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Hausdorff dimension of continued fractions with fast growing partial quotients

by

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Abstract. Let $x \in [0,1)$ be a real number with continued fraction expansion $[a_1(x), a_2(x), a_3(x), \dots]$, and let $p_n(x)/q_n(x)$ denote its nth convergent. For a real number $\alpha > 0$ and a function $\psi : \mathbb{N} \to \mathbb{R}^+$ satisfying $\liminf_{n\to\infty} \psi(n) > 0$, we study the sets

$$E_\psi(\alpha) := \{x \in [0,1) : a_{n+1}(x) \geq q_n^\alpha(x)\psi(n) \text{ for infinitely many } n\},$$

$$\widetilde{E}_\psi(\alpha) := \{x \in [0,1) : a_{n+1}(x) \geq q_n^\alpha(x)\psi(n) \text{ for all sufficiently large } n\}.$$

We obtain exact formulae for the Hausdorff dimensions of these sets in terms of the exponential and double exponential growth rates of ψ. For $E_\psi(\alpha)$, the Hausdorff dimension is determined by the solution of a pressure equation associated with the Gauss map, exhibiting a continuous transition between the classical results of Good (1941) and Sun and Wu (2014). For $\widetilde{E}_\psi(\alpha)$, the Hausdorff dimension depends on the upper double exponential growth rate of ψ and admits a simple explicit expression.

1. Introduction. For any $x \in [0,1)$, we denote its continued fraction expansion by

$$x = [a_1(x), a_2(x), a_3(x), \dots],$$

where $a_1(x), a_2(x), a_3(x), \dots$ are positive integers, called the partial quotients of x. For more details on continued fractions, we refer the reader to [3, 9].

The metric theory of continued fractions, which studies the growth of the partial quotients of real numbers, is a cornerstone of metric number theory. A fundamental result in this field is the Borel–Bernstein theorem (see [2, 1]). It states that for any function $\phi : \mathbb{N} \to \mathbb{R}^+$, the Lebesgue measure of the set

$$E(\phi) = \{x \in [0,1) : a_n(x) \geq \phi(n) \text{ for infinitely many } n\} \tag{1.1}$$

is either zero or one according as the series $\sum_{n=1}^{\infty} \frac{1}{\phi(n)}$ converges or diverges.

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In particular, when ϕ is an exponential function, $E(\phi)$ has Lebesgue measure zero. Good [8] further studied $E(\phi)$ in the exponential case and obtained an upper bound for its Hausdorff dimension, without determining the exact value. Later, in 2008, Wang and Wu [19] gave a complete characterization of the Hausdorff dimension of $E(\phi)$ for arbitrary functions ϕ.

The study of the fractal dimension of continued fractions originated in the pioneering work of Jarník [10]. He was motivated by problems in Diophantine approximation and studied the set of continued fractions with bounded partial quotients. Good [8] further investigated sets with restricted partial quotients, and established that for any $\alpha > 0$,

$$\dim_{\mathrm{H}}\{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \text{ for infinitely many } n\} = \frac{2}{\alpha+2}, \tag{1.2}$$

where q_n is the denominator of the convergent defined in (2.2), and $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension (see [4] for the definition). In 2014, Sun and Wu [17] proved that

$$\dim_{\mathrm{H}}\{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \text{ for all sufficiently large } n\} = \frac{1}{\alpha+2}. \tag{1.3}$$

Related results can be found in Tan and Zhou [18].

Let $\alpha > 0$, and let $\psi : \mathbb{N} \to \mathbb{R}^+$ satisfy $\liminf_{n\to\infty} \psi(n) > 0$. Define

$$E_\psi(\alpha) := \{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x)\psi(n) \text{ for infinitely many } n\}. \tag{1.4}$$

Clearly, the set $E_\psi(\alpha)$ is the combination of the two sets in (1.1) and (1.2). In this paper, we investigate the Hausdorff dimension of $E_\psi(\alpha)$. The motivation of this work arises from the proof of the upper bound for the Hausdorff dimension of the set

$$E(c,d) = \{x \in [0,1) : a_n(x) \geq c^{d^n} \text{ for infinitely many } n\}$$

with $c, d > 1$. It is known that the Hausdorff dimension of $E(c,d)$ is $1/(d+1)$ (see, e.g., [13, 7]). Indeed, Feng et al. [7] gave a simple proof for the lower bound, while Łuczak [13] established the upper bound using more involved arguments. More recently, Shang and Fang [16] provided a simple proof for the upper bound. In their approach, sets of the form (1.4) for some function ψ play a crucial role. This observation naturally motivates a systematic study of the Hausdorff dimension of the set $E_\psi(\alpha)$ for any functions ψ.

We determine the Hausdorff dimension of $E_\psi(\alpha)$ using thermodynamic formalism. To this end, we first introduce the pressure function (see [15]) of continued fractions:

$$P(\theta) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{a_1,\ldots,a_n\in\mathbb{N}} q_n^{-2\theta}(a_1,\ldots,a_n) \quad \text{for } \theta > 1/2,$$

where $q_n(a_1,\ldots,a_n)$ is given by (2.7). It was shown in [11] that the pressure

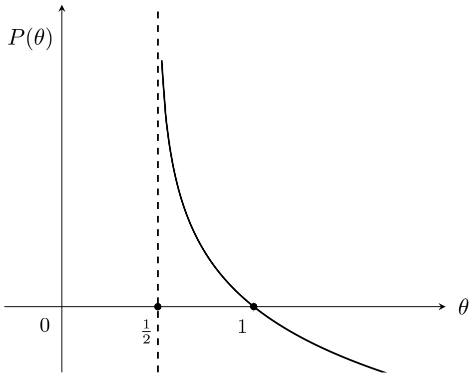

Fig. 1. The sketch of the pressure function

function has a singularity at $1/2$, and is strictly decreasing, convex, and real-analytic on the open interval $(1/2, \infty)$. The graph of the pressure function $P(\theta)$ is illustrated in Figure 1. For a comprehensive analysis of the pressure function, we refer the reader to [14, 6].

By convention, we set $1/\infty = 0$. The main result is stated as follows.

THEOREM 1.1. *Let $\alpha > 0$ be a real number, and let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying $\liminf_{n\to\infty} \psi(n) > 0$. Define*

$$\log B_\psi := \liminf_{n\to\infty} \frac{\log \psi(n)}{n}. \tag{1.5}$$

Then the following statements hold:

(1) *If $B_\psi = 1$, then*

$$\dim_{\mathrm{H}} E_\psi(\alpha) = \frac{2}{\alpha + 2}.$$

(2) *If $1 < B_\psi < \infty$, then*

$$\dim_{\mathrm{H}} E_\psi(\alpha) = s(\alpha, B_\psi),$$

where $s(\alpha, B_\psi)$ is the unique solution of the pressure equation

$$P\left(\left(1 + \frac{\alpha}{2}\right)s\right) = s \log B_\psi. \tag{1.6}$$

(3) *If $B_\psi = \infty$, then*

$$\dim_{\mathrm{H}} E_\psi(\alpha) = \min\left\{\frac{1}{\alpha + 2}, \frac{1}{d_\psi + 1}\right\},$$

where d_ψ is given by

$$\log d_\psi := \liminf_{n\to\infty} \frac{\log \log \psi(n)}{n}. \tag{1.7}$$

We now make some comments on equation (1.6).

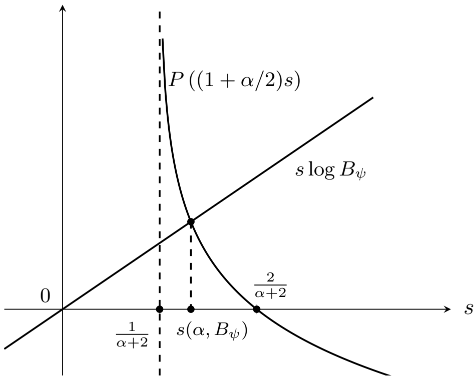

Fig. 2. The sketch for the solution of the pressure equation (1.6)

REMARK 1.2. The solution of the pressure equation in (1.6) is the unique intersection of the pressure function $P((1+\alpha/2)s)$ and the linear function $s\log B_\psi$. This solution, denoted by $s(\alpha, B_\psi)$, is illustrated in Figure 2. In addition, using the analytic properties of the pressure function, we establish the continuity of the solution $s(\alpha, B)$, with

$$\lim_{B\to 1^+} s(\alpha, B) = \frac{2}{\alpha+2} \quad \text{and} \quad \lim_{B\to\infty} s(\alpha, B) = \frac{1}{\alpha+2}. \tag{1.8}$$

We remark that these limiting values coincide with the Hausdorff dimensions of the sets investigated by Good [8] (see (1.2)) and by Sun and Wu [17] (see (1.3)).

We are also interested in a particular subset of $E_\psi(\alpha)$. More precisely, we focus on the Hausdorff dimension of the set of points $x \in [0,1)$ for which the inequality

$$a_{n+1}(x) \geq q_n^\alpha(x)\psi(n)$$

holds for all sufficiently large n. The following result characterizes this set.

THEOREM 1.3. *Let $\alpha > 0$ be a real number, and let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying $\liminf_{n\to\infty} \psi(n) > 0$. Define*

$$\widetilde{E}_\psi(\alpha) := \{x \in [0,1) : a_{n+1}(x) \geq q_n^\alpha(x)\psi(n) \text{ for all sufficiently large } n\}.$$

Then

$$\dim_{\mathrm{H}} \widetilde{E}_\psi(\alpha) = \min\left\{\frac{1}{\alpha+2}, \frac{1}{D_\psi+1}\right\},$$

where D_ψ is given by

$$\log D_\psi := \limsup_{n\to\infty} \frac{\log\log(\max\{\psi(n), \mathrm{e}\})}{n}. \tag{1.9}$$

The paper is organized as follows. For notational convenience, we write "i.m." to denote "for infinitely many" and "$n \gg 1$" to indicate "for all sufficiently large n". In Section 2, we review some basic properties of continued fractions. Section 3 considers the Hausdorff dimension of sets defined by the condition $a_{n+1}(x) \geq q_n^{\alpha}(x)c^{d^n}$, both for infinitely many n and for all sufficiently large n. The proofs of Theorems 1.1 and 1.3 are presented in Section 4.

2. Preliminaries. This section begins by reviewing the fundamental concepts and properties of continued fractions, and then introduces several key lemmas.

2.1. Continued fractions. It is known that the continued fraction expansion can be induced by the Gauss transformation $T : [0,1) \to [0,1)$ given by

$$T(0) := 0 \quad \text{and} \quad T(x) := \frac{1}{x} \pmod 1 \quad \text{for } x \in (0,1).$$

Denote by $\lfloor \cdot \rfloor$ the integer part of a number. Every real number $x \in (0,1)$ admits a continued fraction expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots + \cfrac{1}{a_n(x) + \ddots}}} =: [a_1(x), a_2(x), \ldots, a_n(x), \ldots], \tag{2.1}$$

where $a_1(x) = \lfloor 1/x \rfloor$ and $a_n(x) = a_1(T^{n-1}(x))$. The number x is irrational if and only if its continued fraction expansion (2.1) is infinite.

Denote by

$$\frac{p_n(x)}{q_n(x)} := [a_1(x), a_2(x), \ldots, a_n(x)] \tag{2.2}$$

the n-th convergent of the continued fraction expansion of x, where $p_n(x)$ and $q_n(x)$ are positive and coprime. It is well known that for any $n \geq 1$, $p_n(x)$ and $q_n(x)$ satisfy the following recursive formulae (see [12, p. 4]):

$$\begin{aligned} p_n(x) &= a_n(x)p_{n-1}(x) + p_{n-2}(x), \\ q_n(x) &= a_n(x)q_{n-1}(x) + q_{n-2}(x), \end{aligned} \tag{2.3}$$

where $p_{-1} \equiv 1$, $p_0 \equiv 0$ and $q_{-1} \equiv 0$, $q_0 \equiv 1$. It follows from (2.3) that

$$q_n(x) \geq 2^{\frac{n-1}{2}}, \tag{2.4}$$

and

$$a_1(x) \cdots a_n(x) \leq q_n(x) \leq 2^n a_1(x) \cdots a_n(x). \tag{2.5}$$

Wang et al. [20, Lemma 2.1] note the following relationship:

$$q_n^2(x) \le \prod_{i=1}^{n} \left|T'(T^{i-1}x)\right| \le 4q_n^2(x). \tag{2.6}$$

For any integer $n \ge 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, the set

$$I_n(a_1, \ldots, a_n) := \{x \in [0,1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\}$$

is called a *cylinder of order* n for the continued fraction expansion. Note that all points in $I_n(a_1, \ldots, a_n)$ share the same values of $p_n(x)$ and $q_n(x)$. Hence, for $x \in I_n(a_1, \ldots, a_n)$, we write

$$p_n = p_n(x) = p_n(a_1, \ldots, a_n) \quad \text{and} \quad q_n = q_n(x) = q_n(a_1, \ldots, a_n). \tag{2.7}$$

The structure of a cylinder is described as follows. Denote by $|I|$ the diameter of an interval I.

Proposition 2.1 ([9, p. 18]). *For any $(a_1, \ldots, a_n) \in \mathbb{N}^n$, the cylinder $I_n(a_1, \ldots, a_n)$ is an interval with endpoints p_n/q_n and $(p_n+p_{n-1})/(q_n+q_{n-1})$. More precisely,*

$$I_n(a_1, \ldots, a_n) = \begin{cases} \left[\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right) & \text{if } n \text{ is even,} \\ \left(\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}\right] & \text{if } n \text{ is odd.} \end{cases}$$

As a result, the length of $I_n(a_1, \ldots, a_n)$ is given by

$$|I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}. \tag{2.8}$$

Combining (2.3) and (2.8), we conclude that

$$\left(2^n \prod_{k=1}^{n} a_k\right)^{-2} \le |I_n(a_1, \ldots, a_n)| \le \left(\prod_{k=1}^{n} a_k\right)^{-2}. \tag{2.9}$$

2.2. Some lemmas. We collect several useful lemmas for estimating the Hausdorff dimension of certain sets arising from continued fraction expansions.

The first lemma can be found in the work of Fan et al. [5].

Lemma 2.2 ([5, Lemma 3.2]). *Let $\{s_n\}_{n\ge1}$ be a sequence of positive integers tending to infinity with $s_n \ge 3$ for all $n \ge 1$. Then for any positive number $N \ge 2$,*

$$\dim_{\mathrm{H}}\{x \in [0,1) : s_n \le a_n(x) < Ns_n, \forall n \ge 1\} = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \log s_k}{2\sum_{k=1}^{n} \log s_k + \log s_{n+1}}.$$

The following lemma follows from the work of Wang et al. [20]. By considering the function

$$f(x) = \frac{\alpha}{2} \log |T'x| + \log B$$

in [20, Theorem 1.1], we obtain the following result.

LEMMA 2.3. *For any $\alpha > 0$ and $B > 1$,*

$$\dim_{\mathrm{H}}\Big\{x \in [0,1) : a_{n+1}(x) \geq B^n \prod_{i=1}^{n} |T'(T^{i-1}x)|^{\alpha/2},\ i.m.\ n \in \mathbb{N}\Big\} = s(\alpha, B),$$

where $s(\alpha, B)$ is the unique solution of the pressure equation

$$P\left(\left(1+\frac{\alpha}{2}\right)s\right) = s \log B.$$

Based on Lemma 2.3, we can derive the following result.

LEMMA 2.4. *Let $\alpha > 0$ and $B > 1$ be real numbers, and let $\mathcal{N}$ be an infinite subset of the natural numbers $\mathbb{N}$. Then*

$$\begin{aligned}&\dim_{\mathrm{H}}\{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \cdot B^n,\ i.m.\ n \in \mathcal{N}\}\\ &\quad = \dim_{\mathrm{H}}\{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \cdot B^n,\ i.m.\ n \in \mathbb{N}\} = s(\alpha, B).\end{aligned}$$

Proof. Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$. Observe that

$$\begin{aligned}&\{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \cdot B^n,\ \text{i.m. } n \in \mathcal{N}\}\\ &\qquad\qquad \subseteq \{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \cdot B^n,\ \text{i.m. } n \in \mathbb{N}\}.\end{aligned}$$

Therefore, it suffices to prove the upper bound

$$\dim_{\mathrm{H}}\{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \cdot B^n,\ \text{i.m. } n \in \mathbb{N}\} \leq s(\alpha, B), \tag{2.10}$$

and the lower bound

$$\dim_{\mathrm{H}}\{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \cdot B^n,\ \text{i.m. } n \in \mathcal{N}\} \geq s(\alpha, B). \tag{2.11}$$

For the upper bound, from formula (2.6), we obtain the estimate

$$\frac{1}{2} \prod_{i=1}^{n} |T'(T^{i-1}x)|^{1/2} \leq q_n(x) \leq \prod_{i=1}^{n} |T'(T^{i-1}x)|^{1/2}.$$

Therefore, for any $\varepsilon > 0$ and sufficiently large n, we obtain

$$(B-\varepsilon)^n \prod_{i=1}^{n} |T'(T^{i-1}x)|^{\alpha/2} \leq q_n^{\alpha}(x) \cdot B^n \leq B^n \prod_{i=1}^{n} \left|T'(T^{i-1}x)\right|^{\alpha/2}.$$

Consequently,

$$\{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \cdot B^n, \text{ i.m. } n \in \mathbb{N}\}$$
$$\subseteq \Big\{x \in [0,1) : a_{n+1}(x) \geq (B-\varepsilon)^n \prod_{i=1}^{n} |T'(T^{i-1}x)|^{\alpha/2}, \text{ i.m. } n \in \mathbb{N}\Big\}.$$

By Lemma 2.3 and letting $\varepsilon \to 0^+$, we obtain (2.10).

For the lower bound, from (2.6) we get the inclusion

$$\Big\{x \in [0,1) : a_{n+1}(x) \geq B^n \prod_{i=1}^{n} |T'(T^{i-1}x)|^{\alpha/2}, \text{ i.m. } n \in \mathcal{N}\Big\}$$
$$\subseteq \{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \cdot B^n, \text{ i.m. } n \in \mathcal{N}\}.$$

The problem reduces to proving that

(2.12)
$$\dim_{\mathrm{H}}\Big\{x \in [0,1) : a_{n+1}(x) \geq B^n \prod_{i=1}^{n} |T'(T^{i-1}x)|^{\alpha/2}, \text{ i.m. } n \in \mathcal{N}\Big\} \geq s(\alpha, B).$$

In [20, Section 5], Wang et al. construct recursively a sequence $\{n_k\} \subseteq \mathbb{N}$ such that the following conditions hold:

$$n_k - n_{k-1} \geq \max\{m_k(u), km_k(v), n_{k-1} + t_k : u \in T_k, v \in \overline{T}_k\},$$
$$2^{((n_k - n_{k-1} - 1)\varepsilon)/2}$$
$$\geq \max\{2^5 (q_{t_{k+1}}^2(v)(\beta+1)^{2m_k(v)} \mathrm{e}^{(1+\varepsilon)(S_{t_{k+1}} f(y) + (t_{k+1} + m_k(v)) f_\beta)})^{k+1} :$$
$$y \in I_{t_{k+1}}(v),\ v \in \overline{T}_k\},$$

and

$$\frac{n_k - n_{k-1}}{k} \geq \max\Big\{S_{t_{k+1}} f(y) + S_{n_{k-1}} f(y'_{k-1}) + \sum_{j=1}^{n_{k-1}} \mathrm{Var}_j(f) + \sum_{j=1}^{t_k} \mathrm{Var}_j(f) :$$
$$y \in I_{t_{k+1}}(w^{(k-1)}|_{t_{k+1}}), y'_{k-1} \in I_{n_{k-1}}(w^{(k-1)}), I_{n_{k-1}}(w^{(k-1)}) \in \mathbb{G}_{k-1}\Big\}.$$

Here,

$$f(x) = \frac{\alpha}{2} \log |T'x| + \log B, \quad t_k = t_0 + k,$$
$$f_\beta = \sup\{f(x) : a_1(x) \leq \beta\} + \mathrm{Var}_1(f) \text{ for } \beta \in \mathbb{N},$$
$$\mathrm{Var}_n(f) = \sup_{x,y\colon I_n(x) = I_n(y)} |f(x) - f(y)|, \quad S_n f(x) = \sum_{i=1}^{n} f(T^{i-1}x).$$

Other notation not explicitly defined here can be found in the original reference [20]. Following the argument in [19, Corollary 3.3], we choose an integer sequence $\{n_k\} \subseteq \mathcal{N}$ that also satisfies the conditions above. This ensures that (2.12) holds, and hence (2.11) follows. ∎

3. Auxiliary results. To prove Theorems 1.1 and 1.3, we first estimate the Hausdorff dimension of the following sets. For any $\alpha > 0$, $c > 1$, and $d > 1$, we define

$$E(\alpha; c, d) := \{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \cdot c^{d^n}, \text{ i.m. } n\},$$
$$\widetilde{E}(\alpha; c, d) := \{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha}(x) \cdot c^{d^n}, \ n \gg 1\}.$$

Proposition 3.1. *Let $\alpha > 0$, $c > 1$, and $d > 1$ be real numbers. Then*

$$\dim_{\mathrm{H}} E(\alpha; c, d) = \dim_{\mathrm{H}} \widetilde{E}(\alpha; c, d) = \min\left\{\frac{1}{\alpha+2}, \frac{1}{d+1}\right\}.$$

Proof. The inclusion $\widetilde{E}(\alpha; c, d) \subseteq E(\alpha; c, d)$ is clear, and it yields the inequality $\dim_{\mathrm{H}} \widetilde{E}(\alpha; c, d) \leq \dim_{\mathrm{H}} E(\alpha; c, d)$. To prove

$$\dim_{\mathrm{H}} \widetilde{E}(\alpha; c, d) = \dim_{\mathrm{H}} E(\alpha; c, d),$$

it suffices to show that the upper bound for $\dim_{\mathrm{H}} E(\alpha; c, d)$ coincides with the lower bound for $\dim_{\mathrm{H}} \widetilde{E}(\alpha; c, d)$.

(1) *Upper bound for* $\dim_{\mathrm{H}} E(\alpha; c, d)$. We first establish a Łuczak-type result, which serves as a key ingredient in the proof of the upper bound for $\dim_{\mathrm{H}} E(\alpha; c, d)$. The argument is inspired by the recent work of Shang and Fang [16].

For any $r \in (1/d, 1)$ and $b \in (1/r, d)$, here $(\cdot,\cdot)$ denotes an open interval. If $x \in E(\alpha; c, d)$, we assert that for infinitely many n,

$$a_{n+1}(x) > (a_1(x) \cdots a_n(x))^{\max\{\alpha, rb-1\}} c^{(1-r)b^n}. \tag{3.1}$$

This is essentially Claim 1 in [13].

Next, we prove that inequality (3.1) holds. Let $x \in E(\alpha; c, d)$. Then, for infinitely many n,

$$a_{n+1}(x) \geq q_n^{\alpha}(x) c^{d^n} > q_n^{\alpha}(x) c^{(1-r)b^n}.$$

Combining this with (2.5) yields

$$a_{n+1}(x) > (a_1(x) \cdots a_n(x))^{\alpha} c^{(1-r)b^n}. \tag{3.2}$$

Moreover, it is clear that $a_{n+1}(x) > c^{d^n}$ for infinitely many n. For any fixed integer m, there exists $k \in \mathbb{N}$ with $k > m$ such that

$$a_k(x) > c^{d^{k-1}} \quad \text{and} \quad a_1(x) \cdots a_m(x) \leq c^{d^{k-1} b^{m-k}}.$$

Let $f(n) = c^{d^{k-1} b^{n-k}}$. Then $a_1(x) \cdots a_m(x) \leq f(m)$. Choose the largest n

such that $m \le n < k$ and $a_1(x)\cdots a_n(x) \le f(n)$. Thus, we conclude that

$$\begin{aligned} a_1(x)\cdots a_{n+1}(x) &> f(n+1) = f^r(n+1)f^{1-r}(n+1) \\ &> f^{rb}(n)c^{(1-r)b^n} \\ &\ge (a_1(x)\cdots a_n(x))^{rb}c^{(1-r)b^n}. \end{aligned}$$

This implies that for infinitely many n,

$$a_{n+1}(x) > (a_1(x)\cdots a_n(x))^{rb-1}c^{(1-r)b^n}. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain inequality (3.1).

Let

$$\begin{aligned} &F(\alpha, b, r) \\ &\quad := \{x \in [0,1) : a_{n+1}(x) > (a_1(x)\cdots a_n(x))^{\max\{\alpha, rb-1\}}c^{(1-r)b^n}, \text{ i.m. } n\}. \end{aligned}$$

Then $E(\alpha; c, d) \subseteq F(\alpha, b, r)$. The problem reduces to estimating the upper bound of $\dim_{\mathrm{H}} F(\alpha, b, r)$. For any $n \in \mathbb{N}$ and $(\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n$, let

$$I_n(\sigma_1, \dots, \sigma_n) := \{x \in [0,1) : a_1(x) = \sigma_1, \dots, a_n(x) = \sigma_n\},$$

and

$$J_n(\sigma_1, \dots, \sigma_n) := \bigcup_{j > (\sigma_1\cdots\sigma_n)^{\max\{\alpha, rb-1\}}c^{(1-r)b^n}} I_{n+1}(\sigma_1, \dots, \sigma_n, j). \tag{3.4}$$

Then we obtain

$$F(\alpha, b, r) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{(\sigma_1,\dots,\sigma_n)\in\mathbb{N}^n} J_n(\sigma_1, \dots, \sigma_n). \tag{3.5}$$

This indicates that for any $N \ge 1$, $\{J_n(\sigma_1, \dots, \sigma_n) : (\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n\}$ forms a cover of $F(\alpha, b, r)$. Combining (2.9) and (3.4), we see that

$$\begin{aligned} |J_n(\sigma_1, \dots, \sigma_n)| &= \sum_{j > (\sigma_1\cdots\sigma_n)^{\max\{\alpha, rb-1\}}c^{(1-r)b^n}} |I_{n+1}(\sigma_1, \dots, \sigma_n, j)| \\ &\le \sum_{j > (\sigma_1\cdots\sigma_n)^{\max\{\alpha, rb-1\}}c^{(1-r)b^n}} \frac{1}{j^2 \cdot (\sigma_1\cdots\sigma_n)^2} \\ &= \frac{1}{(\sigma_1\cdots\sigma_n)^{\max\{\alpha+2, rb+1\}} \cdot c^{(1-r)b^n}}. \end{aligned}$$

For any

$$s > \frac{1}{\max\{\alpha+2, rb+1\}},$$

let $t := \max\{\alpha+2, rb+1\}s$, and denote $K_t := \sum_{i=1}^{\infty} i^{-t}$. Then $1 < K_t < \infty$. Denote by $\mathcal{H}^s(\cdot)$ the s-dimensional Hausdorff measure. We deduce from (3.5) that

$$\begin{aligned}\mathcal{H}^s(F(\alpha,b,r)) &\le \liminf_{N\to\infty}\sum_{n=N}^{\infty}\sum_{(\sigma_1,\ldots,\sigma_n)\in\mathbb{N}^n}|J_n(\sigma_1,\ldots,\sigma_n)|^s\\ &\le \liminf_{N\to\infty}\sum_{n=N}^{\infty}\sum_{(\sigma_1,\ldots,\sigma_n)\in\mathbb{N}^n}\frac{1}{(\sigma_1\cdots\sigma_n)^{\max\{\alpha+2,rb+1\}s}\cdot c^{(1-r)sb^n}}\\ &= \liminf_{N\to\infty}\sum_{n=N}^{\infty}\frac{K_t^n}{c^{(1-r)sb^n}}=0,\end{aligned}$$

which yields $\dim_{\mathrm{H}} F(\alpha,b,r)\le s$. Hence,

$$\dim_{\mathrm{H}} F(\alpha,b,r)\le \frac{1}{\max\{\alpha+2,rb+1\}}.$$

Letting $r\to 1^-$ and $b\to d^-$, we obtain

$$\dim_{\mathrm{H}} E(\alpha;c,d)\le \dim_{\mathrm{H}} F(\alpha,b,r)\le \min\left\{\frac{1}{\alpha+2},\frac{1}{d+1}\right\}.$$

(2) *Lower bound for* $\dim_{\mathrm{H}}\widetilde{E}(\alpha;c,d)$. For simplicity, we set

$$p:=\max\{\alpha+1,d\}.$$

For any $\varepsilon>0$, define

$$F_\varepsilon(c,p):=\{x\in[0,1): 3\lfloor c^{(p+\varepsilon)^n}\rfloor\le a_n(x)<6\lfloor c^{(p+\varepsilon)^n}\rfloor,\ n\ge 1\}.$$

Then $F_\varepsilon(c,p)\subseteq\widetilde{E}(\alpha;c,d)$. Indeed, if $x\in F_\varepsilon(c,p)$, from (2.5) we obtain

$$\begin{aligned}q_n\le 2^n a_1\cdots a_n &< 2^n\cdot 6^n\cdot\lfloor c^{p+\varepsilon}\rfloor\cdot\lfloor c^{(p+\varepsilon)^2}\rfloor\cdots\lfloor c^{(p+\varepsilon)^n}\rfloor\\ &< 12^n\cdot c^{(p+\varepsilon)+\cdots+(p+\varepsilon)^n}=12^n\cdot c^{\frac{(p+\varepsilon)^{n+1}-(p+\varepsilon)}{p+\varepsilon-1}}\end{aligned}$$

and consequently

$$q_n^\alpha<12^{\alpha n}\cdot c^{\frac{\alpha}{p+\varepsilon-1}[(p+\varepsilon)^{n+1}-(p+\varepsilon)]}.$$

Moreover, for all sufficiently large n, we get

$$\frac{\alpha}{p+\varepsilon-1}[(p+\varepsilon)^{n+1}-(p+\varepsilon)]+d^n<(p+\varepsilon)^{n+1},$$

and therefore

$$q_n^\alpha\cdot c^{d^n}<c^{(p+\varepsilon)^{n+1}}<3\lfloor c^{(p+\varepsilon)^{n+1}}\rfloor\le a_{n+1}$$

for all sufficiently large n. Thus, $x\in\widetilde{E}(\alpha;c,d)$, which implies $F_\varepsilon(c,p)\subseteq$

$\widetilde{E}(\alpha; c, d)$. By Lemma 2.2 and the Stolz–Cesàro theorem, we obtain

$$\begin{aligned}\dim_{\mathrm{H}} \widetilde{E}(\alpha; c, d) &\geq \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \log(3\lfloor c^{(p+\varepsilon)^k}\rfloor)}{2\sum_{k=1}^{n} \log(3\lfloor c^{(p+\varepsilon)^k}\rfloor) + \log(3\lfloor c^{(p+\varepsilon)^{n+1}}\rfloor)} \\ &= \liminf_{n\to\infty} \frac{\sum_{k=1}^{n}(p+\varepsilon)^k}{2\sum_{k=1}^{n}(p+\varepsilon)^k + (p+\varepsilon)^{n+1}} \\ &= \liminf_{n\to\infty} \frac{(p+\varepsilon)^n}{(p+\varepsilon)^n + (p+\varepsilon)^{n+1}} = \frac{1}{p+1+\varepsilon}.\end{aligned}$$

Letting $\varepsilon \to 0^+$, we conclude that

$$\dim_{\mathrm{H}} \widetilde{E}(\alpha; c, d) \geq \frac{1}{p+1} = \min\left\{\frac{1}{\alpha+2}, \frac{1}{d+1}\right\}. \blacksquare$$

4. Proofs of main results

4.1. Proof of Theorem 1.1. We discuss three cases according to the value of B_ψ: $B_\psi = 1$, $1 < B_\psi < \infty$, and $B_\psi = \infty$. Our approach was inspired by Wang and Wu [19].

(1) Case $B_\psi = 1$. From (1.5), it follows that for any $0 < \varepsilon < \alpha$ and for infinitely many n,

$$\frac{\log \psi(n)}{n} \leq \log(1+\varepsilon),$$

and hence $\psi(n) \leq (1+\varepsilon)^n$ for infinitely many n. We define

$$\mathcal{N} = \{n \in \mathbb{N} : \psi(n) \leq (1+\varepsilon)^n\}.$$

Thus,

$$\{x \in [0,1) : a_{n+1}(x) \geq q_n^\alpha(x)(1+\varepsilon)^n, \text{ i.m. } n \in \mathcal{N}\} \subseteq E_\psi(\alpha).$$

By (1.8) and Lemma 2.4, we deduce that

$$\dim_{\mathrm{H}} E_\psi(\alpha) \geq \lim_{\varepsilon\to 0^+} s(\alpha, 1+\varepsilon) = \frac{2}{\alpha+2}.$$

On the other hand, the condition $\liminf_{n\to\infty} \psi(n) > 0$ implies that there exists a constant $C > 0$ such that $\psi(n) \geq C > 0$ for any $n \geq 1$. From (2.4), we obtain

$$a_{n+1}(x) \geq q_n^\alpha(x)\psi(n) \geq C \cdot q_n^\alpha(x) \geq C \cdot (2^{(n-1)/2})^\varepsilon \cdot q_n^{\alpha-\varepsilon}(x).$$

Hence, $a_{n+1}(x) \geq q_n^{\alpha-\varepsilon}(x)$ for infinitely many n. Thus,

$$E_\psi(\alpha) \subseteq \{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha-\varepsilon}(x), \text{ i.m. } n\}.$$

Combining this with (1.2), we obtain

$$\dim_{\mathrm{H}} E_\psi(\alpha) \leq \frac{2}{\alpha - \varepsilon + 2}.$$

Letting $\varepsilon \to 0^+$ yields

$$\dim_{\mathrm{H}} E_\psi(\alpha) \le \frac{2}{\alpha+2}.$$

(2) Case $1 < B_\psi < \infty$. From (1.5), for any $0 < \varepsilon < B_\psi - 1$, the inequality

$$\frac{\log \psi(n)}{n} \le \log(B_\psi + \varepsilon)$$

holds for infinitely many n, implying $\psi(n) \le (B_\psi+\varepsilon)^n$ for those n. Define

$$\mathcal{N} = \{n \in \mathbb{N} : \psi(n) \le (B_\psi+\varepsilon)^n\}.$$

Then

$$\{x \in [0,1) : a_{n+1}(x) \ge q_n^\alpha(x)(B_\psi+\varepsilon)^n, \text{ i.m. } n \in \mathcal{N}\} \subseteq E_\psi(\alpha).$$

Moreover, there exists $N \in \mathbb{N}$ such that for any $n \ge N$,

$$\frac{\log \psi(n)}{n} \ge \log(B_\psi - \varepsilon),$$

which implies that $\psi(n) \ge (B_\psi-\varepsilon)^n$ for infinitely many n. Consequently,

$$E_\psi(\alpha) \subseteq \{x \in [0,1) : a_{n+1}(x) \ge q_n^\alpha(x)(B_\psi-\varepsilon)^n, \text{ i.m. } n\}.$$

It follows from Lemma 2.4 that

$$s(\alpha, B_\psi+\varepsilon) \le \dim_{\mathrm{H}} E_\psi(\alpha) \le s(\alpha, B_\psi-\varepsilon).$$

Letting $\varepsilon \to 0^+$, we have

$$\dim_{\mathrm{H}} E_\psi(\alpha) = s(\alpha, B_\psi),$$

where $s(\alpha, B_\psi)$ is the unique solution of the pressure equation

$$P\left(\left(1+\frac{\alpha}{2}\right)s\right) = s\log B_\psi.$$

(3) Case $B_\psi = \infty$. In this case, we will discuss three situations corresponding to $d_\psi = 1$, $1 < d_\psi < \infty$ and $d_\psi = \infty$.

(3.1) Subcase $d_\psi = 1$. From (1.7), it follows that for any $\varepsilon > 0$ and for infinitely many n,

$$\frac{\log\log \psi(n)}{n} \le \log(1+\varepsilon),$$

and consequently $\psi(n) \le \mathrm{e}^{(1+\varepsilon)^n}$ for infinitely many n. Define

$$\mathcal{N} = \{n \in \mathbb{N} : \psi(n) \le \mathrm{e}^{(1+\varepsilon)^n}\}.$$

Then

$$\widetilde{E}(\alpha; \mathrm{e}, 1+\varepsilon) \subseteq \{x \in [0,1) : a_{n+1}(x) \ge q_n^\alpha(x)\mathrm{e}^{(1+\varepsilon)^n}, \text{ i.m. } n \in \mathcal{N}\} \subseteq E_\psi(\alpha).$$

From Proposition 3.1, we obtain

$$\dim_{\mathrm{H}} E_\psi(\alpha) \ge \min\left\{\frac{1}{\alpha+2}, \frac{1}{2+\varepsilon}\right\}.$$

Letting $\varepsilon \to 0^+$, it follows that

$$\dim_{\mathrm{H}} E_\psi(\alpha) \geq \frac{1}{\alpha+2}.$$

On the other hand, since $B_\psi = \infty$, for any constant $G > 0$ we have

$$\frac{\log \psi(n)}{n} \geq \log G,$$

i.e. $\psi(n) \geq G^n$ for all sufficiently large n. Hence,

$$\begin{aligned} E_\psi(\alpha) &\subseteq \{x \in [0,1) : a_{n+1}(x) \geq q_n^\alpha(x) \cdot G^n,\ n \gg 1\} \\ &\subseteq \{x \in [0,1) : a_{n+1}(x) \geq q_n^\alpha(x) \cdot G^n,\ \text{i.m. } n\}. \end{aligned}$$

Combining (1.8) with Lemma 2.4, we get

$$\dim_{\mathrm{H}} E_\psi(\alpha) \leq \lim_{G\to\infty} s(\alpha, G) = \frac{1}{\alpha+2}.$$

Therefore,

$$\dim_{\mathrm{H}} E_\psi(\alpha) = \frac{1}{\alpha+2}.$$

(3.2) Subcase $1 < d_\psi < \infty$. From (1.7), for any $0 < \varepsilon < d_\psi - 1$, we have

$$\frac{\log\log \psi(n)}{n} \leq \log(d_\psi + \varepsilon),$$

i.e. $\psi(n) \leq \mathrm{e}^{(d_\psi+\varepsilon)^n}$ for infinitely many n. Define

$$\mathcal{N} = \{n \in \mathbb{N} : \psi(n) \leq \mathrm{e}^{(d_\psi+\varepsilon)^n}\}.$$

Then

$$\widetilde{E}(\alpha; \mathrm{e}, d_\psi+\varepsilon) \subseteq \{x \in [0,1) : a_{n+1}(x) \geq q_n^\alpha(x)\mathrm{e}^{(d_\psi+\varepsilon)^n},\ \text{i.m. } n \in \mathcal{N}\} \subseteq E_\psi(\alpha).$$

Moreover, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\frac{\log\log \psi(n)}{n} \geq \log(d_\psi - \varepsilon),$$

i.e. $\psi(n) \geq \mathrm{e}^{(d_\psi-\varepsilon)^n}$ for all sufficiently large n. Thus,

$$E_\psi(\alpha) \subseteq \widetilde{E}(\alpha; \mathrm{e}, d_\psi - \varepsilon) \subseteq E(\alpha; \mathrm{e}, d_\psi - \varepsilon).$$

By Proposition 3.1, we obtain

$$\min\left\{\frac{1}{\alpha+2}, \frac{1}{d_\psi+1+\varepsilon}\right\} \leq \dim_{\mathrm{H}} E_\psi(\alpha) \leq \min\left\{\frac{1}{\alpha+2}, \frac{1}{d_\psi+1-\varepsilon}\right\}.$$

Letting $\varepsilon \to 0^+$, we conclude that

$$\dim_{\mathrm{H}} E_\psi(\alpha) = \min\left\{\frac{1}{\alpha+2}, \frac{1}{d_\psi+1}\right\}.$$

(3.3) SUBCASE $d_\psi = \infty$. By (1.7), for any constant $K > 0$, we obtain

$$\frac{\log\log\psi(n)}{n} \geq \log K,$$

i.e. $\psi(n) \geq \mathrm{e}^{K^n}$ for all sufficiently large n. Thus,

$$E_\psi(\alpha) \subseteq \widetilde{E}(\alpha; \mathrm{e}, K) \subseteq E(\alpha; \mathrm{e}, K).$$

From the result of Proposition 3.1,

$$\dim_{\mathrm{H}} E_\psi(\alpha) \leq \min\left\{\frac{1}{\alpha+2}, \frac{1}{K+1}\right\}.$$

Upon letting $K \to \infty$, we find that $\dim_{\mathrm{H}} E_\psi(\alpha) \leq 0$. Consequently,

$$\dim_{\mathrm{H}} E_\psi(\alpha) = 0.$$

4.2. Proof of Theorem 1.3. In the proof of this theorem, we consider the value of D_ψ and discuss the following three cases as before: $D_\psi = 1$, $1 < D_\psi < \infty$, and $D_\psi = \infty$.

(1) CASE $D_\psi = 1$. For any $0 < \varepsilon < \alpha$ and all sufficiently large n, formula (1.9) implies that

$$\frac{\log\log(\max\{\psi(n), \mathrm{e}\})}{n} \leq \log(1+\varepsilon),$$

and hence $\psi(n) \leq \mathrm{e}^{(1+\varepsilon)^n}$ for all sufficiently large n. It follows that

$$\widetilde{E}(\alpha; \mathrm{e}, 1+\varepsilon) \subseteq \widetilde{E}_\psi(\alpha).$$

Applying Proposition 3.1 yields

$$\dim_{\mathrm{H}} \widetilde{E}_\psi(\alpha) \geq \min\left\{\frac{1}{\alpha+2}, \frac{1}{2+\varepsilon}\right\} = \frac{1}{\alpha+2}.$$

Conversely, the condition $\liminf_{n\to\infty} \psi(n) > 0$ implies that for all sufficiently large n,

$$a_{n+1}(x) \geq q_n^{\alpha}(x)\psi(n) \geq q_n^{\alpha-\varepsilon}(x).$$

Accordingly,

$$\widetilde{E}_\psi(\alpha) \subseteq \{x \in [0,1) : a_{n+1}(x) \geq q_n^{\alpha-\varepsilon}(x),\ n \gg 1\}.$$

From the result of (1.3), we have

$$\dim_{\mathrm{H}} \widetilde{E}_\psi(\alpha) \leq \frac{1}{\alpha-\varepsilon+2}.$$

Letting $\varepsilon \to 0^+$ yields

$$\dim_{\mathrm{H}} \widetilde{E}_\psi(\alpha) \leq \frac{1}{\alpha+2}.$$

Thus, we obtain

$$\dim_{\mathrm{H}} \widetilde{E}_\psi(\alpha) = \frac{1}{\alpha+2}.$$

(2) Case $1 < D_\psi < \infty$. For any $0 < \varepsilon < D_\psi - 1$, formula (1.9) implies that $\psi(n) \leq \mathrm{e}^{(D_\psi+\varepsilon)^n}$ for all sufficiently large n; and $\psi(n) \geq \mathrm{e}^{(D_\psi-\varepsilon)^n}$ for infinitely many n. Consequently,

$$\widetilde{E}(\alpha; \mathrm{e}, D_\psi + \varepsilon) \subseteq \widetilde{E}_\psi(\alpha) \subseteq E(\alpha; \mathrm{e}, D_\psi - \varepsilon).$$

By Proposition 3.1,

$$\min\left\{\frac{1}{\alpha+2}, \frac{1}{D_\psi + 1 + \varepsilon}\right\} \leq \dim_{\mathrm{H}} \widetilde{E}_\psi(\alpha) \leq \min\left\{\frac{1}{\alpha+2}, \frac{1}{D_\psi + 1 - \varepsilon}\right\}.$$

Letting $\varepsilon \to 0^+$, we conclude that

$$\dim_{\mathrm{H}} \widetilde{E}_\psi(\alpha) = \min\left\{\frac{1}{\alpha+2}, \frac{1}{D_\psi + 1}\right\}.$$

(3) Case $D_\psi = \infty$. For any constant $M > 0$, formula (1.9) implies that $\psi(n) \geq \mathrm{e}^{M^n}$ for infinitely many n. Hence,

$$\widetilde{E}_\psi(\alpha) \subseteq E(\alpha; \mathrm{e}, M).$$

Using Proposition 3.1, we obtain

$$\dim_{\mathrm{H}} \widetilde{E}_\psi(\alpha) \leq \min\left\{\frac{1}{\alpha+2}, \frac{1}{M+1}\right\}.$$

Taking $M \to \infty$, it follows that $\dim_{\mathrm{H}} \widetilde{E}_\psi(\alpha) \leq 0$. Clearly, $\dim_{\mathrm{H}} \widetilde{E}_\psi(\alpha) = 0$.

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