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Dirichlet's Lemma in number fields

by

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Dedicated to the memory of Peter Roquette (1927–2023)

Abstract. Dirichlet's Lemma states that every primitive quadratic Dirichlet character χ can be written in the form $\chi(n) = \left(\frac{\Delta}{n}\right)$ for a suitable quadratic discriminant Δ. In this article we define a group, the separant class group $\mathrm{SCl}(F)$, that measures the extent to which Dirichlet's Lemma fails in general number fields F.

In this article we introduce the separant class group $\mathrm{SCl}(F)$ of a number field F as a finite 2-group measuring the obstruction to Dirichlet characters being induced by Kronecker symbols. We compute its order and show that the separant class group is trivial if and only if F is totally real and has odd class number in the strict sense. In fields with trivial separant class group, there is an analogue of unique factorization into prime discriminants, and genus theory can be made as explicit as over the rationals.

1. Genus theory of quadratic number fields. We begin by reviewing some classical results concerning quadratic number fields, the factorization of discriminants into prime discriminants, and connections with genus theory.

Let Δ denote the discriminant of a quadratic number field K. Quadratic discriminants (together with the discriminant 1 of the trivial extension $\mathbb{Q}/\mathbb{Q}$) form a group $\mathrm{Disc}(\mathbb{Q})$ with respect to the following multiplication: if Δ_1 and Δ_2 are two quadratic discriminants, then $\Delta_1 * \Delta_2 = \Delta_3$, where Δ_3 is the discriminant of the extension $\mathbb{Q}(\sqrt{\Delta_1\Delta_2})$. The neutral element is the discriminant 1 of the trivial extension $\mathbb{Q}/\mathbb{Q}$. Note that $\Delta * \Delta = 1$ (the group $\mathrm{Disc}(\mathbb{Q})$ is an elementary abelian 2-group) and that, e.g., $(-8) * (-4) = 8$. If Δ_1 and Δ_2 are coprime, then clearly $\Delta_1 * \Delta_2 = \Delta_1\Delta_2$.

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Prime discriminants are discriminants having only one prime factor, namely

- the discriminants -4, ± 8 of $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{\pm 2})$;
- the discriminant p of $\mathbb{Q}(\sqrt{p})$ for primes $p \equiv 1 \bmod 4$;
- the discriminant $-q$ of $\mathbb{Q}(\sqrt{-q})$ for primes $q \equiv 3 \bmod 4$.

It is well known and easy to show that every discriminant Δ can be written uniquely as a product of prime discriminants:

$$\Delta = \Delta_1 \cdots \Delta_r \Delta_{r+1} \cdots \Delta_t,$$

where we assume that $\Delta_j < 0$ for $j \le r$ and $\Delta_j > 0$ for $j > r$. In this case,

$$K_{\text{gen}}^+ = \mathbb{Q}(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_t})$$

is the genus class field of K in the strict sense, that is, the maximal abelian extension (necessarily an elementary abelian 2-extension) $L/\mathbb{Q}$ such that L/K is unramified at all finite places. The ordinary genus field K_{gen} of K, i.e., the maximal unramified extension of K that is abelian over $\mathbb{Q}$, is then given by

$$K_{\text{gen}} = \mathbb{Q}(\sqrt{\Delta_1\Delta_2}, \ldots, \sqrt{\Delta_1\Delta_r}, \sqrt{\Delta_{r+1}}, \ldots, \sqrt{\Delta_t})$$

if $\Delta > 0$ (for negative discriminants, $K_{\text{gen}} = K_{\text{gen}}^+$). Observe that the genus field K_{gen} in the usual sense is the maximal real subfield of K_{gen}^+.

Example 1. The quadratic number field $K = \mathbb{Q}(\sqrt{210})$ has discriminant $\Delta = 840$, and the factorization of Δ into prime discriminants is

$$\Delta = 8 \cdot (-3) \cdot 5 \cdot (-7).$$

Thus the genus field of K in the strict sense and in the usual sense is given respectively by

$$K_{\text{gen}}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{-3}, \sqrt{5}, \sqrt{-7}) \quad \text{and} \quad K_{\text{gen}} = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{21}).$$

A quadratic number field has odd class number (in the usual sense) if and only if $K_{\text{gen}} = K$. This condition is obviously necessary; for sufficiency observe that if K has even class number, then its Hilbert 2-class field can be constructed in central steps, i.e., there exists a central unramified quadratic extension of K, which necessarily is elementary abelian over $\mathbb{Q}$ and thus is contained in the genus class field of K (for details, see [5]). The explicit description of the genus field then immediately implies the following classification of quadratic number fields with odd class number:

Theorem 1.1. *The quadratic extensions $\mathbb{Q}(\sqrt{m})$ with odd class number are*

- *the complex quadratic number fields* $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, *and* $\mathbb{Q}(\sqrt{-q})$ *for primes* $q \equiv 3 \bmod 4$*;*
- *the real quadratic number fields* $\mathbb{Q}(\sqrt{p})$ *for arbitrary primes* p*, and the fields* $\mathbb{Q}(\sqrt{pq})$ *for primes* p *and* q*, where* $p = 2$ *or* $p \equiv 3 \bmod 4$ *and* $q \equiv 3 \bmod 4$*.*

Genus characters. Given an ideal class $c \in \mathrm{Cl}^+(K)$, choose an ideal $\mathfrak{a} \in c$ coprime to Δ. Then $\chi_j(c) = \left(\frac{\Delta_j}{N\mathfrak{a}}\right)$ defines a quadratic character on the narrow class group $\mathrm{Cl}^+(K)$ called a *genus character.* The main result concerning genus characters is the following.

THEOREM 1.2 (Principal Genus Theorem). *An ideal class* $c \in \mathrm{Cl}^+(K)$ *is a square if and only if* $\chi_j(c) = 1$ *for* $j = 1, \ldots, t$*.*

In fact, the homomorphism $\mathrm{Cl}^+(K) \to \mu_2^t$ (where $\mu_2 = \{-1, +1\}$) induced by $X = (\chi_1, \ldots, \chi_t)$ gives rise to an exact sequence

$$1 \to \mathrm{Cl}^+(K)^2 \to \mathrm{Cl}^+(K) \xrightarrow{X} \mu_2^t \to \mu_2 \to 1.$$

Work by Goldstein [6] and Sunley [14, 15] (see also Davis [3]) shows that, for a base field F, unique factorization into prime discriminants holds if and only if F is totally real and has class number 1 in the strict sense. Replacing discriminants by separants we will be able to generalize these results to fields with trivial separant class group, i.e., to totally real fields with odd class number in the strict sense (see Theorem 6.1). The notion of separants goes back to the author's dissertation [10], where it was used to generalize Rédei's construction of unramified cyclic quartic extensions.

The construction of cyclic quartic unramified extensions of $\mathbb{Q}(\sqrt{\Delta})$ is also based on the factorization of the discriminant Δ into prime discriminants:

THEOREM 1.3. *Let* Δ *be the discriminant of a quadratic number field* K*. Then the following assertions are equivalent:*

(1) *K admits a cyclic quartic extension L/K containing $\mathbb{Q}(\sqrt{\Delta_1}, \sqrt{\Delta_2})$ and unramified at all finite primes.*
(2) *There exists a factorization $\Delta = \Delta_1\Delta_2$ into coprime discriminants Δ_1 and Δ_2, and the primes dividing Δ_1 split completely in $\mathbb{Q}(\sqrt{\Delta_2})$ and vice versa.*

Our results will allow us to generalize this construction to totally real base fields F with odd class number in the strict sense.

2. Dirichlet's Lemma in $\mathbb{Q}$. Let us now turn to Kronecker symbols $\left(\frac{\Delta}{p}\right)$, which describe the splitting of primes in the quadratic number field with discriminant Δ, and quadratic Dirichlet characters defined modulo m. Given a quadratic extension $K/\mathbb{Q}$ with discriminant Δ, we define the *Kronecker*

symbol for primes p by setting

$$\left(\frac{\Delta}{p}\right) = \begin{cases} +1 & \text{if } p \text{ splits in } K/\mathbb{Q}, \\ 0 & \text{if } p \text{ ramifies in } K/\mathbb{Q}, \\ -1 & \text{if } p \text{ is inert in } K/\mathbb{Q}. \end{cases}$$

Observe that the Kronecker symbol only depends on the ideal generated by p in the sense that $\left(\frac{\Delta}{p}\right) = \left(\frac{\Delta}{-p}\right)$. We then extend the Kronecker symbol multiplicatively to all nonzero integers.

Quadratic Dirichlet characters, on the other hand, are homomorphisms from $(\mathbb{Z}/m\mathbb{Z})^\times$ to the (multiplicatively written) group $\mu_2 \simeq \mathbb{Z}/2\mathbb{Z}$. We identify Dirichlet characters modulo m and modulo n if they agree on all integers coprime to mn.

A quadratic Dirichlet character defined modulo m is called *primitive* if it cannot be defined for any proper divisor of m. The three nontrivial quadratic Dirichlet characters defined modulo 8 are

	1	3	5	7
χ_8	$+1$	-1	-1	$+1$
χ_{-8}	$+1$	$+1$	-1	-1
χ_{-4}	$+1$	-1	$+1$	-1

Their values at odd positive integers agree with $\left(\frac{2}{p}\right)$, $\left(\frac{-2}{p}\right)$ and $\left(\frac{-1}{p}\right)$, respectively. The characters χ_8 and χ_{-8} are primitive, the character χ_{-4} is not as it is already defined modulo 4.

The smallest defining modulus of a character is called its *conductor*; for Kronecker characters $\kappa_a = \left(\frac{a}{\cdot}\right)$, the conductor $N = |\Delta|$ agrees with the absolute value of the discriminant Δ of the quadratic number field $\mathbb{Q}(\sqrt{a})$.

The fact that Kronecker symbols $\left(\frac{\Delta}{p}\right)$ for primes $p > 0$ only depend on the residue class of p modulo Δ is essentially equivalent to the quadratic reciprocity law. This observation assigns a primitive quadratic Dirichlet character to each Kronecker symbol, and Dirichlet's Lemma states that every primitive quadratic Dirichlet character arises in this way (see below for a precise statement).

The same construction works for any algebraic number field, but it turns out that, in general, not every primitive quadratic Dirichlet character comes from a Kronecker symbol. For measuring this defect we introduce the separant class group $\mathrm{SCl}(F)$ of a number field F, and we will determine its structure.

We start by recalling Dirichlet's Lemma ([1]), which goes back to Dirichlet's proof that there are infinitely many primes in arithmetic progressions. In this proof, Dirichlet invented the characters that now bear his name (strictly

([1]) The name "Dirichlet's Lemma" was coined by Cohn [2].

speaking Dirichlet provided the tools, and Dedekind came up with the abstract concept of a character when he edited Dirichlet's lectures). Dirichlet characters are group homomorphisms $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$. Dirichlet's main problem was showing that L-series

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

associated to Dirichlet characters χ have the property

$$\lim_{s\to 1} L(s,\chi) \neq 0 \tag{1}$$

for all characters χ different from the principal character. This was not difficult except for quadratic characters χ, i.e., characters that attain only the values $+1$ and -1. Eventually, Dirichlet realized that the limit in (1) for quadratic characters χ could be evaluated directly by showing that every primitive quadratic Dirichlet character χ has the form $\chi(n) = \left(\frac{\Delta}{n}\right)$ for some quadratic discriminant Δ, which allowed him to show that the limit in question is a product of nonzero constants and the class number and the regulator of the quadratic number field $\mathbb{Q}(\sqrt{\Delta})$. Again this is only true cum grano salis, as Dirichlet worked with quadratic forms instead of fields.

A Kronecker character $\kappa = \left(\frac{\Delta}{n}\right)$ is called *modular* if there exists a modulus $N > 0$ such that $\kappa(n) = \kappa(n+N)$ for all natural numbers n. It is easy to see that if κ is modular with respect to N_1 and N_2, then it is modular with respect to $\gcd(N_1, N_2)$.

Similarly, a quadratic number field K is called *modular* if there exists an integer $N > 0$ such that K is contained in the field $\mathbb{Q}(\zeta_N)$ of N-th roots of unity. Again it is easy to see that if K is modular with respect to N_1 and N_2, then it is modular with respect to $\gcd(N_1, N_2)$. The minimal modulus for K is called the *conductor* of K.

The similarity between these two notions of modularity is not superficial: the splitting of primes p in quadratic number fields K with discriminant Δ depends on the Kronecker character $\kappa(p) = \left(\frac{\Delta}{p}\right)$, that in cyclotomic fields $\mathbb{Q}(\zeta_N)$ on the values $\chi(p)$ for certain Dirichlet characters χ defined modulo N. The modularity of K can be shown to imply the modularity of κ, and in fact the conductor of the Kronecker character $\left(\frac{\Delta}{\cdot}\right)$ is equal to the conductor $N = |\Delta|$ of the corresponding quadratic number field K.

Infinite Kronecker characters. The Kronecker symbol $\kappa_4(n) = \left(\frac{-4}{n}\right)$ is a Dirichlet character defined modulo 4 on positive integers n since

$$\kappa_4(n) = \begin{cases} +1 & \text{if } n \equiv 1 \bmod 4, \\ 0 & \text{if } n \equiv 0 \bmod 2, \\ -1 & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

If we think of the Kronecker symbol $\kappa_4(n)$ as a function of the ideal (n) generated by n, then we have to set $\kappa_4(-n) = \kappa_4(n)$. But then $1 \equiv -3 \bmod 4$, yet we have $\kappa_4(1) = 1$ and $\kappa_4(-3) = \kappa_4(3) = -1$. To get around this problem we either consider our characters only on positive integers (and, over number fields, on totally positive algebraic integers), or introduce an infinite Kronecker symbol

$$\kappa_\infty(n) = \left(\frac{\infty}{n}\right) = \begin{cases} +1 & \text{if } n > 0, \\ -1 & \text{if } n < 0. \end{cases}$$

Then $\kappa_4(n)\kappa_\infty(n) = \chi(n)$ for all nonzero integers n, where χ is the nontrivial Dirichlet character modulo 4.

Dirichlet showed that every quadratic Dirichlet character is a Kronecker character; in fact, we even have

Dirichlet's Lemma. *There is a bijection between primitive quadratic Dirichlet characters modulo N and Kronecker characters with conductor N.*

The fact that, conversely, Kronecker characters $\left(\frac{\Delta}{\cdot}\right)$ are Dirichlet characters defined modulo $|\Delta|$ is equivalent to Euler's version of the quadratic reciprocity law (see [1], as well as, for a proof of Dirichlet's Lemma, [12]).

3. Dirichlet's Lemma in number fields. We now define the group $\mathrm{Kr}(F)$ of Kronecker characters and the group $\mathrm{Dir}(F)$ of quadratic Dirichlet characters in arbitrary number fields F; then we will show that each Kronecker character induces a Dirichlet character, and define the separant class group as the quotient group $\mathrm{Dir}(F)/\mathrm{Kr}(F)$.

Dirichlet and Kronecker characters. A quadratic Dirichlet character in F defined modulo an ideal $\mathfrak{m}$ in $\mathcal{O}_F$ is a homomorphism from the coprime residue class group $(\mathcal{O}_F/\mathfrak{m})^\times$ to the group $\mu_2 = \{-1, +1\}$. We identify Dirichlet characters that are defined modulo $\mathfrak{m}$ and $\mathfrak{n}$ and agree on elements coprime to $\mathfrak{mn}$.

Each element $\alpha \in F^\times$ defines a Kronecker character $\kappa_\alpha = \left(\frac{\alpha}{\cdot}\right)$. As in the special case $F = \mathbb{Q}$, κ attains the values $+1$, 0 and -1 according as $\mathfrak{p}$ splits, ramifies or is inert in K/F, where $K = F(\sqrt{\alpha})$.

Since we want to compare Kronecker and Dirichlet characters, and since Dirichlet characters are defined on elements of the number field, we identify two Kronecker characters κ_α and κ_β if they have the same values at all elements coprime to α and β; for example, we identify the Kronecker symbols $\left(\frac{-1}{\cdot}\right)$ and $\left(\frac{-4}{\cdot}\right)$ over $\mathbb{Q}$.

A less trivial example is the Kronecker character $\kappa = \left(\frac{-1}{\cdot}\right)$ in $F = \mathbb{Q}(\sqrt{-5})$. We have $\kappa(\mathfrak{p}) = -1$ for the prime ideal $\mathfrak{p} = (3, 1+\sqrt{-5})$ since this ideal is inert in the Hilbert class field $F(i)$ of F. But κ is trivial on principal

ideals by the decomposition theorem of class field theory, and in particular we identify κ with the trivial character. Thus Kronecker characters are the restrictions of Artin symbols of quadratic extensions K/F to principal ideals.

We will evaluate Kronecker characters $\left(\frac{\alpha}{\beta}\right)$ only at integers β coprime to the relative discriminant $\mathrm{disc}(K/F)$, where $K = F(\sqrt{\alpha})$. For example, the Kronecker symbol $\left(\frac{3}{m}\right)$ is defined only for integers coprime to 6.

The Kronecker characters form a group $\mathrm{Kr}(F)$ with respect to the product $\kappa_\alpha\kappa_\beta = \kappa_{\alpha\beta}$. If $\alpha \in F^\times$ is a square, then κ_α is the trivial character. This observation implies that there is a homomorphism

$$\phi : F^\times/F^{\times 2} \to \mathrm{Kr}(F) \tag{2}$$

defined by $\phi(\alpha F^{\times 2}) = \kappa_\alpha$. Different elements in $F^\times/F^{\times 2}$ do not necessarily give rise to different Kronecker symbols; in other words, ϕ is not necessarily injective.

Recall that in [11] we have introduced the following groups:

$$\mathrm{Sel}(F) = \{\alpha \in F^\times : (\alpha) = \mathfrak{a}^2\}/F^{\times 2},$$
$$\mathrm{Sel}_4(F) = \{\alpha \in F^\times : (\alpha, 2) = (1),\ (\alpha) = \mathfrak{a}^2,\ \alpha \equiv \xi^2 \bmod 4\} \cdot F^{\times 2}/F^{\times 2}.$$

Thus $\mathrm{Sel}(F)$ is the group of all nonzero elements of F that generate the square of an ideal (modulo squares of elements), and $\mathrm{Sel}_4(F)$ is its subgroup of elements coprime to (2) that generate the square of an ideal and are congruent to a square modulo 4. If such elements are congruent to a square modulo 4, they are called 2-*primary*; the quadratic extension $F(\sqrt{\alpha})/F$ is unramified at 2 if and only if α is 2-primary.

The next proposition determines the kernel of the homomorphism ϕ defined in (2):

Proposition 3.1. *We have an exact sequence*

$$1 \to \mathrm{Sel}_4(F) \xrightarrow{\iota} F^\times/F^{\times 2} \xrightarrow{\phi} \mathrm{Kr}(F) \to 1.$$

Observe that if $\alpha \in \mathrm{Sel}_4(F)$ is nontrivial, then $F(\sqrt{\alpha})/F$ is a quadratic extension unramified at all the finite primes; in particular, the class number of F in the strict sense is even in this case.

If the class number of F is odd, then $\mathrm{Sel}_4(F) \simeq E_4/E^2$, where $E = \mathcal{O}_F^\times$ is the unit group of the ring of integers in F, and E_4 is the subgroup of units congruent to a square modulo 4.

Example 2 ($F = \mathbb{Q}(\sqrt{6})$). Let $F = \mathbb{Q}(\sqrt{6})$. Here $\varepsilon = 5 + 2\sqrt{6}$ is the fundamental unit. Since $-\varepsilon = -5 - 2\sqrt{6} \equiv (1 - \sqrt{6})^2 \bmod 4$, we have $-\varepsilon \in E_4 \setminus E^2$. Next $-\varepsilon = (\sqrt{-3} + \sqrt{-2})^2$ implies $F(\sqrt{-\varepsilon}) = F(\sqrt{-3})$, and this is a quadratic extension of F ramified only at the infinite primes. This implies $(-\varepsilon/\alpha)$ is trivial if and only if α is totally positive, which in turn implies that the corresponding Kronecker character is trivial on totally positive elements.

EXAMPLE 3 ($F = \mathbb{Q}(\sqrt{34})$). The field $F = \mathbb{Q}(\sqrt{34})$ has class number 2, and here $F(\sqrt{\varepsilon})/F$ for $\varepsilon = 35 + 6\sqrt{34}$ is unramified everywhere; in fact, $2\varepsilon = (6 + \sqrt{34})^2$ shows that $F(\sqrt{\varepsilon}) = F(\sqrt{2})$ is the genus class field in the wide sense.

From now on we will restrict Kronecker symbols to totally positive elements instead of introducing infinite Kronecker characters. This implies, in particular, that the Kronecker symbol κ_α is trivial whenever $F(\sqrt{\alpha})/F$ is unramified at all finite primes.

Proof of Proposition 3.1. We only have to show that $\ker\phi = \operatorname{im}\iota$. Assume that $\kappa \in \ker\phi$; then $\kappa = \left(\frac{\alpha}{\cdot}\right)$ is the trivial Kronecker character. Since the conductor of the Kronecker symbol is the conductor of the quadratic extension $F(\sqrt{\alpha})/F$, this extension is unramified at all finite primes; but this implies that (α) is the square of an ideal and $\alpha = \beta\gamma^2$ for some 2-primary β, i.e., $\alpha F^{\times 2} \in \mathrm{Sel}_4(F)$.

Now assume that, conversely, $\alpha F^{\times 2} \in \mathrm{Sel}_4(F)$. Then $F(\sqrt{\alpha})/F$ is unramified at all finite primes, hence κ_α is trivial (on totally positive elements). ∎

Next we show that Kronecker characters are Dirichlet characters:

PROPOSITION 3.2. *Each Kronecker character $\kappa_\alpha = \left(\frac{\alpha}{\cdot}\right)$ is a primitive quadratic Dirichlet character with conductor* $\mathrm{disc}(K/F)$*, where $K = F(\sqrt{\alpha})$.*

Proof. By Artin's reciprocity law, the splitting behavior of a prime ideal in a quadratic extension $K = F(\sqrt{\alpha})$ only depends on the coset of $\mathfrak{p}$ in the corresponding ideal group, and the conductor of this group is the relative discriminant of K/F. In particular, for any $\beta \gg 0$ (i.e., β is totally positive) in $F^\times$ coprime to 2α we have $\left(\frac{\alpha}{\beta}\right) = \chi(\beta)$ for some Dirichlet character with conductor $\mathrm{disc}(K/F)$. Since the conductor is the minimal defining modulus, χ is primitive. ∎

This result will allow us to show that certain Dirichlet characters are not Kronecker characters: a necessary condition for the existence of a Kronecker character with conductor $\mathfrak{f}$ is the existence of a quadratic extension K/F with relative discriminant $\mathrm{disc}(K/F) = \mathfrak{f}$.

For the convenience of the readers, Table 1 contains a list of the groups that we will use below.

4. Determination of the separant class group. The separant class group $\mathrm{SCl}(F) = \mathrm{Dir}(F)/\mathrm{Kr}(F)$ of a number field F is by definition an elementary abelian 2-group, and so it is sufficient to determine its order, or its dimension as an $\mathbb{F}_2$-vector space. This is accomplished by our main theorem:

THEOREM 4.1. *There is an isomorphism*

$$\mathrm{SCl}(F) \simeq \mathrm{Cl}_F\{4\}/\mathrm{Cl}_F\{4\}^2,$$

Table 1. List of groups

E	$E = \mathcal{O}_F^\times$ unit group
$\mathrm{Sel}(F)$	$\{\alpha \in F^\times : (\alpha) = \mathfrak{a}^2\}/F^{\times\,2}$
$\mathrm{Dir}(F)$	group of Dirichlet characters
$\mathrm{Kr}(F)$	group of Kronecker characters
$\mathrm{SCl}(F)$	separant class group $\mathrm{Dir}(F)/\mathrm{Kr}(F)$
E_4	subgroup of units congruent to a square modulo 4
$\mathrm{Sel}_4(F)$	$\{\alpha \in F^\times : (\alpha, 2) = (1),\ (\alpha) = \mathfrak{a}^2,\ \alpha \equiv \xi^2 \bmod 4\} \cdot F^{\times\,2}/F^{\times\,2}$
$\mathrm{Dir}_4(F)$	Dirichlet characters defined mod 4
$\mathrm{Kr}_4(F)$	Kronecker characters with conductor dividing 4
$\mathrm{SCl}_4(F)$	$\mathrm{Dir}_4(F)/\mathrm{Kr}_4(F)$
M_4	residue classes mod 4: $(\mathcal{O}_F/4\mathcal{O}_F)^\times$
$\mathrm{Cl}_F\{4\}$	ray class group modulo (4) in F

where $\mathrm{Cl}_F\{4\}$ *is the ray class group of* F *defined modulo* (4)*. In particular we have* $\#\mathrm{SCl}(F) = 2^{\rho^+ + s}$*, where* $2^{\rho^+} = \#\,\mathrm{Cl}_2^+(F)/\mathrm{Cl}_2^+(F)^2$ *(so* ρ^+ *denotes the* 2*-rank of the class group of* F *in the strict sense), and* s *the number of pairs of complex embeddings of* F*. Thus* F *has trivial separant class group if and only if* F *is totally real with odd class number in the strict sense.*

We will prove this theorem in two steps: First we discuss the case of characters defined modulo 4, and then the general case.

Dirichlet characters with conductor dividing (4). Our proof uses a few facts concerning Kronecker and Dirichlet characters whose conductors divide (4). Let $\mathrm{Kr}_4(F)$ denote the group of Kronecker characters with conductor dividing (4), $\mathrm{Dir}_4(F)$ the group of Dirichlet characters defined modulo (4), and $M_4 = (\mathcal{O}_F/4\mathcal{O}_F)^\times$ the group of coprime residue classes modulo 4. For the proof of Thm. 4.1 we only need the fact that the homomorphism $\iota_2 : M_4/M_4^2 \to \mathrm{Dir}_4(F)$ is surjective.

Proposition 4.2. *There is an isomorphism* $\iota_2 : M_4/M_4^2 \to \mathrm{Dir}_4(F)$*. Both groups have order* 2^n*, where* $n = (F : \mathbb{Q})$*.*

Proof. Quadratic Dirichlet characters defined modulo 4 are homomorphisms from M_4 to the multiplicatively written group $\mathbb{Z}/2\mathbb{Z}$. Since such homomorphisms are trivial on squares, they induce homomorphisms $M_4/M_4^2 \to \mathbb{Z}/2\mathbb{Z}$. Thus $\mathrm{Dir}_4(F) = \mathrm{Hom}(M_4/M_4^2, \mathbb{Z}/2\mathbb{Z})$, which means that $\mathrm{Dir}_4(F)$ is the dual of M_4/M_4^2 (for example as $\mathbb{Z}/2\mathbb{Z}$-vector spaces); in particular, $\mathrm{Dir}_4(F) \simeq M_4/M_4^2$ as abelian groups.

The isomorphism between M_4/M_4^2 and $\mathrm{Dir}_4(F)$ given above is not canonical, but depends on the choice of a basis. In this case, however, there is also a canonical isomorphism that we will now construct.

Let α represent a coprime residue class modulo 4, and let κ_α denote the corresponding Kronecker character. Write $\kappa_\alpha = \chi_0\chi_1$ as the product of a primitive quadratic Dirichlet character χ_0 defined modulo 4 and a Dirichlet character χ_1 defined modulo an odd ideal. Then set $\iota_2(\alpha M_4^2) = \chi_0$.

The map ι_2 is well defined: If $\alpha \equiv \beta \bmod 4$, $\kappa_\alpha = \chi_0\chi_1$ and $\kappa_\beta = \chi_0'\chi_1'$, then $\kappa_\alpha\kappa_\beta = \kappa_{\alpha\beta}$ is defined modulo an odd ideal since $\alpha/\beta \equiv 1 \bmod 4$. Thus $\chi_0 = \chi_0'$.

We claim that ι_2 is injective. Assume that $\iota_2(\alpha M_4^2) = \iota_2(\beta M_4^2)$. Then $\kappa_{\alpha\beta}$ has conductor coprime to 2, hence $\alpha\beta \equiv \xi^2 \bmod 4$, which implies that α and β generate the same class in M_4/M_4^2.

In [11, p. 281] we have shown that sending $\alpha \bmod 2$ to $1 + 2\alpha \bmod 4$ induces an isomorphism $\mathcal{O}/2\mathcal{O} \to M_4/M_4^2$. Since $\mathcal{O}/2\mathcal{O} \simeq (\mathbb{Z}/2\mathbb{Z})^n$ for $n = (F:\mathbb{Q})$, this finally shows that $M_4/M_4^2 \simeq (\mathbb{Z}/2\mathbb{Z})^n$. ∎

Kronecker characters with conductor dividing (4). Next we determine the number of Kronecker characters with conductor dividing (4).

Proposition 4.3. *We have an exact sequence*

$$1 \to \mathrm{Sel}_4(F) \xrightarrow{\iota} \mathrm{Sel}(F) \xrightarrow{\nu} \mathrm{Kr}_4(F) \to 1.$$

Proof. The homomorphism ι is an injection since $\mathrm{Sel}_4(F) \subseteq \mathrm{Sel}(F)$. We set $\nu(\alpha F^{\times 2}) = \kappa_\alpha$. Then $\ker\nu$ consists of the cosets $\alpha F^{\times 2}$ such that $\kappa_\alpha = 1$; this holds if and only if $\alpha \equiv \xi^2 \bmod 4$, i.e., if $\alpha F^{\times 2} \in \mathrm{Sel}_4(F)$.

Finally, assume that $\kappa_\alpha \in \mathrm{Kr}_4(F)$. Then $(\alpha) = \mathfrak{a}^2$, hence $\kappa_\alpha \in \mathrm{im}\,\nu$. ∎

We know from [11] that $\#\mathrm{Sel}_4(F) = 2^{\rho^+}$, where ρ^+ denotes the 2-rank of $\mathrm{Cl}_2^+(F)$, and that $\#\mathrm{Sel}(F) = 2^{\rho+r+s}$, where ρ denotes the 2-rank of $\mathrm{Cl}_2(F)$, and r and $2s$ denote the number of real and complex embeddings of F.

Corollary 4.4. *We have* $\#\mathrm{Kr}_4(F) = 2^{r+s-(\rho^+-\rho)}$.

The separant class group modulo (4). Define the *separant class group modulo* 4 as the quotient

$$\mathrm{SCl}_4(F) = \mathrm{Dir}_4(F)/\mathrm{Kr}_4(F).$$

Then

$$\#\mathrm{SCl}_4(F) = 2^{n-r-s-\rho+\rho^+} = 2^{s+\rho^+-\rho}. \tag{3}$$

This implies

Corollary 4.5. *Every Dirichlet character defined modulo* 4 *is represented by a Kronecker character if and only if* F *is totally real and* $\mathrm{Cl}_2(F)$ *and* $\mathrm{Cl}_2^+(F)$ *have the same rank.*

In fact, we have $s \ge 0$ and $\rho^+ \ge \rho$, hence $2^{s+\rho^+-\rho} = 1$ if and only if $s = 0$ and $\rho^+ = \rho$.

COROLLARY 4.6. *If F has odd class number, then every Kronecker character with conductor dividing 4 has the form κ_ε for a unit $\varepsilon \in E_F$.*

Proof. If F has odd class number, then $\mathrm{Sel}(F) \simeq E/E^2$ and $\mathrm{Sel}_4(F) \simeq E_4/E^2$ by [11, Prop. 3.1]. Thus we get the exact sequence

$$1 \to E_4/E^2 \xrightarrow{\iota} E/E^2 \xrightarrow{\nu} \mathrm{Kr}_4(F) \to 1,$$

which implies the claim. ∎

In Table 2, the $\mathrm{Kr}_4(F)$ column lists the elements α for which κ_α generates $\mathrm{Kr}_4(F)$.

Table 2. Separant class groups modulo 4. Here $\dim \mathrm{Kr}_4(F) = r + s - (\rho^+ - \rho)$ and $\dim \mathrm{SCl}_4(F) = s + \rho^+ - \rho$.

F	$r+s$	ρ	ρ^+	$\mathrm{Kr}_4(F)$	$\mathrm{SCl}_4(F)$
$\mathbb{Q}$	1	0	0	-1	1
$\mathbb{Q}(i)$	1	0	0	i	$\langle\chi_2\rangle$
$\mathbb{Q}(\sqrt{-5})$	1	1	1	2	$\langle\chi_2\rangle$
$\mathbb{Q}(\sqrt{2})$	2	0	0	$-1,\ 1+\sqrt{2}$	1
$\mathbb{Q}(\sqrt{3})$	2	0	1	$2+\sqrt{3}$	$\langle\chi_2\rangle$
$\mathbb{Q}(\sqrt{6})$	2	0	1	-1	$\langle\chi_4\rangle$
$\mathbb{Q}(\sqrt{10})$	2	1	1	$-1,\ 3+\sqrt{10}$	1
$\mathbb{Q}(\sqrt{15})$	2	1	2	$4+\sqrt{15}$	$\langle\chi_2\rangle$
$\mathbb{Q}(\sqrt{34})$	2	1	1	$-1, 5+\sqrt{34}$	1

We next determine the conductor of the Kronecker character $\kappa_{-1} = \left(\frac{-1}{\cdot}\right)$:

PROPOSITION 4.7. *The Kronecker character $\kappa_{-1} = \left(\frac{-1}{\cdot}\right)$ in $F = \mathbb{Q}(\sqrt{m})$ is a Dirichlet character with conductor $\mathfrak{f}$ given by*

$$\mathfrak{f} = \begin{cases} (4) & \text{if } m \equiv 1 \bmod 4, \\ (2) & \text{if } m \equiv 2 \bmod 4, \\ (1) & \text{if } m \equiv 3 \bmod 4. \end{cases}$$

This follows easily from the conductor-discriminant formula, and it is also easily verified directly. If $m \equiv 2 \bmod 4$ and $\alpha = a + b\sqrt{m} \gg 0$, for example, we have

$$\left(\frac{-1}{\alpha}\right) = (-1)^{\frac{a^2 - mb^2 - 1}{2}} = (-1)^{\frac{a^2-1}{2}}(-1)^{\frac{mb^2}{2}} = (-1)^b,$$

hence

$$\left(\frac{-1}{\alpha}\right) = \begin{cases} +1 & \text{if } b \equiv 0 \bmod 2, \text{ i.e., } \alpha \equiv 1 \bmod 2, \\ -1 & \text{if } b \equiv 1 \bmod 2, \text{ i.e., } \alpha \equiv 1 + \sqrt{m} \bmod 2. \end{cases}$$

The Kronecker character $\kappa_\varepsilon = (\frac{\varepsilon}{\cdot})$, where $\varepsilon > 0$ is the fundamental unit of F, is trivial if and only if $F(\sqrt{\varepsilon})/F$ is unramified. This is the case for $m = 34$, where $\sqrt{35+6\sqrt{34}} = 3\sqrt{2}+\sqrt{17}$. For $m = 6$, the element $\sqrt{-\varepsilon} = \sqrt{-2}+\sqrt{-3}$ generates a quadratic extension unramified at all finite primes, hence $\kappa_{-1} = \kappa_\varepsilon$.

Ray class groups. Before we give a proof of Theorem 4.1 we recall a few basic facts about ray class groups. The *ray class group modulo* $\mathfrak{m}$ is defined by the exact sequence

$$1 \to P_\mathfrak{m}/P_\mathfrak{m}^1 \to I_\mathfrak{m}/P_\mathfrak{m}^1 \to I_\mathfrak{m}/P_\mathfrak{m} \to 1.$$

Here $I_\mathfrak{m}$ is the group of fractional ideals coprime to $\mathfrak{m}$ and $P_\mathfrak{m}$ its subgroup of principal ideals in $I_\mathfrak{m}$; the ray $P_\mathfrak{m}^1$ consists of all ideals (α) generated by elements $\alpha \equiv 1 \bmod \mathfrak{m}$ (congruence in the multiplicative sense). Observe that $I_\mathfrak{m}/P_\mathfrak{m} \simeq \mathrm{Cl}(F)$ since every ideal class contains ideals coprime to $\mathfrak{m}$, and that $I_\mathfrak{m}/P_\mathfrak{m}^1 \simeq \mathrm{Cl}_F\{\mathfrak{m}\}$ by definition. Below we will use the abbreviation $R_\mathfrak{m} = P_\mathfrak{m}/P_\mathfrak{m}^1$.

The connection with units is provided by the exact sequence

$$1 \to E/E_\mathfrak{m} \to (\mathcal{O}_F/\mathfrak{m})^\times \to R_\mathfrak{m} \to 1,$$

where $E_\mathfrak{m}$ is the group of units $\varepsilon \equiv 1 \bmod \mathfrak{m}$. These sequences imply the well known formula

$$\#\,\mathrm{Cl}_F\{\mathfrak{m}\} = h \cdot \frac{\Phi(\mathfrak{m})}{(E : E_\mathfrak{m})}.$$

The following simple lemma will be used repeatedly:

LEMMA 4.8. *If the sequence*

$$1 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 1$$

of abelian groups is exact, then so is

$$1 \to A/A\cap B^2 \xrightarrow{\bar\iota} B/B^2 \xrightarrow{\bar\pi} C/C^2 \to 1.$$

Proof. We assume that ι is the inclusion map. We only have to show that $\bar\iota$ is injective and that $\ker\bar\pi = \operatorname{im}\bar\iota$.

Now $\iota(a) \in B^2$ implies that $a \in A \cap B^2$, hence $\bar\iota$ is injective.

Next $\pi(b) \in C^2$ implies $\pi(b) = \pi(b_1)^2$ with $\pi(b_1) = c$, hence $b/b_1^2 \in \ker\pi = \operatorname{im}\iota$. Thus $b/b_1^2 = a$, hence $b = ab_1^2$ and $bB^2 = aB^2 \in \operatorname{im}\bar\iota$. ∎

COROLLARY 4.9. *The sequence*

$$1 \to E/E_4 \to M_4/M_4^2 \to R_4/R_4^2 \to 1 \tag{4}$$

is exact.

This follows by applying Lemma 4.8 to the exact sequence

$$1 \to E/E_{(4)} \to M_4 \to R_4 \to 1$$

by observing that $\varepsilon E_{(4)}$ is in M_4^2 if and only if $\varepsilon \in E_4$. Recall that $E_{(4)}$ is the group of units $\equiv 1 \bmod (4)$.

COROLLARY 4.10. *The sequence*

$$1 \to P_F/P_F \cap I_F^2 \to I_F/I_F^2 \to \mathrm{Cl}(F)/\mathrm{Cl}(F)^2 \to 1$$

is exact.

Here I_F denotes the group of nonzero fractional ideals and P_F its subgroup of principal ideals. The exactness of the sequence follows by applying Lemma 4.8 to the exact sequence

$$1 \to P_F \to I_F \to \mathrm{Cl}(F) \to 1$$

that defines the ideal class group.

COROLLARY 4.11. *The sequence*

$$1 \to P_4/P_4 \cap P_4^1 I_4^2 \to \mathrm{Cl}_F\{4\}/\mathrm{Cl}_F\{4\}^2 \to \mathrm{Cl}(F)/\mathrm{Cl}(F)^2 \to 1$$

is exact.

This follows by applying Lemma 4.8 to the exact sequence

$$1 \to R_4 \to \mathrm{Cl}_F\{4\} \to \mathrm{Cl}(F) \to 1$$

that defines the ray class group modulo 4.

4.1. Proof of the Main Theorem. The main step in the proof of Thm. 4.1 is verifying the exactness of the first two rows of the commutative diagram in Fig. 1 below. Recall that we know that

$$\begin{aligned} \mathrm{Kr}_4(F) &\simeq \mathrm{Sel}(F)/\mathrm{Sel}_4(F), & \#\mathrm{Kr}_4(F) &= 2^{r+s-(\rho^+-\rho)}, \\ \mathrm{Dir}_4(F) &\simeq M_4/M_4^2, & \#\mathrm{Dir}_4(F) &= 2^n, \\ \mathrm{SCl}_4(F) &\simeq P_4/P_4 \cap P_4^1 I_4^2, & \#\mathrm{SCl}_4(F) &= 2^{s+\rho^+-\rho}. \end{aligned}$$

Observe that we have computed the order of $\mathrm{SCl}_4(F)$ in (3); the isomorphism on the left follows from the exactness of the left row in the fundamental square, but is easily verified directly by showing that the sequence

$$1 \to \mathrm{Sel}(F)/\mathrm{Sel}_4(F) \to M_4/M_4^2 \to P_4/P_4 \cap P_4^1 I_4^2 \to 1$$

is exact.

The exactness of the fundamental diagram now implies

$$\mathrm{SCl}(F) \simeq \mathrm{Cl}_F\{4\}/\mathrm{Cl}_F\{4\}^2,$$

as well as

$$\#\mathrm{SCl}(F) = \#\mathrm{SCl}_4(F) \cdot (\mathrm{Cl}(F) : \mathrm{Cl}(F)^2) = 2^{\rho^++s}.$$

The vertical sequences in the left and middle columns of Fig. 1 are the definitions of the groups at the bottom; the exactness of the sequence in

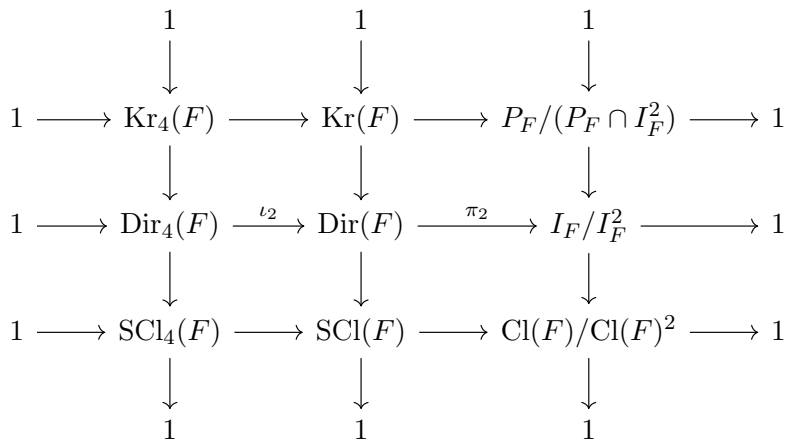

Fig. 1. Fundamental diagram for the determination of $\mathrm{SCl}(F)$.

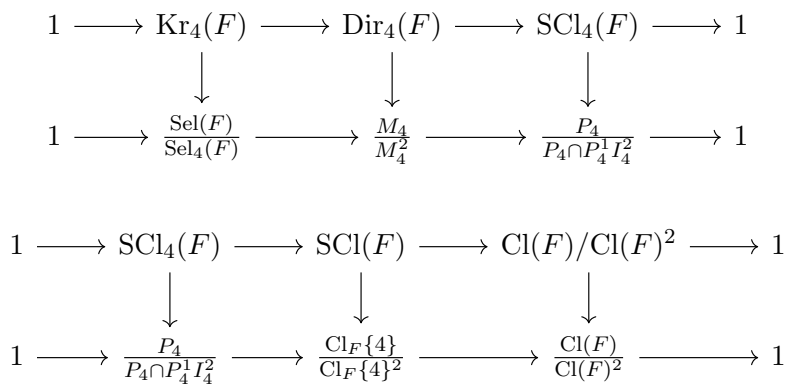

Fig. 2. The sequences in the left column and the bottom row; vertical maps are isomorphisms.

the right column is Cor. 4.10. The snake lemma then implies that the sequence in the third row is exact. This implies that $\mathrm{SCl}(F)$ has the same order as $\mathrm{Cl}_F\{4\}/\mathrm{Cl}_F\{4\}^2$, and hence is isomorphic to this group since both are elementary abelian.

Exactness of the top row. Since $\mathrm{Kr}_4(F) \simeq \mathrm{Sel}(F)/\mathrm{Sel}_4(F)$ we have to show that the sequence

$$1 \to \mathrm{Sel}(F)/\mathrm{Sel}_4(F) \to \mathrm{Kr}(F) \to P_F/P_F \cap I_F^2 \to 1$$

is exact.

Define $j : \mathrm{Sel}(F)/\mathrm{Sel}_4(F) \to \mathrm{Kr}(F)$ as the map sending $\alpha F^{\times 2}$ to the Kronecker character κ_α. This map is well defined since Kronecker characters κ_α with $\alpha \in \mathrm{Sel}_4(F)$ are trivial. Let λ denote the map $\mathrm{Kr}(F) \to P_F/P_F \cap I_F^2$

induced by sending κ_α to (α). Clearly $\operatorname{im} j \subseteq \ker \lambda$. Assume now that $\kappa_\alpha \in \ker \lambda$; then $(\alpha) = \mathfrak{a}^2$ is the square of an ideal, hence $\alpha F^{\times 2} \in \operatorname{im} j$. Since λ is clearly surjective, this proves our claim.

Exactness of the middle row. Next we prove the exactness of

$$1 \to M_4/M_4^2 \xrightarrow{\iota_2} \mathrm{Dir}(F) \xrightarrow{\pi_2} I_F/I_F^2 \to 1.$$

The map ι_2 is defined as in Prop. 4.2; recall that ι_2 is injective, that it makes the left upper square commutative, and that $\operatorname{im} \iota_2 = \mathrm{Dir}_4(F)$.

Let χ be a primitive quadratic Dirichlet character with conductor $\mathfrak{m}$; then we set $\pi_2(\chi) = \mathfrak{m} I_F^2$. This is a homomorphism: If χ and ψ are primitive quadratic Dirichlet characters with conductor $\mathfrak{m}$ and $\mathfrak{n}$, then $\chi\psi$ has conductor $\mathfrak{mn}/\mathfrak{a}^2$ for a suitable ideal $\mathfrak{a}$. This is easy to see for odd prime ideals dividing the conductor; for prime ideals $\mathfrak{l}$ above 2 it follows from the fact that the conductor of χ is an even power if and only if $\chi \in \mathrm{Dir}_4(F)$.

We now claim that π_2 is surjective. Clearly every prime ideal with odd norm is the conductor of a primitive Dirichlet character. If $\mathfrak{l}$ is a prime ideal above 2, choose an α divisible exactly by $\mathfrak{l}$, and take the $\mathfrak{l}$-part of the Kronecker symbol κ_α. Since I_F/I_F^2 is generated by prime ideals, π_2 is onto.

If $\chi \in \ker \pi_2$, then the conductor of χ is the square of an ideal. Since χ is primitive, this implies that χ is defined modulo an ideal dividing 4, hence it is in the image of ι_2 by Prop. 4.2.

If we splice together the sequences in the left column and the bottom row we obtain the exact sequence

$$\mathrm{Sel}(F)/\mathrm{Sel}_4(F) \hookrightarrow M_4/M_4^2 \to \mathrm{Cl}_F\{4\}/\mathrm{Cl}_F\{4\}^2 \twoheadrightarrow \mathrm{Cl}(F)/\mathrm{Cl}(F)^2$$

already derived in [11].

If we write χ as the product $\chi_e\chi_o$ of a character χ_e with even conductor and χ_o with odd conductor and send $[\chi] \in \mathrm{SCl}(F)$ to the ideal class of the conductor of χ_o, this seems to lead to delicate technical calculations involving conductors that are powers of prime ideals above (2) (see Hasse [7, Satz 16_2]).

Examples. The examples in the following table are those where $\mathrm{SCl}(F)$ is strictly larger than $\mathrm{SCl}_4(F)$.

F	$\mathbb{Q}(\sqrt{-5})$	$\mathbb{Q}(\sqrt{10})$	$\mathbb{Q}(\sqrt{15})$	$\mathbb{Q}(\sqrt{34})$
$\mathrm{SCl}(F)$	$\langle \chi_2, \chi_3 \rangle$	$\langle \chi_3 \rangle$	$\langle \chi_2, \chi_3 \rangle$	$\langle \chi_\mathfrak{p} \rangle$

The separant class groups defined modulo 4 were already described above. The difference between $\mathrm{SCl}_4(F)$ and $\mathrm{SCl}(F)$ comes from nontrivial 2-class groups.

- $F = \mathbb{Q}(\sqrt{-5})$. We already know that there is no Kronecker character with conductor 2 (see Table 2). There is also no Kronecker character with

conductor $\mathfrak{p}$ or $\mathfrak{p}'$, the two prime ideals above 3, since there is no quadratic extension ramified exactly at one of these primes. The product of the Dirichlet characters $\chi_{\mathfrak{p}}$ and $\chi_{\mathfrak{p}'}$ is the Kronecker character $\left(\frac{-3}{\cdot}\right)$, hence these two characters are equivalent in $\mathrm{SCl}(F)$.

- $F=\mathbb{Q}(\sqrt{10})$ and $F=\mathbb{Q}(\sqrt{15})$. The prime ideals $\mathfrak{p}$ above 3 are not principal, so the Dirichlet characters $\chi_{\mathfrak{p}}$ with conductor 3 are not Kronecker characters.
- $F=\mathbb{Q}(\sqrt{34})$. The fundamental unit $\varepsilon=35+6\sqrt{34}$ generates E_4/E^2. The Dirichlet character with conductor 3 is not a Kronecker character.

5. Separants. The relative discriminants $\mathrm{disc}(K/F)$ of quadratic extensions K/F of number fields contain information about the ramified primes. Separants, which we will define below, contain more information: they are (cosets of) algebraic numbers whose square roots generate the quadratic extension in question. In this section we will define separants for number fields F with odd class number, and we will prove their most basic properties.

For quadratic extensions K/F, we let $\mathrm{Ram}(K/F)$ denote the set of ramified places of F, and $\mathrm{Ram}_f(K/F)$ its subset of finite ramified places, i.e., of prime ideals in F that are ramified in K/F.

Proposition 5.1. *Let K/F be a quadratic extension of number fields. If F has odd class number and $K=F(\sqrt{\mu})$, then there exists an integer $\delta\in F^\times$ unique up to squares of units such that*

- $F(\sqrt{\mu})=F(\sqrt{\delta})$;
- $\mathrm{disc}(K/F)^h=(\delta)$.

Proof. Assume that F has odd class number h, and consider a quadratic extension $K=F(\sqrt{\mu})$. The relative discriminant $\mathfrak{d}=\mathrm{disc}(K/F)$ is an ideal, and we know that $(4\mu)=\mathrm{disc}(1,\sqrt{\mu})=\mathfrak{a}^2\mathfrak{d}$ for some integral ideal $\mathfrak{a}$. Raising everything to the h-th power we get $(4^h\mu^h)=(\alpha^2\delta)$, where $\mathfrak{a}^h=(\alpha)$ and $\mathfrak{d}^h=(\delta)$; moreover, we can (and will) choose δ in such a way that $F(\sqrt{\mu})=F(\sqrt{\delta})$. ∎

The coset $\mathrm{sep}(K/F)=\delta\cdot E_F^2$, where $E_F=\mathcal{O}_F^\times$ is the unit group of F, is called the *separant* of the quadratic extension K/F. The separants form a group $\mathrm{Sep}(F)$ with respect to the following multiplication: if $K_1=F(\sqrt{\alpha})$ and $K_2=F(\sqrt{\beta})$ are quadratic extensions with separants $\delta_1E_F^2$ and $\delta_2E_F^2$, let $K_3=F(\sqrt{\alpha\beta})$ be the third quadratic subfield of K_1K_2/F, and set

$$\delta_1E_F^2*\delta_2E_F^2=\delta_3E_F^2,$$

where $\delta_3E_F^2=\mathrm{sep}(K_3/F)$. Observe that this is the usual product if and only if the separants are coprime. Over the field $F=\mathbb{Q}$, the separant of a quadratic number field is just its discriminant because $E_{\mathbb{Q}}^2=1$.

EXAMPLE 4. Consider $F = \mathbb{Q}(\sqrt{-23})$ and $K = F(\sqrt{\mu})$ for $\mu = -5 + 2\sqrt{-23}$. Since $(\mu) = \mathfrak{p}^2\mathfrak{q}$ for the prime ideals $\mathfrak{p} = (3, 1 - \sqrt{-23})$ and $\mathfrak{q} = (13, 4 + \sqrt{-23})$, we have $\mathrm{Ram}_f(K/F) = \{\mathfrak{q}\}$ and $\operatorname{disc} K/F = \mathfrak{q}$. Also we find $\mathfrak{q}^3 = (\delta)$ for $\delta = -37 - 6\sqrt{-23}$, and $\mathrm{sep}(K/F) = \delta E_F^2$.

Separants that cannot be written as a product of separants are called *irreducible*; separants $\delta \cdot E_F^2$ for which (δ) is a power of a prime ideal are called *prime*. We only use algebraic integers for δ, hence (δ) is a power of a prime ideal with nonnegative exponent.

EXAMPLE 5. Consider the quadratic extension $K = F(\sqrt{1+2i})$ of $F = \mathbb{Q}(i)$. It has relative integral basis $\left\{1, \frac{1+\sqrt{1+2i}}{1+i}\right\}$, and its relative discriminant is therefore the ideal $(1+i)^2(1+2i)$. The generator $\delta = 2i(1+2i)$ of this ideal has the property that $K = F(\sqrt{\delta})$, hence it is the separant of this extension: $\mathrm{sep}(K/F) = 2i(1+2i)$. Since there is no quadratic extension of F with relative discriminant $\mathfrak{d} = (2)$, this separant cannot be factored into prime separants.

In my early attempts at defining a separant class group, I called $2i$ an ideal separant, defined as the greatest common divisor of the separants $2i(1+2i)$ and $2i(3+2i)$. Only later did I realize that this ideal separant may be identified with the nontrivial Dirichlet character modulo 2 in $\mathbb{Z}[i]$.

The modulus associated to a separant $\mathrm{sep}(K/F)$ is the conductor of the extension K/F, including the infinite places; two separants are called *coprime* if they are coprime as ideals, and *strongly coprime* if their associated conductors are coprime.

Given the separant $\mathrm{sep}(K/F) = \delta E_F^2$, the square root of δ generates the quadratic extension $K = F(\sqrt{\delta})$, and thus the splitting of primes in K/F is determined by the quadratic Kronecker symbol $(\delta/\mathfrak{p})$. This symbol is well defined and does not depend on the choice of δ.

PROPOSITION 5.2. *If F has odd class number in the strict sense, then Kronecker characters can be identified with separants of quadratic extensions K/F.*

As the example $F = \mathbb{Q}(\sqrt{6})$ discussed above shows, it is not enough to assume that F has odd class number in the usual sense.

Proof of Proposition 5.2. The natural homomorphism $\kappa : F^\times/F^{\times 2} \to \mathrm{Kr}(F)$ defined by $\kappa(\alpha F^{\times 2}) = \left(\frac{\alpha}{\cdot}\right)$ is clearly surjective. The kernel consists of all $\alpha F^{\times 2}$ such that $\kappa(\alpha F^{\times 2})$ has conductor (1). This is only possible if α is (up to squares) a unit congruent to a square modulo 4. Since F has odd class number in the strict sense, the only such units are squares.

The separant δE_F^2 corresponds to the Kronecker symbol $\left(\frac{\delta}{\cdot}\right)$. The homomorphism $\lambda : \mathrm{Sep}(F) \to \mathrm{Kr}(F)$ with $\lambda(\delta E^2) = \kappa_\delta$ is injective by what we

have already shown. We claim that it is also surjective. In fact, let κ_α denote any Kronecker character, and let $\delta = \mathrm{sep}(F(\sqrt{\alpha})/F)$. Then $\delta F^{\times 2} = \alpha F^{\times 2}$, and $\kappa_\alpha(\beta) = \kappa_\delta(\beta)$ for all elements $\beta \in F^\times$ coprime to $\alpha\delta$. ∎

6. Unique factorization into prime separants. We now explain how to generalize the prime discriminant factorization considered by Goldstein and Sunley to fields with trivial separant class group. Below, $K^{(2)}_{\mathrm{gen}}$ denotes the genus 2-class field of K over the field F (with odd class number), i.e., the maximal elementary abelian 2-extension of F unramified over K; the extension $K^{(2)+}_{\mathrm{gen}}$ is defined similarly, but we allow ramification at infinite primes.

THEOREM 6.1. *Let F be an algebraic number field. Then the following assertions are equivalent:*

(1) *The separant class group of F is trivial:* $\mathrm{SCl}(F) = 1$.
(2) *F is totally real, and its class number in the strict sense is odd.*
(3) *F has odd class number, and for every Dirichlet character χ defined modulo 4 there is a unit $\varepsilon \in E_F$ such that $\chi = \left(\frac{\varepsilon}{\cdot}\right)$.*
(4) *For each prime ideal $\mathfrak{p}$ with odd norm there is a unique quadratic extension K/F with* $\mathrm{Ram}_f(K/F) = \{\mathfrak{p}\}$.
(5) *F has odd class number in the strict sense, and every separant in* $\mathrm{Sep}(F)$ *can be written uniquely (up to order) as a product of prime separants.*
(6) *F has odd class number in the strict sense, and for every quadratic extension K/F we have $(K^{(2)+}_{\mathrm{gen}} : F) = 2^t$, where $t = \#\,\mathrm{Ram}_f(K/F)$.*

Proof. (1)⇔(2) is a consequence of the formula $\#\mathrm{SCl}(F) = 2^{\rho^+ + s}$, where $\rho^+ = \mathrm{rk}\ \mathrm{Cl}_2^+(F)$ and $2s$ is the number of complex embeddings of F.

(1)⇔(3) is just a reformulation of the exactness of the sequence

$$1 \to \mathrm{SCl}_4(F) \to \mathrm{SCl}(F) \to \mathrm{Cl}(F)/\mathrm{Cl}(F)^2 \to 1.$$

(1)⇒(4): Let $\mathfrak{p}$ be a prime ideal with odd norm. Since F has odd class number, $\mathfrak{p}^h = (\pi)$ for some $\pi \in \mathcal{O}_F$. Since $\mathrm{SCl}(F) = 1$, every element of M_4/M_4^2 is represented by a unit (Cor. 4.6); thus there is a unit ε such that $\varepsilon\pi \equiv \xi^2 \bmod 4$. In the quadratic extension $K = F(\sqrt{\pi\varepsilon})$, the only ramified prime ideal is $\mathfrak{p}$.

If K_1/F and K_2/F are quadratic extensions in which exactly $\mathfrak{p}$ ramifies, then the third quadratic subextension of K_1K_2/F is unramified, hence $\mathrm{Cl}(F)$ and therefore $\mathrm{SCl}(F)$ is nontrivial.

(4)⇒(3): We first show that the class number of F is odd. Let $\mathfrak{p}$ be a prime ideal with odd norm in F, and let K/F be a quadratic extension in which exactly $\mathfrak{p}$ ramifies. Then $K = F(\sqrt{\alpha})$ with $(\alpha) = \mathfrak{p}\mathfrak{a}^2$ for some ideal $\mathfrak{a}$. This implies that the ideal class of $\mathfrak{p}$ is a square. Since the $[\mathfrak{p}]$ generate the

class group, every ideal class is a square. Since the class group is finite, its order must be odd.

Now let χ be a Dirichlet character defined modulo 4. By the proof of Prop. 4.2, χ is in the image of M_4/M_4^2, i.e. there is a Kronecker character $\kappa = \left(\frac{\alpha}{\cdot}\right)$ such that $\kappa = \chi\psi$, where ψ is defined modulo an odd ideal. By assumption, ψ is a Kronecker character; but then so is χ. By Cor. 4.6, this Kronecker character has the form $\kappa = \left(\frac{\varepsilon}{\cdot}\right)$ for some unit $\varepsilon \in E_F$. This implies the claim.

(1)⇒(5): Since $\mathrm{SCl}(F) = 1$, the class number of F in the strict sense is odd.

Let δE_F^2 be a separant. The Kronecker character κ_δ can be written uniquely as a product of primitive quadratic Dirichlet characters with prime power conductor: $\kappa_\delta = \chi_1 \cdots \chi_t$. Since $\mathrm{SCl}(F) = 1$, each factor χ_j is a Kronecker character $(\delta_j/\cdot)$ for separants δ_j. Since the conductors of these Kronecker characters are coprime prime ideal powers we must have $\delta E_F^2 = \delta_1 \cdots \delta_t E_F^2$.

We claim that this factorization is unique. To this end, assume that there are two factorizations $\delta E_F^2 = \delta_1 \cdots \delta_t E_F^2 = \delta_1' \cdots \delta_t' E_F^2$ of δ into prime separants.

Since $K_{\mathrm{gen}}^{(2)+} = F(\sqrt{\delta_1}, \ldots, \sqrt{\delta_t})$ is the maximal elementary abelian 2-extension of F unramified over K at all finite primes, we must have $\sqrt{\delta_j'} \in K_{\mathrm{gen}}^{(2)+}$. But since δ_j' and the δ_j are prime separants, this is only possible if $\delta_j' E_F^2$ is among the $\delta_j E_F^2$.

(5)⇒(6): Let K/F be a quadratic extension with separant δ. By assumption we can write $\delta E_F^2 = \delta_1 \cdots \delta_t E_F^2$ as a product of prime separants that are coprime except for infinite prime factors. But then $L = F(\sqrt{\delta_1}, \ldots, \sqrt{\delta_t})/K$ is unramified at all finite primes, hence equal to the genus field in the strict sense.

(6)⇒(4) is Prop. 6.2 below. ∎

Here is the result we have used in the proof:

PROPOSITION 6.2. *Let F be a field with odd class number in the strict sense. If, for every quadratic extension K/F, we have $(K_{\mathrm{gen}}^{(2)+} : F) = 2^t$, where $t = \#\operatorname{Ram}_f(K/F)$, then for each odd prime ideal $\mathfrak{p}$ there is a unique quadratic extension k/F in which $\mathfrak{p}$ is the only (finite) ramified prime ideal.*

Proof. Let $\mathfrak{p}$ be a prime ideal in F with odd norm. Let K/F be a quadratic extension in which $\mathfrak{p}$ is ramified (choose an ideal $\mathfrak{a}$ in the inverse class of $[\mathfrak{p}]$ coprime to $\mathfrak{p}$ and write $\mathfrak{p}\mathfrak{a} = (\alpha)$; then $K = F(\sqrt{\alpha})$ is such an extension). Then $K_{\mathrm{gen}}^{(2)+}/F$ is an elementary abelian extension in which every ramified prime ideal has ramification index 2. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_{t-1}$ denote the ramified prime ideals different from $\mathfrak{p}$. Their ramification fields have degree

2^{t-1} over F. Thus their intersection is a quadratic extension k/F in which the $\mathfrak{q}_j$ are unramified. Since F admits no quadratic extension unramified at all finite primes, we must have $\mathrm{Ram}_f(k/F) = \{\mathfrak{p}\}$.

The extension k is necessarily unique: if k' is another such extension, the quadratic subextension of kk'/F different from k and k' would be unramified at all finite primes, contradicting the fact that F has odd class number in the strict sense. ∎

As another application, consider $k = \mathbb{Q}(\sqrt{130})$ and its genus class field $k_{\mathrm{gen}} = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{13})$. We have $\mathrm{SCl}(F) = 1$ for $F = \mathbb{Q}(\sqrt{2})$, and the unique factorization of the separant of $F_1 = F(\sqrt{65})$ is $65 = 5 \cdot 13$. Since we also have $\mathrm{SCl}(L) = 1$ for $L = F(\sqrt{5})$, the relative discriminant of k_{gen}/L factors into prime separants. Now 26 is represented by the form $x^2 - 10y^2$, so we find $26 = 6^2 - 10$ and $13 = 2 \cdot 3^2 - 5$. Thus $3\sqrt{2} + \sqrt{5}$ is a factor of $\mathrm{sep}(k_{\mathrm{gen}}/L)$. Since $\mathrm{SCl}(L) = 1$, there must exist a unit $\varepsilon \in E_L$ such that $(3\sqrt{2} + \sqrt{5})\varepsilon \equiv \xi^2 \bmod 4$. We quickly find that $\alpha = (3\sqrt{2}+\sqrt{5})(3+\sqrt{10})(2+\sqrt{5})$ is 2-primary (coprime to (2) and congruent to a square modulo 4), hence its square root generates an unramified quadratic extension of k_{gen} whose Galois group over k is the quaternion group of order 8. The minimal polynomial of $\sqrt{\alpha}$ is $f(x) = x^8 - 180x^6 + 1854x^4 - 2340x^2 + 169$.

7. Genus theory. Genus theory of quadratic extensions k/F of number fields with trivial separant class group can be made as explicit as over $\mathbb{Q}$: the factorization of $\delta = \mathrm{sep}(k/F)$ into prime separants gives us the genus class field as well as the genus characters.

LEMMA 7.1. *Let F be a number field with odd class number in the strict sense. Let $K = F(\sqrt{\delta})$ and assume that $L = F(\sqrt{\delta_1}, \sqrt{\delta_2})$ is a quadratic extension of K, where δ, δ_1 and δ_2 are separants. Then L/K is unramified at all finite primes if and only if $\delta E_F^2 = \delta_1\delta_2 E_F^2$.*

Proof. Let $K_1 = F(\sqrt{\delta_1})$, $K_2 = F(\sqrt{\delta_2})$ and $K = F(\sqrt{\delta_1\delta_2}) = F(\sqrt{\delta})$ be the three quadratic subextensions of L/F. By the conductor-discriminant formula we have

$$\mathrm{disc}(L/F) = \mathrm{disc}(K/F) \cdot \mathrm{disc}(K_1/F) \cdot \mathrm{disc}(K_2/F).$$

Since L/K is unramified, we have $\mathrm{disc}(L/F) = \mathrm{disc}(K/F)^2$. This implies $\mathrm{disc}(K_1/F)\cdot\mathrm{disc}(K_2/F) = \mathrm{disc}(K/F)$. Since separants are h-th powers of the relative discriminants, we must have $\delta_1 \cdot \delta_2 = \varepsilon\delta$. Since $F(\sqrt{\delta_1\delta_2}) = F(\sqrt{\delta})$, the unit ε must be a square, and this implies that $\delta_1 E^2 \cdot \delta_2 E^2 = \delta E^2$ as claimed. The same calculation yields the converse. ∎

Let F be a number field with $\mathrm{SCl}(F) = 1$. For a quadratic extension K/F with separant $\mathrm{sep}(K/F) = \delta E_F^2$ let $\delta E_F^2 = \delta_1 \cdots \delta_t E_F^2$ denote the factorization of $\mathrm{sep}(K/F)$ into prime separants. We already know that this

factorization determines the genus field of K/F, and that the genus field in the strict sense is $K^+_{\text{gen}} = F(\sqrt{\delta_1}, \dots, \sqrt{\delta_t})$.

Now let $c \in \mathrm{Cl}^+(K)$ be an ideal class, and pick an ideal $\mathfrak{a} \in c$ coprime to δ. Then $\chi_j(c) = \left(\frac{\delta_j}{N_{K/F}\mathfrak{a}}\right)$ is a well defined quadratic character on the class group. The characters χ_j are called *genus characters* of K/F.

THEOREM 7.2 (Principal Genus Theorem). *Let F be a number field with trivial separant class group. Then an ideal class $c \in \mathrm{Cl}^+(K)$ is a square if and only if $\chi_j(c) = +1$ for all genus characters χ_j.*

Proof. Write $c = [\mathfrak{p}]$ for some prime ideal $\mathfrak{p}$ unramified in K/F. Clearly $\chi_j(c) = 1$ for all genus characters is equivalent to $\mathfrak{p}$ splitting completely in the genus class field. Since F has odd class number in the strict sense, K^+_{gen} is the maximal elementary abelian 2-extension contained in the Hilbert class field in the strict sense. By class field theory, $\mathfrak{p}$ splits completely in the maximal elementary abelian 2-extension contained in the Hilbert class field in the strict sense if and only if the ideal class $[\mathfrak{p}]$ is a square in the class group $\mathrm{Cl}_2^+(K)$. ∎

EXAMPLE 6. Consider the quadratic extension $K = F(\sqrt{93 + 50\sqrt{2}})$ of $F = \mathbb{Q}(\sqrt{2})$. The separant $\delta = \mathrm{sep}(K/F) = 2(93 + 50\sqrt{2})E_F^2$ has the factorization

$$2(93 + 50\sqrt{2}) = -\sqrt{2}^2\,(-11 - 4\sqrt{2})(7 + 2\sqrt{2})$$

into prime separants (modulo squares of units). Thus

$$K_{\text{gen}} = K\Big(\sqrt{7 + 2\sqrt{2}}\Big) \quad \text{and} \quad K^+_{\text{gen}} = K\Big(\sqrt{-1},\ \sqrt{7 + 2\sqrt{2}}\Big),$$

hence $\mathrm{Cl}_2(K)$ has rank 1 and $\mathrm{Cl}_2^+(K)$ has rank 2. Further, `pari` tells us that in fact $\mathrm{Cl}(K) = [2]$ and $\mathrm{Cl}^+(K) = [2, 2]$.

8. Hilbert's first supplementary law. Hilbert's version of the first supplementary law in number fields (see [11, Theorem 6.5]) is the following:

PROPOSITION 8.1. *Let $\mathfrak{a}$ be an ideal with odd norm in a number field F with class number h. Then $\mathfrak{a}^h = (\alpha)$, and we can choose α in such a way that $\alpha \gg 0$ and $\alpha \equiv \xi^2 \bmod 4$ if and only if $\left(\frac{\beta}{\mathfrak{a}}\right) = +1$ for all elements $\beta \in \mathrm{Sel}(F)$.*

Here we will briefly mention a few variants of Hilbert's first supplementary law in number fields with odd class number (the proofs are all similar), where we can replace the Selmer group by the unit group.

PROPOSITION 8.2. *Let F be a number field with odd class number h, and $\mathfrak{a}$ an ideal with odd norm. Then $\mathfrak{a}^h = (\alpha)$ for some $\alpha \in F^\times$, and in the*

following table, the conditions on $\mathfrak{a}$ in the left column are equivalent to the conditions on α in the right column:

Conditions on $\mathfrak{a}$	*Conditions on* α
$(E/\mathfrak{a})=1$	$\alpha\equiv\xi^2 \bmod 4,\ \alpha\gg 0$
$(E_4/\mathfrak{a})=1$	$\alpha\gg 0$
$(E^+/\mathfrak{a})=1$	$\alpha\equiv\xi^2 \bmod 4$

Here E^+ is the group of totally positive units in $\mathcal{O}_F$.

If $\mathrm{SCl}(F)=1$, then $E^+=E^2$, so $(E^+/\mathfrak{a})=+1$ for all ideals with odd norm, and thus every ideal $\mathfrak{a}^h$, where h denotes the odd class number of F, is generated by an element $\alpha\equiv\xi^2 \bmod 4$ as predicted by our results above.

In quadratic number fields $F=\mathbb{Q}(\sqrt{m})$ with $m\equiv 3 \bmod 4$ we have $-1\in E_4$ and $\left(\frac{-1}{\mathfrak{a}}\right)=(-1)^{(N\mathfrak{a}-1)/2}$, hence $\left(\frac{-1}{\alpha}\right)=+1$ for totally positive elements with norm $\equiv 1 \bmod 4$. Thus if F has odd class number h and if $\mathfrak{a}$ is an ideal with norm $\equiv 1 \bmod 4$, then $\mathfrak{a}^h$ has a totally positive generator, and ideals with norm $\equiv 3 \bmod 4$ do not. In $\mathbb{Q}(\sqrt{3})$, the prime ideal above $13\equiv 1 \bmod 4$ is generated by $4+\sqrt{3}\gg 0$, and the prime ideal with norm 11 by $1+2\sqrt{3}$. Next $2(2+\sqrt{3})=(1+\sqrt{3})^2$, hence $\left(\frac{\varepsilon}{\mathfrak{a}}\right)=\left(\frac{2}{N\mathfrak{a}}\right)$ for the fundamental unit $\varepsilon=2+\sqrt{3}$, and $(E/\mathfrak{p})=+1$ if and only if $N\mathfrak{p}\equiv 1 \bmod 8$. This implies that the prime ideal above 13 does not have a totally positive 2-primary generator (we have $(4+\sqrt{3})(2+\sqrt{3})=11+6\sqrt{3}$, which is not congruent to a square modulo 4), but that the prime ideal above 73 does, and in fact $11+4\sqrt{3}$ is such a generator.

Final remarks. Separants were used in [10] to construct unramified 2-extensions of quadratic extensions K/F of fields F with $\mathrm{SCl}(F)=1$. The simplest result was that if F is a number field with $\mathrm{SCl}(F)=1$ and k/F a quadratic extension with separant δE_F^2, then there is an unramified cyclic quartic extension K/k if and only if δ admits a *C_4-factorization*, i.e., $\delta E_F^2=\delta_1\delta_2 E_F^2$ with $(\delta_1/\mathfrak{p}_2)=(\delta_2/\mathfrak{p}_1)=1$ for all prime ideals $\mathfrak{p}_i\,|\,\delta_j$. In this case, the extension can be constructed explicitly by solving the equation $\delta_1x^2+\delta_2y^2=z^2$ in $F^\times$.

I expect that the main results of Rédei (Rédei matrix, triple symbol reciprocity; see Stevenhagen [13]), Koch [8], Fröhlich [4] and the existence of governing fields generalize to all number fields with trivial separant class group. Kuramoto [9] has recently discussed the Rédei symbol in real quadratic fields with class number 1.

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