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AN OPEN QUESTION FOR d-TILTING BUNDLES ON GEIGLE–LENZING PROJECTIVE SPACES

BY

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Abstract. We construct a family of 2-tilting bundles on a Geigle–Lenzing projective space of type $(2, 2, p, q)$ via the action of iterated 2-APR mutations. As an application, we give some counterexamples to an open question raised by Herschend et al. [Mem. Amer. Math. Soc. 285 (2023), no. 1412].

1. Introduction. Let k be an algebraically closed field and let $d, n \geq 1$ be integers. Following [HIMO], the *Geigle–Lenzing* (*GL*) *complete intersection* is the k-algebra

$$R := k[T_0, \dots, T_d, X_1, \dots, X_n]/(X_i^{p_i} - \ell_i \mid 1 \leq i \leq n),$$

where $\ell_1, \dots, \ell_n \in k[T_0, \dots, T_d]$ are linear forms in general position, and $p_1, \dots, p_n \geq 2$ are integers called *weights*. It is graded by an abelian group

$$\mathbb{L} := \langle \vec{x}_1, \dots, \vec{x}_n, \vec{c} \rangle / \langle p_i \vec{x}_i - \vec{c} \mid 1 \leq i \leq n \rangle.$$

The category $\mathsf{coh}\,\mathbb{X}$ of coherent sheaves on a *Geigle–Lenzing* (*GL*) *projective space* $\mathbb{X}$ of dimension d is defined by applying Serre's construction [S] to the $\mathbb{L}$-graded k-algebra R. The GL projective spaces are higher-dimensional analogs of weighted projective lines in the sense of Geigle–Lenzing [GL]. They link various subjects, including representation theory, singularity theory, homological mirror symmetry and (non-commutative) algebraic geometry; see e.g. [CRW, FU, HIMO, KST1, KST2, KLM].

Denote by $\mathsf{CM}^{\mathbb{L}} R$ the category of $\mathbb{L}$-graded Cohen–Macaulay modules, by $\underline{\mathsf{CM}}^{\mathbb{L}} R$ its associated stable category, and by $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$ the bounded derived category of $\mathsf{coh}\,\mathbb{X}$. In [HIMO], Herschend, Iyama, Minamoto and Oppermann studied when $\underline{\mathsf{CM}}^{\mathbb{L}} R$ or $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$ has a d-tilting object. This property holds significance within the framework of higher Auslander–Reiten theory [HIO, I1]. In fact, the endomorphism algebra of a d-tilting bundle on $\mathbb{X}$ is a d-representation infinite algebra in the sense of [HIO]. They also investigated

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the conditions under which a d-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$ can lift to a d-tilting bundle on $\mathbb{X}$. Naturally, they posed the following problem.

PROBLEM 1.1 ([HIMO, Problem 7.39]). *Let $V \in \mathsf{CM}^{\mathbb{L}}R$ be a d-tilting bundle on $\mathbb{X}$. Is V a d-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$?*

In the case $d = 1$ or $n \leq d+1$, this is true [HIMO, KLM]. For the general case, a partial positive answer to Problem 1.1 was given in [HIMO]; see Theorem 2.7. Based on the observation that the action of d-APR mutations preserves d-tilting bundles, we state Lemma 2.9. In the case $d = 2$ and $n = 4$, we construct a family of 2-tilting bundles on a GL projective space of weight type $(2, 2, p, q)$, and thus obtain a family of 2-representation infinite algebras. As an application, we find some counterexamples to Problem 1.1. Furthermore, we also show that the converse of Theorem 2.7 is not true in general; see Example 4.2.

Let $\vec{s} := \sum_{i=1}^{4} \vec{x}_i$, $\vec{\omega} := \vec{c} - \vec{s}$ and $\vec{\delta} := 2\vec{c} + 2\vec{\omega}$. For each element $\vec{\ell} = \sum_{i=1}^{4} \ell_i \vec{x}_i$ in the interval $[\vec{s}, \vec{s} + \vec{\delta}]$, define

$$U^{\vec{\ell}} := \rho(R/(X_i^{\ell_i} \mid 1 \leq i \leq 4)),$$

where ρ denotes the composition $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{L}} R) \to \mathsf{D}^{\mathbb{L}}_{\mathrm{sg}}(R) \xrightarrow{\sim} \underline{\mathsf{CM}}^{\mathbb{L}}R$ and the set $\mathcal{S}$ consists of elements $\vec{x} = \sum_{i=1}^{4} \lambda_i \vec{x}_i + \lambda \vec{c}$ (in normal form) in $\mathbb{L}$ satisfying

$$0 < 2\lambda + \#\{1 \leq i \leq 4 \mid \lambda_i \geq 1\} \leq \#\{3 \leq i \leq 4 \mid \lambda_i \in \{0, 1\}\}.$$

The main theorem of this paper is the following.

MAIN THEOREM 1.2 (Theorem 3.2, Propositions 3.6 and 3.7). *Let $d=2$, $n = 4$ and $(R, \mathbb{L})$ be a GL complete intersection of type $(2, 2, p, q)$ with integers $p, q \geq 2$, and $\mathbb{X}$ the corresponding GL projective space. Let J be an upset with respect to $[\vec{s}, \vec{s} + \vec{\delta}]$ and $I := [\vec{s}, \vec{s} + \vec{\delta}] \setminus J$. We put*

$$M := \Big(\bigoplus_{\vec{\ell} \in I} U^{\vec{\ell}}\Big) \oplus \Big(\bigoplus_{\vec{z} \in J} U^{\vec{z}}(-\vec{\omega})\Big) \oplus \Big(\bigoplus_{\vec{x} \in \mathcal{S}} R(\vec{x})\Big).$$

Then M gives a 2-tilting bundle on $\mathbb{X}$. Moreover, we have the following:

(a) *If $p = 2$, then M is not a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$ if and only if $J = [\vec{s} + m\vec{x}_4, \vec{s} + (q-2)\vec{x}_4]$ for some $m \in \{1, \ldots, q-2\}$.*
(b) *If $p = q = 3$, then M is not a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$ if and only if $J = [\vec{s} + \vec{x}_i, \vec{s} + \vec{x}_3 + \vec{x}_4]$ for $i = 3, 4$.*

Recall from [HIMO] that we have the following diagram which shows connections between the relevant notions (here the downward arrow denotes restriction):

$$\begin{array}{ccc}
\left\{\begin{array}{c}d\text{-tilting bundles on } \mathbb{X}\\ \text{contained in } \mathsf{CM}^{\mathbb{L}} R\end{array}\right\} & \overset{\text{[HIMO, Theorem 7.23]}}{=\!=\!=} & \left\{\begin{array}{c}\text{slices in } d\text{-cluster tilting}\\ \text{subcategories of } \mathsf{CM}^{\mathbb{L}} R\end{array}\right\} \\
 & & \downarrow \\
\{d\text{-tilting objects in } \underline{\mathsf{CM}}^{\mathbb{L}} R\} & \overset{\text{[HIMO, Theorem 4.53]}}{\hookrightarrow} & \left\{\begin{array}{c}\text{slices in } d\text{-cluster tilting}\\ \text{subcategories of } \underline{\mathsf{CM}}^{\mathbb{L}} R\end{array}\right\}
\end{array}$$

This diagram implies that the bottom inclusion is strict for the counterexamples given in Theorem 1.2.

The paper is organized as follows. In Section 2, we collect some preliminaries on GL projective spaces, d-tilting bundles and d-APR mutations. In Section 3, we prove Theorem 1.2 via the action of iterated d-APR mutations. In Section 4, we give two counterexamples: one to Problem 1.1, and the other to the converse of Theorem 2.7. A key tool for proving Theorem 1.2 is shown in Appendix A.

2. Preliminaries. In this section, we recall some basic notions and facts appearing in the present paper. We refer to [HIMO, KLM] on Geigle–Lenzing complete intersections and projective spaces, and to [I1, J] on higher-dimensional Auslander–Reiten theory. Throughout this paper, we fix an algebraically closed field k and integers $d, n \geq 1$.

2.1. d-cluster tilting subcategories. Let us start by recalling the notion of functorially finite subcategory of an additive category. Let $\mathcal{A}$ be an additive category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$. We say that $\mathcal{C}$ is *contravariantly finite* if for every object $A \in \mathcal{A}$, there exists a morphism $f : C \to A$ with $C \in \mathcal{C}$ such that the induced morphism

$$\mathrm{Hom}_{\mathcal{A}}(C', C) \to \mathrm{Hom}_{\mathcal{A}}(C', A)$$

is surjective for all $C' \in \mathcal{C}$. Such a morphism f is called a *right $\mathcal{C}$-approximation* of A. Dually, we define a *left $\mathcal{C}$-approximation* and a *covariantly finite* subcategory. We say that $\mathcal{C}$ is *functorially finite* if it is both contravariantly and covariantly finite.

Definition 2.1 ([I2, Definition 1.1] and [IY, Section 3]). A full subcategory $\mathcal{C}$ of an exact category or triangulated category $\mathcal{A}$ is called *d-cluster tilting* if it is a functorially finite subcategory of $\mathcal{A}$ such that

$$\begin{aligned}\mathcal{C} &= \{X \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(\mathcal{C}, X) = 0 \text{ for any } 1 \leq i \leq d-1\} \\ &= \{X \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(X, \mathcal{C}) = 0 \text{ for any } 1 \leq i \leq d-1\}\end{aligned}$$

and, in the exact case, $\mathcal{C}$ is both a generating and a cogenerating subcategory of $\mathcal{A}$, that is, for any object $A \in \mathcal{A}$, there exists an admissible epimorphism $C \to A$ with $C \in \mathcal{C}$ and an admissible monomorphism $A \to C'$ with $C' \in \mathcal{C}$.

2.2. Geigle–Lenzing complete intersections and projective spaces. Fix an n-tuple $(p_1, \dots, p_n)$ of integers $p_i \geq 2$, called *weights*. Consider the k-algebra

$$R := k[T_0, \dots, T_d, X_1, \dots, X_n]/(X_i^{p_i} - \ell_i \mid 1 \leq i \leq n),$$

where $\ell_1, \dots, \ell_n \in k[T_0, \dots, T_d]$ are linear forms in general position, meaning that any set of at most $d+1$ linear forms ℓ_i is linearly independent. Let $\mathbb{L}$ be the abelian group on generators $\vec{x}_1, \dots, \vec{x}_n, \vec{c}$ subject to the relations

$$p_1\vec{x}_1 = \cdots = p_n\vec{x}_n = \vec{c}.$$

The element $\vec{c}$ is called the *canonical element* of $\mathbb{L}$. Each element $\vec{x}$ in $\mathbb{L}$ can be uniquely written in a *normal form* as

$$\vec{x} = \sum_{i=1}^{n} \lambda_i \vec{x}_i + \lambda\vec{c} \tag{2.1}$$

with $0 \leq \lambda_i < p_i$ and $\lambda \in \mathbb{Z}$. We can regard R as an $\mathbb{L}$-graded k-algebra by setting $\deg X_i := \vec{x}_i$ and $\deg T_j := \vec{c}$ for any i and j so that $R = \bigoplus_{\vec{x}\in\mathbb{L}} R_{\vec{x}}$, where $R_{\vec{x}}$ denotes the homogeneous component of degree $\vec{x}$. The pair $(R, \mathbb{L})$ is called the *Geigle–Lenzing* (*GL*) *complete intersection* associated with the linear forms $\ell_1, \dots, \ell_n$ and weights $p_1, \dots, p_n$. In particular, R is regular if $n \leq d+1$, and R is a hypersurface if $n = d+2$ by [HIMO, Proposition 3.8].

Let $\mathbb{L}_+$ be the submonoid of $\mathbb{L}$ generated by $\vec{x}_1, \dots, \vec{x}_n, \vec{c}$. Then we equip $\mathbb{L}$ with the structure of a partially ordered set: $\vec{x} \leq \vec{y}$ if and only if $\vec{y} - \vec{x} \in \mathbb{L}_+$. Each element $\vec{x}$ of $\mathbb{L}$ satisfies exactly one of the following two possibilities:

$$\vec{x} \geq 0 \quad \text{or} \quad \vec{x} \leq d\vec{c} + \vec{\omega}, \tag{2.2}$$

where $\vec{\omega} = (n-d-1)\vec{c} - \sum_{i=1}^{n} \vec{x}_i$ is called the *dualizing element* of $\mathbb{L}$. Note that $R_{\vec{x}} \neq 0$ if and only if $\vec{x} \geq 0$ if and only if $\lambda \geq 0$ in its normal form in (2.1). The interval in $\mathbb{L}$ is denoted by $[\vec{x}, \vec{y}] := \{\vec{z} \in \mathbb{L} \mid \vec{x} \leq \vec{z} \leq \vec{y}\}$. Let $I \subset J \subset \mathbb{L}$. We say that I is an *upset* with respect to J if $(I + \mathbb{L}_+) \cap J \subset I$ holds; we also call I a *J-upset*. Moreover, we say that an upset with respect to J is *non-trivial* if it is neither $\emptyset$ nor J. In particular, when $J = \mathbb{L}$, this recovers the upset defined in [HIMO, Definition 3.43].

Denote by $\mathsf{CM}^{\mathbb{L}} R$ the category of $\mathbb{L}$-graded (maximal) Cohen–Macaulay R-modules, which is Frobenius, and thus its stable category $\underline{\mathsf{CM}}^{\mathbb{L}} R$ is a triangulated category by a general result of Happel [H]. By results of Buchweitz [B] and Orlov [O], there exists a triangle equivalence $\mathsf{D}^{\mathbb{L}}_{\mathrm{sg}}(R) \simeq \underline{\mathsf{CM}}^{\mathbb{L}} R$, where $\mathsf{D}^{\mathbb{L}}_{\mathrm{sg}}(R) := \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{L}} R)/\mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{L}} R)$ is the *singularity category* of R. Throughout, we identify objects in $\underline{\mathsf{CM}}^{\mathbb{L}} R$ with objects in $\mathsf{CM}^{\mathbb{L}} R$ without nonzero projective direct summands. The following presents a basic property in the Cohen–Macaulay representation theory. Here, we denote by D the k-dual, that is, $D(-) := \mathrm{Hom}_k(-, k)$.

THEOREM 2.2 (Auslander–Reiten–Serre duality [HIMO, Theorem 4.3]). *There exists a functorial isomorphism for any* $X, Y \in \underline{\mathsf{CM}}^{\mathbb{L}} R$*:*

$$\operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathbb{L}} R}(X, Y) \simeq D \operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathbb{L}} R}(Y, X(\vec{\omega})[d]). \tag{2.3}$$

The category of *coherent sheaves* on a *Geigle–Lenzing* (*GL*) *projective space* $\mathbb{X}$ of dimension d is defined as the quotient category

$$\mathsf{coh}\,\mathbb{X} := \mathsf{mod}^{\mathbb{L}} R / \mathsf{mod}^{\mathbb{L}}_0 R$$

of the category $\mathsf{mod}^{\mathbb{L}} R$ of finitely generated $\mathbb{L}$-graded modules modulo its Serre subcategory $\mathsf{mod}^{\mathbb{L}}_0 R$ consisting of finite-dimensional modules. It follows from [HIMO] that $\mathsf{coh}\,\mathbb{X}$ is a Noetherian abelian category of global dimension d.

Denote by $\pi : \mathsf{mod}^{\mathbb{L}} R \to \mathsf{coh}\,\mathbb{X}$ the natural functor, also called *sheafification*. The object $\mathcal{O} := \pi(R)$ is called the *structure sheaf* of $\mathbb{X}$, and $\mathbb{L}$ acts on $\mathsf{coh}\,\mathbb{X}$ by a grading shift. Each line bundle has the form $\mathcal{O}(\vec{x})$ for a uniquely determined $\vec{x}$ in $\mathbb{L}$, and for any $\vec{x}, \vec{y} \in \mathbb{L}$ and $i \in \mathbb{Z}$, we have

$$\operatorname{Ext}^i_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = \begin{cases} R_{\vec{y}-\vec{x}} & \text{if } i = 0, \\ D(R_{\vec{x}-\vec{y}+\vec{\omega}}) & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Denote by $\mathsf{vect}\,\mathbb{X}$ the full subcategory of $\mathsf{coh}\,\mathbb{X}$ formed by all vector bundles, and by $\mathsf{line}\,\mathbb{X}$ its full subcategory of finite direct sums of line bundles $\mathcal{O}(\vec{x})$ with $\vec{x} \in \mathbb{L}$. The natural functor $\pi : \mathsf{mod}^{\mathbb{L}} R \to \mathsf{coh}\,\mathbb{X}$ restricts to a fully faithful functor $\mathsf{CM}^{\mathbb{L}} R \to \mathsf{vect}\,\mathbb{X}$, and further induces an equivalence $\mathsf{proj}^{\mathbb{L}} R \simeq \mathsf{line}\,\mathbb{X}$. Therefore $\mathsf{CM}^{\mathbb{L}} R$ has two exact structures: one is inherited from $\mathsf{mod}^{\mathbb{L}} R$, and the other is inherited from $\mathsf{coh}\,\mathbb{X}$. The following result investigates a close relationship between them.

PROPOSITION 2.3 ([HIMO, Proposition 5.19(c)]). *For every integer* i *with* $0 \le i \le d-1$*, there exists a functorial isomorphism*

$$\operatorname{Ext}^i_{\mathsf{mod}^{\mathbb{L}} R}(X, Y) \simeq \operatorname{Ext}^i_{\mathbb{X}}(X, Y) \tag{2.5}$$

for any $X \in \mathsf{mod}^{\mathbb{L}} R$ *and* $Y \in \mathsf{CM}^{\mathbb{L}} R$*.*

2.3. d-tilting objects and slices. Let $\mathcal{T}$ be a triangulated category with a suspension functor [1]. In this paper, we only treat the case $\mathcal{T} = \underline{\mathsf{CM}}^{\mathbb{L}} R$ or $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$.

DEFINITION 2.4. An object $V \in \mathcal{T}$ is called *tilting* in $\mathcal{T}$ if

(a) V is *rigid*, that is, $\operatorname{Hom}_{\mathcal{T}}(V, V[i]) = 0$ for all $i \neq 0$,
(b) $\mathsf{thick}\,V = \mathcal{T}$, where $\mathsf{thick}\,V$ denotes the smallest thick subcategory of $\mathcal{T}$ containing V.

If moreover $\mathrm{End}_{\mathcal{T}}(V)$ has global dimension at most d, such a V is called *d-tilting*. A d-tilting object V in $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$ is called a *d-tilting bundle* on $\mathbb{X}$ if $V \in \mathsf{vect}\,\mathbb{X}$.

Definition 2.5 (cf. [HIMO, Definition 7.16]). Let $\mathcal{U}$ be a d-cluster tilting subcategory of $\mathsf{CM}^{\mathbb{L}} R$ (resp. $\underline{\mathsf{CM}}^{\mathbb{L}} R$). An object $V \in \mathcal{U}$ is called a *slice* in $\mathcal{U}$ if the following conditions are satisfied:

(a) $\mathcal{U} = \mathsf{add}\,\{V(\ell\vec{\omega}) \mid \ell \in \mathbb{Z}\}$,
(b) $\mathrm{Hom}_{\mathcal{U}}(V, V(\ell\vec{\omega})) = 0$ for any $\ell > 0$.

In this case, any $\vec{\omega}$-orbit of indecomposable objects in $\mathcal{U}$ contains exactly one element in $\mathsf{add}\,V$.

Recall from [HIMO, Theorem 4.53] that a d-tilting object U in $\underline{\mathsf{CM}}^{\mathbb{L}} R$ induces a d-cluster tilting subcategory $\underline{\mathcal{U}} = \mathsf{add}\,\{U(\ell\vec{\omega}) \mid \ell \in \mathbb{Z}\}$ of $\underline{\mathsf{CM}}^{\mathbb{L}} R$ and further induces a d-cluster tilting subcategory

$$\mathcal{U} = \mathsf{add}\,\{U(\ell\vec{\omega}),\ R(\vec{x}) \mid \ell \in \mathbb{Z},\, \vec{x} \in \mathbb{L}\}$$

of $\mathsf{CM}^{\mathbb{L}} R$. In fact, U is a slice in $\underline{\mathcal{U}}$, which follows from combining [HIMO, Proposition 4.28] and [HIO, Proposition 2.3(b)].

The following theorem investigates the relationship between d-tilting bundles and slices in d-cluster tilting subcategories.

Theorem 2.6 ([HIMO, Theorem 7.23]). *For $V \in \mathsf{CM}^{\mathbb{L}} R$, V is a d-tilting bundle on $\mathbb{X}$ if and only if V is a slice in a d-cluster tilting subcategory of $\mathsf{CM}^{\mathbb{L}} R$.*

The following result gives a partial answer to Problem 1.1.

Theorem 2.7 ([HIMO, Theorem 7.40(b)]). *Let $V \in \mathsf{CM}^{\mathbb{L}} R$ be a d-tilting bundle on $\mathbb{X}$. Take a decomposition $V = P \oplus W$, where $P \in \mathsf{proj}^{\mathbb{L}} R$ is a maximal projective direct summand of V. If $\underline{\mathrm{End}}^{\mathbb{L}}_R(W) = \mathrm{End}^{\mathbb{L}}_R(W)$, then V is a d-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}} R$.*

2.4. d-APR mutations of d-tilting bundles. In this section, we introduce the notion of a d-APR (Auslander–Platzeck–Reiten) mutation of d-tilting bundles (cf. [MY, Definition 2.4] and [IO, Definition 4.12]) as a higher-dimensional analog of APR-mutation in the sense of [G, Definition 4.2].

Definition 2.8. Let $V \in \mathsf{CM}^{\mathbb{L}} R$ be a d-tilting bundle on $\mathbb{X}$, and $V = V' \oplus V''$ be a direct sum decomposition such that $\mathrm{Hom}_{\mathbb{X}}(V'', V') = 0$. Put

$$\mu^{+}_{V'}(V) := V'(-\vec{\omega}) \oplus V'' \quad \text{and} \quad \mu^{-}_{V''}(V) := V' \oplus V''(\vec{\omega}).$$

If moreover V' (resp. V'') is indecomposable, we call $\mu^{+}_{V'}(V)$ (resp. $\mu^{-}_{V''}(V)$) a *d-APR mutation*.

Lemma 2.9. *In the setup of Definition* 2.8, *$\mu^{+}_{V'}(V)$ and $\mu^{-}_{V''}(V)$ are both d-tilting bundles on $\mathbb{X}$.*

Proof. We only prove the case of $\mu^+_{V'}(V)$, as the other case can be shown similarly. By Theorem 2.6, V is a slice in the d-cluster tilting subcategory $\mathcal{U} = \mathsf{add}\{V(\ell\vec{\omega}) \mid \ell \in \mathbb{Z}\}$ of $\mathsf{CM}^{\mathbb{L}} R$, and it is enough to show that $\mu^+_{V'}(V)$ is also a slice in $\mathcal{U}$. Clearly it satisfies the condition (a) of Definition 2.5. It remains to prove the condition (b), that is, $\mathrm{Hom}_{\mathbb{X}}(\mu^+_{V'}(V), \mu^+_{V'}(V)(\ell\vec{\omega})) = 0$ for any $\ell > 0$. This follows from the definition of the slice V and $\mathrm{Hom}_{\mathbb{X}}(V'', V') = 0$. ∎

By an argument parallel to [IO, Lemma 4.14], we have the following.

THEOREM 2.10. *Let $\mathcal{U}$ be a d-cluster tilting subcategory of $\mathsf{CM}^{\mathbb{L}} R$ containing a basic d-tilting bundle V on $\mathbb{X}$. If the quiver of V contains no oriented cycles, then the action of iterated d-APR mutations on the set of basic d-tilting bundles on $\mathbb{X}$ contained in $\mathcal{U}$ is transitive.*

Proof. By Theorem 2.6, V is a slice in the d-cluster tilting subcategory $\mathcal{W} = \mathsf{add}\{V(\ell\vec{\omega}) \mid \ell \in \mathbb{Z}\}$. By [HIMO, Proposition 5.21], $\mathcal{U}$ is closed under the degree shift by $\pm\vec{\omega}$; we have $\mathcal{W} \subset \mathcal{U}$ and thus $\mathcal{U} = \mathcal{W}$. Take a decomposition $V = \bigoplus_{i\in I} U_i$ into indecomposable modules in $\mathsf{CM}^{\mathbb{L}} R$. Let $V_1 := \bigoplus_{i\in I} U_i(a_i\vec{\omega})$ and $V_2 := \bigoplus_{i\in I} U_i(b_i\vec{\omega})$ be any two basic d-tilting bundles on $\mathbb{X}$ contained in $\mathcal{U}$, where $a_i, b_i \in \mathbb{Z}$. Note that the quiver of V contains no oriented cycles, and similarly for the quiver of $\mathcal{U}$. Thus we can number the indecomposable direct summands of $V_1 = W_1 \oplus \cdots \oplus W_n$ in such a way that $\mathrm{Hom}(W_i, W_j) = 0$ for any $i > j$. Therefore we may mutate and obtain

$$\mu^-_{W_1} \circ \cdots \circ \mu^-_{W_n}(V_1) = \mu^-_{V_1}(V_1) = V_1(\vec{\omega}).$$

Hence, without loss of generality, we can assume $a_i \geq b_i$ for all $i \in I$. We set $J := \{i \in I \mid a_i - b_i \text{ is maximal}\}$ and take the decomposition of V_1 into the direct sum of

$$V_1' = \bigoplus_{j\in J} U_j(a_j\vec{\omega}) \quad \text{and} \quad V_1'' = \bigoplus_{t\in I\setminus J} U_t(a_t\vec{\omega}).$$

Then for any $j \in J$ and $t \in I \setminus J$, we have $m := (a_j - b_j) - (a_t - b_t) > 0$ and thus

$$\mathrm{Hom}_{\mathbb{X}}(U_t(a_t\vec{\omega}), U_j(a_j\vec{\omega})) = \mathrm{Hom}_{\mathbb{X}}(U_t(b_t\vec{\omega}), U_j((m + b_j)\vec{\omega})) = 0$$

by the definition of the slice V_2. We number the indecomposable direct summands of $V_1' = U_1' \oplus \cdots \oplus U_\ell'$ with $\mathrm{Hom}(U_i', U_j') = 0$ for any $i > j$. Therefore, we may mutate and obtain

$$\mu^+_{U_\ell'} \circ \cdots \circ \mu^+_{U_1'}(V_1) = \mu^+_{V_1'}(V_1) = \bigoplus_{j\in J} U_j((a_j - 1)\vec{\omega}) \oplus \bigoplus_{t\notin J} U_t(a_t\vec{\omega}).$$

Repeating this procedure, we find that V_1 and V_2 are iterated mutations of each other. ∎

When convenient, we denote $\mathrm{Ext}^i_{\mathsf{coh}\,\mathbb{X}}(-,-)$ by $\mathrm{Ext}^i(-,-)$, or even more simply by ${}^i(-,-)$ for $i \geq 0$, and denote $\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathbb{L}} R}(-,-)$ by $\underline{\mathrm{Hom}}(-,-)$.

3. Proof of Theorem 1.2. Throughout this section, we assume $d = 2$, $n = 4$, let $(R, \mathbb{L})$ be a GL complete intersection of type $(2, 2, p, q)$ with integers $p, q \geq 2$ and $\mathbb{X}$ be the corresponding GL projective space. Recall from [HIMO, Observation 3.3] that R can be normalized as a hypersurface singularity

$$R = k[X_1, \dots, X_4]/(X_1^2 + X_2^2 + X_3^p + X_4^q).$$

Let $\vec{s} := \sum_{i=1}^4 \vec{x}_i$, $\vec{\omega} := \vec{c} - \vec{s}$ and $\vec{\delta} := 2\vec{c} + 2\vec{\omega} = (p-2)\vec{x}_3 + (q-2)\vec{x}_4$. For each $\vec{\ell} = \sum_{i=1}^4 \ell_i \vec{x}_i$ in the interval $[\vec{s}, \vec{s} + \vec{\delta}]$, we put

$$U^{\vec{\ell}} := \rho(R/(X_i^{\ell_i} \mid 1 \leq i \leq 4)) \quad \text{and} \quad U^{\mathrm{CM}} := \bigoplus_{\vec{\ell} \in [\vec{s}, \vec{s}+\vec{\delta}]} U^{\vec{\ell}},$$

where ρ is the composition $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{L}} R) \to \mathsf{D}^{\mathbb{L}}_{\mathrm{sg}}(R) \xrightarrow{\sim} \underline{\mathsf{CM}}^{\mathbb{L}} R$ and the set $\mathcal{S}$ consists of elements $\vec{x} = \sum_{i=1}^4 \lambda_i \vec{x}_i + \lambda \vec{c}$ in normal form satisfying

$$0 < 2\lambda + \#\{1 \leq i \leq 4 \mid \lambda_i \geq 1\} \leq \#\{3 \leq i \leq 4 \mid \lambda_i \in \{0, 1\}\}. \tag{3.1}$$

Proposition 3.1 ([HIMO, Theorem 4.76, Corollary 7.32]).

(a) *U^{CM} is a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}} R$.*

(b) *The object*

$$T := \pi(U^{\mathrm{CM}}) \oplus \Big(\bigoplus_{\vec{x} \in \mathcal{S}} \mathcal{O}(\vec{x}) \Big)$$

is a 2-tilting bundle on $\mathbb{X}$.

The following result presents a family of 2-tilting bundles on $\mathbb{X}$ by the action of iterated 2-APR mutations to the 2-tilting bundle T in Proposition 3.1. Let J be an upset restricted to $[\vec{s}, \vec{s} + \vec{\delta}]$ and $I := [\vec{s}, \vec{s} + \vec{\delta}] \setminus J$. We put

$$M := \Big(\bigoplus_{\vec{\ell} \in I} U^{\vec{\ell}} \Big) \oplus \Big(\bigoplus_{\vec{z} \in J} U^{\vec{z}}(-\vec{\omega}) \Big) \oplus \Big(\bigoplus_{\vec{x} \in \mathcal{S}} R(\vec{x}) \Big).$$

Main Theorem 3.2. *$\pi(M)$ is a 2-tilting bundle on $\mathbb{X}$.*

To prove Theorem 3.2, we need the following observations. By an argument similar to that of [I2, Lemma 3.5], we have the following lemma. Here, $\mathsf{add}\, X$ denotes the full subcategory of $\mathsf{coh}\, \mathbb{X}$ consisting of direct summands of finite direct sums of X.

Lemma 3.3. *Let $X \in \mathsf{coh}\, \mathbb{X}$ and let*

$$0 \to X_0 \to X_1 \to X_2 \to X_3 \to 0$$

be an exact sequence in $\mathsf{coh}\, \mathbb{X}$ with $X_i \in \mathsf{add}\, X$.

(a) *If $W \in \mathsf{coh}\, \mathbb{X}$ satisfies $\mathrm{Ext}^1(W, X) = 0$, then there exists an exact sequence*

$$0 \to (W, X_0) \to \cdots \to (W, X_3) \to {}^2(W, X_0) \to \cdots \to {}^2(W, X_3) \to 0.$$

(b) *If* $Y \in \mathsf{coh}\,\mathbb{X}$ *satisfies* $\mathrm{Ext}^1(X,Y)=0$, *then there exists an exact sequence*

$$0 \to (X_3,Y) \to \cdots \to (X_0,Y) \to {}^2(X_3,Y) \to \cdots \to {}^2(X_0,Y) \to 0.$$

The following result is a key tool to prove Theorem 3.2. We give a complete proof in Appendix A. Here, we put $(p_1,p_2,p_3,p_4) := (2,2,p,q)$.

PROPOSITION 3.4. *For each* $\vec{\ell} \in [\vec{s}, \vec{s}+\vec{\delta}]$, *there exists an exact sequence*

$$\eta: \quad 0 \to \mathcal{O} \to \pi(U^{\vec{\ell}}) \to \bigoplus_{1\le i\le 4} \mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i) \xrightarrow{\gamma} \mathcal{O}(3\vec{c}-\vec{\ell}) \to 0 \tag{3.2}$$

in $\mathsf{coh}\,\mathbb{X}$, *where* $\vec{\ell}=\sum_{i=1}^4 \ell_i\vec{x}_i$ *in normal form and* $\gamma=(X_i^{p_i-\ell_i})_{1\le i\le 4}$.

Recall from [HIMO, Proposition 5.19(b)] that for any $\vec{x} \in \mathbb{L}$ and $\vec{\ell} \in [\vec{s}, \vec{s}+\vec{\delta}]$,

$$\mathrm{Ext}^1(\pi(U^{\vec{\ell}}),\mathcal{O}(\vec{x}))=0 \quad \text{and} \quad \mathrm{Ext}^1(\mathcal{O}(\vec{x}),\pi(U^{\vec{\ell}}))=0.$$

PROPOSITION 3.5. *For any* $\vec{s} \le \vec{\ell}, \vec{z} \le \vec{s}+\vec{\delta}$ *with* $\vec{\ell} \not\ge \vec{z}$,

$$\mathrm{Hom}(\pi(U^{\vec{\ell}}),\pi(U^{\vec{z}}))=0.$$

Proof. By Proposition 3.4, there exists an exact sequence

$$\eta_{\vec{\ell}}: \quad 0 \to \mathcal{O} \to \pi(U^{\vec{\ell}}) \to \bigoplus_{1\le i\le 4} \mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i) \xrightarrow{\gamma_{\vec{\ell}}} \mathcal{O}(3\vec{c}-\vec{\ell}) \to 0,$$

where $\vec{\ell}=\sum_{i=1}^4 \ell_i\vec{x}_i$ and $\gamma_{\vec{\ell}}=(X_i^{p_i-\ell_i})_{1\le i\le 4}$. Write $\vec{z}=\sum_{i=1}^4 z_i\vec{x}_i$ in normal form. Factor the sequence $\eta_{\vec{z}}$ into the two short exact sequences

$$\eta_{\vec{z},1}: \quad 0 \to \mathcal{O} \to \pi(U^{\vec{z}}) \to K \to 0,$$
$$\eta_{\vec{z},2}: \quad 0 \to K \to \bigoplus_{1\le i\le 4} \mathcal{O}(2\vec{c}-\vec{z}+z_i\vec{x}_i) \xrightarrow{\gamma_{\vec{z}}} \mathcal{O}(3\vec{c}-\vec{z}) \to 0.$$

The proof is divided into the following three steps.

STEP 1. We show that $\mathrm{Hom}(\pi(U^{\vec{\ell}}),\mathcal{O})=0$. By Lemma 3.3, applying $\mathrm{Hom}(-,\mathcal{O})$ to the sequence $\eta_{\vec{\ell}}$ we obtain an exact sequence

$$\bigoplus_{1\le i\le 4} (\mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i),\mathcal{O}) \to (\pi(U^{\vec{\ell}}),\mathcal{O}) \to (\mathcal{O},\mathcal{O}) \xrightarrow{\partial} {}^2(\mathcal{O}(3\vec{c}-\vec{\ell}),\mathcal{O}) \to \bigoplus_{1\le i\le 4} {}^2(\mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i),\mathcal{O}).$$

Since $-2\vec{c}+\vec{\ell}-\ell_i\vec{x}_i \le 2\vec{c}+\vec{\omega}$, we infer that the leftmost term is zero. By Auslander–Reiten–Serre duality,

$$\mathrm{Ext}^2(\mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i),\mathcal{O}) \simeq D\,\mathrm{Hom}(\mathcal{O},\mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i+\vec{\omega}))=0$$

for any $i \in \{1, \dots, 4\}$. Similarly, we have

$$\operatorname{Ext}^2(\mathcal{O}(3\vec{c} - \vec{\ell}), \mathcal{O}) \simeq D \operatorname{Hom}(\mathcal{O}, \mathcal{O}(3\vec{c} - \vec{\ell} + \vec{\omega})) = k.$$

Since $\operatorname{Hom}(\mathcal{O}, \mathcal{O}) = k$, the epimorphism ∂ between one-dimensional spaces is an isomorphism. Thus we have $\operatorname{Hom}(\pi(U^{\vec{\ell}}), \mathcal{O}) = 0$.

Step 2. We show that $\operatorname{Hom}(\pi(U^{\vec{\ell}}), K) = 0$. By Lemma 3.3, for $1 \le i \le 4$, we apply $\operatorname{Hom}(-, \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i))$ to the sequence $\eta_{\vec{\ell}}$ and obtain an exact sequence

$$\begin{aligned} 0 \to\ & (\mathcal{O}(3\vec{c} - \vec{\ell}), \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i)) \\ \to\ & \bigoplus_{1 \le j \le 4} (\mathcal{O}(2\vec{c} - \vec{\ell} + \ell_j\vec{x}_j), \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i)) \\ \to\ & (\pi(U^{\vec{\ell}}), \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i)) \to (\mathcal{O}, \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i)). \end{aligned}$$

Clearly, we have $\operatorname{Hom}(\mathcal{O}, \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i)) = 0$. By the assumption $\vec{\ell} \not\ge \vec{z}$, we have $\vec{\ell} - \vec{z} \le 2\vec{c} + \vec{\omega}$ by (2.2), and thus $2\vec{c} - \vec{z} + z_i\vec{x}_i - 3\vec{c} + \vec{\ell} \le 2\vec{c} + \vec{\omega}$, implying $\operatorname{Hom}(\mathcal{O}(3\vec{c} - \vec{\ell}), \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i)) = 0$. This implies that

$$\begin{aligned} (3.3) \qquad & (\pi(U^{\vec{\ell}}), \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i)) \\ & \simeq \bigoplus_{1 \le j \le 4} (\mathcal{O}(2\vec{c} - \vec{\ell} + \ell_j\vec{x}_j), \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i)). \end{aligned}$$

Applying $\operatorname{Hom}(\pi(U^{\vec{\ell}}), -)$ to $\eta_{\vec{z},2}$, we have an exact sequence

$$0 \to (\pi(U^{\vec{\ell}}), K) \to \bigoplus_{1 \le i \le 4} (\pi(U^{\vec{\ell}}), \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i)) \xrightarrow{\gamma_*} (\pi(U^{\vec{\ell}}), \mathcal{O}(3\vec{c} - \vec{z})).$$

Notice that the inequality $2\vec{c} - \vec{z} + z_i\vec{x}_i - 2\vec{c} + \vec{\ell} - \ell_j\vec{x}_j \ge 0$ is equivalent to

$$\sum_{t \ne j} (\ell_t - z_t)\vec{x}_t + z_i\vec{x}_i - z_j\vec{x}_j \ge 0.$$

This holds if and only if $i = j$ and $\ell_t \ge z_t$ for any $t \ne j$. Thus, by (3.3),

$$(3.4) \qquad \bigoplus_{1 \le i \le 4} (\pi(U^{\vec{\ell}}), \mathcal{O}(2\vec{c} - \vec{z} + z_i\vec{x}_i)) \ne 0$$

if and only if there exists a subset I of $H := \{1, \dots, 4\}$ with $|I| = 3$ such that $\ell_t \ge z_t$ for any $t \in I$ and $\ell_j < z_j$ for $j \in H \setminus I$ (since $\vec{\ell} \not\ge \vec{z}$). In this case, it is isomorphic to k. We may assume that (3.4) holds, otherwise we are done. Applying $\operatorname{Hom}(-, \mathcal{O}(3\vec{c} - \vec{z}))$ to the sequence $\eta_{\vec{\ell}}$, we obtain an exact sequence

$$0 \to (\mathcal{O}(3\vec{c}-\vec{\ell}), \mathcal{O}(3\vec{c}-\vec{z})) \to \bigoplus_{1\le i\le 4} (\mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i), \mathcal{O}(3\vec{c}-\vec{z})) \\ \to (\pi(U^{\vec{\ell}}), \mathcal{O}(3\vec{c}-\vec{z})) \to (\mathcal{O}, \mathcal{O}(3\vec{c}-\vec{z})) = 0.$$

Since $\vec{\ell} \not\geq \vec{z}$, we have $\mathrm{Hom}(\mathcal{O}(3\vec{c}-\vec{\ell}), \mathcal{O}(3\vec{c}-\vec{z})) = 0$, implying

$$\mathrm{Hom}(\pi(U^{\vec{\ell}}), \mathcal{O}(3\vec{c}-\vec{z})) \simeq \bigoplus_{1\le i\le 4} \mathrm{Hom}(\mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i), \mathcal{O}(3\vec{c}-\vec{z})). \tag{3.5}$$

By the assumption, there exists a subset I of $H = \{1, \dots, 4\}$ with $|I| = 3$ such that $\ell_t \geq z_t$ for any $t \in I$ and $\ell_j < z_j$ for $j \in H \setminus I$. Then the inequality $3\vec{c} - \vec{z} - 2\vec{c} + \vec{\ell} - \ell_i\vec{x}_i \geq 0$ holds if and only if $i = j$. In this case, (3.5) is isomorphic to k. Thus the nonzero morphism γ_* between one-dimensional spaces is an isomorphism, implying that $\mathrm{Hom}(\pi(U^{\vec{\ell}}), K) = 0$.

Step 3. Applying $\mathrm{Hom}(\pi(U^{\vec{\ell}}), -)$ to $\eta_{\vec{z},1}$, we obtain an exact sequence

$$0 \to (\pi(U^{\vec{\ell}}), \mathcal{O}) \to (\pi(U^{\vec{\ell}}), \pi(U^{\vec{z}})) \to (\pi(U^{\vec{\ell}}), K) \to 0.$$

By Steps 1 and 2, we have $\mathrm{Hom}(\pi(U^{\vec{\ell}}), \pi(U^{\vec{z}})) = 0$. ∎

Proof of Theorem 3.2. First, we claim that $\mathrm{Hom}(\mathcal{O}(\vec{x}), \pi(U^{\vec{\ell}})) = 0$ for any $\vec{x} \in \mathcal{S}$ and $\vec{\ell} \in [\vec{s}, \vec{s}+\vec{\delta}]$. By Lemma 3.3, applying $\mathrm{Hom}(\mathcal{O}(\vec{x}), -)$ to the exact sequence (3.2), we obtain an exact sequence

$$(\mathcal{O}(\vec{x}), \mathcal{O}) \to (\mathcal{O}(\vec{x}), \pi(U^{\vec{\ell}})) \\ \to \bigoplus_{1\le i\le 4} (\mathcal{O}(\vec{x}), \mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i)) \xrightarrow{\gamma_*} (\mathcal{O}(\vec{x}), \mathcal{O}(3\vec{c}-\vec{\ell})).$$

Write $\vec{x} = \sum_{i=1}^n \lambda_i\vec{x}_i + \lambda\vec{c}$ in normal form. Notice that $\vec{x} \in \mathcal{S}$ implies $-1 \leq \lambda \leq 1$. The proof of the claim is divided into the three cases depending on the value of λ,

Case 1. $\lambda = 1$. In this case, $\vec{x} = \vec{c}$. By (2.2), it is straightforward to check that $\mathrm{Hom}(\mathcal{O}(\vec{c}), \mathcal{O}) = 0$ and $\mathrm{Hom}(\mathcal{O}(\vec{c}), \mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i)) = 0$ for each $i = 1, \dots, 4$, which implies that $\mathrm{Hom}(\mathcal{O}(\vec{x}), \pi(U^{\vec{\ell}})) = 0$.

Case 2. $\lambda = 0$. In this case, $\vec{x} = \sum_{i=1}^4 \lambda_i\vec{x}_i$. Since $\vec{x} \in \mathcal{S}$, we have $0 < \#\{1 \leq i \leq 4 \mid \lambda_i \geq 1\}$, yielding $\vec{x} > 0$ and thus $\mathrm{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}) = 0$. On the other hand, since $2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i-\vec{x} \leq 2\vec{c}+\vec{\omega}$, we have $\mathrm{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i)) = 0$ for any $i \in \{1, \dots, 4\}$. This implies that $\mathrm{Hom}(\mathcal{O}(\vec{x}), \pi(U^{\vec{\ell}})) = 0$.

Case 3. $\lambda = -1$. In this case, $\vec{x} = \sum_{i=1}^4 \lambda_i\vec{x}_i - \vec{c}$. Since $\vec{x} \in \mathcal{S}$, we have $2 < \#\{1 \leq i \leq 4 \mid \lambda_i \geq 1\}$, which yields $\vec{c} + \sum_{i=1}^4 (\lambda_i - 1)\vec{x}_i \geq 0$. Thus, we have $-\vec{x} \leq \vec{c} + \vec{\omega} \leq 2\vec{c} + \vec{\omega}$, implying $\mathrm{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}) = 0$.

On the other hand, if $2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i-\vec{x}\not\geq 0$ holds for all $i\in\{1,\dots,4\}$, then $\mathrm{Hom}(\mathcal{O}(\vec{x}),\mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i))=0$ holds for all $i\in\{1,\dots,4\}$ and thus $\mathrm{Hom}(\mathcal{O}(\vec{x}),\pi(U^{\vec{\ell}}))=0$. Otherwise $2\vec{c}-\vec{\ell}+\ell_t\vec{x}_t-\vec{x}\geq 0$ holds for some $t\in\{1,\dots,4\}$, by an easy calculation, which is equivalent to

$$\sum_{j\neq t}(\ell_j+\lambda_j)\vec{x}_j+\lambda_t\vec{x}_t\leq 3\vec{c},$$

yielding $\lambda_t=0$ and $\lambda_j+\ell_j\leq p_j$ for any $j\neq t$. Since $2<\#\{1\leq i\leq 4\mid \lambda_i\geq 1\}$, this implies that $\lambda_j\geq 1$ for any $j\neq t$. Therefore, $2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i-\vec{x}\geq 0$ is only possible for $i=t$. In this case, we have

$$\bigoplus_{1\leq i\leq 4}\mathrm{Hom}(\mathcal{O}(\vec{x}),\mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i))=k\quad\text{and}\quad\mathrm{Hom}(\mathcal{O}(\vec{x}),\mathcal{O}(3\vec{c}-\vec{\ell}))=k,$$

which shows that the nonzero morphism γ_* is an isomorphism. This also implies that $\mathrm{Hom}(\mathcal{O}(\vec{x}),\pi(U^{\vec{\ell}}))=0$, and thus the claim follows.

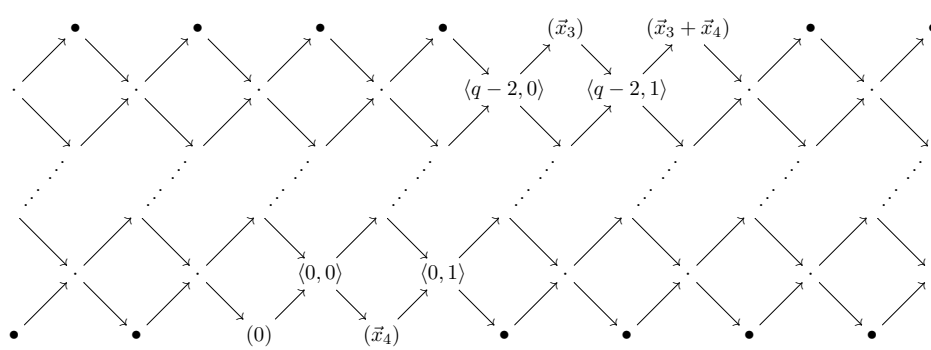
Combining the preceding claim and Proposition 3.5, we deduce that for any $\vec{x}\in\mathcal{S}$, $\vec{\ell}\in I$ and $\vec{z}\in J$, $\mathrm{Hom}(\pi(U^{\vec{\ell}})\oplus\mathcal{O}(\vec{x}),\pi(U^{\vec{z}}))=0$. Hence, the assertion follows immediately from Lemma 2.9. ∎

As an application, we find some counterexamples to Problem 1.1. From now on, we focus on a GL hypersurface $(R,\mathbb{L})$ of type $(2,2,2,q)$ or $(2,2,3,3)$, where $q\geq 2$. Such a GL hypersurface is of Cohen–Macaulay finite type, as discussed in [HIMO].

Case $(2,2,2,q)$. In this case, there exists a triangle equivalence

$$\underline{\mathsf{CM}}^{\mathbb{L}}R\simeq\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,k\mathbb{A}_{q-1}),$$

where $k\mathbb{A}_{q-1}$ denotes the path algebra of the equioriented quiver $\mathbb{A}_{q-1}$. By adding all indecomposable projective modules $R(\vec{x})$ for $\vec{x}\in\mathbb{L}$ and arrows connected to them into the Auslander–Reiten quiver $\mathfrak{A}(\underline{\mathsf{CM}}^{\mathbb{L}}R)$, we obtain the following Auslander–Reiten quiver $\mathfrak{A}(\mathsf{CM}^{\mathbb{L}}R)$ (see [HIMO, Theorems 4.13, 4.46] for details):



Here, a vertex labeled $\langle i, j\rangle$ denotes the position of $U^{\vec{s}+(q-2-i)\vec{x}_4}(j\vec{x}_4)$ for $0 \le i \le q-2$ and $j \in \mathbb{Z}$. The position of $R(\vec{x})$ is denoted by $(\vec{x})$, or by $\bullet$ when $\vec{x}$ is not specified. For any integers $1 \le a < b \le 3$, the positions of $R(\vec{x})$ and $R(\vec{x} + \bar{x}_{ab})$ are overlapping, where $\bar{x}_{ab} := \vec{c} - \vec{x}_a - \vec{x}_b$. This means that every arrow starting or ending at vertex $(\vec{x})$ or $\bullet$ represents four arrows in the actual Auslander–Reiten quiver (each connected to one of the $R(\vec{x})$ and $R(\vec{x} + \bar{x}_{ab})$).

PROPOSITION 3.6. *Assume that $(R, \mathbb{L})$ is given by weight type $(2, 2, 2, q)$ with $q \ge 2$. Then M is not a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$ if and only if $J = [\vec{s} + m\vec{x}_4, \vec{s} + (q-2)\vec{x}_4]$ for some $m \in \{1, \dots, q-2\}$.*

Proof. First, we assume for contradiction that $J = \emptyset$ or $[\vec{s}, \vec{s} + (q-2)\vec{x}_4]$. By Proposition 3.1, $U^{\mathrm{CM}} := \bigoplus_{\vec{s} \le \vec{\ell} \le \vec{s}+(q-2)\vec{x}_4} U^{\vec{\ell}}$ is a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$ and also $U^{\mathrm{CM}}(-\vec{\omega})$ is a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$.

On the other hand, assume $J = [\vec{s} + m\vec{x}_4, \vec{s} + (q-2)\vec{x}_4]$ for some $m \in \{1, \dots, q-2\}$ and thus $I = [\vec{s}, \vec{s} + (m-1)\vec{x}_4]$. Notice that the irreducible morphism $U^{\vec{s}+m\vec{x}_4} \to U^{\vec{s}+(m-1)\vec{x}_4}$ induces the irreducible morphism $U^{\vec{s}+(m-1)\vec{x}_4} \to \tau^{-1}U^{\vec{s}+m\vec{x}_4}$, where τ denotes the Auslander–Reiten translation. By [HIMO, Proposition 4.28], we have

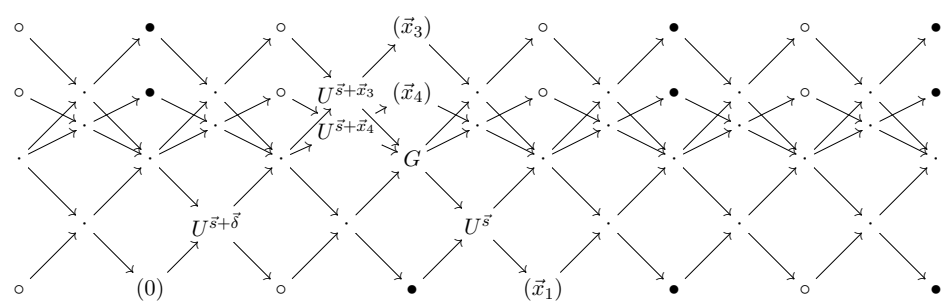
$$\underline{\mathrm{Hom}}(U^{\vec{s}+(m-1)\vec{x}_4}, U^{\vec{s}+m\vec{x}_4}(-\vec{\omega})[-1]) = \underline{\mathrm{Hom}}(U^{\vec{s}+(m-1)\vec{x}_4}, \tau^{-1}U^{\vec{s}+m\vec{x}_4}) = k,$$

and thus M is not a tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$. Hence the assertion follows. ∎

CASE $(2, 2, 3, 3)$. In this case, there exists a triangle equivalence

$$\underline{\mathsf{CM}}^{\mathbb{L}}R \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, k\mathbb{D}_4),$$

where $k\mathbb{D}_4$ denotes the path algebra of type $\mathbb{D}_4$. Adding all indecomposable projective modules $R(\vec{x})$ for $\vec{x} \in \mathbb{L}$ and arrows connected to them into the Auslander–Reiten quiver $\mathfrak{A}(\underline{\mathsf{CM}}^{\mathbb{L}}R)$, we obtain the Auslander–Reiten quiver $\mathfrak{A}(\mathsf{CM}^{\mathbb{L}}R)$, which is depicted below (see [HIMO, Theorems 4.13, 4.46]):



Here, we write $(\vec{x})$ for the position of $R(\vec{x})$, and use $\bullet$ or $\circ$ when $\vec{x}$ is not specified. The positions of $R(\vec{x})$ and $R(\vec{x} + \vec{x}_1 - \vec{x}_2)$ are overlapping. This means that every arrow starting or ending at one of the vertices $(\vec{x})$, $\bullet$ and $\circ$

represents two arrows in the actual Auslander–Reiten quiver (each connected to either $R(\vec{x})$ or $R(\vec{x}+\vec{x}_1-\vec{x}_2)$).

PROPOSITION 3.7. *Assume that $(R,\mathbb{L})$ is given by weight type $(2,2,3,3)$. Then M is not a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$ if and only if $J=[\vec{s}+\vec{x}_i,\vec{s}+\vec{x}_3+\vec{x}_4]$ for $i=3,4$.*

Proof. We assume for contradiction that J is one of the four cases

$$\emptyset,\ \{\vec{s}+\vec{\delta}\},\ (\vec{s},\vec{s}+\vec{\delta}) \quad \text{or} \quad [\vec{s},\vec{s}+\vec{\delta}],$$

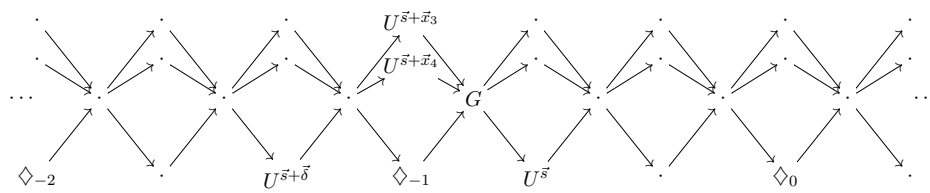
where $\vec{\delta}=\vec{x}_3+\vec{x}_4$. For the case $J=\emptyset$ or $[\vec{s},\vec{s}+\vec{\delta}]$, by Proposition 3.1, M is a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$. Next we only show the case of $J=\{\vec{s}+\vec{\delta}\}$, as one can show the case of $J=(\vec{s},\vec{s}+\vec{\delta}]$ by a similar argument. In this case,

$$M=\Big(\bigoplus_{\vec{\ell}\in[\vec{s},\vec{s}+\vec{\delta})} U^{\vec{\ell}}\Big)\oplus(U^{\vec{s}+\vec{\delta}}(-\vec{\omega}))\oplus\Big(\bigoplus_{\vec{x}\in\mathcal{S}}R(\vec{x})\Big).$$

STEP 1. We show that M is rigid, that is, $\underline{\mathrm{Hom}}(M,M[n])=0$ for any $n\neq 0$. By Propositions 3.5 and 2.3,

$$\mathrm{Hom}^{\mathbb{L}}_R(U^{\vec{\ell}},U^{\vec{s}+\vec{\delta}})=\mathrm{Hom}_{\mathbb{X}}(\pi(U^{\vec{\ell}}),\pi(U^{\vec{s}+\vec{\delta}}))=0,$$

which implies that $\underline{\mathrm{Hom}}(U^{\vec{\ell}},U^{\vec{s}+\vec{\delta}})=0$. By Proposition 3.1, we further have $\underline{\mathrm{Hom}}(U^{\vec{\ell}},U^{\vec{s}+\vec{\delta}}[i])=0$ for any $\vec{\ell}\in[\vec{s},\vec{s}+\vec{\delta})$ and $i\in\mathbb{Z}$. By Auslander–Reiten–Serre duality, $\underline{\mathrm{Hom}}(U^{\vec{s}+\vec{\delta}}(-\vec{\omega}),U^{\vec{\ell}}[i])=0$. On the other hand, keeping the above notations, the Auslander–Reiten quiver of $\underline{\mathsf{CM}}^{\mathbb{L}}R$ has the form



where the symbol $\diamondsuit_i$ denotes the object $U^{\vec{s}+\vec{\delta}}(-\vec{\omega})[i]$ for $i\in\mathbb{Z}$. We compute the dimensions of $\underline{\mathrm{Hom}}(U^{\vec{\ell}},-)$ for each $\vec{\ell}\in[\vec{s},\vec{s}+\vec{\delta})$ as in Figure 1.

Combining everything, we have shown that $\underline{\mathrm{Hom}}(U^{\vec{\ell}},U^{\vec{s}+\vec{\delta}}(-\vec{\omega})[i])=0$ for any $\vec{\ell}\in[\vec{s},\vec{s}+\vec{\delta})$ and $i\neq 0$. Hence, M is rigid.

STEP 2. We show that $\mathsf{thick}\,M=\underline{\mathsf{CM}}^{\mathbb{L}}R$. Note that there are the two triangles

$$U^{\vec{s}+\vec{\delta}}(-\vec{\omega})[-1]\to G\to U^{\vec{s}}\to U^{\vec{s}+\vec{\delta}}(-\vec{\omega}), \tag{3.6}$$

$$U^{\vec{s}+\vec{\delta}}\to U^{\vec{s}+\vec{x}_3}\oplus U^{\vec{s}+\vec{x}_4}\to G\to U^{\vec{s}+\vec{\delta}}[1]. \tag{3.7}$$

Thus, $G\in\mathsf{thick}\,M$ by (3.6), and further $U^{\vec{s}+\vec{\delta}}\in\mathsf{thick}\,M$ by (3.7). By Proposition 3.1, U^{CM} is a tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$. Since $U^{\mathrm{CM}}\in\mathsf{thick}\,M$, we have $\mathsf{thick}\,M=\mathsf{thick}\,U^{\mathrm{CM}}=\underline{\mathsf{CM}}^{\mathbb{L}}R$.

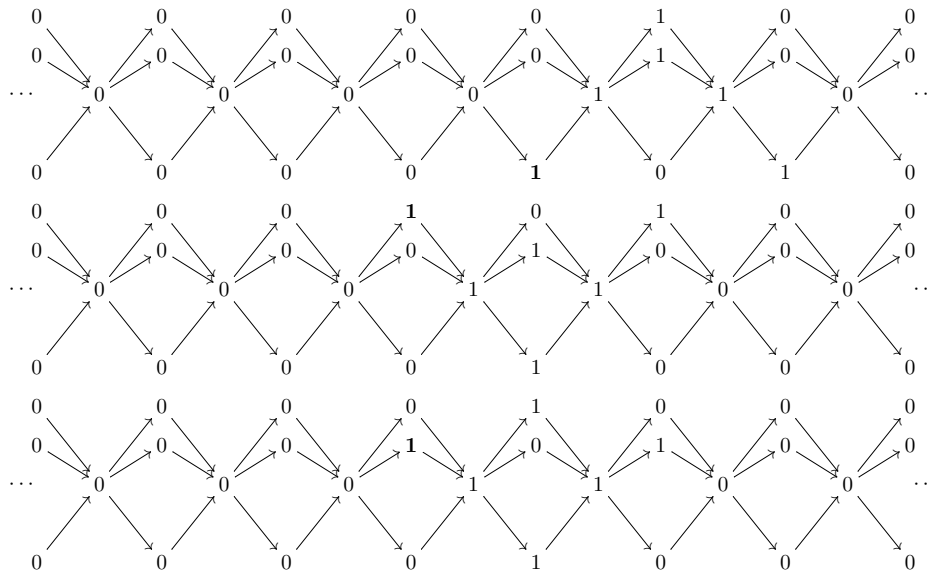
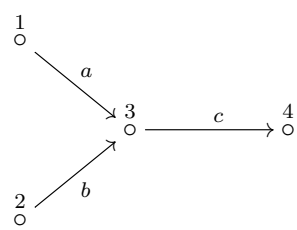


Fig. 1. The dimensions of $\underline{\mathrm{Hom}}(U^{\vec{s}}, -)$, $\underline{\mathrm{Hom}}(U^{\vec{s}+\vec{x}_3}, -)$ and $\underline{\mathrm{Hom}}(U^{\vec{s}+\vec{x}_4}, -)$, respectively

Step 3. The endomorphism algebra of M is presented by the quiver



with relations $ac = 0$ and $bc = 0$. Thus, gl.dim $\mathrm{End}(M) = 2$.

Hence, M is a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$, yielding a contradiction.

Finally, we only show the case for $J = [\vec{s} + \vec{x}_3, \vec{s} + \vec{x}_3 + \vec{x}_4]$, as one can show the other case similarly. By Figure 1, we have

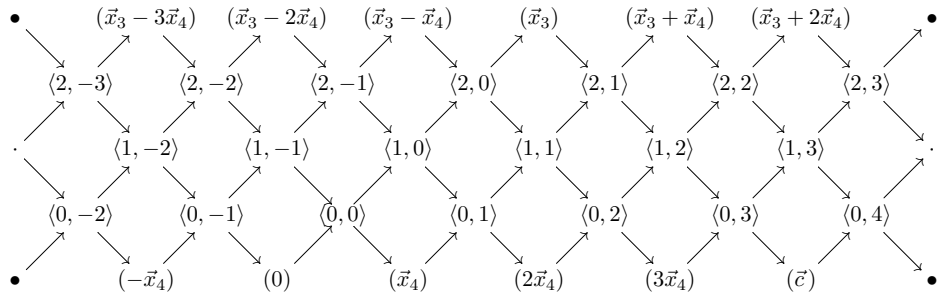
$$\mathrm{Hom}(U^{\vec{s}+\vec{x}_4}, U^{\vec{s}+\vec{x}_3}(-\vec{\omega})[-1]) = \mathrm{Hom}(U^{\vec{s}+\vec{x}_4}, \tau^{-1}U^{\vec{s}+\vec{x}_3}) = k,$$

where τ is the Auslander–Reiten translation. Thus, M is not tilting. ∎

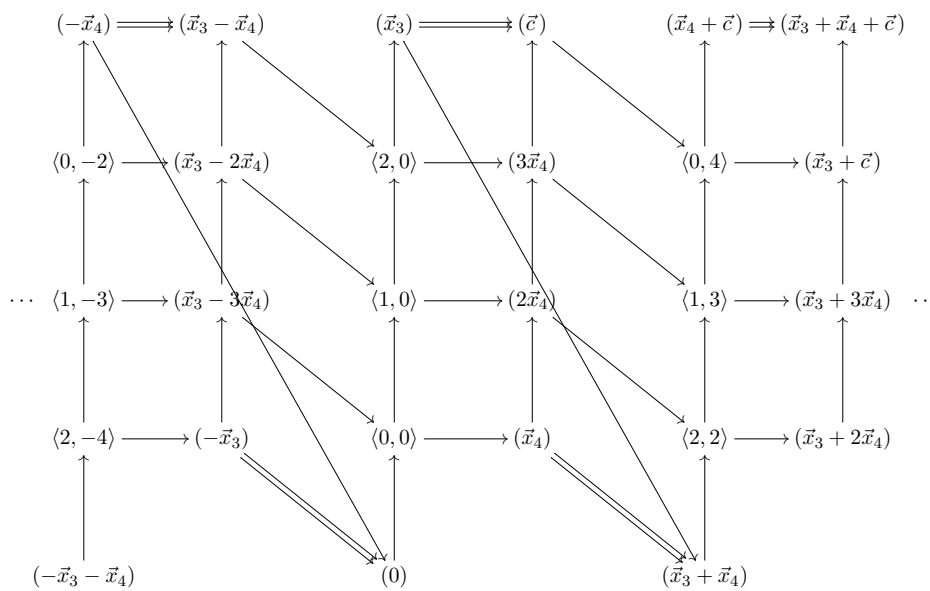
4. Examples. In this section, we will give two counterexamples: one to Problem 1.1, and the other to the converse of Theorem 2.7.

Let us first consider the former.

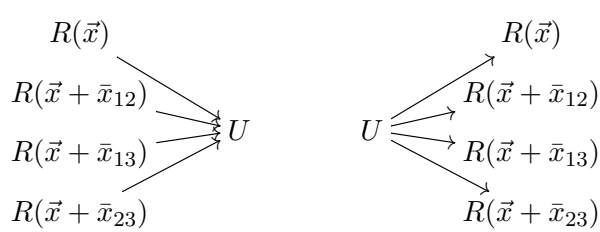
Example 4.1. Let $(R, \mathbb{L})$ be a GL hypersurface of weight type $(2, 2, 2, 4)$, and $\mathbb{X}$ be the corresponding GL projective space. Keeping the earlier notations, the Auslander–Reiten quiver $\mathfrak{A}(\mathsf{CM}^{\mathbb{L}}\, R)$ is depicted below.



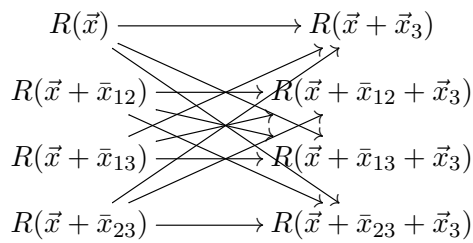
The object $U^{\mathrm{CM}} = \bigoplus_{i=0}^{2}\langle i,0\rangle$ is a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$, and thus $\mathcal{U} := \mathsf{add}\,\{U^{\mathrm{CM}}(\ell\vec{\omega}),\ R(\vec{x}) \mid \ell \in \mathbb{Z},\ \vec{x} \in \mathbb{L}\}$ is a 2-cluster tilting subcategory of $\mathsf{CM}^{\mathbb{L}}R$. Notice that $\langle i,j\rangle(\vec{\omega}) = \langle 2-i, i+j-4\rangle$ and $\langle i,j\rangle(-\vec{\omega}) = \langle 2-i, i+j+2\rangle$. One can compute the following Auslander–Reiten quiver of $\mathcal{U}$ from $\mathfrak{A}(\mathsf{CM}^{\mathbb{L}}\,R)$:



With regard to the arrows, the following situations occur for $U = \langle i,j\rangle$ and $\vec{x} \in \mathbb{L}$:



$$R(\vec{x}) \longrightarrow R(\vec{x}+\vec{x}_4)$$
$$R(\vec{x}+\bar{x}_{12}) \longrightarrow R(\vec{x}+\bar{x}_{12}+\vec{x}_4)$$
$$R(\vec{x}+\bar{x}_{13}) \longrightarrow R(\vec{x}+\bar{x}_{13}+\vec{x}_4)$$
$$R(\vec{x}+\bar{x}_{23}) \longrightarrow R(\vec{x}+\bar{x}_{23}+\vec{x}_4)$$



For simplicity, we denote them by

$$(\vec{x}) \to U, \qquad U \longrightarrow (\vec{x}),$$
$$(\vec{x}) \to (\vec{x}+\vec{x}_4), \qquad (\vec{x}) \Longrightarrow (\vec{x}+\vec{x}_3).$$

In this example, one can check that the set $\mathcal{S}$ in (3.1) is given by $H \cup \{\vec{x}+\bar{x}_{ab} \mid \vec{x} \in H,\ 1 \le a < b \le 3\}$, where $H := \{\vec{x}_3, \vec{x}_3+\vec{x}_4, \vec{x}_4, 2\vec{x}_4, 3\vec{x}_4, \vec{c}\}$. The 2-tilting bundle T of Proposition 3.1 is given by

$$T = U^{\mathrm{CM}} \oplus \bigoplus_{\vec{x}\in\mathcal{S}} R(\vec{x}).$$

Take a decomposition $T = \langle 0,0\rangle \oplus F$. Since $\mathrm{Hom}(F, \langle 0,0\rangle) = 0$, we may mutate and obtain a 2-tilting bundle

$$M = \langle 1,0\rangle \oplus \langle 2,0\rangle \oplus \langle 2,2\rangle \oplus \bigoplus_{\vec{x}\in\mathcal{S}} R(\vec{x})$$

on $\mathbb{X}$. However, $\langle 1,0\rangle \oplus \langle 2,0\rangle \oplus \langle 2,2\rangle$ is not a tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$ since

$$\underline{\mathrm{Hom}}(\langle 1,0\rangle, \langle 2,2\rangle[-1]) = \underline{\mathrm{Hom}}(\langle 1,0\rangle, \langle 0,1\rangle) = k.$$

Finally, we give a counterexample to the converse of Theorem 2.7.

Example 4.2. Let $(R,\mathbb{L})$ be a GL hypersurface of type $(2,2,3,3)$, and let $\mathbb{X}$ be the corresponding GL projective space. Keeping the earlier notations, we define

$$M = \Big(\bigoplus_{\vec{\ell}\in[\vec{s},\vec{s}+\vec{\delta})} U^{\vec{\ell}}\Big) \oplus (U^{\vec{s}+\vec{\delta}}(-\vec{\omega})) \oplus \Big(\bigoplus_{\vec{x}\in\mathcal{S}} R(\vec{x})\Big),$$

where $\vec{\delta} := \vec{x}_3+\vec{x}_4$. Combining Theorem 3.2 and Proposition 3.7, M is both a 2-tilting bundle on $\mathbb{X}$ and a 2-tilting object in $\underline{\mathsf{CM}}^{\mathbb{L}}R$. Take a decomposition $M = P \oplus W$, where $P := \bigoplus_{\vec{x}\in\mathcal{S}} R(\vec{x})$.

Now, we claim that $\underline{\mathrm{End}}^{\mathbb{L}}_R(W) \neq \mathrm{End}^{\mathbb{L}}_R(W)$. Notice that there exists a non-zero morphism $U^{\vec{s}+\vec{x}_3} \to R(\vec{x}_3)$. Moreover, there exists a sequence $R(\vec{x}_3) \to R(\vec{x}_3+\vec{x}_4) \to U^{\vec{s}+\vec{\delta}}(-\vec{\omega})$ of nonzero morphisms, and thus the composition is injective by [HIMO, Proposition 3.28(c)]. Then the composition

$$U^{\vec{s}+\vec{x}_3} \to R(\vec{x}_3) \to R(\vec{x}_3+\vec{x}_4) \to U^{\vec{s}+\vec{\delta}}(-\vec{\omega})$$

is also nonzero. But we have $\underline{\mathrm{Hom}}(U^{\vec{s}+\vec{x}_3}, U^{\vec{s}+\vec{\delta}}(-\vec{\omega})) = 0$ by Figure 1 in Proposition 3.7. Therefore, we obtain the claim.

Appendix A. The proof of Proposition 3.4. It is well-known that for hypersurface singularities, matrix factorizations play a key role when we treat Cohen–Macaulay modules. Let $R = S/(f)$ be an $\mathbb{L}$-graded hypersurface singularity for $S := k[X_1, \dots, X_4]$ and $f := X_1^2 + X_2^2 + X_3^p + X_4^q$.

The aim of this section is to give a complete proof of Proposition 3.4 in terms of matrix factorizations. The proof still works for any weight quadrangle. We refer to [E, Y] for the definitions and some basic facts on matrix factorizations; in particular, see [KeLM, HIMO] for the matrix factorizations of GL hypersurface singularities.

Definition A.1 ([HIMO]). An ($\mathbb{L}$-*graded*) *matrix factorization* of f is a pair

$$(\phi, \psi) = (\phi : Q \to P,\ \psi : P(-\vec{c}) \to Q)$$

of morphisms in $\mathsf{proj}^{\mathbb{L}} S$, also denoted by $Q \underset{\psi}{\overset{\phi}{\rightleftharpoons}} P$, such that

$$\psi\phi(-\vec{c}) = f : Q(-\vec{c}) \to Q \quad \text{and} \quad \phi\psi = f : P(-\vec{c}) \to P.$$

For example, any $P \in \mathsf{proj}^{\mathbb{L}} S$ gives $\mathbb{L}$-graded matrix factorizations

$$P \underset{f}{\overset{1}{\rightleftharpoons}} P \quad \text{and} \quad P(-\vec{c}) \underset{1}{\overset{f}{\rightleftharpoons}} P.$$

For two matrix factorizations $Q \underset{\psi}{\overset{\phi}{\rightleftharpoons}} P$ and $Q' \underset{\psi'}{\overset{\phi'}{\rightleftharpoons}} P'$ of f, a morphism from the former to the latter is a pair $(\alpha, \beta) \in \mathrm{Hom}_S^{\mathbb{L}}(P, P') \times \mathrm{Hom}_S^{\mathbb{L}}(Q, Q')$ making the following diagram commutative:

$$\begin{array}{ccccc} P(-\vec{c}) & \xrightarrow{\psi} & Q & \xrightarrow{\phi} & P \\ \downarrow{\scriptstyle \alpha(-\vec{c})} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} \\ P'(-\vec{c}) & \xrightarrow{\psi'} & Q' & \xrightarrow{\phi'} & P' \end{array}$$

The composition of morphisms is defined in an obvious way. Denote by $\mathsf{MF}^{\mathbb{L}}(S, f)$ the category of $\mathbb{L}$-graded matrix factorizations of f over S.

For any $\mathbb{L}$-graded matrix factorization $Q \underset{\psi}{\overset{\phi}{\rightleftharpoons}} P$, the cokernel $\mathrm{Cok}\,\phi$ is annihilated by f, hence it belongs to $\mathsf{mod}^{\mathbb{L}} R$. More precisely, $\mathrm{Cok}\,\phi$ belongs to $\mathsf{CM}^{\mathbb{L}} R$. It further follows from [E] that there exists a full dense functor

$$\mathrm{Cok} : \mathsf{MF}^{\mathbb{L}}(S, f) \to \mathsf{CM}^{\mathbb{L}} R, \quad (Q \underset{\psi}{\overset{\phi}{\rightleftharpoons}} P) \mapsto \mathrm{Cok}\,\phi. \tag{A.1}$$

Observe that for any $P \in \mathsf{proj}^{\mathbb{L}} S$, the additive functor Cok sends $P \underset{f}{\overset{1}{\rightleftharpoons}} P$ to zero and $P(-\vec{c}) \underset{1}{\overset{f}{\rightleftharpoons}} P$ to P/fP, and all projective R-modules are obtained

in this way. Consider the two full subcategories

$$\mathcal{P} := \mathsf{add}\,\{P \underset{f}{\overset{1}{\rightleftharpoons}} P,\ P(-\vec{c}) \underset{1}{\overset{f}{\rightleftharpoons}} P\} \quad \text{and} \quad \mathcal{P}_0 := \mathsf{add}\,\{P \underset{f}{\overset{1}{\rightleftharpoons}} P \mid P \in \mathsf{proj}^{\mathbb{L}}\, S\}$$

of $\mathsf{MF}^{\mathbb{L}}(S,f)$. Let $[\mathcal{P}]$ (resp. $[\mathcal{P}_0]$) denote the ideal of the category $\mathsf{MF}^{\mathbb{L}}(S,f)$ consisting of all morphisms which factor through objects in $\mathcal{P}$ (resp. $\mathcal{P}_0$), and $\underline{\mathsf{MF}}^{\mathbb{L}}(S,f) := \mathsf{MF}^{\mathbb{L}}(S,f)/[\mathcal{P}]$ (resp. $\mathsf{MF}^{\mathbb{L}}(S,f)/[\mathcal{P}_0]$) the corresponding factor category.

The following result states Eisenbud's matrix factorization theorem [E], which is adapted to the $\mathbb{L}$-graded context, following Yoshino's approach [Y].

Proposition A.2. *The functor* (A.1) *gives equivalences*

$$\mathrm{Cok}\colon \mathsf{MF}^{\mathbb{L}}(S,f)/[\mathcal{P}_0] \simeq \mathsf{CM}^{\mathbb{L}}\, R \quad \textit{and} \quad \mathrm{Cok}\colon \underline{\mathsf{MF}}^{\mathbb{L}}(S,f) \simeq \underline{\mathsf{CM}}^{\mathbb{L}} R \simeq \mathsf{D}^{\mathbb{L}}_{\mathrm{sg}}(R).$$

Moreover, the suspension functor of the triangulated category of $\underline{\mathsf{CM}}^{\mathbb{L}} R$ *is induced by the* (*functorial*) *expression* $(Q \underset{\psi}{\overset{\phi}{\rightleftharpoons}} P)[1] = (P \underset{\phi}{\overset{\psi}{\rightleftharpoons}} Q(\vec{c}))$.

This provides an approach to computing objects and morphisms in $\mathsf{CM}^{\mathbb{L}}\, R$ and $\underline{\mathsf{CM}}^{\mathbb{L}} R$. More precisely, for any $(Q \underset{\psi}{\overset{\phi}{\rightleftharpoons}} P) \in \mathsf{MF}^{\mathbb{L}}(S,f)$ and the corresponding object $X = \mathrm{Cok}\,\phi$, by iterating the formation of matrix factorizations, we obtain a sequence

$$\cdots \xrightarrow{\psi} Q(-\vec{c}) \xrightarrow{\phi} P(-\vec{c}) \xrightarrow{\psi} Q \xrightarrow{\phi} P \to X \to 0. \tag{A.2}$$

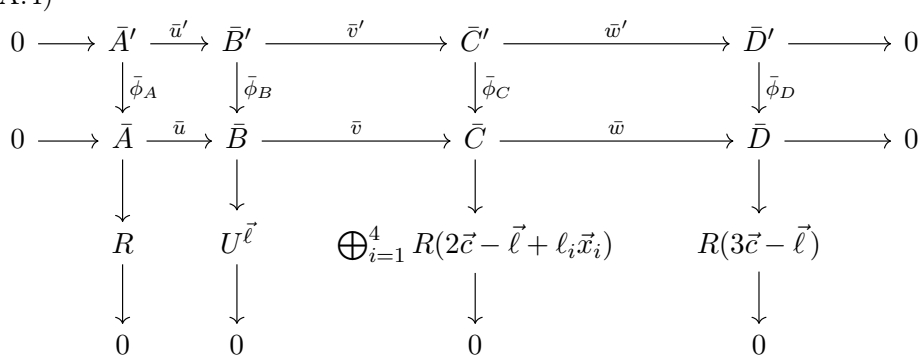
By reducing modulo (f), we obtain a projective resolution

$$\cdots \xrightarrow{\bar{\psi}} \bar{Q}(-\vec{c}) \xrightarrow{\bar{\phi}} \bar{P}(-\vec{c}) \xrightarrow{\bar{\psi}} \bar{Q} \xrightarrow{\bar{\phi}} \bar{P} \to X \to 0 \tag{A.3}$$

of X in $\mathsf{mod}^{\mathbb{L}}\, R$, which is 2-periodic up to the degree shift by $\vec{c}$. Here, the bar always stands for reduction modulo (f).

Fix $\vec{\ell} = \sum_{i=1}^{4} \ell_i \vec{x}_i$ in the interval $[\vec{s}, \vec{s} + \vec{\delta}]$; we write $X_i^{\circ} := X_i^{\ell_i}$ and $X_i^{\bullet} := X_i^{p_i - \ell_i}$ for $1 \le i \le 4$, where $(p_1, p_2, p_3, p_4) := (2, 2, p, q)$. To prove Proposition 3.4, we need to construct the following diagram:

(A.4)



Here, the projective presentations of R, $\bigoplus_{i=1}^4 R(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i)$ and $R(3\vec{c}-\vec{\ell})$ are chosen as follows (all reduced modulo (f); see [HIMO, Example 4.77] for the projective presentation of $U^{\vec{\ell}}$):

$$
\begin{aligned}
\bar{A} &= R\oplus R(-\ell_1\vec{x}_1),\\
\bar{A}' &= R(-\vec{c})\oplus R(-\ell_1\vec{x}_1),\\
\bar{B} &= R\oplus R(\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2)\oplus R(\vec{c}-\ell_1\vec{x}_1-\ell_3\vec{x}_3)\oplus R(\vec{c}-\ell_1\vec{x}_1-\ell_4\vec{x}_4)\\
&\quad\oplus R(\vec{c}-\ell_2\vec{x}_2-\ell_3\vec{x}_3)\oplus R(\vec{c}-\ell_2\vec{x}_2-\ell_4\vec{x}_4)\oplus R(\vec{c}-\ell_3\vec{x}_3-\ell_4\vec{x}_4)\\
&\quad\oplus R(2\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_3\vec{x}_3-\ell_4\vec{x}_4),\\
\bar{B}' &= R(-\ell_1\vec{x}_1)\oplus R(-\ell_2\vec{x}_2)\oplus R(-\ell_3\vec{x}_3)\oplus R(-\ell_4\vec{x}_4)\\
&\quad\oplus R(\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_3\vec{x}_3)\oplus R(\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_4\vec{x}_4)\\
&\quad\oplus R(\vec{c}-\ell_1\vec{x}_1-\ell_3\vec{x}_3-\ell_4\vec{x}_4)\oplus R(\vec{c}-\ell_2\vec{x}_2-\ell_3\vec{x}_3-\ell_4\vec{x}_4),\\
\bar{C} &= R(2\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_3\vec{x}_3)\oplus R(2\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_4\vec{x}_4)\\
&\quad\oplus R(2\vec{c}-\ell_1\vec{x}_1-\ell_3\vec{x}_3-\ell_4\vec{x}_4)\oplus R(2\vec{c}-\ell_2\vec{x}_2-\ell_3\vec{x}_3-\ell_4\vec{x}_4)\\
&\quad\oplus R(\vec{c}-\ell_2\vec{x}_2-\ell_3\vec{x}_3)\oplus R(\vec{c}-\ell_2\vec{x}_2-\ell_4\vec{x}_4)\oplus R(\vec{c}-\ell_3\vec{x}_3-\ell_4\vec{x}_4)\\
&\quad\oplus R(2\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_3\vec{x}_3-\ell_4\vec{x}_4),\\
\bar{C}' &= R(\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_3\vec{x}_3)\oplus R(\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_4\vec{x}_4)\\
&\quad\oplus R(\vec{c}-\ell_1\vec{x}_1-\ell_3\vec{x}_3-\ell_4\vec{x}_4)\oplus R(\vec{c}-\ell_2\vec{x}_2-\ell_3\vec{x}_3-\ell_4\vec{x}_4)\\
&\quad\oplus R(\vec{c}-\ell_2\vec{x}_2-\ell_3\vec{x}_3)\oplus R(\vec{c}-\ell_2\vec{x}_2-\ell_4\vec{x}_4)\oplus R(\vec{c}-\ell_3\vec{x}_3-\ell_4\vec{x}_4)\\
&\quad\oplus R(2\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_3\vec{x}_3-\ell_4\vec{x}_4),\\
\bar{D} &= R(3\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_3\vec{x}_3-\ell_4\vec{x}_4)\oplus R(2\vec{c}-\ell_2\vec{x}_2-\ell_3\vec{x}_3-\ell_4\vec{x}_4),\\
\bar{D}' &= R(2\vec{c}-\ell_1\vec{x}_1-\ell_2\vec{x}_2-\ell_3\vec{x}_3-\ell_4\vec{x}_4)\oplus R(2\vec{c}-\ell_2\vec{x}_2-\ell_3\vec{x}_3-\ell_4\vec{x}_4)
\end{aligned}
$$

together with the morphisms

$$\bar{\phi}_A=\bar{\phi}_D=\begin{bmatrix}0&0\\0&1\end{bmatrix},\qquad \bar{\phi}_C=\begin{bmatrix}0_{4\times4}&0_{4\times4}\\0_{4\times4}&I_4\end{bmatrix}$$

and

$$\bar{\phi}_B=\begin{bmatrix}
X_1^\circ & X_2^\bullet & -X_3^\bullet & X_4^\bullet & 0 & 0 & 0 & 0\\
-X_2^\circ & X_1^\bullet & 0 & 0 & -X_3^\bullet & X_4^\bullet & 0 & 0\\
X_3^\circ & 0 & X_1^\bullet & 0 & -X_2^\bullet & 0 & X_4^\bullet & 0\\
-X_4^\circ & 0 & 0 & X_1^\bullet & 0 & -X_2^\bullet & X_3^\bullet & 0\\
0 & X_3^\circ & X_2^\circ & 0 & X_1^\circ & 0 & 0 & X_4^\bullet\\
0 & -X_4^\circ & 0 & X_2^\circ & 0 & X_1^\circ & 0 & X_3^\bullet\\
0 & 0 & -X_4^\circ & -X_3^\circ & 0 & 0 & X_1^\circ & X_2^\bullet\\
0 & 0 & 0 & 0 & -X_4^\circ & -X_3^\circ & -X_2^\circ & X_1^\bullet
\end{bmatrix}^T.$$

The morphisms $\bar{u}, \bar{u}', \bar{v}, \bar{v}', \bar{w}$ and $\bar{w}'$ are given by

$$\bar{u} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_1^\circ & X_2^\bullet & -X_3^\bullet & X_4^\bullet & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\bar{u}' = \begin{bmatrix} X_1^\bullet & -X_2^\bullet & X_3^\bullet & -X_4^\bullet & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\bar{v} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_3^\bullet & -X_4^\bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ X_2^\bullet & 0 & -X_4^\bullet & 0 & 0 & 0 & 0 & 0 \\ 0 & X_2^\bullet & -X_3^\bullet & 0 & 0 & 0 & 0 & 0 \\ X_1^\bullet & 0 & 0 & -X_4^\bullet & 1 & 0 & 0 & 0 \\ 0 & X_1^\bullet & 0 & -X_3^\bullet & 0 & 1 & 0 & 0 \\ 0 & 0 & X_1^\bullet & -X_2^\bullet & 0 & 0 & 1 & 0 \\ X_4^\circ & X_3^\circ & X_2^\circ & X_1^\circ & 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$\bar{v}' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -X_3^\bullet & X_4^\bullet & 0 & 0 \\ 0 & 0 & 0 & 0 & -X_2^\bullet & 0 & X_4^\bullet & 0 \\ 0 & 0 & 0 & 0 & 0 & -X_2^\bullet & X_3^\bullet & 0 \\ 1 & 0 & 0 & 0 & X_1^\circ & 0 & 0 & X_4^\bullet \\ 0 & 1 & 0 & 0 & 0 & X_1^\circ & 0 & X_3^\bullet \\ 0 & 0 & 1 & 0 & 0 & 0 & X_1^\circ & X_2^\bullet \\ 0 & 0 & 0 & 1 & -X_4^\circ & -X_3^\circ & -X_2^\circ & X_1^\bullet \end{bmatrix}^T,$$

$$\bar{w} = \begin{bmatrix} X_4^\bullet & X_3^\bullet & X_2^\bullet & X_1^\bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -X_4^\bullet & -X_3^\bullet & -X_2^\bullet & X_1^\circ \end{bmatrix},$$

$$\bar{w}' = \begin{bmatrix} X_4^\bullet & X_3^\bullet & X_2^\bullet & X_1^\bullet & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -X_4^\bullet & -X_3^\bullet & -X_2^\bullet & X_1^\circ \end{bmatrix}.$$

LEMMA A.3. *Let $I = \{2, 3, 4\}$. Then there exists an exact sequence*

$$0 \to \mathcal{O}(-\ell_1\vec{x}_1) \xrightarrow{g} \bigoplus_{i \in I} \mathcal{O}(\vec{c} - \ell_1\vec{x}_1 - \ell_i\vec{x}_i) \xrightarrow{h} \bigoplus_{i,j \in I,\ i<j} \mathcal{O}(2\vec{c} - \ell_1\vec{x}_1 - \ell_i\vec{x}_i - \ell_j\vec{x}_j) \xrightarrow{t} \mathcal{O}(3\vec{c} - \vec{\ell}) \to 0 \tag{A.5}$$

in $\mathsf{coh}\,\mathbb{X}$ *with the morphisms*

$$g := \begin{bmatrix} X_2^\bullet \\ -X_3^\bullet \\ X_4^\bullet \end{bmatrix}, \quad h := \begin{bmatrix} X_3^\bullet & X_2^\bullet & 0 \\ -X_4^\bullet & 0 & X_2^\bullet \\ 0 & -X_4^\bullet & -X_3^\bullet \end{bmatrix}, \quad t := \begin{bmatrix} X_4^\bullet \\ X_3^\bullet \\ X_2^\bullet \end{bmatrix}^T.$$

Proof. Since $(X_i)_{i\in I}$ is an R-regular sequence, we infer that $(X_i^\bullet)_{i\in I}$ is also an R-regular sequence. Hence the corresponding Koszul complex

$$\begin{aligned} (\mathrm{A}.6)\qquad 0 \to R(-3\vec{c}+\sum_{i\in I}\ell_i\vec{x}_i) \\ \to \bigoplus_{i,j\in I,\ i<j} R(-2\vec{c}+\ell_i\vec{x}_i+\ell_j\vec{x}_j) \to \bigoplus_{i\in I} R(-\vec{c}+\ell_i\vec{x}_i) \to R \to 0 \end{aligned}$$

of R is exact except in the rightmost position whose homology is $(R/(X_i^\bullet)_{i\in I})$. Since this belongs to $\mathsf{mod}_0^{\mathbb{L}} R$, the image of (A.6) in $\mathsf{coh}\,\mathbb{X}$ is exact. Then by applying degree shift $3\vec{c}-\vec{\ell}$ to the image of (A.6), we prove the assertion. ∎

For any morphism $\bar{g} : \bar{X} \to \bar{Y}$ in $\mathrm{proj}^{\mathbb{L}}R$, we denote by $\widetilde{g} : \widetilde{X} \to \widetilde{Y}$ the image of $\bar{g}$ under π. Note that $\mathrm{Hom}_R^{\mathbb{L}}(\bar{X},\bar{Y}) \simeq \mathrm{Hom}_{\mathbb{X}}(\widetilde{X},\widetilde{Y})$ by (2.5).

LEMMA A.4. *The diagram* (A.4) *is commutative with exact sequences in* $\mathsf{coh}\,\mathbb{X}$.

Proof. The commutativity of diagram (A.4) can be checked easily. We only show that the image of the second top row is exact in $\mathsf{coh}\,\mathbb{X}$, as the exactness of the top row can be shown similarly. Let $a = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] : M \to \widetilde{A}$ be a morphism in $\mathsf{coh}\,\mathbb{X}$ such that $\widetilde{u}a = 0$. Then we have

$$a_1 + X_1^\circ a_2 = 0 \ \text{ and } \ X_i^\bullet a_2 = 0 \text{ for each } 2 \le i \le 4.$$

Since $X_i^\bullet$ is a monomorphism, we have $a_2 = 0$ and then $a_1 = 0$. Thus, $a = 0$. This shows that $\widetilde{u}$ is a monomorphism in $\mathsf{coh}\,\mathbb{X}$. On the other hand, let $a' = (a'_1, a'_2)$ be a morphism in $\mathsf{coh}\,\mathbb{X}$ such that $a'\widetilde{w} = 0$. Then we have $a'_j X_i^\bullet = 0$ for any $j = 1, 2$ and $2 \le i \le 4$. Notice that the morphism

$$(X_i^\bullet)_{2\le i\le 4} : \bigoplus_{2\le i\le 4} \mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i) \to \mathcal{O}(3\vec{c}-\vec{\ell})$$

is an epimorphism. Then $a'_1(X_i^\bullet)_{2\le i\le 4} = 0$ yields $a'_1 = 0$. Similarly, we have $a'_2 = 0$. Thus, $a' = 0$. This shows that $\widetilde{w}$ is an epimorphism in $\mathsf{coh}\,\mathbb{X}$.

It remains to show that $\widetilde{u}$ is the kernel of $\widetilde{v}$, and $\widetilde{w}$ is the cokernel of $\widetilde{v}$ in $\mathsf{coh}\,\mathbb{X}$. Clearly, we have $\widetilde{v}\widetilde{u} = 0$ and $\widetilde{w}\widetilde{v} = 0$. Let $b = [b_1, \ldots, b_8]^T : N \to \widetilde{B}$ be a morphism in $\mathsf{coh}\,\mathbb{X}$ such that $\widetilde{v}b = 0$. Then we have

$$X_3^\bullet b_2 + X_2^\bullet b_3 = 0,\ X_4^\bullet b_2 - X_2^\bullet b_4 = 0,\ X_4^\bullet b_3 + X_3^\bullet b_4 = 0, \tag{A.7}$$

and $b_5 = \cdots = b_8 = 0$. By the exactness of (A.5), the three equalities in (A.7) imply that there exists a morphism $\xi : N \to \mathcal{O}(-\ell_1\vec{x}_1)$ such that

$$b_2 = X_2^\bullet\xi, \quad b_3 = -X_3^\bullet\xi \quad \text{and} \quad b_4 = X_4^\bullet\xi.$$

Then we obtain a morphism

$$\xi' := \begin{bmatrix} b_1 - X_1^\circ\xi \\ \xi \end{bmatrix} : N \to \widetilde{A} = \mathcal{O} \oplus \mathcal{O}(-\ell_1\vec{x}_1)$$

such that $b = \widetilde{u}\xi'$. Hence $\widetilde{u}$ is the kernel of $\widetilde{v}$ in $\mathsf{coh}\,\mathbb{X}$.

On the other hand, let $c := [c_1, \dots, c_8] : \widetilde{C} \to W$ be a morphism in $\mathsf{coh}\,\mathbb{X}$ such that $c\widetilde{v} = 0$. Then we have

$$c_1X_3^\bullet - c_2X_4^\bullet = 0, \quad c_1X_2^\bullet - c_3X_4^\bullet = 0 \quad \text{and} \quad c_2X_2^\bullet - c_3X_3^\bullet = 0. \tag{A.8}$$

By the exactness of (A.5), the three equalities in (A.8) imply that there exists a morphism $\zeta : \mathcal{O}(3\vec{c} - \vec{\ell}) \to W$ such that

$$c_1 = \zeta X_4^\bullet, \quad c_2 = \zeta X_3^\bullet \quad \text{and} \quad c_3 = \zeta X_2^\bullet.$$

Moreover, we obtain a morphism

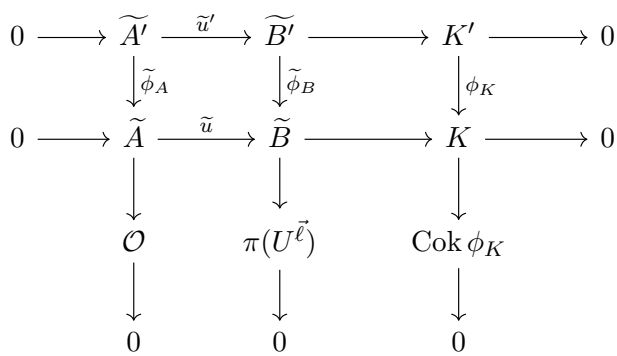
$$\zeta' := [\zeta,\ \zeta X_1^\bullet - c_4] : \widetilde{D} = \mathcal{O}(3\vec{c} - \vec{\ell}) \oplus \mathcal{O}(2\vec{c} - \vec{\ell} + \ell_1\vec{x}_1) \to W$$

such that $c = \zeta'\widetilde{w}$ (since $c\widetilde{v} = 0$). Thus $\widetilde{w}$ is the cokernel of $\widetilde{v}$ in $\mathsf{coh}\,\mathbb{X}$, as claimed. ∎

Now we are ready to prove Proposition 3.4.

Proof of Proposition 3.4. By the canonical factorization of the morphisms $\widetilde{v}$ and $\widetilde{v'}$, we obtain the following two commutative diagrams:

(A.9)



and

(A.10)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \bigoplus_{i=1}^4 \mathcal{O}(\vec{c}-\vec{\ell}+\ell_i\vec{x}_i) & & \mathcal{O}(2\vec{c}-\vec{\ell}) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & \widetilde{C'} & \xrightarrow{\widetilde{w}'} & \widetilde{D'} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_K} & & \downarrow{\scriptstyle \widetilde{\phi}_C} & & \downarrow{\scriptstyle \widetilde{\phi}_D} & & \\
0 & \longrightarrow & K & \longrightarrow & \widetilde{C} & \xrightarrow{\widetilde{w}} & \widetilde{D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Cok}\phi_K & & \bigoplus_{i=1}^4 \mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i) & & \mathcal{O}(3\vec{c}-\vec{\ell}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns in $\mathsf{coh}\,\mathbb{X}$. Applying the snake lemma to the diagrams (A.9) and (A.10), we obtain the exact sequences

$$\mathcal{O} \to \pi(U^{\vec{\ell}}) \to \operatorname{Cok}\phi_K \to 0, \tag{A.11}$$

$$\begin{aligned}\bigoplus_{1\le i\le 4} \mathcal{O}(\vec{c}-\vec{\ell}+\ell_i\vec{x}_i) &\xrightarrow{\gamma} \mathcal{O}(2\vec{c}-\vec{\ell}) \to \operatorname{Cok}\phi_K \\ &\xrightarrow{\iota} \bigoplus_{1\le i\le 4} \mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i) \xrightarrow{\gamma} \mathcal{O}(3\vec{c}-\vec{\ell}) \to 0.\end{aligned} \tag{A.12}$$

We observe that the induced morphism $R \to U^{\vec{\ell}}$ in (A.4) is injective by [HIMO, Proposition 3.28]. Thus, its image $\mathcal{O} \to \pi(U^{\vec{\ell}})$ in (A.11) is also injective. On the other hand, the induced morphism γ in (A.12) is given by $(X_i^\bullet)_{1\le i\le 4}$ since the diagram

$$\begin{array}{ccc}
\widetilde{C} & \xrightarrow{\widetilde{w}} & \widetilde{D} \\
\downarrow{\scriptstyle [J_4 \;\; 0_{4\times 4}]} & & \downarrow{\scriptstyle [1 \;\; 0]} \\
\bigoplus_{i=1}^4 \mathcal{O}(2\vec{c}-\vec{\ell}+\ell_i\vec{x}_i) & \xrightarrow{(X_i^\bullet)_{1\le i\le 4}} & \mathcal{O}(3\vec{c}-\vec{\ell})
\end{array}$$

commutes, where J_4 denotes the 4×4 anti-diagonal identity matrix. The surjectivity of γ in the leftmost position of (A.12) implies that the morphism ι is injective. Connecting the exact sequences in (A.11) and (A.12), we obtain the sequence (3.2). Thus, the assertion follows. ∎

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REFERENCES

[B] R.-O. Buchweitz, *Maximal Cohen–Macaulay Modules and Tate Cohomology*, Math. Surveys Monogr. 262, Amer. Math. Soc., Providence, RI, 2021.

[CRW] J. Chen, S. Ruan and W. Weng, *Brieskorn–Pham singularities via ACM bundles on Geigle–Lenzing projective spaces*, arXiv:2501.15375v1 (2025).

[E] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. 260 (1980), 35–64.

[FU] M. Futaki and K. Ueda, *Homological mirror symmetry for Brieskorn–Pham singularities*, Selecta Math. (N.S.) 17 (2011), 435–452.

[GL] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, in: Singularities, Representation of Algebras, and Vector Bundles, Lecture Notes in Math. 1273, Springer, Berlin, 1987, 265–297.

[G] S. Geng, *Mutation of tilting bundles of tubular type*, J. Algebra 550 (2020), 186–209.

[H] D. Happel, *Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras*, London Math. Soc. Lecture Note Ser. 119, Cambridge Univ. Press, Cambridge, 1988.

[HIMO] M. Herschend, O. Iyama, H. Minamoto and S. Oppermann, *Representation theory of Geigle–Lenzing complete intersections*, Mem. Amer. Math. Soc. 285 (2023), no. 1412, vii+141 pp.

[HIO] M. Herschend, O. Iyama and S. Oppermann, *n-representation infinite algebras*, Adv. Math. 252 (2014), 292–342.

[I1] O. Iyama, *Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories*, Adv. Math. 210 (2007), 22–50.

[I2] O. Iyama, *Cluster tilting for higher Auslander algebras*, Adv. Math. 226 (2011), 1–61.

[IO] O. Iyama and S. Oppermann, *n-representation-finite algebras and n-APR tilting*, Trans. Amer. Math. Soc. 363 (2011), 6575–6614.

[IY] O. Iyama and Y. Yoshino, *Mutation in triangulated categories and rigid Cohen–Macaulay modules*, Invent. Math. 172 (2008), 117–168.

[J] G. Jasso, *n-abelian and n-exact categories*, Math. Z. 283 (2016), 703–759.

[KST1] H. Kajiura, K. Saito and A. Takahashi, *Matrix factorizations and representations of quivers II: Type ADE case*, Adv. Math. 211 (2007), 327–362.

[KST2] H. Kajiura, K. Saito and A. Takahashi, *Triangulated categories of matrix factorizations for regular systems of weights with* $\epsilon = -1$, Adv. Math. 220 (2009), 1602–1654.

[KeLM] D. E. Kędzierski, H. Lenzing and H. Meltzer, *Matrix factorizations for domestic triangle singularities*, Colloq. Math. 140 (2015), 239–278.

[KLM] D. Kussin, H. Lenzing and H. Meltzer, *Triangle singularities, ADE-chains, and weighted projective lines*, Adv. Math. 237 (2013), 194–251.

[MY] Y. Mizuno and K. Yamaura, *Higher APR tilting preserves n-representation infiniteness*, J. Algebra 447 (2016), 56–73.

[O] D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, in: Algebra, Arithmetic, and Geometry: in Honor of Yu. I. Manin. Vol. II, Progr. Math. 270, Birkhäuser Boston, Boston, MA, 2009, 503–531.

[S] J. P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. 61 (1955), 197–278.
[Y] Y. Yoshino, *Cohen–Macaulay Modules over Cohen–Macaulay Rings*, London Math. Soc. Lecture Note Ser. 146, Cambridge Univ. Press, Cambridge, 1990.

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