{2})^{n_0(1-s-\varepsilon)/\ell} > \frac{c_1}{c_2}. \tag{4.2}$$

Fix a tuple $\boldsymbol{n} = (n_1, \dots, n_\ell) \in \mathbb{N}_0^\ell$ with $n_1 + \cdots + n_\ell \geq n_0$. Suppose that $\mathcal{D}_\alpha^{\boldsymbol{n}} \cap S_{\beta,A} \neq \emptyset$, and we take $z \in \mathcal{D}_\alpha^{\boldsymbol{n}} \cap S_{\beta,A}$.

For $p \in \mathcal{P}$, we have

$$p \in \prod_{j \in \mathcal{I}_p} \mathfrak{p}_j^{e_j},$$

and hence

$$p^{\max\{\lceil n_j/e_j \rceil : j \in \mathcal{I}_p\}} \in \prod_{j \in \mathcal{I}_p} \mathfrak{p}_j^{n_j}.$$

It follows that

$$\prod_{p \in \mathcal{P}} p^{\max\{\lceil n_j/e_j \rceil : j \in \mathcal{I}_p\}} \in \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}.$$

Note that $z\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_\ell^{n_\ell} \subset \mathcal{O}_K$. So we have

$$z \prod_{p \in \mathcal{P}} p^{\max\{\lceil n_j/e_j \rceil : j \in \mathcal{I}_p\}} \in \mathcal{O}_K.$$

This means that we can write

$$z = \frac{v}{\prod_{p \in \mathcal{P}} p^{\max\{\lceil n_j/e_j \rceil : j \in \mathcal{I}_p\}}} \quad \text{with } v \in \mathcal{O}_K.$$

Since $z \in S_{\beta,A}$, by Proposition 2.1, the complex number z has an eventually periodic coding $a_1 \dots a_k (a_{k+1} \dots a_{k+m})^\infty \in A^{\mathbb{N}}$ with

$$m \leq c_1 \Big(\prod_{p \in \mathcal{P}} p^{\max\{\lceil n_j/e_j \rceil : j \in \mathcal{I}_p\}} \Big)^{s+\varepsilon}. \tag{4.3}$$

Note that $A \subset \mathcal{O}_K$. We have

$$\begin{aligned} z &= \sum_{j=1}^{k} \frac{a_j}{\beta^j} + \left(1 + \frac{1}{\beta^m} + \frac{1}{\beta^{2m}} + \cdots\right) \sum_{j=1}^{m} \frac{a_{k+j}}{\beta^{k+j}} \\ &= \sum_{j=1}^{k} \frac{a_j}{\beta^j} + \frac{\beta^m}{\beta^m - 1} \sum_{j=1}^{m} \frac{a_{k+j}}{\beta^{k+j}} \\ &\in \frac{\mathcal{O}_K}{\beta^k(\beta^m - 1)}. \end{aligned}$$

That is, $z\beta^k(\beta^m - 1) \in \mathcal{O}_K$.

We claim that

$$\beta^m - 1 \in \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_\ell^{n_\ell}. \tag{4.4}$$

Write

$$\beta^k(\beta^m-1)\mathcal{O}_K=\prod_{j=1}^{\ell}\mathfrak{p}_j^{d_j}\prod_{j=\ell+1}^{r}\mathfrak{p}_j^{d_j}$$

as a product of powers of distinct prime ideals, where $d_1,\dots,d_\ell\in\mathbb{N}_0$ and $d_{\ell+1},\dots,d_r\in\mathbb{N}$. Then

$$(\mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_\ell^{n_\ell})+(\beta^k(\beta^m-1)\mathcal{O}_K)=\prod_{j=1}^{\ell}\mathfrak{p}_j^{\min\{n_j,d_j\}}.$$

It follows that

$$(z\mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_\ell^{n_\ell})+(z\beta^k(\beta^m-1)\mathcal{O}_K)=z\prod_{j=1}^{\ell}\mathfrak{p}_j^{\min\{n_j,d_j\}}.$$

Note that $z\mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_\ell^{n_\ell}\subset\mathcal{O}_K$ and $z\beta^k(\beta^m-1)\in\mathcal{O}_K$. We obtain

$$z\prod_{j=1}^{\ell}\mathfrak{p}_j^{\min\{n_j,d_j\}}\subset\mathcal{O}_K.$$

By the minimality of $\boldsymbol{n}_z$, we have $d_j\ge n_j$ for all $1\le j\le\ell$. It follows that

$$\mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_\ell^{n_\ell}\mid\beta^k(\beta^m-1)\mathcal{O}_K.$$

Since α and β are relatively prime, we conclude that

$$\mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_\ell^{n_\ell}\mid(\beta^m-1)\mathcal{O}_K,$$

which implies the claim (4.4).

By (4.4), we have $\mathrm{Ord}_{\mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_\ell^{n_\ell}}(\beta)\mid m$. By Proposition 3.2, we obtain

$$m\ge\mathrm{Ord}_{\mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_\ell^{n_\ell}}(\beta)\ge c_2\prod_{p\in\mathcal{P}}p^{\max\{\lceil n_j/e_j\rceil:\,j\in\mathcal{I}_p\}}.\tag{4.5}$$

Together with (4.3), we obtain

$$\Big(\prod_{p\in\mathcal{P}}p^{\max\{\lceil n_j/e_j\rceil:\,j\in\mathcal{I}_p\}}\Big)^{1-s-\varepsilon}\le\frac{c_1}{c_2}.$$

Note that $p\ge2$ for all $p\in\mathcal{P}$, and $e_j=1$ or 2 for all $1\le j\le\ell$. We have

$$\begin{aligned}\prod_{p\in\mathcal{P}}p^{\max\{\lceil n_j/e_j\rceil:\,j\in\mathcal{I}_p\}}&\ge2^{\max\{\lceil n_j/e_j\rceil:\,1\le j\le\ell\}}\ge2^{\max\{n_j/2:\,1\le j\le\ell\}}\\&\ge2^{\frac{n_1+\cdots+n_\ell}{2\ell}}\ge(\sqrt{2})^{n_0/\ell}.\end{aligned}$$

Thus, we get

$$(\sqrt{2})^{n_0(1-s-\varepsilon)/\ell}\le\frac{c_1}{c_2},$$

which contradicts (4.2).

Therefore, for any $\boldsymbol{n} = (n_1, \dots, n_\ell) \in \mathbb{N}_0^\ell$ with $n_1 + \cdots + n_\ell \geq n_0$, we have $\mathcal{D}_\alpha^{\boldsymbol{n}} \cap S_{\beta,A} = \emptyset$. The proof is complete. ∎

Proof of Theorem 1.2(ii). The proof is essentially identical with that of Theorem 1.2(i). We need a better upper bound in (4.3).

Assume that $\mathcal{O}_K$ is a unique factorization domain. Then every prime ideal is a principal ideal. This means that for $1 \leq j \leq \ell$, we can find $\alpha_j \in \mathcal{O}_K$ such that $\mathfrak{p}_j = \alpha_j \mathcal{O}_K$. Thus, for $\boldsymbol{n} = (n_1, \dots, n_\ell) \in \mathbb{N}_0^\ell$, each $z \in \mathcal{D}_\alpha^{\boldsymbol{n}}$ has the form

$$z = \frac{v}{\alpha_1^{n_1} \cdots \alpha_\ell^{n_\ell}} \quad \text{for some } v \in \mathcal{O}_K. \tag{4.6}$$

We also assume that $\overline{\alpha}$ and α are relatively prime. By Remark 1.3(d), all primes $p_1, \dots, p_\ell$ are distinct, and $\mathfrak{p}_j \neq \overline{\mathfrak{p}_j}$ for all $1 \leq j \leq \ell$, where $\overline{\mathfrak{p}_j}$ is the conjugate prime ideal of $\mathfrak{p}_j$. By [7, Theorem 4.11], we have $e_j = f_j = 1$ for all $1 \leq j \leq \ell$. It follows that

$$p_j = N(\mathfrak{p}_j) = N(\alpha_j \mathcal{O}_K) = |\alpha_j|^2. \tag{4.7}$$

The remaining argument is the same as in the proof of Theorem 1.2(i).

Assume that $s = \dim_{\mathrm{H}} S_{\beta,A} < 2$. Choose $\varepsilon > 0$ such that $s + \varepsilon < 2$. Take a large integer $n_0 \in \mathbb{N}$ such that

$$(\sqrt{2})^{n_0(2-s-\varepsilon)} > \frac{c_1}{c_2}, \tag{4.8}$$

where c_1 and c_2 are the constants appearing in Propositions 2.1 and 3.2, respectively. Fix $\boldsymbol{n} = (n_1, \dots, n_\ell) \in \mathbb{N}_0^\ell$ with $n_1 + \cdots + n_\ell \geq n_0$. Suppose that $\mathcal{D}_\alpha^{\boldsymbol{n}} \cap S_{\beta,A} \neq \emptyset$, and we take $z \in \mathcal{D}_\alpha^{\boldsymbol{n}} \cap S_{\beta,A}$.

By Proposition 2.1 and (4.6), the complex number z has an eventually periodic coding $a_1 \dots a_k (a_{k+1} \dots a_{k+m})^\infty$ with

$$m \leq c_1 |\alpha_1^{n_1} \cdots \alpha_\ell^{n_\ell}|^{s+\varepsilon} = c_1 (p_1^{n_1} \cdots p_\ell^{n_\ell})^{(s+\varepsilon)/2}, \tag{4.9}$$

where the equality follows from (4.7). As in the proof of Theorem 1.2(i), we have

$$\beta^m - 1 \in \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_\ell^{n_\ell}.$$

Note that all primes $p_1, \dots, p_\ell$ are distinct, and $e_j = 1$ for all $1 \leq j \leq \ell$. By Proposition 3.2, we have

$$m \geq c_2 p_1^{n_1} \cdots p_\ell^{n_\ell}.$$

Together with (4.9), we obtain

$$(p_1^{n_1} \cdots p_\ell^{n_\ell})^{(2-s-\varepsilon)/2} \leq \frac{c_1}{c_2}.$$

It follows that

$$(\sqrt{2})^{n_0(2-s-\varepsilon)} \le (p_1^{n_1}\cdots p_\ell^{n_\ell})^{(2-s-\varepsilon)/2} \le \frac{c_1}{c_2},$$

which contradicts (4.8).

Thus, for any $\boldsymbol{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}_0^\ell$ with $n_1 + \cdots + n_\ell \ge n_0$, we have $\mathcal{D}_\alpha^{\boldsymbol{n}} \cap S_{\beta,A} = \emptyset$. Therefore, by (4.1), we conclude that $\#(\mathcal{D}_\alpha \cap S_{\beta,A}) < +\infty$. ∎

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