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A study of the boundary spirallike mappings via horospheres

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Abstract. We study boundary spirallike mappings defined on a bounded strongly convex domain via horospheres.

1. Introduction. Let $\mathbb{B}^d := \{z \in \mathbb{C}^d : \|z\| < 1\}$ and $f : \mathbb{B}^d \to \mathbb{C}^d$ be a univalent map. The geometric properties of the set $f(\mathbb{B}^d)$ have been studied extensively over the past several decades in geometric function theory (see [GK03] and references therein for details). A univalent map $f : \mathbb{B}^d \to \mathbb{C}^d$ with $f(0) = 0$ is called *spirallike with respect to a matrix* $A \in \mathrm{GL}(d, \mathbb{C})$ if $e^{tA}(f(\mathbb{B}^d)) \subseteq f(\mathbb{B}^d)$ for all $t \geq 0$. In particular, if $A = -\mathrm{I}_d$, then the univalent map f is said to be *starlike*. A necessary and sufficient condition for a univalent map defined on a ball to be spirallike or in particular starlike was given in [G75, Theorem 2]. Later, in [HK98] an analogous condition is also provided when the univalent map is defined on a certain balanced bounded domain instead of the unit ball. In [LS10] the authors studied a certain class of univalent maps defined on the open unit ball whose image is not starlike with respect to some interior point but rather with respect to some boundary point, i.e, $0 \in \partial f(\mathbb{B}^d)$ and $tz \in f(\mathbb{B}^d)$ for all $t \in (0, 1]$ and $z \in f(\mathbb{B}^d)$.

In this paper, we introduce a more general notion of boundary starlike mapping, namely boundary spirallike mapping, which is defined as follows:

Definition 1.1. Let $\Omega, D \subset \mathbb{C}^d$ be two domains and $0 \in \overline{\Omega}$ and $A \in \mathrm{GL}(d, \mathbb{C})$. Suppose that $e^{tA}\Omega \subset \Omega$ for all $t \geq 0$. Then Ω is said to be *spirallike with respect to* A if $0 \in \Omega$, and *boundary spirallike with respect to* A if $0 \in \partial\Omega$. A univalent map $f : D \to \mathbb{C}^d$ is said to be boundary spirallike with respect to A if $f(D)$ is a boundary spirallike domain with respect to A.

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REMARK 1.2. If $A = -\mathrm{I}_d$ in the above definition, then Ω is said to be *boundary starlike*.

The set of all boundary spirallike mappings with respect to the matrix A is denoted by $S_A^{\partial}(D)$. The aim of this article is to provide a characterization of a certain class of mappings in $S_A^{\partial}(D)$. With this notation, the following result was proved in [LS10, Theorem 1].

RESULT 1.3. *Let $f : \mathbb{B}^d \to \mathbb{C}^d$ be a biholomorphism such that*

(i) $f(0) = \mathbf{e}_1 := (1, 0, \ldots, 0)$,
(ii) $\lim_{\mathbb{B}^d \ni z \to \mathbf{e}_1} f(z) = 0$ *and for every open neighborhood $U_{\mathbf{e}_1}$ of the point $\mathbf{e}_1$ there exists $\delta > 0$ such that $\{e^{-t}\mathbf{e}_1 : t > \delta\} \subset f(U_{\mathbf{e}_1} \cap \mathbb{B}^d)$.*

Let $E_k := \{z \in \mathbb{B}^d : |z_1 - 1|^2 < k(1 - \|z\|^2)\}$ for $k > 0$. Then $f \in S_{-\mathrm{I}}^{\partial}(\mathbb{B}^d)$ if and only if $f \in S_{-\mathrm{I}}^{\partial}(E_k)$ for all $k > 0$.

Let us briefly recall the notion of horosphere for a Kobayashi hyperbolic domain $D \subset \mathbb{C}^d$ which we will use crucially throughout the paper. For the definition of Kobayashi hyperbolic domains see Section 2. For further details about such domains we refer the reader to [K98].

DEFINITION 1.4. Let $D \subset \mathbb{C}^d$ be a Kobayashi hyperbolic domain, $p \in D$ and $\xi \in \partial D$. Then for $R > 0$ the *small horosphere* $E_p(\xi, R)$ and the *big horosphere* $F_p(\xi, R)$ with center ξ, pole $p \in D$ and radius $R > 0$ are defined as follows:

$$E_p(\xi, R) := \Big\{ z \in D : \limsup_{w \to \xi} [K_D(z, w) - K_D(p, w)] < \tfrac{1}{2} \ln R \Big\},$$

$$F_p(\xi, R) := \Big\{ z \in D : \liminf_{w \to \xi} [K_D(z, w) - K_D(p, w)] < \tfrac{1}{2} \ln R \Big\}.$$

REMARK 1.5. In [A88] Abate introduced the notion of small and big horosphere. It is not always true that $\lim_{w \to \xi}[K_D(z, w) - K_D(p, w)]$ exists for a given domain D and its boundary point $\xi \in \partial D$ for every boundary point of a bounded strongly convex domain (see Result 2.3). In particular, it does exist for the unit ball $\mathbb{B}^d$.

It turns out that the domains E_k appearing in Result 1.3 are, in fact, horospheres centered at the boundary point $\mathbf{e}_1$. The study of boundary spirallike mappings is important in the context of embedding a univalent map into a Loewner chain (see [AB$^+$13, Theorem 8.4] for details). The embedding problem for a univalent map defined on a strongly convex domain is studied in [CG26]. Motivated by this observation, we generalize Result 1.3 to the case where the biholomorphism is defined on a bounded strongly convex domain. We further extend the analysis to a broader class of mappings, namely spirallike mappings with respect to a given invertible matrix. Before stating our first main result, let us recall that $A \in \mathrm{GL}(d, \mathbb{C})$ is a *Hurwitz matrix*

if $\operatorname{Re}\lambda < 0$ for all eigenvalues λ of A. If A is Hurwitz, then $\|e^{tA}z\| \to 0$ as $t \to \infty$ for all $z \in \mathbb{C}^d$ (see [RS05, p. 311]). We now state our first main result.

THEOREM 1.6. *Let $D \subset \mathbb{C}^d$ be a bounded strongly convex domain with $\mathcal{C}^3$ smooth boundary, let $p \in D$ and let $A \in \mathrm{GL}(d,\mathbb{C})$ be a Hurwitz matrix. Let $f : D \to \Omega := f(D) \subset \mathbb{C}^d$ be a biholomorphism such that the following conditions hold:*

(i) *there exist $x \in \partial D$ and $z_0 \in D$ such that $f(z_0) = x$,*
(ii) $\lim_{D\ni z\to x} f(z) = 0$ *and for every open neighborhood U_x of x there exists $M_{U_x} > 0$ such that $\{e^{tA}x : t > M_{U_x}\} \subset f(U_x \cap D)$.*

Then $f \in S_A^{\partial}(D)$ if and only if the domain $\Omega_k := f(E_p(x,k))$ is boundary spirallike with respect to A for all $k > 0$.

REMARK 1.7. Result 1.3 is recovered from Theorem 1.6 by taking

$$D = \mathbb{B}^d, \quad p = 0, \quad x = \mathbf{e}_1, \quad z_0 = 0, \quad A = -\mathrm{I}_d.$$

Theorem 1.6 provides a geometric characterization of A-boundary spirallike mappings in terms of images of horospheres. This characterization is also useful in studying the embedding problem for Loewner chains. We record the following consequence of Theorem 1.6; for details on Loewner chains and the problem of embedding univalent mappings into Loewner chains, we refer the reader to [GK03, AB$^+$13].

COROLLARY 1.8. *Let $D \subset \mathbb{C}^d$, $A \in \mathrm{GL}(d,\mathbb{C})$, f, Ω_k be as in Theorem 1.6. If Ω_k is boundary spirallike with respect to A for all $k > 0$, then f can be embedded into the L^∞-Loewner chain f_t defined by $f_t = e^{-tA}f(z)$.*

Proof. By Theorem 1.6, the assumption that Ω_k is boundary spirallike with respect to A for every $k > 0$ implies that $f(D)$ is boundary spirallike with respect to A. Thus, $e^{tA}f(D) \subset f(D)$ for every $t \geq 0$. For $t \geq 0$, we set

$$f_t(z) := e^{-tA}f(z), \quad z \in D.$$

Clearly, f_t is univalent for all $t \geq 0$. Note that for $0 \leq s \leq t$, we have $e^{(t-s)A}f(D) \subset f(D)$. Consequently, $f_s(D) \subset f_t(D)$ whenever $0 \leq s \leq t$. Fix $T > 0$ and a compact set $K \Subset D$. Since f is continuous, $M_K := \sup_{z\in K}\|f(z)\| < \infty$. For $z \in K$ and $0 \leq s \leq t \leq T$, we have

$$\begin{aligned}\|f_t(z) - f_s(z)\| &= \|e^{-tA}f(z) - e^{-sA}f(z)\| \\ &= \left\|\int_s^t \frac{d}{d\xi}(e^{-\xi A}f(z))\,d\xi\right\| = \left\|\int_s^t -Ae^{-\xi A}f(z)\,d\xi\right\| \\ &\leq M_K\Big(\sup_{0\leq\xi\leq T}\|Ae^{-\xi A}\|\Big)(t-s).\end{aligned}$$

Hence, if $C_{K,T} := M_K \sup_{0\leq\xi\leq T}\|Ae^{-\xi A}\|$, then $\|f_t(z) - f_s(z)\| \leq \int_s^t C_{K,T}\,d\xi$ for all $z \in K$ and $0 \leq s \leq t \leq T$. Since the constant function $C_{K,T}$ belongs

to $L^\infty([0,T])$, the family $(f_t)_{t\geq 0}$ is an L^∞-Loewner chain. Finally, $f_0 = f$, and hence f is embedded into the L^∞-Loewner chain $(f_t)_{t\geq 0}$. ∎

Next, as an application of Theorem 1.6, we obtain an algebraic characterization of holomorphic maps on strongly convex domains that are boundary spirallike with respect to a matrix A. Before stating our next result, we recall some notation from [LS10].

Let $D \subset \mathbb{C}^d$ be a domain and $f : D \to \mathbb{C}^d$ be a biholomorphic mapping. Suppose that $E \subset D$ is a subdomain such that ∂E is a $\mathcal{C}^1$ hypersurface in a neighborhood of a point $z \in D \cap \partial E$. Let $G = f(E)$. Fix $z \in D \cap \partial E$. We denote by $n(f(z))$ the outward unit normal vector to ∂G at $f(z)$, defined to be a unit normal vector to ∂G at $f(z)$ such that for every sufficiently small neighborhood U_z of z, there exists $\delta > 0$ such that

$$f(z) + s \cdot n(f(z)) \notin f(E \cap U_z) \quad \text{for every } s \in (0, \delta).$$

Since ∂G is a $\mathcal{C}^1$ hypersurface near $f(z)$, the outward unit normal is uniquely determined. We use the notations $n(f(z))$ and $n(w)$, where $w = f(z)$, interchangeably.

In our setting, $D \Subset \mathbb{C}^d$ is a strongly convex domain with $\mathcal{C}^{4,\alpha}$ boundary for $\alpha \in (0,1)$. Given $z_0 \in D$, we consider

$$k_0 := \exp\Big(2 \cdot \lim_{w\to x} [K_D(z_0, w) - K_D(p, w)]\Big),$$

where the existence of the limit follows from Result 2.3. Then, it follows that $z_0 \in \partial E_p(x, k_0) \cap D$. Accordingly, $n(f(z_0))$ denotes the outward unit normal vector to $\partial(f(E_p(x, k_0)))$ at the point $f(z_0)$. Hence, in our setting, $n(f(z))$ is defined in this way for all $z \in D$. With this notation in place, our next result gives the desired characterization.

THEOREM 1.9. *Let $D \subset \mathbb{C}^d$ be a bounded strongly convex domain with $\mathcal{C}^{4,\alpha}$ boundary for some $\alpha \in (0,1)$, let $p \in D$, and let $A \in \mathrm{GL}(d, \mathbb{C})$ be a Hurwitz matrix. Let $f : D \to f(D) \subset \mathbb{C}^d$ be a biholomorphism satisfying hypotheses* (i) *and* (ii) *of Theorem* 1.6. *Assume further that*

$$\mathrm{Re}\,\langle Af(z), n(f(z))\rangle \neq 0 \quad \textit{for all } z \in D.$$

Then $f \in S_A^\partial(D)$ if and only if $\mathrm{Re}\,\langle -Af(z), n(f(z))\rangle > 0$ for all $z \in D$.

REMARK 1.10. The additional boundary regularity assumption in Theorem 1.9 is required because, in the proof, we need the hypersurface $\partial E_p(x,k)\setminus\{x\}$ to be of class $\mathcal{C}^1$ for every $k > 0$. For strongly convex domains, the assumed boundary regularity guarantees this property; see Result 2.3.

2. Preliminaries and technical results. In this section, we discuss the notions of Kobayashi distance, mention some known results from the literature and prove a lemma that we need to prove our main results.

Recall that, for a domain $D \subset \mathbb{C}^d$, a point $z \in D$, and a vector $X \in \mathbb{C}^d$, the *Kobayashi–Royden (pseudo)metric* κ_D is defined by

$$\kappa_D(z; X) = \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D),\ \varphi(0) = z,\ \alpha\varphi'(0) = X\}.$$

It follows from [R71, Proposition 3] that for any domain $D \subset \mathbb{C}^d$ the map $\kappa_D : D \times \mathbb{C}^d \to [0, \infty)$ is upper semicontinuous.

The Kobayashi–Royden (pseudo)metric defines the Kobayashi (pseudo) distance, which is defined as follows:

For $x, y \in D$, the *Kobayashi (pseudo)distance* is defined as follows:

$$K_D(x, y) = \inf_{\gamma} \int_0^1 \kappa_D(\gamma(t); \gamma'(t))\, dt, \tag{2.1}$$

where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : [0, 1] \to D$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Definition 2.1. A domain $D \subset \mathbb{C}^d$ is called a *Kobayashi hyperbolic domain* if (D, K_D) forms a metric space.

We now mention a result due to Abate which will be used in the proof of Theorem 1.6.

Result 2.2 ([A89, Proposition 2.4.15]). *Let D be a bounded domain in $\mathbb{C}^n$, and fix a point $z_0 \in D$. Let $f : D \to D$ be a holomorphic map such that there exists a sequence $\{w_\nu\} \subset D$ converging to a point $x \in \partial D$ for which $\{f(w_\nu)\}$ converges to a point $y \in \partial D$, and*

$$\lim_{\nu \to \infty} \big(K_D(z_0, w_\nu) - K_D(z_0, f(w_\nu))\big) \leq \tfrac{1}{2} \log \alpha < +\infty$$

for a suitable $\alpha \in \mathbb{R}^+$. Then, for every $R > 0$, $f(E_{z_0}(x, R)) \subset F_{z_0}(y, \alpha R)$.

We now recall a result obtained by collecting several known results from the literature, which will be used crucially in the proofs of Theorem 1.9 and Lemma 2.4.

Result 2.3. *Let $D \Subset \mathbb{C}^d$ be a strongly convex domain with $\mathcal{C}^3$ smooth boundary, $p \in D$ and $x \in \partial D$.*

(i) *For every $z \in D$ and every boundary point, $\xi \in \partial D$ the limit*

$$\lim_{w \to \xi} [K_D(z, w) - K_D(p, w)]$$

exists. In particular, $E_p(\xi, R) = F_p(\xi, R)$ for all $R > 0$. Moreover, $\overline{E_p(\xi, R)} \cap \partial D = \{\xi\}$ for all $R > 0$ and $\xi \in \partial D$.

(ii) $\bigcup_{R>0} E_p(x, R) = D$.

Furthermore, if D has $\mathcal{C}^{4,\alpha}$ smooth boundary with $\alpha \in (0, 1)$, then for all $R > 0$, $E_p(x, R)$ is a convex domain and $\partial E_p(x, R) \cap D$ is a $\mathcal{C}^1$ smooth submanifold.

The first assertion in Result 2.3(i) follows from [A89, Theorem 2.6.47, Corollary 2.6.48], and the corresponding boundary property is given by [A88, Theorem 1.7]. The conclusion in Result 2.3(ii) is proved in [A88, Lemma 1.1(v)].

The convexity of the horospheres $E_p(\xi, R)$ follows from [A89, Corollary 2.6.49]. Finally, to deduce the $\mathcal{C}^1$-regularity of $\partial E_p(x,k)\setminus\{x\}$ we observe the following:

By [HW22, Proposition 4.2], the boundary spherical representation Ψ_x maps each horosphere in D centered at x onto the corresponding horosphere in $\mathbb{B}^d$, i.e., $\Psi_x\big(E_p(x,R)\big) = E_{\Psi_x(p)}(\Psi_x(x), R)$, $R > 0$. Moreover, by [W24, Theorem C], under the assumption that ∂D is of class $\mathcal{C}^{4,\alpha}$, $\alpha \in (0,1)$, the map Ψ_x is $\mathcal{C}^1$ diffeomorphism away from the distinguished boundary point x. Consequently, for every $R > 0$,

$$\partial E_p(x,R) \cap D$$

is a $\mathcal{C}^1$-smooth hypersurface.

We now establish a lemma that will be used in the proof of Theorem 1.9. The proof is similar to that of [LS10, Lemma, p. 363]; nevertheless, for completeness, we include a proof here.

LEMMA 2.4. *Let $D \subset \mathbb{C}^d$ be a bounded strongly convex domain with $\mathcal{C}^{4,\alpha}$ boundary for $\alpha \in (0,1)$. Let $f\colon D \to \mathbb{C}^d$ and Ω_k for $k > 0$ be as Theorem 1.6. Let $w_0 \in \partial\Omega_k \setminus \{0\}$ for some $k > 0$. Then for all $q \in (0,1)$ there exists an $\varepsilon > 0$ such that*

$$C_+(w_0,q,\varepsilon) \cap \Omega_k = \emptyset \quad \textit{and} \quad C_-(w_0,q,\varepsilon) \subset \Omega_k,$$

where

$$C_+(w_0,q,\varepsilon) := \{w \in \mathbb{C}^d : \mathrm{Re}\langle w - w_0, n(w_0)\rangle > q\|w - w_0\|,\ \|w - w_0\| < \varepsilon\},$$
$$C_-(w_0,q,\varepsilon) := \{w \in \mathbb{C}^d : \mathrm{Re}\langle w - w_0, -n(w_0)\rangle > q\|w - w_0\|,\ \|w - w_0\| < \varepsilon\}.$$

Proof. Recall that for $k > 0$, $E_p(x,k) = \{z \in D : h_x(z) < \frac{1}{2}\ln k\}$, where $h_x(z) := \lim_{w\to x}[K_D(z,w) - K_D(w,p)]$. It follows from Result 2.3 that $E_p(x,k)$ is a convex domain and $\partial E_p(x,k) \cap D$ is $\mathcal{C}^1$ smooth for all $k > 0$. Fix $w_0 \in \partial\Omega_k \setminus \{0\}$, where $\Omega_k := f(E_p(x,k))$. We first observe that there exists $z_0 \in \partial E_p(x,k) \cap D$ such that $f(z_0) = w_0$. Indeed, since $w_0 \in \partial\Omega_k$, there exists a sequence $\{w_j\} \subset \Omega_k$ such that $w_j \to w_0$. For every $j \in \mathbb{N}$, let $z_j := f^{-1}(w_j) \in E_p(x,k)$. Since D is bounded, after passing to a subsequence if necessary, there exists $z_0 \in \overline{E_p(x,k)}$ such that $z_j \to z_0$. By Result 2.3 we have $\overline{E_p(x,k)} \cap \partial D = \{x\}$. We claim that $z_0 \neq x$. Indeed, if $z_0 = x$, then hypothesis (ii) of Theorem 1.6 gives $w_j = f(z_j) \to 0$. But $w_0 \neq 0$. Hence, $z_0 \neq x$. Consequently, $z_0 \in D$. Since f is continuous on D,

we obtain

$$f(z_0) = \lim_{j\to\infty} f(z_j) = \lim_{j\to\infty} w_j = w_0.$$

Moreover, $z_0 \notin E_p(x,k)$, since otherwise $w_0 = f(z_0) \in \Omega_k$, contradicting $w_0 \in \partial\Omega_k$. Therefore, $z_0 \in \partial E_p(x,k) \cap D$.

Since f is biholomorphic, it is a $\mathcal{C}^\infty$ diffeomorphism in a neighborhood of z_0. As $\partial E_p(x,k)$ is $\mathcal{C}^1$ smooth near z_0, it follows that $\partial\Omega_k$ is a $\mathcal{C}^1$ hypersurface in a neighborhood of w_0. In particular, $n(w_0)$ is well defined as the outward unit normal to $\partial\Omega_k$ at w_0. We now establish the property of $C^+(w_0,q,\varepsilon)$.

Fix $q \in (0,1)$ and let

$$c_q := \frac{q}{\sqrt{1-q^2}} > 0.$$

After translating w_0 to the origin and applying a unitary change of coordinates, we may assume that $w_0 = 0$ and $n(w_0) = (0,\dots,0,1)$. Identifying $\mathbb{C}^d \simeq \mathbb{R}^{2d}$, we write any $w \in \mathbb{C}^d$ in the form $w = (y,s)$, where $y \in \mathbb{R}^{2d-1}$ and $s \in \mathbb{R}$ is the coordinate in the direction $n(0)$. Then

$$s = \operatorname{Re}\langle w, n(0)\rangle.$$

Since $\partial\Omega_k$ is $\mathcal{C}^1$ near w_0 and its tangent hyperplane at w_0 is orthogonal to $n(0)$, there exist $r > 0$ and a $\mathcal{C}^1$ function $\rho\colon B_{\mathbb{R}^{2d-1}}(0,r) \to \mathbb{R}$ such that

$$\rho(0) = 0, \qquad \nabla\rho(0) = 0,$$

and, in a sufficiently small neighborhood U of the origin,

$$\partial\Omega_k \cap U = \{(y,s) \in U : s = \rho(y)\} \quad \text{and} \quad \Omega_k \cap U = \{(y,s) \in U : s < \rho(y)\}.$$

Here the latter choice of inequality follows from the fact that $n(0) = \mathbf{e_d}$ is the outward unit normal. Since $\rho(0) = 0$, $\nabla\rho(0) = 0$, and ρ is $\mathcal{C}^1$, we have

$$\rho(y) = o(\|y\|) \quad \text{as } y \to 0.$$

Thus, there exists $r_0 > 0$ such that

$$|\rho(y)| < \frac{c_q}{2}\|y\| \quad \text{whenever } 0 < \|y\| < r_0. \tag{2.2}$$

Shrinking r_0 if necessary, choose $\varepsilon > 0$ such that $B(0,\varepsilon) \subset U$ and every $w = (y,s) \in B(0,\varepsilon)$ satisfies $\|y\| < r_0$. Let $w = (y,s) \in C_+(w_0,q,\varepsilon)$. By the definition of $C_+(w_0,q,\varepsilon)$,

$$s = \operatorname{Re}\langle w, n(0)\rangle > q\|w\| = q\sqrt{\|y\|^2 + s^2}.$$

Squaring the above inequality we get $s^2 > q^2(\|y\|^2 + s^2)$. Consequently,

$$(1-q^2)s^2 > q^2\|y\|^2, \quad \text{so} \quad s > c_q\|y\|.$$

Therefore, from (2.2), we obtain

$$s > c_q\|y\| > \frac{c_q}{2}\|y\| > \rho(y).$$

Thus $w \notin \Omega_k$, because locally in $\Omega_k \cap U$ lies on the side $s < \rho(y)$. Hence, $C_+(w_0, q, \varepsilon) \cap \Omega_k = \emptyset$. The property of $C_-(w_0, q, \varepsilon)$ can be proved by a similar argument. ∎

3. Proofs of main results. Here we present the proofs of Theorems 1.6 and 1.9.

Proof of Theorem 1.6. Let $D \subset \mathbb{C}^d$ be a bounded strongly convex domain with $\mathcal{C}^3$-smooth boundary, $A \in \mathrm{GL}(d, \mathbb{C})$ a Hurwitz matrix, and $f : D \to \mathbb{C}^d$ a biholomorphism satisfying hypotheses (i) and (ii). We first show that $f \in S_A^\partial(D)$ implies that $\Omega_k := f(E_p(x,k))$ is boundary spirallike with respect to A for all $k > 0$. We have $z_0 \in D$ and $f(z_0) = x \in \partial D$. Note that $x \in \overline{E_p(x,k)}$ for all $k > 0$. Hence, there exists a sequence $\{z_j\} \subset E_p(x,k)$ such that $z_j \to x$. By hypothesis (ii), $f(z_j) \to 0$, and hence $0 \in \overline{\Omega_k}$. Indeed, $0 \notin \Omega_k$. Otherwise, there exists $a \in E_p(x,k)$ such that $f(a) = 0$. Since f^{-1} is continuous on $f(D)$,

$$z_j = f^{-1}(f(z_j)) \to f^{-1}(0) = a,$$

which contradicts $z_j \to x \in \partial D$. Therefore $0 \notin \Omega_k$, and consequently $0 \in \partial\Omega_k$ for all $k > 0$. We now show that $e^{tA}\Omega_k \subset \Omega_k$ for all $t \geq 0$ and all $k > 0$. Fix $k > 0$ and $t > 0$ and consider the map $g_t \colon D \to D$ defined by

$$g_t(z) := f^{-1}(e^{tA} f(z)).$$

Since we have assumed that $f \in S_A^\partial(D)$, the map g_t is well defined and $g_t \in \mathrm{Hol}(D, D)$. For sufficiently large $m \in \mathbb{N}$, let

$$w_m := e^{mtA}x, \qquad z_m := f^{-1}(w_m).$$

We have $f(z_0) = x \in \Omega$ and from hypothesis (ii), it follows that $w_m \in \Omega$ for large enough m. Therefore, there exists $m_0 \in \mathbb{N}$ such that $z_m = f^{-1}(w_m) \in D$ for all $m \geq m_0$. Note that

$$\begin{aligned} g_t(z_m) &= f^{-1}(e^{tA}w_m) = f^{-1}(e^{tA}(e^{mtA}x)) = f^{-1}(e^{t(1+m)A}x) = f^{-1}(w_{m+1}) \\ &= z_{m+1}. \end{aligned}$$

Since A is a Hurwitz matrix, $w_m \to 0$ as $m \to \infty$. From (ii), it follows that for every open neighborhood U of x there exists $m_U > 0$ such that

$$\{e^{mtA}x : m \geq m_U\} \subset f(U \cap D).$$

Therefore, for every neighborhood U of x there exists $m_U > 0$ such that $w_m \in f(U \cap D)$ for all $m > m_U$. Consequently, $z_m \in U \cap D$ for all $m > m_U$. Since U is an arbitrary neighborhood, it follows that $g_t(z_m) = z_{m+1} \to x$ as $m \to \infty$. We now fix a point $p \in D$ and consider the following sequence:

$$a_m := K_D(p, z_m) - K_D(p, g_t(z_m)).$$

Note that from the triangle inequality we have

$$|a_m| \leq K_D(z_m, z_{m+1}). \tag{3.1}$$

Since holomorphic maps are non-expanding with respect to the Kobayashi distance, we have the following for all $m \geq m_0 + 1$:

$$K_D(z_m, z_{m+1}) = K_D(g_t(z_{m-1}), g_t(z_m)) \leq K_D(z_{m-1}, z_m).$$

Iterating the above relation, we conclude that

$$K_D(z_m, z_{m+1}) \leq K_D(z_{m_0}, z_{m_0+1}) \quad \text{for all } m \geq m_0 + 1.$$

Therefore, (3.1) yields $\liminf_{m\to\infty} a_m \in \mathbb{R}$. Let $l_t := \liminf_{m\to\infty} a_m$. We now show that $l_t \leq 0$. Indeed, suppose that $l_t > 0$. It follows from the definition of limit inferior that after increasing m_0 if necessary, there exists $\varepsilon \in (0, l_t)$ such that

$$K_D(p, z_m) - K_D(p, z_{m+1}) > \varepsilon \quad \text{for all } m \geq m_0.$$

Equivalently,

$$K_D(p, z_{m+1}) < K_D(p, z_m) - \varepsilon \quad \text{for all } m \geq m_0.$$

Iterating this inequality, we obtain

$$K_D(p, z_{m_0+j}) < K_D(p, z_{m_0}) - j\varepsilon$$

for every $j \geq 1$. Choosing j sufficiently large, the right-hand side of the above equations becomes negative, which is impossible since the Kobayashi distance is non-negative. Hence, $l_t \leq 0$. We choose a subsequence a_{m_l} such that $l_t = \lim_{l\to\infty}[K_D(p, z_{m_l}) - K_D(p, z_{m_l+1})]$. Therefore, we obtain a sequence $\{z_{m_l}\}_{l\in\mathbb{N}}$ such that $z_{m_l} \to x$ as $l \to \infty$ and

$$l_t := \lim_{l\to\infty}[K_D(p, z_{m_l}) - K_D(p, g_t(z_{m_l}))] \leq \tfrac{1}{2}\ln 1.$$

We now infer from Result 2.2 that

$$g_t(E_p(x,k)) \subset F_p(x,k) \quad \text{for all } k > 0.$$

Since D is a bounded, strongly convex domain with $\mathcal{C}^3$ boundary, we conclude from Result 2.3 that $F_p(x,k) = E_p(x,k)$. Consequently, from the expression of the map g_t we get $e^{tA}(f(E_p(x,k))) \subset f(E_p(x,k))$ for all $k > 0$. This proves the forward implication.

We now prove the converse. Suppose that Ω_k is A-boundary spirallike for all $k > 0$. Note that from hypothesis (ii) we have $0 \in \overline{\Omega}$. Moreover, $0 \notin \Omega$. Indeed, if $0 \in \Omega$, then there exists $z \in D$ such that $f(z) = 0$. Since

$$D = \bigcup_{k>0} E_p(x,k),$$

there exists $k_1 > 0$ such that $z \in E_p(x,k_1)$. Hence $0 = f(z) \in \Omega_{k_1}$, which contradicts the fact that Ω_{k_1} is boundary spirallike. Therefore, $0 \notin \Omega$. Since (ii) gives $0 \in \overline{\Omega}$, we conclude that $0 \in \partial\Omega$. Clearly, for every $w \in f(D)$ there

exist $k_0 > 0$ and $z \in E_p(x, k_0)$ such that $w = f(z)$. Therefore, from the assumption that Ω_k is A-boundary spirallike, for all $k > 0$ we obtain

$$e^{tA}w = e^{tA}(f(z)) \in f(E_p(x, k_0)) \quad \text{for all } t \geq 0.$$

Consequently, $f(D)$ is boundary spirallike with respect to A. ∎

We now present the proof of Theorem 1.9.

Proof of Theorem 1.9. We first prove that the condition

$$\mathrm{Re}\langle -Af(z), n(f(z))\rangle > 0 \quad \text{for all } z \in D$$

implies that $f : D \to f(D)$ is A-boundary spirallike. In view of Theorem 1.6, it suffices to show that $\Omega_k := f(E_p(x, k))$ is A-boundary spirallike for all $k > 0$. Note that it follows from (ii) that $0 \in \overline{\Omega}_k$ for all $k > 0$. We first observe that $0 \in \partial\Omega_k$ for every $k > 0$. We now show that $0 \in \partial\Omega_k$ for all $k > 0$. Indeed if $0 \in \Omega_k$, then there exists $a \in E_p(x, k)$ such that $f(a) = 0$. Since $x \in \overline{E_p(x, k)}$, we may choose a sequence $\{z_j\} \subset E_p(x, k)$ such that $z_j \to x$. By (ii),

$$f(z_j) \to 0 = f(a).$$

Since f^{-1} is continuous on $f(D)$, it follows that

$$z_j = f^{-1}(f(z_j)) \to f^{-1}(0) = a \in D,$$

which contradicts $z_j \to x \in \partial D$. Hence $0 \notin \Omega_k$, and therefore $0 \in \partial\Omega_k$ for all $k > 0$. We now argue by contradiction. Suppose, to the contrary, there exists $k_0 > 0$ such that Ω_{k_0} is not A-boundary spirallike. Hence, there exists $w \in \Omega_{k_0}$ such that the set $\{e^{tA}w : t > 0\}$ is not contained in Ω_{k_0}. Let

$$t_0 := \inf\{t > 0 : e^{tA}w \notin \Omega_{k_0}\}, \quad w_0 := e^{t_0 A}w.$$

Clearly, $t_0 \in (0, \infty)$ and $w_0 \neq 0$. Let us show that $w_0 \in \partial\Omega_{k_0}$. Note that since f is a biholomorphism, Ω_{k_0} is open and connected. Indeed, if $w_0 \in \Omega_{k_0}$, then there exists $\varepsilon > 0$ such that $e^{sA}w \in \Omega_{k_0}$ for all $s \in [t_0 - \varepsilon, t_0 + \varepsilon)$. It follows from the definition of t_0 that there exists $t_\varepsilon \in [t_0, t_0 + \varepsilon)$ such that $e^{t_\varepsilon A}w \notin \Omega_{k_0}$. This gives a contradiction. Therefore, we conclude that $w_0 \notin \Omega_{k_0}$. On the other hand, if we assume that $w_0 \in \mathbb{C}^d \setminus \overline{\Omega}_{k_0}$, then it follows that there exists $\varepsilon_0 > 0$ such that $e^{sA}w \notin \Omega_{k_0}$ for all $s \in (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$. This also leads to a contradiction with the definition of $t_0 > 0$. Consequently, we conclude that $w_0 \in \partial\Omega_{k_0} \setminus \{0\}$.

Again, from the construction of $t_0 > 0$, we have $e^{tA}w \in \Omega_{k_0}$ for all $t \in (0, t_0)$. Equivalently,

$$e^{-\tau A}w_0 = e^{(t_0 - \tau)A}w \in \Omega_{k_0} \quad \text{for all } \tau \in (0, t_0).$$

Fix $q \in (0, 1)$. Since $w_0 \in \partial\Omega_{k_0} \setminus \{0\}$, Lemma 2.4 implies that there exists $\varepsilon_q > 0$ such that

$$C_+(w_0, q, \varepsilon_q) \cap \Omega_{k_0} = \emptyset, \tag{3.2}$$

where

$$C_+(w_0, q, \varepsilon_q) = \{\zeta \in \mathbb{C}^d : \operatorname{Re}\langle \zeta - w_0, n(w_0)\rangle > q\|\zeta - w_0\|,\ \|\zeta - w_0\| < \varepsilon_q\}.$$

Note that $\lim_{\tau\to 0^+} e^{-\tau A}w_0 = w_0$. Hence, there exists $\delta_q > 0$ such that

$$\|e^{-\tau A}w_0 - w_0\| < \varepsilon_q \quad \text{for all } 0 < \tau < \delta_q.$$

Therefore, for every $0 < \tau < \min\{t_0, \delta_q\}$, we have

$$e^{-\tau A}w_0 \in \Omega_{k_0} \quad \text{and} \quad \|e^{-\tau A}w_0 - w_0\| < \varepsilon_q.$$

Hence (3.2) implies that

$$e^{-\tau A}w_0 \notin C_+(w_0, q, \varepsilon_q),$$

since otherwise $C_+(w_0, q, \varepsilon_q) \cap \Omega_{k_0} \neq \emptyset$. Consequently,

$$\operatorname{Re}\langle e^{-\tau A}w_0 - w_0, n(w_0)\rangle \leq q\|e^{-\tau A}w_0 - w_0\| \quad \text{for } 0 < \tau < \min\{t_0, \delta_q\}. \tag{3.3}$$

Note that $Aw_0 \neq 0$ implies that $e^{-\tau A}w_0 \neq w_0$ for all sufficiently small $\tau > 0$. Hence, (3.3) yields

$$\operatorname{Re}\left\langle \frac{e^{-\tau A}w_0 - w_0}{\|e^{-\tau A}w_0 - w_0\|}, n(w_0)\right\rangle \leq q. \tag{3.4}$$

Now, $e^{-\tau A}w_0 - w_0 = -\tau Aw_0 + o(\tau)$ as $\tau \to 0^+$. Therefore,

$$\lim_{\tau\to 0^+} \frac{e^{-\tau A}w_0 - w_0}{\|e^{-\tau A}w_0 - w_0\|} = -\frac{Aw_0}{\|Aw_0\|}.$$

Letting $\tau \to 0^+$ in (3.4), we obtain

$$\operatorname{Re}\left\langle -\frac{Aw_0}{\|Aw_0\|}, n(w_0)\right\rangle \leq q.$$

Since $q \in (0, 1)$ was arbitrary, we conclude that

$$\operatorname{Re}\left\langle -\frac{Aw_0}{\|Aw_0\|}, n(w_0)\right\rangle \leq 0,$$

and consequently

$$\operatorname{Re}\langle -Aw_0, n(w_0)\rangle \leq 0. \tag{3.5}$$

We next show that there exists $b_0 \in D$ such that $w_0 = f(b_0)$. Choose a sequence $\{t_j\} \subset (0, t_0)$ such that $t_j \to t_0$. Since

$$e^{t_j A}w \in \Omega_{k_0} = f(E_p(x, k_0)),$$

for every $j \in \mathbb{N}$, there exists $z_j \in E_p(x, k_0)$ such that $f(z_j) = e^{t_j A}w$. Since D is bounded, after passing to a subsequence if necessary, we conclude that there exists

$$b_0 \in \overline{E_p(x, k_0)} \subset \overline{D}$$

such that $z_j \to b_0$. Since D is strongly convex with $\mathcal{C}^{4,\alpha}$ boundary, it follows from Result 2.3 that

$$\overline{E_p(x,k_0)} \cap \partial D = \{x\}.$$

Observe that $b_0 \neq x$. Indeed, if $b_0 = x$, then by hypothesis (ii), it follows that $f(z_j) \to 0$. But according to the construction we have $f(z_j) = e^{t_j A}w \to e^{t_0 A}w = w_0 \neq 0$. which is not possible. Therefore, $b_0 \in D$. By the continuity of f on D, we obtain

$$f(b_0) = \lim_{j\to\infty} f(z_j) = \lim_{j\to\infty} e^{t_j A}w = w_0.$$

Moreover, since $w_0 \notin \Omega_{k_0}$, we necessarily have $b_0 \notin E_p(x,k_0)$, and hence $b_0 \in \partial E_p(x,k_0) \cap D$. Thus, $w_0 = f(b_0)$ for some $b_0 \in D$. Substituting this into (3.5), we obtain

$$\operatorname{Re}\langle -Af(b_0), n(f(b_0))\rangle \leq 0,$$

which contradicts the hypothesis. Hence, the sufficiency part of the proof is complete.

Conversely, suppose that $f \in S_A^{\partial}(D)$. In view of Theorem 1.6, the domain $\Omega_k := f(E_p(x,k))$ is A-boundary spirallike for every $k > 0$. We shall prove that

$$\operatorname{Re}\langle -Af(z), n(f(z))\rangle > 0 \quad \text{for all } z \in D.$$

Fix $z_0 \in D$ arbitrarily and let $w_0 := f(z_0)$. Recall that from Result 2.3, it follows that there exists $h_{x,p}(z_0) \in \mathbb{R}$ such that

$$h_{x,p}(z_0) = \lim_{w\to x}(K_D(z_0,w) - K_D(p,w)).$$

Choose $k_0 = e^{2h_{x,p}(z_0)}$. Then, by the definition of $E_p(x,k)$, we have $z_0 \in \partial E_p(x,k_0) \setminus \{x\}$. Since $z_0 \in \partial E_p(x,k_0) \cap D$, we have $w_0 = f(z_0) \in \partial\Omega_{k_0}$. Moreover, since $f \in S_A^{\partial}(D)$, we have $0 \in \partial f(D)$ and hence $0 \notin f(D)$. Therefore, $w_0 \neq 0$, and consequently $w_0 \in \partial\Omega_{k_0} \setminus \{0\}$. Note that by hypothesis,

$$\operatorname{Re}\langle Af(z), n(f(z))\rangle \neq 0 \quad \text{for all } z \in D.$$

We show that

$$\operatorname{Re}\langle -Aw_0, n(w_0)\rangle \geq 0. \tag{3.6}$$

Suppose to the contrary that $\operatorname{Re}\langle -Aw_0, n(w_0)\rangle < 0$. Clearly, $Aw_0 \neq 0$. Hence, if

$$q_0 := \operatorname{Re}\left\langle \frac{Aw_0}{\|Aw_0\|}, n(w_0)\right\rangle,$$

then from our assumption $q_0 > 0$. Choose $q \in (0,1)$ such that $0 < q < q_0$. By Lemma 2.4, there exists $\epsilon_q > 0$ such that

$$C_+(w_0,q,\epsilon_q) \cap \Omega_{k_0} = \emptyset,$$

where $C_+(w_0,q,\epsilon_q):=\{\zeta\in\mathbb{C}^d:\operatorname{Re}\langle\zeta-w_0,n(w_0)\rangle>q\|\zeta-w_0\|,\ \|\zeta-w_0\|<\epsilon_q\}$. Now arguing as in the previous part we get

$$\lim_{t\to0^+}\frac{e^{tA}w_0-w_0}{\|e^{tA}w_0-w_0\|}=\frac{Aw_0}{\|Aw_0\|}.$$

Consequently,

$$\lim_{t\to0^+}\operatorname{Re}\left\langle\frac{e^{tA}w_0-w_0}{\|e^{tA}w_0-w_0\|},n(w_0)\right\rangle=q_0.$$

Since $q<q_0$, there exists $\delta_1>0$ such that

$$\operatorname{Re}\left\langle\frac{e^{tA}w_0-w_0}{\|e^{tA}w_0-w_0\|},n(w_0)\right\rangle>q\quad\text{ for all } t\in(0,\delta_1).$$

Hence,

$$\operatorname{Re}\langle e^{tA}w_0-w_0,n(w_0)\rangle>q\|e^{tA}w_0-w_0\|\quad\text{ for all } t\in(0,\delta_1).$$

Moreover, $\|e^{tA}w_0-w_0\|\to0$ as $t\to0^+$. Thus, there exists $\delta_2>0$ such that

$$\|e^{tA}w_0-w_0\|<\epsilon_q$$

for all $t\in(0,\delta_2)$. Therefore, if $0<t<\delta:=\min\{\delta_1,\delta_2\}$, then $e^{tA}w_0\in C_+(w_0,q,\epsilon_q)$. Noting that $C_+(w_0,q,\epsilon_q)$ is open and $C_+(w_0,q,\epsilon_q)\cap\Omega_{k_0}=\emptyset$, it follows that

$$e^{tA}w_0\notin\overline{\Omega}_{k_0}\quad\text{ for all } t\in(0,\delta). \tag{3.7}$$

Indeed, if the above conclusion is not true, then for small perturbations of t it will follow that $e^{tA}w_0$ will intersect both Ω_{k_0} and $C_+(w_0,q,\epsilon_q)$. That is not possible. We now obtain a contradiction using the A-boundary spirallikeness of Ω_{k_0}. Since Ω_{k_0} is A-boundary spirallike, we have $e^{tA}\Omega_{k_0}\subset\Omega_{k_0}$ for all $t>0$. By continuity of the exponential, we have $e^{tA}\overline{\Omega}_{k_0}\subset\overline{\Omega}_{k_0}$ for each fix $t>0$. In particular, this holds for every $t\in(0,\delta)$. Consequently, $e^{tA}w_0\in\overline{\Omega}_{k_0}$. This contradicts (3.7). The contradiction arises from the assumption that $q_0>0$. Therefore,

$$\operatorname{Re}\langle-Aw_0,n(w_0)\rangle\geq0.$$

Since $w_0=f(z_0)$, we obtain

$$\operatorname{Re}\langle-Af(z_0),n(f(z_0))\rangle\geq0.$$

On the other hand, by the hypothesis of the theorem, $\operatorname{Re}\langle-Af(z_0),n(f(z_0))\rangle\neq0$. Combining this with the preceding inequality, we conclude that

$$\operatorname{Re}\langle-Af(z_0),n(f(z_0))\rangle>0.$$

Since $z_0\in D$ was arbitrary, it follows that $\operatorname{Re}\langle-Af(z),n(f(z))\rangle>0$ for all $z\in D$. This proves the converse implication and completes the proof. ∎

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