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# Average genus number for pure fields of prime degree

by

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**Abstract.** Let $\ell \geq 3$ be prime. Let $\mathcal{F}_\ell$ be the collection of (isomorphism classes of) pure number fields $\mathbb{Q}(\sqrt[\ell]{a})$ of degree $\ell$, ordered by the absolute value of their discriminant. In 2018, Benli proved a counting theorem for $\mathcal{F}_\ell$, generalizing a previous theorem of Cohen and Morra when $\ell = 3$. We prove that the proportion of pure fields of degree $\ell$ with genus number one is asymptotic to $(A_\ell \log X)^{-1}$ and that the average genus number for pure fields of degree $\ell$ is asymptotic to $B_\ell (\log X)^{\ell-1}$. Both $A_\ell$ and $B_\ell$ are expressed explicitly as a product over primes.

**1. Introduction.** The subject of counting number fields, and more generally arithmetic statistics, has been one of intense interest over the last few decades. Given a collection $\mathcal{F}$ of number fields and a function $f\colon \mathcal{F} \to \mathbb{R}_{\geq 0}$, one can ask for an asymptotic (or the limiting value) for the expression

$$\frac{\sum_{K\in\mathcal{F}, |\Delta_K|\leq X} f(K)}{\sum_{K\in\mathcal{F}, |\Delta_K|\leq X} 1}.$$

Here our fields $K$ are ordered by the absolute value of the discriminant $\Delta_K$, but alternative height functions could be considered. Examples of functions $f(K)$ of very high interest are the class number $h_K$, the indicator function for $h_K = 1$, or the size of the $p$-torsion of the class group $\#(\mathrm{Cl}_K[p])$. A prototypical example is the famous Davenport–Heilbronn theorems (see [DH71, BST13, TT13, BTT24]) which allow one to obtain the average of $\#(\mathrm{Cl}_K[3])$ over the collection of quadratic fields of fixed signature. However, many such questions are completely out of reach. Recently, McGown–Tucker, Kim, McGown–Tucker–Thorne, and Frei–Loughran–Newton studied the statistics of the genus number $g_K$ (see [MT23, MTT23, K20, FLN23]). This quan-

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tity is supported at the bad primes, in the language of the Cohen–Lenstra heuristics (see [CL84]), and is much more amenable to study than $h_K$.

Fröhlich (see [F59]) defined the genus field of a number field $K$ to be the maximal extension $K^*$ of $K$ that is unramified at all finite primes and is a compositum of the form $Kk^*$ where $k^*$ is absolutely abelian. The genus number of $K$ is defined as the degree of this extension; that is, $g_K = [K^* : K]$. By the Kronecker–Weber Theorem, the field $k^*$ is contained in a cyclotomic field, which makes the genus field more tractable than the full Hilbert class field.

It is most common to take $\mathcal{F}$ to consist of all degree $d$ extensions of $\mathbb{Q}$ having a specified Galois closure $G$. In this case, Malle's conjecture (see [M02, M04]) predicts that the count of such extensions with $|\Delta_K| \leq X$ is asymptotic to $CX^{\alpha}(\log X)^{\beta}$ for specific constants $0 < \alpha \leq 1$, $\beta \geq 0$, and some constant $C > 0$. (One can also consider the more general problem where $\mathbb{Q}$ is replaced by any base field $K$.)

On the other hand, there are other interesting collections of fields that one might consider. In this paper we will be concerned with the collection of pure fields of prime degree $\ell$. Let $\mathcal{F}_\ell$ denote the collection of all (isomorphism classes of) degree $\ell$ number fields of the form $K = \mathbb{Q}(\sqrt[\ell]{a})$ with $a \in \mathbb{Z}$. Note that $a \in \mathbb{Z}$ is not an $\ell$-th power; moreover, we may assume that $a$ is positive and $\ell$-free, which we will tacitly do throughout. Define the counting function

$$N_\ell(X) = \#\{K \in \mathcal{F}_\ell : |\Delta_K| \leq X\}.$$

The following is due to Benli (see [B18]):

THEOREM 1.1. *For all primes $\ell \geq 3$, one has*

$$N_\ell(X) \sim C_\ell X^{\frac{1}{\ell-1}} (\log X)^{\ell-2}, \tag{1.1}$$

*for an explicit constant $C_\ell$.*

Actually, she proves something stronger, but at present we will only be concerned with asymptotics. We note that Cohen and Morra had previously treated the case of $\ell = 3$ (see [CM11]). The aim of this paper is to determine the proportion of fields with genus number one and the average genus number for the family $\mathcal{F}_\ell$. More precisely, we seek an asymptotic as $X$ tends to infinity for these quantities.

THEOREM 1.2. *The proportion of pure fields of prime degree $\ell \geq 3$ with genus number one equals*

$$\frac{\#\{K \in \mathcal{F}_\ell : g_K = 1, |\Delta_K| \leq X\}}{\#\{K \in \mathcal{F}_\ell : |\Delta_K| \leq X\}} \sim \frac{1}{A_\ell \log X},$$

*where*

$$A_\ell = \frac{1}{(\ell-1)(\ell-2)} \prod_p A_{\ell,p}, \quad A_{\ell,p} = \left(1-\frac{1}{p}\right) \begin{cases} 1+\frac{\ell-1}{p}, & p \equiv 1 \pmod{\ell}, \\ 1, & p \not\equiv 1 \pmod{\ell}. \end{cases}$$

We remark that the $A_{\ell,p}$ can also be written as a single formula in terms of the $\ell$-th power residue symbol, and the same is true for $B_{\ell,p}$ in the following theorem.

THEOREM 1.3. *The average genus number over pure fields of prime degree $\ell \geq 3$ satisfies*

$$\frac{\sum_{\substack{K \in \mathcal{F}_\ell \\ |\Delta_K| \leq X}} g_K}{\sum_{\substack{K \in \mathcal{F}_\ell \\ |\Delta_K| \leq X}} 1} \sim B_\ell (\log X)^{\ell-1}, \quad B_\ell = \frac{(\ell-2)!}{(2\ell-3)!(\ell-1)^{\ell-1}} \prod_p B_{\ell,p},$$

*where*

$$B_{\ell,p} = \left(1-\frac{1}{p}\right)^{\ell-1} \begin{cases} \left(1+\frac{\ell(\ell-1)}{p}\right)\left(1+\frac{\ell-1}{p}\right)^{-1}, & p \equiv 1 \pmod{\ell}, \\ 1, & p \not\equiv 1 \pmod{\ell}. \end{cases}$$

In Section 2 we will give some notation and definitions that we will use throughout the paper. In Section 3 we will summarize previous results concerning statistics of genus numbers. In Section 4 we state some preliminary lemmas that will be required in the proofs of our results. In Section 5 we give a brief proof of the asymptotic for $N_\ell(X)$ given in (1.1). Although our argument differs from the one given in [B18], the main reason for presenting the proof in its entirety is that this will serve as an outline of the proof strategy we will employ in later sections. Finally, in Sections 6 and 7 we prove Theorems 1.2 and 1.3, respectively.

**2. Notation and definitions.** Here we collect some definitions and notation, some of which were already introduced in the previous section. Throughout the paper, $\ell$ will always denote a fixed odd prime. The letter $p$, wherever it appears, will always denote a prime.

Given $n \in \mathbb{Z}^+$, we will write $\mathrm{rad}(n)$ to denote the radical of $n$, and $\omega(n)$ for the number of distinct prime divisors of $n$. We write $\widehat{\omega}(n)$ to denote the number of primes dividing $n$ that are congruent to 1 modulo $\ell$. We will refer to $n$ as *$\ell$-free* if it is not divisible by an $\ell$-th power; that is, $n$ is $\ell$-free if $\nu_p(n) < \ell$ for all primes $p$, where $\nu_p$ denotes the $p$-adic valuation.

The letter $K$ will always denote a number field and $\Delta_K$ will denote its discriminant. We will write $\mathcal{F}_\ell$ for the collection of pure fields of degree $\ell$, and $N_\ell(X)$ for the number of $K \in \mathcal{F}_\ell$ such that $|\Delta_K| \leq X$.

Recall that the class group (resp., narrow class group) of $K$, denoted by $\mathrm{Cl}_K$ (resp., $\mathrm{Cl}_K^+$), is the group of all nonzero fractional ideals modulo the subgroup of all principal ideals (resp., totally positive principal ideals).

We write $h_K$ (resp., $h_K^+$) to denote the class number (resp., narrow class number), defined as $h_K = |\mathrm{Cl}_K|$ (resp., $h_K^+ = |\mathrm{Cl}_K^+|$).

Definition 2.1. The genus field of a number field $K$ is the maximal extension $K^*$ of $K$ that is unramified at all finite primes and is a compositum of the form $Kk^*$ where $k^*$ is absolutely abelian. The genus number of $K$ is defined as $g_K = [K^* : K]$.

Note that by class field theory, $g_K \mid h_K^+$. Moreover, since $h_K$ and $h_K^+$ differ by a power of 2, it follows from Lemma 4.3 that for $K \in \mathcal{F}_\ell$, one has $g_K \mid h_K$. In fact, this holds whenever $K$ has odd prime degree.

**3. Prior work.** Here we will indicate previous works that have considered the study of the genus number of a number field from a statistical perspective. Some background and notation necessary for this section will not be needed for the rest of the paper.

The study was initiated in [MT23], where the family of all cubic fields was considered. It was shown that the number of cubic fields $K/\mathbb{Q}$ with genus number one such that $|\Delta_K| \le X$ and $\mathrm{sgn}(\Delta_K) = \pm 1$ equals

$$\frac{29}{54n^{\pm}\zeta(2)} \prod_{p \equiv 2 \,(\mathrm{mod}\, 3)} \left(1 + \frac{1}{p(p+1)}\right) X + O\big(X^{16/17+\varepsilon}\big),$$

where $n^+ = 6$, $n^- = 2$, and hence the proportion of cubic fields with genus number one (of positive or negative discriminant) equals

$$\frac{29\,\zeta(3)}{27\,\zeta(2)} \prod_{p \equiv 2 \,(\mathrm{mod}\, 3)} \left(1 + \frac{1}{p(p+1)}\right).$$

In the above, $\zeta(s)$ denotes the Riemann zeta function. The average genus number of a cubic field (in the positive or negative discriminant case) is given by

$$\lim_{X \to \infty} \frac{\sum_{0 < \pm \Delta_K \le X} g_K}{\sum_{0 < \pm \Delta_K \le X} 1} = \frac{119\zeta(3)}{108\zeta(2)} \prod_{p \equiv 1 \,(\mathrm{mod}\, 3)} \left(1 + \frac{3}{p(p+1)}\right) \prod_{p \equiv 2 \,(\mathrm{mod}\, 3)} \left(1 + \frac{1}{p(p+1)}\right).$$

Moreover, it was shown that given any $k \ge 0$, there is a positive proportion of cubic fields with genus number equal to $3^k$ (and hence with class number divisible by $3^k$).

Recently, this result was improved in [Y26], where secondary terms for the number of cubic fields with $g_K = 3^k$ were established. Namely, the number of cubic fields with $g_K = 3^k$, $|\Delta_K| \le X$, and $\mathrm{sgn}(\Delta_K) = \pm 1$ equals

$$\alpha_k^{\pm} X + \beta_k^{\pm} X^{5/6} + O(X^{2/3+\varepsilon}),$$

for constants $\alpha_k^{\pm}, \beta_k^{\pm} > 0$. More generally, this paper gives uniform estimates with a secondary term for all moments of the genus number (including the average), also allowing for additional local specifications.

As $S_3$-cubic fields make up 100% of all cubic fields, from the perspective of [MT23, Y26] it was only necessary to treat the case of $S_3$-cubic fields. However, interesting subfamilies of cubic fields include the cyclic cubic fields and the pure cubic fields. We note that the collection of pure cubic fields $\mathcal{F}_3$ is the subfamily of $S_3$-cubic fields whose Galois closure contains $\mathbb{Q}(\sqrt{-3})$.

In [K20], the cases of cyclic and dihedral fields of any prime degree $\ell$ were considered. The number of cyclic fields of prime degree $\ell$ and $|\Delta_K| \leq X$ with genus number one was shown to equal

$$X^{\frac{1}{\ell-1}}(\log X)^{-1} + O\big(X^{\frac{1}{\ell-1}}(\log X)^{-2}\big),$$

and in regards to the average genus number, it was shown that

$$\sum_{|\Delta_K|\leq X} g_K = B\frac{2\ell-2}{\ell(\ell-1)^\ell}\,\frac{X^{\frac{1}{\ell-1}}}{(\log X)^{\ell-1}} + O\bigg(\frac{X^{\frac{1}{\ell-1}}}{(\log X)^{\ell-2}}\bigg),$$

for some constant $B > 0$. The dihedral case is more difficult and therefore results are conditional on a conjecture for the average $\ell$-part of the class group across discriminants in residue classes (see [K20, Conjecture 5.6]); under this hypothesis, it was shown that a positive proportion of $D_\ell$-extensions (of degree $\ell$) have genus number one and that

$$\sum_{|\Delta_K|\leq X} g_K \ll X^{\frac{2}{\ell-1}}.$$

In [MTT23], the family of all quintic fields was considered. An asymptotic for the counting function of all quintic fields with $g_K = 1$ with a power saving error term was given; as a consequence, it was shown that the proportion of quintic fields with $i$ pairs of complex embeddings having genus number one equals

$$\frac{506874}{506875}\prod_{p\equiv 1 \,(\mathrm{mod}\, 5)}\frac{p^4+p^3+2p^2+2p}{p^4+p^3+2p^2+2p+1}.$$

Moreover, it was shown that given any $k \geq 0$, there is a positive proportion of quintic fields with genus number equal to $5^k$. Finally, the average genus number taken over all quintic fields of a given signature is given by

$$\lim_{X\to\infty}\frac{\sum_{|\Delta_K|\leq X} g_K}{\sum_{|\Delta_K|\leq X} 1} = \frac{506879}{506875}\prod_{p\equiv 1 \,(\mathrm{mod}\, 5)}\frac{p^4+p^3+2p^2+2p+5}{p^4+p^3+2p^2+2p+1}.$$

As in the cubic case, the $S_5$-quintic fields constitute 100% of all quintic fields, and hence that work only considered $S_5$-quintic extensions. As $\mathcal{F}_5$

(the collection of pure quintic fields) is a subcollection of those quintic fields whose Galois closure is the Frobenius group of order 20, these fields were not treated in that work. Indeed, one can show that the Galois closure of $K \in \mathcal{F}_\ell$ (the collection of pure fields of prime degree $\ell$) is isomorphic to a semidirect product $\mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/(\ell-1)\mathbb{Z}$.

Finally, in [FLN23], the case of all abelian extensions, over an arbitrary case field $F$, was considered. It is shown that

$$\sum_{N(\mathfrak{f}_{K/F}) \le X} g_{K/F} \sim cX(\log X)^{\rho(F,G)-1},$$

where $g_{K/F}$ is the relative genus number, $\mathfrak{f}_{K/F}$ denotes the conductor of $K/F$, and the sum is taken over all abelian extensions $K/F$ with Galois group isomorphic to a fixed abelian group $G$, and $N(\mathfrak{f}_{K/F}) \le X$. Here $\rho(F,G)$ is a constant that depends only on the group structure of $G$ and the roots of unity contained in $F$. Their main result is a generalization of this statement, in which the constant $c$ is given explicitly. In addition, it is shown that every genus number appears 0% of the time, in contrast with the $S_3$ and $S_5$ cases.

**4. Preliminaries.** In what follows it is helpful to remember that $\mathbb{Q}(\sqrt[\ell]{a}) = \mathbb{Q}(\sqrt[\ell]{-a})$, and hence one only needs to consider $a > 0$. The following gives a formula for the discriminant of the fields in $\mathcal{F}_\ell$ (see [B64] or [M93, Theorem 1]).

LEMMA 4.1. *For $K = \mathbb{Q}(\sqrt[\ell]{a}) \in \mathcal{F}_\ell$ with $\mathrm{rad}(a) = n$, we have*

$$|\Delta_K| = \begin{cases} \ell^{\ell-2} n^{\ell-1}, & a^{\ell-1} \equiv 1 \pmod{\ell^2}, \\ \ell^\ell n^{\ell-1}, & a^{\ell-1} \not\equiv 1 \pmod{\ell^2}. \end{cases}$$

The following result allows one to parametrize $\mathcal{F}_\ell$. The reader may wish to consult a reference on Kummer theory, such as in [L02, Chapter VI, Sections 7–9]. We give the proof for the sake of completeness.

LEMMA 4.2. *Suppose $a, b \in \mathbb{Z}_{\ge 2}$ are $\ell$-free. Then $\mathbb{Q}(\sqrt[\ell]{a})$ and $\mathbb{Q}(\sqrt[\ell]{b})$ are isomorphic if and only if $a = b^i$ in $\mathbb{Q}^\times/\left(\mathbb{Q}^\times\right)^\ell$. Moreover, the function*

$$\{a \in \mathbb{Z}_{\ge 2} : a \text{ is } \ell\text{-free}\} \to \mathcal{F}_\ell$$

*given by $a \mapsto \mathbb{Q}(\sqrt[\ell]{a})$ is a surjective $(\ell-1)$-to-1 map.*

*Proof.* Let $a = p_1^{e_1} \cdots p_k^{e_k}$. Suppose for the direct implication that $\mathbb{Q}(\sqrt[\ell]{a}) \cong \mathbb{Q}(\sqrt[\ell]{b})$. Since both fields are the unique real fields amongst their conjugates, we have $\mathbb{Q}(\sqrt[\ell]{a}) = \mathbb{Q}(\sqrt[\ell]{b})$. By Lemma 4.1, we can assume that $\mathrm{rad}(a) = \mathrm{rad}(b)$ and write $b = p_1^{f_1} \cdots p_k^{f_k}$. Set

$$t = (a^{f_1}/b^{e_1})^{1/\ell} = p_2^{(f_1e_2 - e_1f_2)/\ell} \cdots p_k^{(f_1e_k - e_1f_k)/\ell} \in \mathbb{Q}(\sqrt[\ell]{a}).$$

We claim $t \in \mathbb{Q}$. If $k = 1$, then the proof is immediate, as $t = 1$. In the case where $k \geq 2$ and $t \notin \mathbb{Q}$, we find $\mathbb{Q}(t) = \mathbb{Q}(\sqrt[\ell]{a})$ but $\mathrm{rad}(t^\ell) \neq \mathrm{rad}(a)$, a contradiction.

Because $t \in \mathbb{Q}$, we now have $f_1 e_j - e_1 f_j \equiv 0 \pmod{\ell}$ for all $2 \leq j \leq k$. It follows that $a = b^c$ in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^\ell$, where $c = e_1 f_1^{-1} \pmod{\ell}$. The reverse implication is easy, and the rest of the lemma follows. ∎

The following formula for the genus number appears as Theorem 7 in [I76, Chapter 7]. It is also established in [I81, Sections 4 and 5].

LEMMA 4.3. *Let $K = \mathbb{Q}(\sqrt[\ell]{m}) \in \mathcal{F}_\ell$. Then*

$$g_K = \prod_{\substack{p \mid m \\ p \neq \ell}} (\ell, p-1) = \ell^{\widehat{\omega}(m)},$$

*where $\widehat{\omega}(n)$ denotes the number of $p \mid n$ such that $p \equiv 1 \pmod{\ell}$.*

The following result is standard (see, for example, [MV07, Chapter 4]).

LEMMA 4.4 (Mertens' Second Theorem in residue classes). *Suppose $T \subseteq (\mathbb{Z}/m\mathbb{Z})^\times$. As $X \to \infty$, we have*

$$\sum_{\substack{p \leq X \\ p \in T \pmod{m}}} \frac{1}{p} = \frac{|T|}{\phi(m)} \log\log X + M_T + O_T\left(\frac{1}{\log X}\right).$$

The following result, first proved in [M74], is classical; see, for example, [MV07, Chapter 2].

LEMMA 4.5 (Merten's Third Theorem).

$$\prod_{p \leq X} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log X}.$$

The following result is well-known (see [W61, Theorem 1]).

LEMMA 4.6 (Wirsing's Theorem). *Let $f(n) \geq 0$ be a multiplicative function. Assume that there is a constant $\tau > 0$ such that*

$$\sum_{p \leq X} f(p) \sim \tau \cdot \frac{X}{\log X} \tag{4.1}$$

*and constants $\gamma_1 > 0$ and $0 \leq \gamma_2 < 2$ such that $f(p^k) \leq \gamma_1 \gamma_2^k$ for all $k \geq 2$. Then*

$$\sum_{n \leq X} f(n) \sim \frac{X}{\log X} \cdot \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \prod_{p \leq X} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right).$$

REMARK 4.7. Our applications of Wirsing's Theorem will always be to a sum of the form $\sum_{n \leq X} \mu^2(n) f(n)$. Note that this simplifies both the hypothesis and the conclusion of Lemma 4.6.

**5. Counting pure fields of prime degree $\ell$.** Fix a prime $\ell \geq 3$.

LEMMA 5.1. *One has*

$$(5.1)\quad (\ell-1)N_\ell(X)+1 = \sum_{n\leq a(\ell)X^{\frac{1}{\ell-1}}} \mu^2(n) \sum_{\substack{\mathrm{rad}(a)=n \\ a \text{ is } \ell\text{-free} \\ a^{\ell-1}\equiv 1 \,(\mathrm{mod}\,\ell^2)}} 1 + \sum_{n\leq b(\ell)X^{\frac{1}{\ell-1}}} \mu^2(n) \sum_{\substack{\mathrm{rad}(a)=n \\ a \text{ is } \ell\text{-free} \\ a^{\ell-1}\not\equiv 1 \,(\mathrm{mod}\,\ell^2)}} 1,$$

*where* $a(\ell)=\ell^{-1+\frac{1}{\ell-1}}$ *and* $b(\ell)=\ell^{-1-\frac{1}{\ell-1}}$.

*Proof.* According to Lemma 4.2, we can parametrize fields in $\mathcal{F}_\ell$ by $\ell$-free numbers $a\in\mathbb{Z}_{\geq 2}$, granted that we overcount by a factor of $\ell-1$. According to Lemma 4.1, the discriminant of such a field is determined by $n=\mathrm{rad}(a)$, with two cases depending on the residue class of $a^{\ell-1}$ modulo $\ell^2$. ∎

DEFINITION 5.2. We define the following multiplicative function (supported on squarefree integers $n$), as well as the associated summatory function (over squarefree integers):

$$f_\chi(n)=\sum_{\substack{\mathrm{rad}(a)=n \\ a \text{ is } \ell\text{-free}}} \chi(a), \qquad F_\chi(Y)=\sum_{n\leq Y}\mu^2(n)f_\chi(n).$$

Taking the character to be the principal character of modulus 1, we similarly define

$$f(n)=\sum_{\substack{\mathrm{rad}(a)=n \\ a \text{ is } \ell\text{-free}}} 1=(\ell-1)^{\omega(n)}, \qquad F(Y)=\sum_{n\leq Y}\mu^2(n)f(n).$$

LEMMA 5.3. *One has*

$$(5.2)\quad (\ell-1)N_\ell(X)+1 = \frac{1}{\ell}\sum_{\substack{\chi \bmod \ell^2 \\ \chi^\ell=\chi_0}} \left[F_\chi\left(a(\ell)X^{\frac{1}{\ell-1}}\right)-F_\chi\left(b(\ell)X^{\frac{1}{\ell-1}}\right)\right]+F\left(b(\ell)X^{\frac{1}{\ell-1}}\right).$$

*Proof.* One begins with the expression derived in Lemma 5.1. Using the orthogonality of Dirichlet characters in the form

$$\sum_{\chi \bmod \ell^2}\chi(a^{\ell-1})=\begin{cases}\phi(\ell^2), & a^{\ell-1}\equiv 1 \ (\mathrm{mod}\ \ell^2),\\ 0, & \text{otherwise,}\end{cases}$$

the first summand on the right-hand side of (5.1) becomes

$$\frac{1}{\phi(\ell^2)}\sum_{\chi \bmod \ell^2}\sum_{n\le a(\ell)X^{\frac{1}{\ell-1}}}\mu^2(n)\sum_{\substack{\mathrm{rad}(a)=n\\ a \text{ is } \ell\text{-free}}}\chi^{\ell-1}(a) = \frac{1}{\ell}\sum_{\substack{\chi \bmod \ell^2\\ \chi^\ell=\chi_0}}\sum_{n\le a(\ell)X^{\frac{1}{\ell-1}}}\mu^2(n)\sum_{\substack{\mathrm{rad}(a)=n\\ a \text{ is } \ell\text{-free}}}\chi(a). \tag{5.3}$$

The sums above are taken over all Dirichlet characters modulo $\ell^2$. In (5.3), the notation $\chi_0$ refers to the principal character modulo $\ell^2$.

Using the notation in Definition 5.2, (5.3) equals

$$\frac{1}{\ell}\sum_{\substack{\chi \bmod \ell^2\\ \chi^\ell=\chi_0}}F_\chi\big(a(\ell)X^{\frac{1}{\ell-1}}\big),$$

which is the first summand on the right-hand side of (5.1). In a similar manner, the second summand on the right-hand side of (5.1) equals

$$\sum_{n\le b(\ell)X^{\frac{1}{\ell-1}}}\mu^2(n)\sum_{\substack{\mathrm{rad}(a)=n\\ a \text{ is } \ell\text{-free}}}1 - \sum_{n\le b(\ell)X^{\frac{1}{\ell-1}}}\mu^2(n)\sum_{\substack{\mathrm{rad}(a)=n\\ a \text{ is } \ell\text{-free}\\ a^{\ell-1}\equiv 1 \,(\mathrm{mod}\, \ell^2)}}1 = F\big(b(\ell)X^{\frac{1}{\ell-1}}\big) - \frac{1}{\ell}\sum_{\substack{\chi \bmod \ell^2\\ \chi^\ell=\chi_0}}F_\chi\big(b(\ell)X^{\frac{1}{\ell-1}}\big). \ \blacksquare$$

To give an asymptotic for $N_\ell(X)$, it suffices to give an asymptotic for $F_\chi(Y)$ when $\chi=\chi_0$, including the case of $F(Y)$, and to bound $F_\chi(Y)$ when $\chi\neq\chi_0$. Indeed, our asymptotic will be order of magnitude $Y(\log Y)^{\ell-2}$ and our bound will be $F_\chi(Y)\ll Y(\log Y)^{1-2/\ell}$ when $\chi\neq\chi_0$. Since $1-2/\ell<\ell-2$, the latter bound is lower order of magnitude and hence we will be able to deduce an asymptotic for the right-hand side of (5.2).

Lemma 5.4. *We have the following asymptotics:*

$$F(Y)\sim D_\ell Y(\log Y)^{\ell-2}, \tag{5.4}$$
$$D_\ell = \frac{1}{(\ell-2)!}\prod_p\left(1+\frac{\ell-1}{p}\right)\left(1-\frac{1}{p}\right)^{\ell-1},$$

$$F_{\chi_0}(Y)\sim c(\ell)D_\ell Y(\log Y)^{\ell-2},\quad c(\ell)=\left(1+\frac{\ell-1}{\ell}\right)^{-1}=\frac{\ell}{2\ell-1}. \tag{5.5}$$

*Proof.* We clearly have $\sum_{p\le Y}(\ell-1) \sim (\ell-1)Y/\log Y$ by the Prime Number Theorem and therefore applying Lemma 4.6 to $F(Y)$ implies

$$\sum_{n\le Y}\mu^2(n)f(n) \sim \frac{Y}{\log Y}\,\frac{e^{-\gamma(\ell-1)}}{(\ell-2)!}\prod_{p\le Y}\left(1+\frac{\ell-1}{p}\right).$$

We multiply both sides of the asymptotic by the function

$$e^{\gamma(\ell-1)}(\log Y)^{\ell-1}\prod_{p\le Y}\left(1-\frac{1}{p}\right)^{\ell-1},$$

which is asymptotic to 1 by Lemma 4.5, to obtain the asymptotic for $F(Y)$. The proof for $F_{\chi_0}(Y)$ is identical except that the term $(1+(\ell-1)/\ell)$ is initially missing from the product over $p\le Y$. The result follows. ∎

Lemma 5.5. *When $\chi\neq\chi_0$, one has*

$$F_\chi(Y) \ll Y(\log Y)^{1-2/\ell}. \tag{5.6}$$

*Proof.* Our calculation will require the information given in the following table:

| $f_\chi(p)$ | $\#n\in(\mathbb{Z}/\ell^2\mathbb{Z})^\times$ | Congruences |
|---|---|---|
| $\ell-1$ | $(\ell-1)$ | $p^{\ell-1}\equiv 1 \pmod{\ell^2}$ |
| $-1$ | $(\ell-1)^2$ | $p^{\ell-1}\not\equiv 1 \pmod{\ell^2}$ |

We use the bound

$$|F_\chi(Y)| \le \sum_{n\le Y}\mu^2(n)|f_\chi(n)|$$

and apply Lemma 4.6 to the latter sum to obtain

$$F_\chi(Y) \ll \frac{Y}{\log Y}\prod_{\substack{p\le Y\\ p^{\ell-1}\not\equiv 1 \,(\mathrm{mod}\,\ell^2)}}\left(1+\frac{1}{p}\right)\prod_{\substack{p\le Y\\ p^{\ell-1}\equiv 1 \,(\mathrm{mod}\,\ell^2)}}\left(1+\frac{\ell-1}{p}\right).$$

Notice that $f_\chi(p)=\ell-1$ when $p^{\ell-1}\equiv 1 \pmod{\ell^2}$ and $f_\chi(p)=-1$ otherwise, which is where the quantities $1/p$ and $(\ell-1)/p$ arise. Taking logarithms of the two products above, using the Taylor expansion at $x=0$ for $\log(1+x)$ gives

$$\sum_{\substack{p\le Y\\ p^{\ell-1}\not\equiv 1 \,(\mathrm{mod}\,\ell^2)}}\frac{1}{p}+\sum_{\substack{p\le Y\\ p^{\ell-1}\equiv 1 \,(\mathrm{mod}\,\ell^2)}}\frac{\ell-1}{p}+\sum_{p\le Y}O\left(\frac{1}{p^2}\right).$$

Then by applying Lemma 4.4 we obtain

$$\left(\frac{(\ell-1)^2}{\phi(\ell^2)}\cdot 1\right)\log\log Y+\left(\frac{\ell-1}{\phi(\ell^2)}\cdot(\ell-1)\right)\log\log Y+O(1)$$
$$=2\left(1-\frac{1}{\ell}\right)\log\log Y+O(1).$$

The result follows. ∎

*Proof of Theorem 1.1.* We use Lemmas 5.3–5.5. In particular, we plug (5.4)–(5.6) into (5.2). We treat the principal and non-principal characters separately. In regard to the principal character, we see that we may apply Lemma 5.4 to three terms on the right-hand side of (5.2). This contribution is

$$\frac{1}{\ell}\big[F_{\chi_0}(a(\ell)X^{\frac{1}{\ell-1}})-F_{\chi_0}(b(\ell)X^{\frac{1}{\ell-1}})\big]+F(b(\ell)X^{\frac{1}{\ell-1}})$$
$$\sim\frac{1}{\ell}\big[c(\ell)a(\ell)-c(\ell)b(\ell)\big]D_\ell X^{\frac{1}{\ell-1}}\frac{(\log X)^{\ell-2}}{(\ell-1)^{\ell-2}}+b(\ell)D_\ell X^{\frac{1}{\ell-1}}\frac{(\log X)^{\ell-2}}{(\ell-1)^{\ell-2}},$$

where we have used

$$\log\big(a(\ell)X^{\frac{1}{\ell-1}}\big)^{\ell-2}\sim\frac{1}{(\ell-1)^{\ell-2}}(\log X)^{\ell-2}, \tag{5.7}$$

and the same for $b(\ell)$. The contribution on the right-hand side of (5.2) coming from non-principal characters is bounded by

$$\frac{1}{\ell}\sum_{\substack{\chi \bmod \ell^2\\ \chi^\ell=\chi_0\\ \chi\neq\chi_0}} O\big(X^{\frac{1}{\ell-1}}(\log X)^{1-2/\ell}\big)\ll X^{\frac{1}{\ell-1}}(\log X)^{1-2/\ell}.$$

This establishes (1.1) with

$$C_\ell=\frac{1}{(\ell-1)^{\ell-1}}\left(\frac{a(\ell)-b(\ell)}{\ell}c(\ell)+b(\ell)\right)D_\ell \tag{5.8}$$
$$=\frac{\ell^{-\frac{1}{\ell-1}}}{\ell!(\ell-1)^{\ell-2}}\left(1+\frac{\ell^{\frac{2}{\ell-1}}-1}{2\ell-1}\right)\prod_p\left(1+\frac{\ell-1}{p}\right)\left(1-\frac{1}{p}\right)^{\ell-1}.$$ ∎

We note that our formula for the constant (5.8) differs slightly from the one in [B18], but we believe it to be correct and it matches the formula in [CM11] when specialized to $\ell=3$.

## 6. Pure fields with genus number 1

*Proof of Theorem 1.2.* We let $K\in\mathcal{F}_\ell$ be of the form $K=\mathbb{Q}(\sqrt[\ell]{a})$ where $\mathrm{rad}(a)=n$. Lemma 4.3 tells us that $g_K=\ell^{\widehat{\omega}(n)}$. We define the new counting

function

$$\widehat{N}_\ell(X) = \#\{K \in \mathcal{F}_\ell : g_K = 1, |\Delta_K| \le X\}.$$

In order to only count fields $K$ with genus number 1, we use the indicator function $g(n)$ that is 1 if $p \not\equiv 1 \pmod{\ell}$ for all primes $p \mid n$ and 0 otherwise. We impose this restriction on the sums over $n$ in (5.1) to obtain the following analogue of Lemma 5.1:

$$\begin{aligned}(6.1)\qquad &(\ell-1)\widehat{N}_\ell(X)+1\\ &= \sum_{n\le a(\ell)X^{\frac{1}{\ell-1}}} \mu^2(n)g(n) \sum_{\substack{\mathrm{rad}(a)=n\\ a \text{ is } \ell\text{-free}\\ a^{\ell-1}\equiv 1 \,(\mathrm{mod}\, \ell^2)}} 1 + \sum_{n\le b(\ell)X^{\frac{1}{\ell-1}}} \mu^2(n)g(n) \sum_{\substack{\mathrm{rad}(a)=n\\ a \text{ is } \ell\text{-free}\\ a^{\ell-1}\not\equiv 1 \,(\mathrm{mod}\, \ell^2)}} 1.\end{aligned}$$

Building on Definition 5.2, we create new functions, multiplying $f_\chi(n)$ and $f(n)$ by the multiplicative function $g(n)$. That is, we define $\widehat{f_\chi}(n) = g(n)f_\chi(n)$, $\widehat{f}(n) = g(n)f(n)$, and

$$(6.2)\qquad \widehat{F}_\chi(Y) = \sum_{n\le Y} \mu^2(n)\widehat{f}_\chi(n), \quad \widehat{F}(Y) = \sum_{n\le Y} \mu^2(n)\widehat{f}(n).$$

In a manner analogous to Lemma 5.3, we obtain

$$\begin{aligned}(6.3)\qquad &(\ell-1)\widehat{N}_\ell(X)+1\\ &\qquad = \frac{1}{\ell} \sum_{\substack{\chi \bmod \ell^2\\ \chi^\ell=\chi_0}} \big[\widehat{F}_\chi\big(a(\ell)X^{\frac{1}{\ell-1}}\big) - \widehat{F}_\chi\big(b(\ell)X^{\frac{1}{\ell-1}}\big)\big] + \widehat{F}\big(b(\ell)X^{\frac{1}{\ell-1}}\big).\end{aligned}$$

As was carried out in Lemmas 5.4 and 5.5, we split into cases.

Case $\chi = \chi_0$: First, we note that by the Prime Number Theorem in Arithmetic Progressions, one has

$$\sum_{\substack{p\le Y\\ p\not\equiv 1 \,(\mathrm{mod}\, \ell)}} (\ell-1) \sim (\ell-2)Y/\log Y.$$

Having verified the hypothesis, we apply Lemma 4.6 to the function $\mu^2(n)g(n)f(n)$ to conclude that

$$\sum_{n\le Y} \mu^2(n)g(n)f(n) \sim \frac{Y}{\log Y} \frac{e^{-\gamma(\ell-2)}}{(\ell-3)!} \prod_{\substack{p\le Y\\ p\not\equiv 1 \,(\mathrm{mod}\, \ell)}} \left(1+\frac{\ell-1}{p}\right).$$

Using an argument analogous to the one in Lemma 5.4, we multiply by the function

$$e^{\gamma(\ell-2)}(\log Y)^{\ell-2} \prod_{p\le Y} \left(1-\frac{1}{p}\right)^{\ell-2} \sim 1,$$

which leads to

$$\widehat{F}(Y) \sim \widehat{D}_\ell Y (\log Y)^{\ell-3}, \quad \widehat{D}_\ell = \frac{1}{(\ell-3)!} \prod_p \widehat{D}_{\ell,p},$$

where

$$\widehat{D}_{\ell,p} = \left(1 - \frac{1}{p}\right)^{\ell-2} \begin{cases} 1, & p \equiv 1 \pmod{\ell}, \\ 1 + \frac{\ell-1}{p}, & p \not\equiv 1 \pmod{\ell}. \end{cases}$$

Similarly, we have

$$\widehat{F}_\chi(Y) \sim c(\ell) \widehat{D}_\ell Y (\log Y)^{\ell-3}.$$

Case $\chi \neq \chi_0$: We proceed as in the proof of Lemma 5.5. We use the inequality

$$|\widehat{F}_\chi(Y)| \leq \sum_{n \leq Y} \mu^2(n) |\widehat{f}_\chi(n)|$$

and apply Lemma 4.6. The difference is that $\widehat{f}_\chi(p)$ can take four different values. The following table gives the values of $\widehat{f}_\chi(p)$ for various congruence classes modulo $\ell^2$.

| $\widehat{f}_\chi(p)$ | $\#n \in (\mathbb{Z}/\ell^2\mathbb{Z})^\times$ | Congruences |
|---|---|---|
| 0 | 1 | $p \equiv 1 \pmod{\ell}$, $p^{\ell-1} \equiv 1 \pmod{\ell^2}$ |
| 0 | $\ell - 1$ | $p \equiv 1 \pmod{\ell}$, $p^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ |
| $\ell - 1$ | $\ell - 2$ | $p \not\equiv 1 \pmod{\ell}$, $p^{\ell-1} \equiv 1 \pmod{\ell^2}$ |
| $-1$ | $(\ell-1)(\ell-2)$ | $p \not\equiv 1 \pmod{\ell}$, $p^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ |

Indeed, this leads to a bound of the form

$$\widehat{F}_\chi(Y) \ll \frac{Y}{\log Y} (\log Y)^\alpha.$$

From the table, one reads off the value

$$\alpha = \frac{(\ell-1)(\ell-2) + (\ell-1)(\ell-2)}{\phi(\ell^2)} = 2\left(1 - \frac{2}{\ell}\right),$$

and therefore

$$\widehat{F}_\chi(Y) \ll Y (\log Y)^{1-4/\ell}.$$

We have thus obtained $\widehat{N}_\ell(Y) \sim \widehat{C}_\ell Y (\log Y)^{\ell-3}$, where one can write down the constant $\widehat{C}_\ell$ exactly as in (5.8). (The expression is the same but with $D_\ell$ replaced by $\widehat{D}_\ell$.) This proves the theorem with a constant of proportionality $\widehat{C}_\ell / C_\ell = (\ell-1)\widehat{D}_\ell / D_\ell =: 1/A_\ell$. The extra factor of $\ell - 1$ arises from the analogue of (5.7). ∎

## 7. The average genus number

*Proof of Theorem 1.3.* The proof is similar to the proof of Theorem 1.2. We define $S_\ell(X)$ to be the sum of $g_K$ over $K \in \mathcal{F}_\ell$ with $|\Delta_K| \le X$, that is,

$$S_\ell(X) = \sum_{\substack{K\in\mathcal{F}_\ell \\ |\Delta_K|\le X}} g_K.$$

Define the multiplicative function $h(n) = \ell^{\widehat{\omega}(n)}$. In order to write an expression for $S_\ell(X)$, we insert $h(n)$ immediately to the right of both sums over $n$ in (5.1). The analogue of the formula in Lemma 5.1 in this context is the following:

$$\begin{aligned}(7.1)\qquad &(\ell-1)S_\ell(X)+1 \\ &= \sum_{n\le a(\ell)X^{\frac{1}{\ell-1}}} \mu^2(n)h(n) \sum_{\substack{\mathrm{rad}(a)=n \\ a \text{ is } \ell\text{-free} \\ a^{\ell-1}\equiv 1 \,(\mathrm{mod}\,\ell^2)}} 1 + \sum_{n\le b(\ell)X^{\frac{1}{\ell-1}}} \mu^2(n)h(n) \sum_{\substack{\mathrm{rad}(a)=n \\ a \text{ is } \ell\text{-free} \\ a^{\ell-1}\not\equiv 1 \,(\mathrm{mod}\,\ell^2)}} 1.\end{aligned}$$

We write $\widetilde{f}_\chi(n) = h(n)f_\chi(n)$ and $\widetilde{f}(n) = h(n)f(n)$. The definitions of $\widetilde{F}_\chi(Y)$ and $\widetilde{F}(Y)$ are then analogous to (6.2). Explicitly,

$$\widetilde{f}_\chi(n) = \ell^{\widehat{\omega}(n)} \sum_{\substack{\mathrm{rad}(a)=n \\ a \text{ is } \ell\text{-free}}} \chi(a).$$

Following the argument from Lemma 5.3, we have

$$\begin{aligned}(7.2)\qquad &(\ell-1)S_\ell(X)+1 \\ &\qquad = \frac{1}{\ell} \sum_{\substack{\chi \bmod \ell^2 \\ \chi^\ell=\chi_0}} \big[\widetilde{F}_\chi\big(a(\ell)X^{\frac{1}{\ell-1}}\big) - \widetilde{F}_\chi\big(b(\ell)X^{\frac{1}{\ell-1}}\big)\big] + \widetilde{F}\big(b(\ell)X^{\frac{1}{\ell-1}}\big).\end{aligned}$$

As in the proof of Theorem 1.2 we split into cases, mimicking the approach of Lemmas 5.4 and 5.5.

Case $\chi = \chi_0$: Using the Prime Number Theorem in Arithmetic Progressions, we observe that

$$\sum_{p\le Y} h(p)(\ell-1) = \sum_{\substack{p\le Y \\ p\equiv 1 \,(\mathrm{mod}\,\ell)}} \ell(\ell-1) + \sum_{\substack{p\le Y \\ p\not\equiv 1 \,(\mathrm{mod}\,\ell)}} (\ell-1) \sim (2\ell-2)\frac{Y}{\log Y}.$$

Applying Lemma 4.6 gives

$$\widetilde{F}(Y) \sim \widetilde{D}_\ell Y(\log Y)^{2\ell-3}, \qquad \widetilde{D}_\ell = \frac{1}{(2\ell-3)!}\prod_p \widetilde{D}_{\ell,p},$$

where

$$\widetilde{D}_{\ell,p} = \left(1-\frac{1}{p}\right)^{2\ell-2} \begin{cases} 1+\frac{\ell(\ell-1)}{p}, & p \equiv 1 \pmod{\ell}, \\ 1+\frac{\ell-1}{p}, & p \not\equiv 1 \pmod{\ell}. \end{cases}$$

Similarly,

$$\widetilde{F}_\chi(Y) \sim c(\ell)\widetilde{D}_\ell Y(\log Y)^{2\ell-3}.$$

Case $\chi \neq \chi_0$: The bound works the same as in Section 5; there are four cases to consider. We have

$$\widetilde{f}_\chi(n) = \ell^{\widehat{\omega}(n)} \begin{cases} \ell-1, & n^{\ell-1} \equiv 1 \pmod{\ell^2}, \\ -1, & n^{\ell-1} \not\equiv 1 \pmod{\ell^2}. \end{cases}$$

The cases (when $p \neq \ell$) are summarized in the following table:

| $\widetilde{f}_\chi(p)$ | $\#n \in (\mathbb{Z}/\ell^2\mathbb{Z})^\times$ | Congruences |
|---|---|---|
| $\ell(\ell-1)$ | 1 | $p \equiv 1 \pmod{\ell}$, $p^{\ell-1} \equiv 1 \pmod{\ell^2}$ |
| $-\ell$ | $\ell-1$ | $p \equiv 1 \pmod{\ell}$, $p^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ |
| $\ell-1$ | $\ell-2$ | $p \not\equiv 1 \pmod{\ell}$, $p^{\ell-1} \equiv 1 \pmod{\ell^2}$ |
| $-1$ | $(\ell-1)(\ell-2)$ | $p \not\equiv 1 \pmod{\ell}$, $p^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ |

Thus we have the bound

$$\widetilde{F}_\chi(Y) \ll \frac{Y}{\log Y}(\log Y)^\alpha$$

with

$$\begin{aligned} \alpha &= \frac{\ell(\ell-1)\cdot 1 + \ell\cdot(\ell-1) + (\ell-1)\cdot(\ell-2) + 1\cdot(\ell-1)(\ell-2)}{\phi(\ell^2)} \\ &= 4\left(1-\frac{1}{\ell}\right). \end{aligned}$$

This completes the bound on the error term, which suffices since $3 - 4/\ell < 2\ell - 3$.

We have thus obtained $S_\ell(Y) \sim \widetilde{C}_\ell Y(\log Y)^{2\ell-3}$, where one can write down the constant $\widetilde{C}_\ell$ exactly as in (5.8). It follows that we recover the constant in the theorem as

$$\widetilde{C}_\ell / C_\ell = (\ell-1)^{-\ell+1}\widetilde{D}_\ell / D_\ell =: B_\ell. \ \blacksquare$$

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