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Abd Elhamid Mehamdia, Yacine Chaib,
Tahar Bechouat, Ziadi Raouf

ANOTHER HYBRID CONJUGATE GRADIENT METHOD AS A CONVEX COMBINATION OF NVHS* AND CD METHODS

Abstract. Conjugate gradient methods play a fundamental role in solving unconstrained optimization problems. We propose a new hybrid conjugate gradient algorithm that forms a convex combination of $\beta_k^{\mathrm{NVHS}^*}$ and β_k^{CD}. The parameter θ_k is determined so as to satisfy the conjugacy condition. Under the strong Wolfe line search conditions, we establish both the descent property and the global convergence of the proposed hybrid method. Numerical experiments demonstrate that the new method is robust and computationally efficient.

1. Introduction. Optimization plays a pivotal role in operational research, lying at the intersection of computer science, mathematics, and economics. It is also a fundamental tool in applied mathematics, with critical applications in industry, engineering, nonparametric estimation, and image restoration; see [6, 7, 11] and [26]–[25]. In this paper, we consider the unconstrained optimization problem

$$\min\{f(x) : x \in \mathbb{R}^n\}, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function.

Conjugate gradient methods are very important methods for solving (1.1), especially when the dimension n is large. The iterative process of a conjugate gradient method for solving (1.1) is given by

$$x_{k+1} = x_k + \alpha_k d_k, \tag{1.2}$$

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where x_k represents the current iterate, $\alpha_k > 0$ is a step size determined by line search, and d_k is the search direction defined by the formula

$$d_0 = -H_0; \quad d_{k+1} = -H_{k+1} + \beta_k d_k; \tag{1.3}$$

here $H_{k+1} = \nabla f(x_{k+1})$ is the gradient of f at x_{k+1} and β_k is a parameter known as the *conjugate gradient coefficient*.

The step length α_k is very important for global convergence of conjugate gradient methods. Usually, two major inexact line searches are the standard *Wolfe line search*,

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k H_k^T d_k, \tag{1.4}$$

$$H_{k+1}^T d_k \geq \sigma H_k^T d_k, \tag{1.5}$$

where $0 < \delta < \sigma < 1$, and the *strong Wolfe conditions line search* consists of (1.4) and

$$|H_{k+1}^T d_k| \leq -\sigma H_k^T d_k. \tag{1.6}$$

For general nonlinear functions, however, different formulas for the scalar parameter β_k lead to distinct nonlinear conjugate gradient methods. These include the Fletcher–Reeves (FR) method [16], the Dai–Yuan (DY) method [9], the Conjugate Descent (CD) method proposed by Fletcher [15], the Polak–Ribière–Polyak (PRP) method [30, 31], the Hestenes–Stiefel (HS) method [18], and the Liu Storey (LS) method [16]:

$$\beta_k^{\mathrm{FR}} = \frac{\|H_{k+1}\|^2}{\|H_k\|^2}, \quad \beta_k^{\mathrm{DY}} = \frac{\|H_{k+1}\|^2}{y_k^T d_k}, \quad \beta_k^{\mathrm{CD}} = \frac{\|H_{k+1}\|^2}{-H_k^T d_k}, \tag{1.7}$$

$$\beta_k^{\mathrm{PRP}} = \frac{H_{k+1}^T y_k}{\|H_k\|^2}, \quad \beta_k^{\mathrm{HS}} = \frac{H_{k+1}^T y_k}{y_k^T d_k}, \quad \beta_k^{\mathrm{LS}} = \frac{H_{k+1}^T y_k}{-H_k^T d_k}, \tag{1.8}$$

where $y_k = H_{k+1} - H_k$ and $\|\cdot\|$ the Euclidean norm.

The conjugate gradient methods with the choice of β_k taken in (1.7) have strong convergence properties, but they may not perform well in practice due to jamming; see [3, 4]. In contrast, methods based on (1.8) generally lack convergence guarantees but are often characterized by greater numerical efficiency. Recent research has focused on developing new CG parameters based on these six fundamental formulas and their hybrids in order to improve computational efficiency and performance. For example, Yao et al. [33] introduced a nonnegative formula called the VHS coefficient, which is defined as follows:

$$\beta_k^{\mathrm{VHS}} = \frac{H_{k+1}^T \left(H_{k+1} - \frac{\|H_{k+1}\|}{\|H_k\|} H_k\right)}{y_k^T d_k}.$$

Yao et al. [33] established that the VHS method produces a sufficient descent direction and is globally convergent when the strong Wolfe conditions (1.4) and (1.6) are satisfied with $\sigma < 1/3$.

Zhang [34] subsequently introduced an enhanced version of the VHS method, called the NHS method, where the parameter β_k is given by

$$\beta_k^{\mathrm{NHS}} = \frac{\|H_{k+1}\| - \frac{\|H_{k+1}\|}{\|H_k\|}|H_{k+1}^T H_k|}{d_k^T y_k}.$$

Zhang [34] established that the NHS method satisfies the sufficient descent condition and is globally convergent under the strong Wolfe conditions (1.4) and (1.6) with $\sigma < 1/2$.

Du et al. [14] proposed a new conjugate gradient method, termed the NVHS* method. In this method, the parameter β_k is defined as

$$\beta_k^{\mathrm{NVHS}^*} = \frac{H_{k+1}^T\big(H_{k+1} - \frac{|H_{k+1}^T H_k|}{\|H_k\|^2} H_k\big)}{d_k^T y_k}.$$

The NVHS* method has sufficient descent condition, and is globally convergent if the strong Wolfe line search is used and $\sigma < 1/3$.

Recently, Touati–Ahmed and Storey [32] introduced a modified conjugate gradient method with the parameter

$$\beta_k^{\mathrm{TaS}} = \min\{\beta_k^{\mathrm{FR}},\ \beta_k^{\mathrm{PRP}}\}.$$

The authors proved that β_k^{TaS} has good convergence properties and numerically outperforms both the β_k^{FR} and β_k^{PRP} algorithms. Soon afterward, Hu and Storey [19] and Gilbert and Nocedal [17] further studied additional hybrid methods. Dai and Yuan [10] combined the DY method with the HS method, proposing the following two-hybrid methods:

$$\beta_k^{\mathrm{hDYz}} = \max\{0, \min\{\beta_k^{\mathrm{HS}}, \beta_k^{\mathrm{DY}}\}\},$$
$$\beta_k^{\mathrm{hDY}} = \max\{-c\beta_k^{\mathrm{DY}}, \min\{\beta_k^{\mathrm{HS}}, \beta_k^{\mathrm{DY}}\}\},$$

where $c = \frac{1-\sigma}{1+\sigma}$. For the standard Wolfe conditions (1.4) and (1.5), under the Lipschitz continuity of the gradient, Dai and Yuan [10] established the global convergence of these hybrid computational schemes. Also, Zhou et al. [35] further combined the LS and CD approaches, introducing the following formulation:

$$\beta_k^{H3} = \max\{0, \min\{\beta_k^{\mathrm{LS}}, \beta_k^{\mathrm{CD}}\}\}.$$

Its global convergence under the strong Wolfe line search was proved by Zhou et al. [35]. The second class of hybrid conjugate gradient methods is based on convex combination of the standard methods. Recently, Andria [2] introduced new hybrid conjugate gradient method based on the HS and DY method (called the HYBRID method) for solving unconstrained optimization problems (1.1), calculating the parameter β_k^c as a convex combination of β_k^{HS} and β_k^{DY}:

$$\beta_k^c = (1-\theta_k)\beta_k^{\mathrm{HS}} + \theta_k\beta_k^{\mathrm{DY}},$$

where θ_k is a scalar parameter satisfying $0 \leq \theta_k \leq 1$. Convergence with the standard Wolfe condition was established and numerical results show that this hybrid computational scheme outperforms the Hestenes–Stiefel and the Dai–Yuan conjugate gradient algorithms. Also, Djordjević [12] proposed the hybridization of LS and CD by their convex combination, such that

$$\beta_k^{\mathrm{HLSCD}} = (1-\theta_k)\beta_k^{\mathrm{LS}} + \theta_k \beta_k^{\mathrm{CD}}.$$

The parameter θ_k in β_k^{HLSCD} is selected such that the conjugacy condition holds. The global convergence of this method is proved under the strong Wolfe line search without the convexity assumption on the objective function.

This paper presents a novel hybrid conjugate gradient method. Under the strong Wolfe line search (SWLS) conditions, we establish the convergence properties of the proposed approach. Numerical experiments demonstrate that our method is efficient, robust, and outperforms six well-known conjugate gradient algorithms. The remainder of this work is organized as follows: Section 2 introduces the modified algorithm, determines the parameter θ_k, and proves the sufficient descent condition. Section 3 establishes the global convergence of the proposed method under SWLS conditions. Numerical results are presented in Section 4, followed by a summary of our contributions in Section 5.

2. Convex combination method. In this section, we develop a new hybrid method by combining the NVHS* and CD approaches. The parameter β_k in the presented method is given by

$$\beta_k^{\mathrm{New}} = (1-\theta_k)\beta_k^{\mathrm{NVHS}^*} + \theta_k \beta_k^{\mathrm{CD}}, \tag{2.1}$$

where θ_k is a scalar parameter satisfying $0 \leq \theta_k \leq 1$. The search direction d_k of our algorithm is computed by

$$d_0 = -H_0, \qquad d_{k+1} = -H_{k+1} + \beta_k^{\mathrm{New}} d_k. \tag{2.2}$$

The traditional conjugacy condition $d_{k+1}^T y_k = 0$, plays an important role in the convergence analyses and numerical calculation.

The coefficients θ_k are chosen to satisfy the conjugacy condition $d_{k+1}^T y_k = 0$. Indeed, multiplying both sides of relation (2.2) by y_k^T, we obtain

$$\theta_k = \frac{(y_k^T H_{k+1} - \beta_k^{\mathrm{NVHS}^*} y_k^T d_k)}{(\beta_k^{\mathrm{CD}} - \beta_k^{\mathrm{NVHS}^*}) y_k^T d_k}.$$

After simplification, we find

$$\theta_k = \frac{\varepsilon - \eta}{\mu - \eta}.$$

We define

$$\varepsilon = \|H_k\|^2 y_k^T H_{k+1}, \qquad \mu = \frac{\|H_{k+1}\|^2 \|H_k\|^2 y_k^T d_k}{-H_k^T d_k},$$
$$\eta = \|H_{k+1}\|^2 \|H_k\|^2 - |H_{k+1}^T H_k| H_{k+1}^T H_k,$$

then

$$\theta_k = \begin{cases} 0 & \text{if } \dfrac{\varepsilon-\eta}{\mu-\eta} \leq 0 \text{ or } \mu-\eta = 0, \\ 1 & \text{if } \dfrac{\varepsilon-\eta}{\mu-\eta} \geq 1, \\ \dfrac{\varepsilon-\eta}{\mu-\eta} & \text{if } 0 < \dfrac{\varepsilon-\eta}{\mu-\eta} < 1. \end{cases} \tag{2.3}$$

2.1. Algorithm. The proposed algorithm is as follows:

STEP 1: *Initialization.* Choose an initial point $x_0 \in \mathbb{R}^n$ and the parameters $0 < \delta < \sigma < 1/3$. Compute $f(x_0)$ and H_0. Set $d_0 = -H_0$.

STEP 2: *Test for continuation of iterations.* If $\|H_k\|_\infty \leq 10^{-6}$, then stop. Otherwise, go to the next step.

STEP 3: *Line search.* Calculate a step size α_k satisfying the strong Wolfe line search conditions (1.4) and (1.6) and update the variables:

$$x_{k+1} = x_k + \alpha_k d_k.$$

STEP 4: *Compute θ_k using* (2.3).

STEP 5: *Compute β_k.* If $0 < \theta_k < 1$, then calculate β_k^{New} as in (2.1). If $\theta_k \geq 1$, then calculate $\beta_k^{\mathrm{New}} = \beta_k^{\mathrm{CD}}$. If $\theta_k \leq 0$, then calculate $\beta_k^{\mathrm{New}} = \beta_k^{\mathrm{NVHS}^*}$.

STEP 6: *Compute the search direction.* Generate $d_{k+1} = -H_{k+1} + \beta_k^{\mathrm{New}} d_k$.

STEP 7: Set $k = k+1$ and go to Step 2.

2.2. Descent condition. Now, we prove that the search direction d_k obtained by the new hybrid conjugate gradient method satisfies the sufficient descent condition.

THEOREM 2.1. *Let the sequences $\{d_k\}_{k\geqslant 0}$ and $\{H_k\}_{k\geqslant 0}$ be generated by the New method. Then the search direction d_k satisfies the sufficient descent direction, for all k,*

$$H_k^T d_k \leq -c\|H_k\|^2, \qquad \forall k \geqslant 0. \tag{2.4}$$

Proof. The following proof is by induction. For $k = 0$, $H_0^T d_0 = -\|H_0\|^2$, we conclude that the sufficient descent condition holds for $k = 0$. Now, we assume (2.4) holds for k and prove that for $k+1$. From (2.1) and (2.2), we get

$$d_{k+1} = -H_{k+1} + ((1-\theta_k)\beta_k^{\mathrm{NVHS}^*} + \theta_k \beta_k^{\mathrm{CD}}) d_k.$$

After a simple calculation, we have

$$d_{k+1} = \theta_k d_{k+1}^{\mathrm{CD}} + (1-\theta_k) d_k^{\mathrm{NVHS}^*}. \tag{2.5}$$

Firstly, let $\theta_k = 0$. Then

$$d_{k+1} = d_k^{\mathrm{NVHS}^*} = -H_{k+1} + \beta_k^{\mathrm{NVHS}^*} d_k. \tag{2.6}$$

Multiplying (2.6) by H_{k+1}^T on the left, we get

$$H_{k+1}^T d_k^{NVHS^*} = -\|H_{k+1}\|^2 + \frac{H_{k+1}^T \big(H_{k+1} - \frac{|H_{k+1}^T H_k|}{\|H_k\|} H_k\big)}{d_k^T y_k} H_{k+1}^T d_k.$$

We see from the definition of $\beta_k^{\mathrm{NVHS}^*}$ and the Cauchy–Schwarz inequality that

$$0 \le \beta_k^{\mathrm{NVHS}^*} \le 2\beta_k^{\mathrm{DY}}. \tag{2.7}$$

Thus from (1.6), we have

$$d_k^T y_k \ge (1-\sigma)(-H_k^T d_k). \tag{2.8}$$

Using (1.6), (2.7), (2.8) and the second strong Wolfe line search condition, we get

$$H_{k+1}^T d_{k+1}^{\mathrm{NVHS}^*} \le -c_1 \|H_{k+1}\|^2, \tag{2.9}$$

with $c_1 = \frac{1-3\sigma}{1-\sigma}$, so there is a constant $c_1 > 0$ with $0 < \sigma < 1/3$.

Now let $\theta_k = 1$. Then

$$d_{k+1} = d_{k+1}^{\mathrm{CD}} = -H_{k+1} + \beta_k^{\mathrm{CD}} d_k. \tag{2.10}$$

Multiplying (2.10) by H_{k+1}^T on the left, we get

$$H_{k+1}^T d_{k+1}^{\mathrm{CD}} = -\|H_{k+1}\|^2 + \frac{\|H_{k+1}\|^2}{-d_k^T H_k} H_{k+1}^T d_k.$$

Applying the strong Wolfe line search condition, we get

$$H_{k+1}^T d_{k+1}^{\mathrm{CD}} \le -\|H_{k+1}\|^2 (1-\sigma);$$

we denote $c_2 = 1-\sigma$, so

$$H_{k+1}^T d_{k+1}^{\mathrm{CD}} \le -c_2 \|H_{k+1}\|^2. \tag{2.11}$$

Finally, we assume that $0 < \theta_k < 1$, i.e. there are two constants a_1 and a_2 such that $0 < a_1 \le \theta_k \le a_2 < 1$. From the relations (2.5), (2.9) and (2.11), we get

$$H_{k+1}^T d_{k+1} \le -c \|H_{k+1}\|^2$$

with $c = a_1 c_1 + (1-a_2) c_2$. ∎

3. Global convergence. To establish the global convergence of our method, we need the following basic assumptions on the objective function.

ASSUMPTION 3.1. The level set

$$S = \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$$

is bounded.

ASSUMPTION 3.2. In some open convex neighborhood $\mathcal{N}$ of S, the function f is continuously differentiable and its gradient is Lipschitz continuous, namely, there exists a constant $L > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in \mathcal{N}. \tag{3.1}$$

From Assumption 3.2, we can deduce that for all $x \in \mathcal{N}$, there exists a constant $\varGamma \geq 0$ such that

$$\|\nabla f(x)\| \leq \varGamma \quad \text{for all } x \in \mathcal{N}. \tag{3.2}$$

Dai et al. [8] established the sufficient condition for the convergence of conjugate gradient methods under the strong Wolfe line search.

LEMMA 3.1. *Let Assumptions* 3.1 *and* 3.2 *hold. Consider the method* (1.2) *and* (1.3), *d_k is a descent direction, and α_k is obtained by the strong Wolfe line search. If*

$$\sum_{k\geq 0} \frac{1}{\|d_k\|^2} = \infty, \tag{3.3}$$

then

$$\liminf_{k\to\infty} \|H_k\| = 0.$$

Now, we also need this lemma to prove the convergence of our method.

LEMMA 3.2. *Let Assumptions* 3.1 *and* 3.2 *hold and take $\{x_k\}_{k\geq 0}$ to be the sequence produced by the new method with step sizes α_k satisfying* (1.4) *and* (1.6). *Then*

$$\alpha_k \geq \frac{(1-\sigma)|H_k^T d_k|}{L\|d_k\|^2}. \tag{3.4}$$

Proof. From (1.6), we have

$$(H_{k+1}^T d_k - H_k^T d_k) \geq (\sigma - 1)H_k^T d_k.$$

Using the Cauchy–Schwarz inequality and (2.1), we obtain

$$(\sigma - 1)H_k^T d_k \leq (H_{k+1} - H_k)^T d_k \leq L\alpha_k\|d_k\|^2.$$

By combining these two inequalities, the result can be achieved. ■

This indicates that α_k obtained by the new method is not equal to zero, hence there exists a constant $\lambda > 0$ such that

$$\alpha_k \geq \lambda \quad \text{for all } k \geq 0. \tag{3.5}$$

The following theorem establishes global convergence of the proposed method under the strong Wolfe line search conditions.

THEOREM 3.1. *Suppose that Assumptions* 3.1 *and* 3.2 *hold. Consider any conjugate gradient method in the form* (1.2) *and* (1.3), *with* β_k *defined by* (2.1), *in which the step length* α_k *is determined to satisfy the strong Wolfe conditions* (1.4) *and* (1.6). *If the search directions satisfy the decent condition* (2.4), *then this method converges in the sense that*

$$\liminf_{k\to\infty} \|H_k\| = 0. \tag{3.6}$$

Proof. We argue by contradiction and assume that there exists a positive constant γ such that

$$\|H_k\| \geq \gamma \quad \text{for all } k. \tag{3.7}$$

From (2.1) and (2.7), we have

$$|\beta_k^{\mathrm{New}}| \leq 2\beta_k^{\mathrm{DY}} + |\beta_k^{\mathrm{CD}}|. \tag{3.8}$$

By the definition of β_k^{DY} and using (2.8), (3.2), (3.7), we get

$$\beta_k^{\mathrm{DY}} \leq \frac{\Gamma^2}{c_1(1-\sigma)\gamma^2}. \tag{3.9}$$

By the definition of β_k^{CD} and using (3.2),

$$|\beta_k^{\mathrm{CD}}| \leq \frac{\Gamma^2}{|-H_k^T d_k|}.$$

Since the sufficient descent condition holds for CD method and (3.7), we can get

$$|\beta_k^{\mathrm{CD}}| \leq \frac{\Gamma^2}{c_2\gamma^2}. \tag{3.10}$$

From (3.8)–(3.10), we have

$$|\beta_k^{\mathrm{New}}| \leq \frac{2\Gamma^2}{c_1(1-\sigma)\gamma^2} + \frac{\Gamma^2}{c_2\gamma^2} = E. \tag{3.11}$$

Thus, it follows from (2.2), (3.2), (3.5) and (3.11) that

$$\|d_{k+1}\| \leq \|H_{k+1}\| + |\beta_k^{\mathrm{New}}|\frac{\|x_{k+1}-x_k\|}{\alpha_k} \leq M, \tag{3.12}$$

with

$$M = \Gamma + E\frac{D}{\upsilon}, \qquad D = \max\{\|y-z\| : y, z \in S\}.$$

Summing for $k \geq 0$ we get

$$\sum_{k\geq 0} \frac{1}{\|d_k\|^2} = \infty.$$

So, applying Lemma 3.1, we conclude that (3.6) is true. This contradicts (3.7), so we have proved (3.6). ■

4. Numerical experiments. In this section, we present some numerical experiments obtained with the new proposed conjugate gradient method with the hybridization parameter β_k given by (2.1). The test problems are selected from [1, 5]. All the algorithms have been coded in MATLAB 2013 and compiler settings on the PC machine (2.5 GHz, 3.8 GB RAM) with Windows XP operating system. We compare the computational results of our method against the CD [16], NVHS* [14], hDYz [10], H3 [35], HYBRID [2] and HLSCD [12]. In this numerical result, all algorithms implement the SWLS condition with $\delta = 10^{-3}$ and $\sigma = 10^{-1}$. The iteration is terminated if one of the following conditions is satisfied: (i) $\|H_k\|_\infty < 10^{-6}$, where $\|\cdot\|_\infty$ is the maximum absolute component of a vector, (ii) the number of iterations exceeded 2000, (iii) the computing time is more than 500 s. We show the performance difference clearly between our method and six conjugate gradient algorithms. We choose the performance profile introduced by Dolan and Moré [13] to compare the performance with respect to the number of iterations and CPU time to rule as follows. Let S be the set of methods and P the set of test problems with n_p, n_s the number of test problems and the number of methods, respectively. For each problem $p \in P$ and a solver $s \in S$, let $\tau_{p,s}$ be the number of iterations or CPU time required to solve problem $p \in P$ by the solver $s \in S$. Then comparison between different solvers based on the performance ratio is given by

$$r_{p,s} = \frac{\tau_{p,s}}{\min\{\tau_{p,i},\ 1 \le i \le n_s\}}.$$

Suppose that a parameter $r_M \ge r_{p,s}$ for all problems and solvers is chosen, and $r_M = r_{p,s}$ if and only if the solver s does not solve problem p. The overall evaluation of the performance of the solvers is then given by the performance profile function given by

$$F_s(t) = \frac{\text{size}\{p : 1 \le p \le n_p,\ r_{p,s} \le t\}}{n_p},$$

where $t \ge 1$ and “size” is the number of elements. The function $F_s : [1, \infty[\to [0, 1]$ is the distribution function for the performance ratio. The value of $F_s(1)$ is the probability that the solver outperforms the others.

Figures 1 and 2 give a performance comparison of the new method with those for the CPU time and the number of iterations, respectively. From these figures, we can conclude that the new method is competitive and is superior to the CD, NVHS*, hDYz, H3, HYBRID, and HLSCD methods for the given test problems. These numerical findings are indeed encouraging.

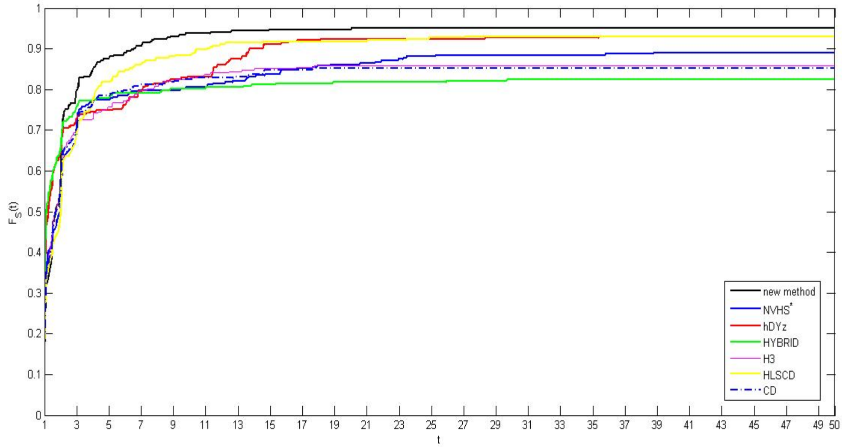

Fig. 1. Performance profile on the CPU time

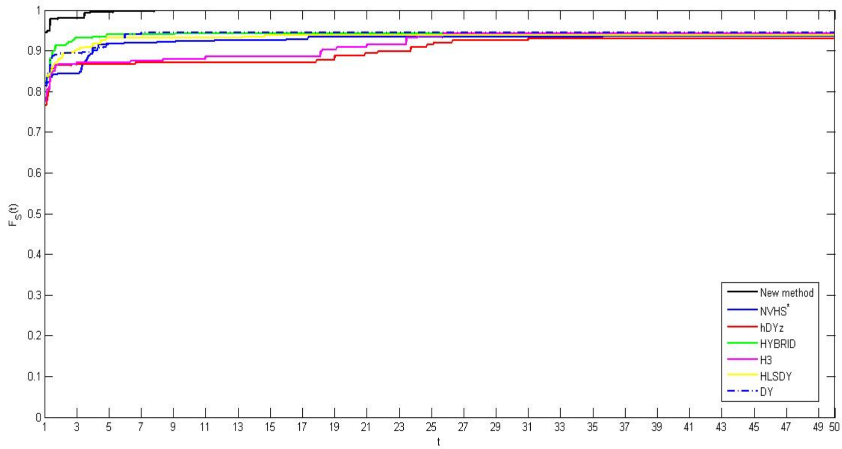

Fig. 2. Performance profile on the number of iterations

Conclusion. We have presented a new hybrid conjugate gradient algorithm by using the convex combination of NVHS* and CD methods. The global convergence properties and the sufficient descent condition of the proposed method have been established, under the strong Wolfe line search conditions. Numerical results also show that our method is very robust and effective.

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Abd Elhamid Mehamdia (corresponding author)
Laboratory Informatics and Mathematics (LIM)
Boumerdès (MBUB)
Muhamed Bougara University
35000 Boumerdès, Algeria
E-mail: a.mehamdia@univ-boumerdes.dz

Yacine Chaib, Tahar Bechouat
Laboratory Informatics and Mathematics (LIM)
Mohamed Cherif Messaadia University
41000 Souk Ahras, Algeria

Ziadi Raouf
Laboratory of Fundamental and Numerical Mathematics (LMFN)
University Setif-1-Ferhat Abbas
16000 Setif, Algeria