COLLOQUIUM MATHEMATICUM

Online First version

ARITHMETIC PROPERTIES OF THE DSOME FUNCTION

BY

NAYANDEEP DEKA BARUAH and PANKAJ GOGOI

Abstract. Recently, Andrews and Ghosh Dastidar [Ramanujan J. 69 (2026), art. 26] studied two interesting functions SOME(n) and DSOME(n), where SOME(n) is the sum of all the odd parts in the partitions of n minus the sum of all the even parts, and DSOME(n) is the sum of all the odd parts in the partitions of n into distinct parts minus the sum of all the even parts. They expressed the generating functions of SOME(n) and DSOME(n) in terms of q-series and found several interesting congruences modulo 4 and 5. In this paper, we express the generating function of DSOME(n) in closed form, which allows us to find some new congruences and internal congruences modulo 4 and 8 for DSOME(n).

1. Introduction. For complex numbers a and q with $|q| < 1$, define the standard *q-products*

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1} (1-aq^k), \qquad (a;q)_\infty := \lim_{n\to\infty} (a;q)_n.$$

In the sequel, we set $f_n := (q^n;q^n)_\infty$ for integers $n \geq 1$.

A *partition* $\lambda := (\lambda_1, \ldots, \lambda_k)$ of a positive integer n is a finite non-increasing sequence of positive integers such that $\lambda_1 + \cdots + \lambda_k = n$. The partition function $p(n)$ is defined as the number of partitions of n. It is well known that the generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1},$$

where by convention, we set $p(0) = 1$. The arithmetic properties of the partition function have been studied extensively after Ramanujan [SR19] found his famous congruences modulo 5, 7 and 11, namely

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

2020 *Mathematics Subject Classification*: Primary 11P83; Secondary 05A17.
Key words and phrases: integer partition, congruence, DSOME function.
Received 19 January 2026; revised 8 June 2026.
Published online 6 November 2026.

 © Instytut Matematyczny PAN, 2026

Recently, Andrews and Ghosh Dastidar [AG26] studied two interesting functions related to partitions, namely, SOME(n) and DSOME(n), where SOME(n) is the sum of all the odd parts in the partitions of n minus the sum of all the even parts, and DSOME(n) is the sum of all the odd parts in the partitions of n into distinct parts minus the sum of all the even parts. They found that the generating functions of SOME(n) and DSOME(n) can be expressed as

$$\sum_{n=0}^{\infty}\text{SOME}(n)q^n=\frac{1}{(q;q)_\infty}\sum_{m\geq 1}\frac{q^m}{(1+q^m)^2}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\text{DSOME}(n)q^n&=(-q;q)_\infty\sum_{m\geq 1}\frac{(-1)^{m-1}q^m}{(1+q^m)^2}\\&=(-q;q)_\infty\sum_{n=1}^{\infty}\bigg(\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}}-\sum_{n\geq 0}\frac{(2n)q^{2n}}{1+q^{2n}}\bigg).\end{aligned}\tag{1.1}$$

They also proved the following beautiful congruences. For $n\geq 0$,

$$\begin{aligned}\text{SOME}(5n+2)&\equiv 0\pmod 5,\\\text{SOME}(5n+4)&\equiv 0\pmod 5,\\\text{SOME}(4n)&\equiv 0\pmod 4,\\\text{DSOME}(4n)&\equiv 0\pmod 4.\end{aligned}$$

Moreover, they also conjectured that if $24\lambda\equiv 1\pmod{5^\alpha}$, then

$$\text{SOME}(\lambda)\equiv 0\pmod{5^\alpha}.$$

In this paper, we express the generating function (1.1) of DSOME(n) in closed form (see Theorem 1.1 below). That allows us to find some new congruences and internal congruences for DSOME(n) modulo 4 and 8. We present the main results in the following four theorems.

THEOREM 1.1. *We have*

$$\sum_{n\geq 0}\text{DSOME}(n)q^n=\frac{1}{8}\bigg(\frac{f_2}{f_1}-\frac{f_1^7}{f_2^3}\bigg).\tag{1.2}$$

THEOREM 1.2. *For all $n\geq 0$ and $1\leq r\leq 4$, we have*

$$\text{DSOME}(25n+5r+1)\equiv 0\pmod 4.$$

THEOREM 1.3. *For all $n\geq 0$ and $1\leq r\leq 4$, we have*

$$\text{DSOME}(625n+125r+26)\equiv 0\pmod 8.$$

THEOREM 1.4. *For all $n\geq 0$ and $1\leq r\leq 4$, we have*

$$\begin{aligned}\text{DSOME}(15625n+651)&\equiv 5\text{DSOME}(25n+1)+\text{DSOME}(625n+26)\\&\quad-5\text{DSOME}(n)\pmod 8\end{aligned}$$

and

$$\mathrm{DSOME}(15625n+3125r+651) \equiv 5\mathrm{DSOME}(25n+5r+1) + \mathrm{DSOME}(625n+125r+26) \pmod 8.$$

We organize the paper in the following way. In Section 2, we present some preliminary definitions and lemmas that are useful for proving our theorems. In Section 3, we prove Theorems 1.1–1.4. In the final section, we propose some conjectural congruences for DSOME(n) modulo 8 and 16.

2. Preliminary definitions and lemmas. The Rogers–Ramanujan identities are given by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;\,q^5)_\infty (q^4;\,q^5)_\infty}, \tag{2.1}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;\,q^5)_\infty (q^3;\,q^5)_\infty}, \tag{2.2}$$

where $G(q)$ and $H(q)$ are called the *Rogers–Ramanujan functions*. Recall from [LR94] that

$$R(q) = q^{1/5}\frac{H(q)}{G(q)},$$

where $R(q)$ is the famous Rogers–Ramanujan continued fraction defined by

$$R(q) := \frac{q^{1/5}}{1}\,\underset{+}{}\,\frac{q}{1}\,\underset{+}{}\,\frac{q^2}{1}\,\underset{+}{}\,\cdots, \qquad |q|<1.$$

Some useful results are presented in the following lemmas. For the first two lemmas, we set

$$T(q) := \frac{q^{1/5}}{R(q)} = \frac{G(q)}{H(q)}. \tag{2.3}$$

LEMMA 2.1 (Baruah and Begum [BB18]). *If $x := T(q)$ and $y := T(q^2)$, then*

$$xy^2 - \frac{q^2}{xy^2} = K, \tag{2.4}$$

$$\frac{x^2}{y} - \frac{y}{x^2} = \frac{4q}{K}, \tag{2.5}$$

$$\frac{y^3}{x} + q^2\frac{x}{y^3} = K + \frac{4q^2}{K} - 2q, \tag{2.6}$$

$$x^3y + \frac{q^2}{x^3y} = K + \frac{4q^2}{K} + 2q, \tag{2.7}$$

$$x^5 - \frac{q^2}{x^5} = \frac{f_1^6}{f_5^6} + 11q, \tag{2.8}$$

where $K = f_2 f_5^5/(f_1 f_{10}^5)$.

In the next lemma, we present two well-known 5-dissections of f_1 and $1/f_1$.

LEMMA 2.2 ([B06, p. 165]). *We have*

$$f_1 = f_{25}\left(T(q^5) - q - \frac{q^2}{T(q^5)}\right), \tag{2.9}$$

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\left(T(q^5)^4 + qT(q^5)^3 + 2q^2T(q^5)^2 + 3q^3T(q^5) + 5q^4 - \frac{3q^5}{T(q^5)} + \frac{2q^6}{T(q^5)^2} - \frac{q^7}{T(q^5)^3} + \frac{q^8}{T(q^5)^4}\right). \tag{2.10}$$

The following gives 2-dissections of $G(q)$ and $H(q)$ due to Watson [W33, p. 60].

LEMMA 2.3. *We have*

$$G(q) = \frac{f_8}{f_2}(G(q^{16}) + qH(-q^4)), \tag{2.11}$$

$$H(q) = \frac{f_8}{f_2}(G(-q^4) + q^3H(q^{16})). \tag{2.12}$$

We use the identities of the following lemma quite frequently in the next sections.

LEMMA 2.4 (Baruah and Begum [BB18, (2.6) and (2.7)]).

$$\frac{f_5^5}{f_1^4f_{10}^3} = \frac{f_5}{f_2^2f_{10}} + 4q\frac{f_{10}^2}{f_1^3f_2}, \tag{2.13}$$

$$\frac{f_2^3f_5^2}{f_1^2f_{10}^2} = \frac{f_5^5}{f_1f_{10}^3} + q\frac{f_{10}^2}{f_2}. \tag{2.14}$$

We conclude this section by proving another useful result.

LEMMA 2.5. *We have*

$$G(q)G(q^4) \pm qH(q)H(q^4) \equiv \frac{1}{f_2} \pmod 2. \tag{2.15}$$

Proof. One of Ramanujan's famous forty identities for $G(q)$ and $H(q)$ (See [BC07] for more details) is given by

$$G(q)G(q^4) - qH(q)H(q^4) = \frac{f_{10}^5}{f_2f_5^2f_{20}^2}. \tag{2.16}$$

However, by the binomial theorem, for positive integers k and ℓ, it can be easily shown that

$$f_k^{2^\ell} \equiv f_{2k}^{2^{\ell-1}} \pmod{2^\ell}. \tag{2.17}$$

In particular,

$$f_1^2 \equiv f_2 \pmod 2 \quad \text{and} \quad f_5^2 \equiv f_{10} \pmod 2.$$

Thus, (2.15) readily follows from (2.16). ∎

REMARK 2.6. We will frequently use (2.17) without further reference.

3. Proofs of Theorems 1.1–1.4

Proof of Theorem 1.1. We recall from [B06, p. 61, (3.3.6)] that

$$\frac{(q;q)_\infty^4}{(-q;q)_\infty^4} = 1 - 8\sum_{n=1}^{\infty}\bigg(\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}} - \sum_{n\geq 0}\frac{2nq^{2n}}{1+q^{2n}}\bigg).$$

Employing the above identity in (1.1), we have

$$\begin{aligned}\sum_{n\geq 0}\mathrm{DSOME}(n)q^n &= (-q;q)_\infty\bigg(\sum_{n\geq 0}\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}} - \sum_{n\geq 0}\frac{2nq^{2n}}{1+q^{2n}}\bigg)\\ &= \frac{1}{8}(-q,q)_\infty\bigg(1-\frac{(q;q)_\infty^4}{(-q;q)_\infty^4}\bigg) = \frac{1}{8}\frac{f_2}{f_1}\bigg(1-\frac{f_1^8}{f_2^4}\bigg)\\ &= \frac{1}{8}\bigg(\frac{f_2}{f_1}-\frac{f_1^7}{f_2^3}\bigg),\end{aligned}$$

which is (1.2). ∎

Proof of Theorem 1.2. From (1.2) and (2.13),

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{DSOME}(n)q^n &= \frac{f_2}{8f_1} - \frac{1}{8}\bigg(\frac{f_1^3f_5^4}{f_2f_{10}^2} - 4q\frac{f_1^4f_{10}^3}{f_2^2f_5}\bigg)\\ &= \frac{f_2}{8f_1} - \frac{1}{8}\bigg(\frac{f_2f_5^8}{f_1f_{10}^4} - 4qf_5^3f_{10}\bigg) + \frac{q}{2}\bigg(f_5^3f_{10} - 4q\frac{f_1f_{10}^6}{f_2f_5^2}\bigg)\\ &= \frac{f_2}{8f_1} - \frac{f_2f_5^8}{8f_1f_{10}^4} + qf_5^3f_{10} - 2q^2\frac{f_{10}^6f_1}{f_5^2f_2},\end{aligned}$$

which, by (2.9) and (2.10), can be recast as

$$\begin{aligned}(3.1)\qquad &\sum_{n=0}^{\infty}\mathrm{DSOME}(n)q^n\\ &= \frac{f_{25}^5f_{50}}{f_5^6}\bigg(\frac{1}{8}-\frac{f_5^8}{8f_{10}^4}\bigg)\bigg(T(q^{10})-q^2-\frac{q^4}{T(q^{10})}\bigg)\bigg(T^4(q^5)+qT^3(q^5)+2qT^2(q^5)\\ &\quad+3q^3T(q^5)+5q^4-3\frac{q^5}{T(q^5)}+2\frac{q^6}{T^2(q^5)}-\frac{q^7}{T^3(q^5)}+\frac{q^8}{T^4(q^5)}\bigg)+qf_5^3f_{10}\\ &\quad-2q^2\frac{f_{25}f_{50}^5}{f_5^2}\bigg(T(q^5)-q-\frac{q^2}{T(q^5)}\bigg)\bigg(T^4(q^{10})+qT^3(q^{10})+2q^2T^2(q^{10})\\ &\quad+3q^6T(q^{10})+5q^8-3\frac{q^{10}}{T(q^{10})}+2\frac{q^{12}}{T^2(q^{10})}-\frac{q^{14}}{T^3(q^{10})}+\frac{q^{16}}{T^4(q^{10})}\bigg).\end{aligned}$$

Extracting the terms involving q^{5n+1} from both sides of (3.1), dividing by q and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty}\text{DSOME}(5n+1)q^n$$
$$= f_1^3f_2+\left(\frac{1}{8}-\frac{f_1^8}{8f_2^4}\right)\frac{f_5^5f_{10}}{f_1^6}\left(x^3y+\frac{q^2}{x^3y}-2q\left(\frac{x^2}{y}-\frac{y}{x^2}\right)-5q\right)$$
$$-2\frac{f_5f_{10}^5}{f_1^2}\left(2q\left(xy^2-\frac{q^2}{xy^2}\right)-q\left(\frac{y^3}{x}+q^2\frac{x}{y^3}\right)-5q^2\right),$$

which by (2.4)–(2.7) can be recast as

$$\sum_{n=0}^{\infty}\text{DSOME}(5n+1)q^n \tag{3.2}$$
$$= f_1^3f_2+\frac{1}{8}\left(1-\frac{f_1^8}{8f_2^4}\right)\frac{f_5^5f_{10}}{f_1^6}\left(K-\frac{4q^2}{K}-3q\right)-2q\frac{f_5f_{10}^5}{f_1^2}\left(K-\frac{4q^2}{K}-3q\right).$$

Now, by (2.13) and (2.14), we have

$$K-\frac{4q^2}{K}-3q=(K+q)\left(1-\frac{4q}{K}\right)=\left(\frac{f_2f_5^5}{f_1f_{10}^5}+q\right)\left(1-4q\frac{f_1f_{10}^5}{f_2f_5^5}\right) \tag{3.3}$$
$$=\frac{f_2^4f_5^2}{f_1^2f_{10}^4}\times\frac{f_1^4f_{10}^2}{f_2^2f_5^4}=\frac{f_1^2f_2^2}{f_5^2f_{10}^2}.$$

Employing (3.3) in (3.2), we have

$$\sum_{n=0}^{\infty}\text{DSOME}(5n+1)q^n=f_1^3f_2+\frac{1}{8}\frac{f_2^2f_5^3}{f_1^4f_{10}}-\frac{1}{8}\frac{f_1^4f_5^3}{f_2^2f_{10}}-2q\frac{f_2^2f_{10}^3}{f_5}. \tag{3.4}$$

Applying (2.13), we find that

$$\sum_{n=0}^{\infty}\text{DSOME}(5n+1)q^n \tag{3.5}$$
$$= f_1^3f_2+\frac{1}{8}\left(\frac{f_{10}}{f_5}+4q\frac{f_2f_{10}^4}{f_1^3f_5^2}\right)-\frac{1}{8}\left(\frac{f_5^7}{f_{10}^3}-4q\frac{f_1f_5^2f_{10}^2}{f_2}\right)-2q\frac{f_2^2f_{10}^3}{f_5}$$
$$= f_1^3f_2+\frac{1}{8}\frac{f_{10}}{f_5}+\frac{q}{2}\left(\frac{f_1f_{10}^6}{f_2f_5^6}+4q\frac{f_{10}^9}{f_1^2f_5^7}\right)-\frac{f_5^7}{8f_{10}^3}+\frac{q}{2}\frac{f_1f_5^2f_{10}^2}{f_2}-2q\frac{f_2^2f_{10}^3}{f_5}$$
$$= \frac{f_{10}}{8f_5}-\frac{f_5^7}{8f_{10}^3}+f_1^3f_2+\frac{q}{2}\left(\frac{f_1f_{10}^6}{f_2f_5^6}+\frac{f_1f_5^2f_{10}^2}{f_2}\right)-2q\frac{f_2^2f_{10}^3}{f_5}+2q^2\frac{f_{10}^9}{f_1^2f_5^7}.$$

Therefore,

$$\sum_{n=0}^{\infty}\text{DSOME}(5n+1)q^n$$
$$\equiv \frac{f_{10}}{8f_5}-\frac{f_5^7}{8f_{10}^3}+f_1^3f_2+q\frac{f_1f_5^2f_{10}^2}{f_2}-2q\frac{f_2^2f_{10}^3}{f_5}+2q^2\frac{f_{10}^9}{f_1^2f_5^7}\pmod 4,$$

which by (2.14) yields

$$\begin{aligned}\sum_{n=0}^{\infty}&\mathrm{DSOME}(5n+1)q^n\\ &=\frac{f_{10}}{8f_5}-\frac{f_5^7}{8f_{10}^3}+f_1^3f_2+\left(\frac{f_2^3f_5^4}{f_1f_{10}^2}-\frac{f_5^7}{f_{10}^3}\right)-2q\frac{f_2^2f_{10}^3}{f_5}+2q^2\frac{f_{10}^9}{f_1^2f_5^7}\\ &\equiv\frac{f_{10}}{8f_5}-\frac{9}{8}\frac{f_5^7}{f_{10}^3}+2f_1f_4+2q\frac{f_4f_{10}^3}{f_5}+2q^2\frac{f_5f_{10}^5}{f_2}\pmod 4.\end{aligned}$$

Employing (2.9) and (2.10), we have

$$\begin{aligned}(3.6)\quad&\sum_{n=0}^{\infty}\mathrm{DSOME}(5n+1)q^n\\ &\equiv\frac{f_{10}}{8f_5}-\frac{9}{8}\frac{f_5^7}{f_{10}^3}+2f_{25}f_{100}\left(T(q^5)-q-\frac{q^2}{T(q^5)}\right)\left(T(q^{20})-q^4-\frac{q^8}{T(q^{20})}\right)\\ &\quad+2q\frac{f_{10}^3f_{100}}{f_5}\left(T(q^{20})-q^4-\frac{q^8}{T(q^{20})}\right)+2q^2\frac{f_5f_{10}^5f_{50}^5}{f_{10}^6}\Bigg(T^4(q^{10})\\ &\quad+q^2T^3(q^{10})+2q^4T^2(q^{10})+3q^6T(q^{10})+5q^8-3\frac{q^{10}}{T(q^{10})}+2\frac{q^{12}}{T^2(q^{10})}\\ &\quad-\frac{q^{14}}{T^3(q^{10})}+\frac{q^{16}}{T^4(q^{10})}\Bigg)\pmod 4.\end{aligned}$$

To derive Theorem 1.2, we need to extract the terms involving q^{5n+r} for $r=1,2,3,4$ in (3.6). We treat each of the cases for r separately in the following.

Case $r=1$. Extracting the terms involving q^{5n+1} from both sides of (3.6), dividing by q, and then replacing q^5 by q, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathrm{DSOME}(25n+6)q^n\\ &\equiv 2f_5f_{20}\left(-T(q^4)+\frac{q}{T(q)}\right)+2\frac{f_2^3f_{20}T(q^4)}{f_1}+2q^3\frac{f_1f_{10}^5}{f_2T^3(q^2)}\pmod 4,\end{aligned}$$

which by (2.3) can be rewritten as

$$\begin{aligned}(3.7)\quad&\sum_{n=0}^{\infty}\mathrm{DSOME}(25n+6)q^n\\ &\equiv 2f_5f_{20}\left(\frac{G(q^4)}{H(q^4)}-q\frac{H(q)}{G(q)}\right)+2f_1f_4f_{20}\frac{G(q^4)}{H(q^4)}+2q^3\frac{f_1f_{10}^5H^3(q^2)}{f_2G^3(q^2)}\\ &\equiv 2\frac{f_5f_{20}}{G(q)H(q^4)}\big(G(q)G(q^4)-qH(q)H(q^4)\big)+2f_1f_4f_{20}\frac{G(q^4)}{H(q^4)}\\ &\quad+2q^3\frac{f_1f_{10}f_{40}H(q^2)H(q^4)}{f_2G(q^2)G(q^4)}\pmod 4,\end{aligned}$$

where we have also used the fact that for any positive integer j,

$$G^2(q^j) \equiv G(q^{2j}) \pmod 2 \quad \text{and} \quad H^2(q^j) \equiv H(q^{2j}) \pmod 2. \tag{3.8}$$

Next, it follows from (2.1) and (2.2) that

$$H(q)G(q) = f_5/f_1. \tag{3.9}$$

Employing (2.15), (3.8), and (3.9) in (3.7), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \text{DSOME}(25n+6)q^n &\equiv 2\frac{f_5 f_{20}}{f_2 G(q)H(q^4)} + 2f_1 f_8 G^2(q^4) + 2q^3 f_1 f_{40}\frac{H(q^8)}{G(q^4)} \\ &\equiv 2f_1 f_2 H(q)G(q^4) + 2f_1 f_8 G(q^4)(G(q^4) + q^3 H(q^{16})) \pmod 4.\end{aligned} \tag{3.10}$$

However, from (2.12), we have

$$H(q) \equiv \frac{f_8}{f_2}(G(q^4) + q^3 H(q^{16})) \pmod 2. \tag{3.11}$$

Employing (3.11) in (3.10), we find that

$$\sum_{n=0}^{\infty} \text{DSOME}(25n+6)q^n \equiv 2f_1 f_2 H(q)G(q^4) + 2f_1 f_2 H(q)G(q^4) \equiv 0 \pmod 4.$$

This completes the proof of the case $r = 1$.

Case $r = 2$. Extracting the terms involving q^{5n+2} from both sides of (3.6), dividing by q^2, and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \text{DSOME}(25n+11)q^n \equiv 2f_5 f_{20}\frac{T(q^4)}{T(q)} + 2\frac{f_1 f_{10}^5}{f_2}\left(T^4(q^2) - \frac{3q^2}{T(q^2)}\right) \pmod 4.$$

Using (2.3), (3.8), and (3.9), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \text{DSOME}(25n+11)q^n &\equiv 2f_5 f_{20}\frac{G(q^4)H(q)}{H(q^4)(q)} + 2\frac{f_1 f_{10}^5}{f_2}\left(\frac{G^4(q^2)}{H^4(q^2)} + q^2\frac{H(q^2)}{G(q^2)}\right) \\ &\equiv 2f_1 f_4 H^2(q)G^2(q^4) + 2\frac{f_1 f_{10}^5}{f_2}\left(\frac{G(q^8)}{H(q^8)} + q^2\frac{H(q^2)}{G(q^2)}\right) \\ &\equiv 2f_1 f_4 H(q^2)G(q^8) \\ &\quad + 2\frac{f_1 f_{10}^5}{f_2 G(q^2)H(q^8)}(G(q^2)G(q^8) + q^2 H(q^2)H(q^8)) \pmod 4.\end{aligned}$$

Using (2.15) and then employing (3.9), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{DSOME}(25n+11)q^n &\equiv 2f_1f_4H(q^2)G(q^8)+2\frac{f_1f_{10}^5}{f_2f_4G(q^2)H(q^8)}\\ &\equiv 2f_1f_4H(q^2)G(q^8)+2\frac{f_1f_4f_{10}f_{40}}{f_2f_8G(q^2)H(q^8)}\\ &\equiv 2f_1f_4H(q^2)G(q^8)+2f_1f_4H(q^2)G(q^8)\\ &\equiv 4f_1f_4H(q^2)G(q^8)\\ &\equiv 0 \pmod 4,\end{aligned}$$

which completes the proof of the case $r=2$.

Case $r=3$. We extract the terms involving q^{5n+3} from both sides of (3.6) and then proceed as in the previous two cases. As a result, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{DSOME}(25n+16)q^n &\equiv 2qf_5f_{20}\frac{T(q)}{T(q^4)}+2q\frac{f_1f_{10}^5}{f_2}\left(T(q^2)+\frac{q^2}{T^4(q^2)}\right)\\ &\equiv 2qf_5f_{20}\frac{H(q^4)G(q)}{G(q^4)H(q)}+2q\frac{f_1f_{10}f_{40}}{f_2}\left(\frac{G(q^2)}{H(q^2)}+q^2\frac{H^4(q^2)}{G^4(q^2)}\right)\\ &\equiv 2qf_5f_{20}\frac{H(q^4)G(q)}{G(q^4)H(q)}+2q\frac{f_1f_{10}f_{40}}{f_2H(q^2)G(q^8)}(G(q^2)G(q^8)+q^2G(q^2)G(q^8))\\ &\equiv 2qf_1f_4G(q^2)H(q^8)+2q\frac{f_1f_{10}f_{40}}{f_2f_4H(q^2)G(q^8)}\\ &\equiv 2qf_1f_4G(q^2)H(q^8)+2qf_1f_4G(q^2)H(q^8)\equiv 0 \pmod 4.\end{aligned}$$

Case $r=4$. For this case, we extract the terms involving q^{5n+4} from both sides of (3.6) and then proceed as in the case of $r=1$ to deduce that

$$\begin{aligned}(3.12)\qquad &\sum_{n=0}^{\infty}\mathrm{DSOME}(25n+21)q^n\\ &\equiv 2f_5f_{20}\left(T(q)+\frac{q}{T(q^4)}\right)+2q\frac{f_1f_2^2f_{20}}{T(q^4)}+2\frac{f_1f_{10}^5}{f_2}T^3(q^2)\\ &\equiv \frac{2f_5f_{20}}{H(q)G(q^4)}(G(q)G(q^4)+qH(q)H(q^4))+2qf_1f_4f_{20}\frac{H(q^4)}{G(q^4)}\\ &\quad+2\frac{f_1f_{10}f_{40}G(q^2)G(q^4)}{f_2H(q^2)H(q^4)}\\ &\equiv 2\frac{f_5f_{20}}{f_2H(q)G(q^4)}+2qf_1f_4^2H(q^8)+2f_1f_8H(q^4)G(q^{16})\\ &\equiv 2f_1f_2G(q)H(q^4)+2f_1f_8H(q^4)(G(q^{16})+qH(q^4)) \pmod 4.\end{aligned}$$

Now, by (2.11), we have

$$G(q) \equiv \frac{f_8}{f_2}(G(q^{16}) + qH(q^4)) \pmod 4. \tag{3.13}$$

Using (3.13) in (3.12), we arrive at

$$\sum_{n=0}^{\infty} \text{DSOME}(25n+21)q^n \equiv 4f_1f_2G(q)H(q^4) \equiv 0 \pmod 4.$$

This completes the proof. ∎

Proof of Theorem 1.3. Employing (2.9) and (2.10) in (3.5), and then extracting the terms involving q^{5n} from the resulting identity, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \text{DSOME}(25n+1)q^n \\
&\quad = \frac{f_2}{8f_1} - \frac{f_5^7}{8f_{10}} + f_5^3f_{10}\bigg(x^3y + \frac{q^2}{x^3y} + 3q\bigg(\frac{x^2}{y} - \frac{y}{x^2}\bigg) - 5q\bigg) \\
&\qquad + \frac{1}{2}\bigg(\frac{f_5f_{10}^5}{f_1^6} + \frac{f_1^2f_5f_{10}^5}{f_2^4}\bigg)\bigg(2q\bigg(xy^2 - \frac{q^2}{xy^2}\bigg) - q\bigg(\frac{y^3}{x} + q^2\frac{x}{y^3}\bigg) - 5q^2\bigg) \\
&\qquad + 2q\frac{f_2^3f_{10}^2}{f_1} + 2q\frac{f_2^9f_5^{10}}{f_1^{19}}\bigg(10\bigg(x^5 - \frac{q^2}{x^5}\bigg) + 15q\bigg),
\end{aligned}$$

where, as defined in Lemma 2.1, $x = T(q)$ and $y = T(q^2)$. Employing (2.4)–(2.8) and (3.3), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \text{DSOME}(25n+1)q^n \\
&\quad = \frac{f_2}{8f_1} - \frac{f_1^7}{8f_2^3} + f_5^3f_{10}\bigg(K + \frac{16q^2}{K} - 3q\bigg) \\
&\qquad + \frac{q}{2}\bigg(\frac{f_5f_{10}^5}{f_1^6} + \frac{f_1^2f_5f_{10}^5}{f_2^4}\bigg)\bigg(K - \frac{4q^2}{K} - 3q\bigg) \\
&\qquad + 2q\frac{f_2^3f_{10}^2}{f_1} + 10q\frac{f_2^9f_5^{10}}{f_1^{19}}\bigg(2\frac{f_1^6}{f_5^6} + 25q\bigg) \\
&\quad = \frac{f_2}{8f_1} - \frac{f_1^7}{8f_2^3} + \frac{f_2^2f_5^8}{f_1f_{10}^4} + 16q^2\frac{f_1f_{10}^6}{f_2f_5^2} - 3qf_5^3f_{10} + \frac{q}{2}\frac{f_1^4f_{10}^3}{f_2^2f_5} + \frac{q}{2}\frac{f_2^2f_{10}^3}{f_1^4f_5} \\
&\qquad + 2q\frac{f_2^3f_{10}^2}{f_1} + 20q\frac{f_2^9f_5^4}{f_1^{13}} + 250q^2\frac{f_2^9f_5^{10}}{f_1^{19}}.
\end{aligned}$$

We now take the congruence modulo 8 on both sides of the above and repeatedly use (2.13) for further simplification. As a result, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \text{DSOME}(25n+1)q^n \\
&\quad \equiv \frac{8}{9}\frac{f_2}{f_1} - \frac{f_1^7}{8f_2^3} - 3qf_5^3f_{10} + \frac{q}{2}\frac{f_1^4f_{10}^3}{f_2^2f_5} + \frac{q}{2}\frac{f_2^2f_{10}^3}{f_1^4f_5} + 6q\frac{f_2^3f_{10}^2}{f_1} + 2q^2\frac{f_1f_5^2f_{20}^2}{f_2}
\end{aligned}$$

$$\equiv \frac{9}{8}\frac{f_2}{f_1}-\frac{1}{8}\bigg(\frac{f_1^3f_5^4}{f_2f_{10}^2}-4q\frac{f_1^4f_{10}^3}{f_2^2f_5}\bigg)-3qf_5^3f_{10}+\frac{q}{2}\frac{f_1^4f_{10}^3}{f_2^2f_5}+\frac{q}{2}\bigg(\frac{f_{10}^5}{f_5^5}+4q\frac{f_2f_{10}^8}{f_1^3f_5^6}\bigg)$$
$$+6qf_1^3f_2f_{10}^2+2q^2\frac{f_1f_5^2f_{20}^2}{f_2}$$
$$\equiv \frac{9}{8}\frac{f_2}{f_1}-\frac{1}{8}\frac{f_1^3f_5^4}{f_2f_{10}^2}-3qf_5^3f_{10}+q\frac{f_1^4f_{10}^3}{f_2^2f_5}+\frac{q}{2}\frac{f_{10}^5}{f_5^5}+6qf_1^3f_2f_{10}^2+4q^2\frac{f_1f_5^2f_{20}^2}{f_2}$$
$$\equiv \frac{9}{8}\frac{f_2}{f_1}-\frac{1}{8}\bigg(\frac{f_2f_5^8}{f_1f_{10}^4}-4qf_5^3f_{10}\bigg)-3qf_5^3f_{10}+q\bigg(f_5^3f_{10}-4q\frac{f_1f_{10}^6}{f_2f_5^2}\bigg)+\frac{q}{2}\frac{f_{10}^5}{f_5^5}$$
$$+6qf_1^3f_2f_{10}^2+4q^2\frac{f_1f_5^2f_{20}}{f_2}$$
$$\equiv \frac{f_2}{f_1}\bigg(\frac{9}{8}-\frac{f_5^8}{8f_{10}^4}\bigg)-\frac{3q}{2}f_5^3f_{10}+\frac{q}{2}\frac{f_{10}^5}{f_5^5}-2qf_1^3f_2f_{10}^2 \pmod 8.$$

With the use of (2.9) and (2.10), we rewrite the above as

$$\sum_{n=0}^{\infty}\mathrm{DSOME}(25n+1)q^n$$
$$\equiv\frac{f_{50}f_{25}^5}{f_5^6}\bigg(\frac{9}{8}-\frac{f_5^8}{8f_{10}^4}\bigg)\bigg(T(q^{10})-q^2-\frac{q^4}{T(q^{10})}\bigg)\bigg(T^4(q^5)+qT^3(q^5)+2q^2T^2(q^5)$$
$$+3q^3T(q^5)+5q^4-\frac{3q^5}{T(q^5)}+\frac{2q^6}{T^2(q^5)}-\frac{q^7}{T^3(q^5)}+\frac{q^8}{T^4(q^5)}\bigg)-\frac{3q}{2}f_5^3f_{10}+\frac{q}{2}\frac{f_{10}^5}{f_5^5}$$
$$-2qf_{10}^2f_{25}^3f_{50}\bigg(T(q^{10})-q^2-\frac{q^4}{T(q^{10})}\bigg)$$
$$\cdot\bigg(T^3(q^5)-3qT^2(q^5)+5q^3-\frac{3q^5}{T^2(q^5)}-\frac{q^6}{T^3(q^5)}\bigg).$$

Extracting the terms involving q^{5n+1} from both sides of the above and then using (2.5), (2.7), (2.14) and (3.3), we deduce that

$$\sum_{n=0}^{\infty}\mathrm{DSOME}(125n+26)q^n$$
$$\equiv\frac{f_{10}f_5^5}{f_1^6}\bigg(\frac{9}{8}-\frac{f_1^8}{8f_2^4}\bigg)\bigg(x^3y+\frac{q^2}{x^3y}-2\bigg(\frac{x^2}{y}-\frac{y}{x^2}\bigg)-5q\bigg)-\frac{3}{2}f_1^3f_2+\frac{1}{2}\frac{f_2^5}{f_1^5}$$
$$-2qf_2^2f_5^3f_{10}\bigg(x^3y+\frac{q^2}{x^3y}+3q\bigg(\frac{x^2}{y}-\frac{y}{x^2}\bigg)-5q\bigg)$$
$$\equiv\frac{f_{10}f_5^5}{f_1^6}\bigg(\frac{9}{8}-\frac{f_1^8}{8f_2^4}\bigg)\bigg(K^2-\frac{4q^2}{K}-3q\bigg)-\frac{3}{2}f_1^3f_2+\frac{1}{2}\frac{f_2^5}{f_1^5}-2f_2^2f_5^3f_{10}(K+q)$$
$$\equiv\frac{9}{8}\frac{f_2^2f_5^3}{f_1^4f_{10}}-\frac{1}{8}\frac{f_1^4f_5^3}{f_2^2f_{10}}-\frac{3}{2}f_1^3f_2+\frac{f_2^5}{2f_1^5}-2f_1^2f_2^4\frac{f_5}{f_{10}}.$$

Now by repeated use of (2.4) and (2.5), we find that

$$\begin{aligned}(3.14)\quad &\sum_{n=0}^{\infty}\text{DSOME}(125n+26)q^n\\
&\equiv \frac{9}{8}\bigg(\frac{f_{10}}{f_5}+4q\frac{f_2f_{10}^4}{f_1^3f_5^2}\bigg)-\frac{1}{8}\bigg(\frac{f_5^7}{f_{10}^3}-4q\frac{f_1f_5^2f_{10}^2}{f_2}\bigg)-\frac{3}{2}\bigg(\frac{f_2^3f_5^4}{f_1f_{10}^2}-4q\frac{f_2^2f_{10}^3}{f_5}\bigg)\\
&\quad+\frac{1}{2}\bigg(\frac{f_2^3f_{10}^2}{f_1f_5^4}+4q\frac{f_2^4f_{10}^5}{f_1^4f_5^5}\bigg)-2\bigg(\frac{f_1^3f_2f_5^4}{f_{10}^2}+q\frac{f_1^4f_{10}^3}{f_5}\bigg)\\
&\equiv \frac{9}{8}\,\frac{f_{10}}{f_5}+\frac{9q}{2}\bigg(\frac{f_1f_{10}^6}{f_2f_5^6}+4q\frac{f_{10}^9}{f_1^2f_5^7}\bigg)-\frac{1}{8}\,\frac{f_5^7}{f_{10}^3}+\frac{q}{2}\,\frac{f_1f_5^2f_{10}^2}{f_2}\\
&\quad-\frac{3}{2}\bigg(\frac{f_5^7}{f_{10}^3}+q\frac{f_1f_5^2f_{10}^2}{f_2}\bigg)+\frac{f_2^3f_{10}^2}{2f_1f_5^4}-2f_1^3f_2-2q\bigg(\frac{f_1^5f_5^2f_{10}^2}{f_2^3}+q\frac{f_1^6f_{10}^7}{f_2^4f_5^3}\bigg)\\
&\equiv \frac{9}{8}\,\frac{f_{10}}{f_5}+\frac{9q}{2}\,\frac{f_1f_{10}^6}{f_2f_5^6}-\frac{13}{8}\,\frac{f_5^7}{f_{10}^3}-3q\frac{f_1f_5^2f_{10}^2}{f_2}+\frac{1}{2}\bigg(\frac{f_{10}}{f_5}+q\frac{f_1f_{10}^6}{f_2f_5^6}\bigg)-2f_1^3f_2\\
&\equiv \frac{13}{8}\bigg(\frac{f_{10}}{f_5}-\frac{f_5^7}{f_{10}^3}\bigg)+2\bigg(\frac{f_2^3f_5^4}{f_1f_{10}^2}-\frac{f_5^7}{f_{10}^3}\bigg)-2f_1^3f_2\\
&\equiv \frac{13}{8}\bigg(\frac{f_{10}}{f_5}-\frac{f_5^7}{f_{10}^3}\bigg)-2\frac{f_{10}}{f_5}\pmod 8.\end{aligned}$$

It follows from the above that

$$\text{DSOME}(625n+125r+26)\equiv 0\pmod 8,\qquad 1\le r\le 4.$$

This completes the proof of Theorem 1.3. ∎

Proof of Theorem 1.4. Using (1.2) in (3.14), we have

$$(3.15)\quad \sum_{n=0}^{\infty}\text{DSOME}(125n+26)q^n\equiv 5\sum_{n=0}^{\infty}\text{DSOME}(n)q^{5n}-2\frac{f_{10}}{f_5}\pmod 8,$$

which yields

$$\sum_{n=0}^{\infty}\text{DSOME}(625n+26)q^n\equiv 5\sum_{n=0}^{\infty}\text{DSOME}(n)q^n-2\frac{f_2}{f_1}\pmod 8.$$

Employing (2.9) and (2.10) and then extracting the terms involving q^{5n+1} from the resulting congruence, we find that

$$\begin{aligned}&\sum_{n\ge 0}\text{DSOME}(3125n+651)q^n\\
&\qquad\equiv 5\sum_{n\ge 0}\text{DSOME}(5n+1)q^n-2\frac{f_5^5f_{10}}{f_1^6}\bigg(K^2-\frac{4q^2}{K}-3q\bigg),\end{aligned}$$

which, by (3.3), implies that

$$\sum_{n\geq 0}\mathrm{DSOME}(3125n+651)q^n \equiv 5\sum_{n\geq 0}\mathrm{DSOME}(5n+1)q^n - 2\frac{f_2^2f_5^3}{f_1^4f_{10}}$$
$$\equiv 5\sum_{n\geq 0}\mathrm{DSOME}(5n+1)q^n - 2\frac{f_{10}}{f_5} \pmod 8.$$

It follows from the above congruence and (3.15) that

$$\sum_{n\geq 0}\mathrm{DSOME}(3125n+651)q^n \tag{3.16}$$
$$\equiv 5\sum_{n\geq 0}\mathrm{DSOME}(5n+1)q^n + \sum_{n\geq 0}\mathrm{DSOME}(125n+26)q^n$$
$$- 5\sum_{n\geq 0}\mathrm{DSOME}(n)q^{5n} \pmod 8.$$

Comparing the coefficients of q^{5n} as well as q^{5n+r}, where $1\leq r\leq 4$, in turn, on both sides of (3.16), we obtain

$$\mathrm{DSOME}(15625n+651) \equiv 5\mathrm{DSOME}(25n+1) + \mathrm{DSOME}(625n+26)$$
$$- 5\mathrm{DSOME}(n) \pmod 8$$

and

$$\mathrm{DSOME}(15625n+3125r+651) \equiv 5\mathrm{DSOME}(25n+5r+1)$$
$$+ \mathrm{DSOME}(625n+125r+26) \pmod 8.$$

Thus, we have completed the proof of Theorem 1.4. ∎

4. Conclusion and remarks. In this paper, we have expressed the generating function of DSOME(n), found by Andrews and Ghosh Dastidar, in closed form. The new expression allows us to find new congruences modulo 4 and 8. It seems that far more congruences are yet to be discovered. We propose the following conjecture based on numerical calculations.

Conjecture 4.1. *For all $n\geq 0$,*

$$\mathrm{DSOME}(50n+21) \equiv 0 \pmod 8$$
$$\mathrm{DSOME}(100n+71) \equiv 0 \pmod{16}.$$

Acknowledgements. The authors wish to thank the anonymous referee and Dazhao Tang for their helpful comments.

Funding. The second author was partially supported by University Grants Commission, Government of India, under the UGC-JRF scheme (Ref. No. 221610056019). The author thanks the funding agency.

REFERENCES

[AG26] G. E. Andrews and M. Ghosh Dastidar, $p(5n+4)$ *again*, Ramanujan J. 69 (2026), art. 26, 9 pp.

[BB18] N. D. Baruah and N. M. Begum, *Exact generating functions for the number of partitions into distinct parts*, Int. J. Number Theory 14 (2018), 1995–2011.

[B06] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Amer. Math. Soc., Providence, RI, 2006.

[BC07] B. C. Berndt, G. Choi, Y.-S. Choi, H. Hahn, B. P. Yeap, A. J. Yee, H. Yesilyurt and J. Yi, *Ramanujan's forty identities for the Rogers–Ramanujan functions*, Mem. Amer. Math. Soc. 188 (2007), no. 880, vi+96 pp.

[SR19] S. Ramanujan, *Some properties of $p(n)$, the number of partitions of n*, Proc. Cambridge Philos. Soc. 19 (1919), 207–210.

[LR94] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. 25 (1894), 318–343.

[W33] G. N. Watson, *Proof of certain identities in combinatory analysis*, J. Indian Math. Soc. 20 (1933), 57–69.

Nayandeep Deka Baruah, Pankaj Gogoi
Department of Mathematical Sciences
Tezpur University
Assam 784028, India
E-mail: nayan@tezu.ernet.in
nayandeeptezu@gmail.com
msp23110@tezu.ac.in
gopankajgo07@gmail.com