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Curvature estimates for hypersurfaces of constant sum Hessian curvature in $\mathbb{H}^{n+1}$

Jianbo Yang and Yueming Lu

Abstract. We study the asymptotic Plateau problem in hyperbolic space for constant sum Hessian curvature. Assuming that $n \geq 2$ and $\alpha > 0$, we establish curvature estimates for $\widetilde{\Gamma}_{n,\alpha}$-convex hypersurfaces with σ_n bounded from below, i.e. $\sigma_n > -A$, $A \in ((n-1)\sigma/\alpha, +\infty)$, and a prescribed asymptotic boundary at the infinity for constant $S_{n,\alpha}$ curvature.

1. Introduction

1.1. Background. Let Σ be a smooth surface in $\mathbb{R}^3$ and let κ_1, κ_2 be the principal curvatures of Σ. If for all $X \in \Sigma$ we have

$$H(X) = \frac{\kappa_1(X) + \kappa_2(X)}{2} = 0,$$

then Σ will be referred to as a *minimal surface*. A fundamental problem concerns the existence of such minimal surfaces. More generally, the existence of minimal submanifolds Σ in arbitrary Riemannian manifolds M constitutes a fundamental problem in Riemannian geometry. The cases $M = \mathbb{R}^n$ and $M = \mathbb{S}^n$ have been thoroughly researched by many authors and require no further discussion here. Of particular interest is the case $M = \mathbb{H}^{n+1}$. We adopt the upper half-space model

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$$

endowed with the hyperbolic metric

$$ds^2 = \frac{1}{x_{n+1}^2} \sum_{i=1}^{n+1} dx_i^2.$$

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More precisely, let $\partial_\infty \mathbb{H}^{n+1}$ be the ideal boundary of $\mathbb{H}^{n+1}$ at infinity. Given a closed embedded smooth $(n-1)$-dimensional submanifold $\Gamma \subset \partial_\infty \mathbb{H}^{n+1}$, one seeks a complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying

$$f(\kappa) = \sigma, \quad \partial\Sigma = \Gamma, \tag{1.1}$$

where f is a smooth symmetric function of n variables, $\kappa = (\kappa_1, \ldots, \kappa_n)$ are the principal curvatures of Σ and $\sigma \in (0,1)$. Moreover, we may consider the Poincaré ball model of hyperbolic space. Indeed, the two models are equivalent; for computational convenience we prefer the upper half-space model.

In the 1980s, the asymptotic Plateau problem (1.1) for area-minimizing varieties was first addressed by Anderson [AD82, AD83] employing geometric measure theory. Following this work, Tonegawa [T96] achieved an extension to constant mean curvature hypersurfaces. PDE theory provides a powerful framework for addressing such problems. Lin [L89, L] provided a purely PDE-theoretic proof of Anderson's existence-uniqueness theorem while also establishing regularity properties of the solutions. Later, the asymptotic Plateau problem for constant mean curvature was investigated via PDE methods in the works of Nelli–Spruck [NS96] and Guan–Spruck [GS00]. Labourie [L91] solved the case of hypersurfaces with constant Gauss curvature in $\mathbb{H}^3$. More generally, Rosenberg–Spruck [RS94] extended this result to hypersurfaces in $\mathbb{H}^{n+1}$. Guan, Spruck, Szapiel, and Xiao [GSS09, GS11, GSX14] resolved the asymptotic Plateau problem for a broad class of functions f defined on the positive cone Γ_n. For general f satisfying some structure conditions defined on an open symmetric convex cone K such that $\Gamma_n \subset K$, the asymptotic Plateau problem was solved by Guan–Spruck [GS10] if $\sigma > \sigma_0$, where $0.3703 < \sigma_0 < 0.3704$.

Building on the seminal work of Guan–Spruck [GS10], we consider specific types of f defined on a certain open symmetric convex cone K such that our results hold for all $\sigma \in (0,1)$. Lu [L23] studied the case where $f = \sigma_{n-1}$, $K = \Gamma_{n-1}$ by using Ren–Wang's concavity inequality [RW19]. In a recent preprint [W], Wang was able to obtain the curvature estimates for problem (1.1) with $f = \sigma_{n-2}$, $K = \Gamma_{n-2}$ by using Ren–Wang's important concavity inequality [RW23]. In [W], Wang also studied the case where $f = (H_k/H_l)^{\frac{1}{k-l}}$, $1 \le k < l \le n$, $K = \Gamma_k$ with curvature bounded from below. Here H_k is the normalized k-th elementary symmetric polynomial. The last remaining interesting case is $f = \sigma_2$, $K = \Gamma_2$. Wang in [W25] was able to solve this case in all dimensions by using the almost-Jacobi inequality due to Shankar–Yuan [SY25]. For the case $f = \sigma_k$ with $2 < k < n-2$, Hong–Zhang [HZ24] was able to derive the curvature estimates for semi-convex hypersurfaces in $\mathbb{H}^{n+1}$. However, the case $f = \sigma_k$, $K = \Gamma_k$ with $2 < k < n-2$ remains open.

Very recently, Chen–Sui–Sun [CSS25] investigated another class of special curvature functions f, i.e.

$$f(\kappa) = \mathcal{F}_p(\kappa) = \prod_{1\le i_1<i_2<\cdots<i_p\le n}(\kappa_{i_1}+\kappa_{i_2}+\cdots+\kappa_{i_p})$$

defined on the p-convex cone

$$K = \mathcal{P}_p = \{\kappa\in\mathbb{R}^n : \forall 1\le i_1<\cdots<i_p\le n,\ \kappa_{i_1}+\cdots+\kappa_{i_p}>0\}.$$

In [CSS25], Chen–Sui–Sun solved the problem (1.1) for the case $f=\mathcal{F}_2$, $K=\mathcal{P}_2$ in $\mathbb{H}^4$. For general dimensions, the authors believe that the key to problem (1.1) lies in further exploring the structure of the curvature function f.

1.2. Main results. Let $n>1, \alpha>0$. In this paper, we consider the curvature estimates for problem (1.1) for the case $f=S_{n,\alpha}$, $K=\widetilde{\Gamma}_{n,\alpha}$ with σ_n bounded from below.

The main result of the paper is the following.

THEOREM 1.1. *For $n\ge 2$, let $\Gamma=\partial\Omega$, where Ω is a bounded smooth domain in $\mathbb{R}^n$ with nonnegative mean curvature and $\sigma\in(0,1)$. Suppose Σ is a C^4 $\widetilde{\Gamma}_{n,\alpha}$-convex ([1]) vertical graph over Ω in $\mathbb{H}^{n+1}$ satisfying*

$$S_{n,\alpha}(\kappa)=\sigma,\qquad \partial\Sigma=\Gamma, \tag{1.2}$$

If ν^{n+1} has a positive lower bound and $\kappa\in\mathscr{J}_A$, then

$$\max_{x\in\Sigma,\,1\le i\le n}|\kappa_i(x)|\le C\Big(1+\max_{x\in\partial\Sigma,\,1\le i\le n}|\kappa_i(x)|\Big),$$

where C is a constant depending only on n, Ω, A, α and σ.

REMARK 1.2. Denote

$$\mathscr{J}_A=\{\kappa\in\widetilde{\Gamma}_{n,\alpha}\mid \sigma_n(\kappa)>-A,\ A>0\}.$$

Since the condition $\sigma_n>-A$ is not homogeneous, $\mathscr{J}_A$ is not a cone. However, when $A\in((n-1)\sigma/\alpha,+\infty)$, we prove that (see Section 2.3)

$$\kappa\in\widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}}\Longrightarrow\kappa\in\mathscr{J}_A,$$
$$\kappa\in\mathscr{J}_A\not\Longrightarrow\kappa\in\widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}}.$$

This implies that the condition $\kappa\in\mathscr{J}_A$ is weaker than the condition $\kappa\in\widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}}$. When $A=(n-1)\sigma/\alpha$, we have

$$\kappa\in\widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}}\iff\kappa\in\mathscr{J}_A.$$

REMARK 1.3. After the completion of this work, we noticed Wang's discussion in [W24]. In [W24], Wang provided the conditions under which the asymptotic Plateau problem for curvature functions f satisfying certain structural conditions is solvable for any $\sigma\in(0,1)$. More importantly, Wang's

([1]) The definition of the cone $\widetilde{\Gamma}_{n,\alpha}$ and $\widetilde{\Gamma}_{n,\alpha}$-convex hypersurface can be found in Section 2.2.

proof did not rely on concavity inequalities. This naturally raises the question of whether the concavity inequalities can be avoided in the proof of Theorem 1.1, and, if so, whether the proof can be simplified.

Note that when $\alpha = 0$, the problem (1.1) for the case $f = \sigma_{n-1}$, $K = \Gamma_{n-1}$ has been settled by Lu [L23]. Thus, in this paper we assume that α is a positive number.

1.3. Outline of proof. Regarding the proof of Theorem 1.1, we select a new test function proposed by Lu [L23] and apply the maximum principle. To bound the largest principal curvature of Σ, we exploit the maximum point of the test function, reducing its second-order differential inequalities through cancellation of:

(1) fourth-order terms,
(2) negative third-order terms,
(3) some other negative second-order terms.

The fourth-order terms can be handled directly using the curvature equation.

The more challenging part is the handling of the negative third-order terms, which is highly dependent on the structure of the sum Hessian operator $S_{n,\alpha}$. In this paper, we employ the concavity inequality established by Li–Ren [LR25] to effectively address the bad third-order terms.

Due to the special structure of $S_{n,\alpha}$, very bad second-order terms $-\kappa_1\sigma_{n-1}$ arise in our proof. To overcome this difficulty, we introduce a lower bound on σ_n, i.e. the condition

$$\kappa \in \mathscr{J}_A := \{\kappa \in \widetilde{\Gamma}_{n,\alpha} \mid \sigma_n(\kappa) > -A\}, \quad A > 0.$$

$\mathscr{J}_A$ is not a cone, but it is larger than the cone associated with equation (1.2), and hence represents a weaker condition. In contrast to the cone $\widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}}$, the weaker condition $\kappa \in \mathscr{J}_A$ brings in the bad term $-C\kappa_1$. Thanks to Lemma 2.2 below, we can eliminate the term $-C\kappa_1$ with the term $\sum_i S^{ii}_{n,\alpha}\kappa_i^2$. However, the problem (1.1) for the case $f = S_{n,\alpha}$, $K = \widetilde{\Gamma}_n$ without imposing any additional conditions remains open. Furthermore, we can also consider the problem (1.1) for the case $f = S_{k,\alpha}$, $K = \widetilde{\Gamma}_k$ with $2 \le k \le n-1$.

2. Preliminaries. In this section, we compile the fundamental formulas for hypersurfaces in hyperbolic space along with key properties of the operator $S_{n,\alpha}$ (see [GS10, LR25]).

2.1. Properties of sum Hessian $S_{n,\alpha}$

LEMMA 2.1. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. We have*

(i) $S_n^{ii}(\lambda) := \dfrac{\partial S_n(\lambda)}{\partial \lambda_i} = S_{n-1}(\lambda|i), \quad i = 1, \dots, n;$

(ii) $S_n^{ii,jj}(\lambda) := \dfrac{\partial^2 S_n(\lambda)}{\partial \lambda_i \partial \lambda_j} = S_{n-2}(\lambda|ij), \quad i, j = 1, \dots, n, \quad i \neq j;$

(iii) $S_n(\lambda) = \lambda_i S_{n-1}(\lambda|i) + S_n(\lambda|i)$, $i = 1, \dots, n$;

(iv) $\sum_{i=1}^{n} S_n(\lambda|i) = \sigma_{n-1}(\lambda)$;

(v) $\sum_{i=1}^{n} \lambda_i S_{n-1}(\lambda|i) = nS_n(\lambda) - \sigma_{n-1}(\lambda)$.

LEMMA 2.2. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \widetilde{\Gamma}_n$, $\lambda_1 \geq \cdots \geq \lambda_n$. If $i = 1, \dots, n-1$, then there exists a positive constant θ depending on n, such that*

$$S_n^{ii} \geq \frac{\theta S_n(\lambda)}{\lambda_i}.$$

The following concavity inequality of Li–Ren [LR25] plays a crucial role in the proof of our main theorem.

LEMMA 2.3 ([LR25, Lemma 3.4]). *If $\lambda \in \widetilde{\Gamma}_n$, then for every $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that*

$$\lambda_1 \Big(K \Big(\sum_j S_n^{jj}(\lambda)\xi_j \Big)^2 - S_n^{pp,qq}(\lambda)\xi_p\xi_q \Big) - S_n^{11}(\lambda)\xi_1^2 + (1+\epsilon)\sum_{j>1} S_n^{jj}\xi_j^2 \geq 0, \tag{2.1}$$

where $\xi = (\xi_1, \dots, \xi_n)$ is an arbitrary vector in $\mathbb{R}^n$.

2.2. An admissible set $\widetilde{\Gamma}_{n,\alpha'}$. We now provide some necessary notation. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Denote

$$S_{n,\alpha}(\lambda) = \sigma_{n-1}(\lambda) + \alpha\sigma_n(\lambda)$$

and

$$S_{n,\alpha}^{ii}(\lambda) = \frac{\partial S_{n,\alpha}(\lambda)}{\partial \lambda_i}, \qquad S_{n,\alpha}^{ii,jj}(\lambda) = \frac{\partial^2 S_{n,\alpha}(\lambda)}{\partial \lambda_i \partial \lambda_j}.$$

We denote Gårding's cone by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda) > 0, \dots, \sigma_k(\lambda) > 0\}, \qquad k = 1, \dots, n.$$

As pointed out in Li–Wang–Ren[LRW19], an admissible set of $S_{n,\alpha}$ is

$$\widetilde{\Gamma}_{n,\alpha} = \Gamma_{n-1} \cap \{\lambda \in \mathbb{R}^n \mid S_{n,\alpha}(\lambda) > 0\}.$$

This means that $\widetilde{\Gamma}_{n,\alpha}$ is a convex set. In the cone $\widetilde{\Gamma}_{n,\alpha}$, $S_{n,\alpha}^{ii}$ is elliptic. Consider the convex cone $\widetilde{\Gamma}_{n,\alpha'}$, where $0 < \alpha < \alpha'$. We claim that $\widetilde{\Gamma}_{n,\alpha'}$ is also an admissible set of $S_{n,\alpha}$. Indeed, let $\lambda \in \widetilde{\Gamma}_{n,\alpha'}$. Then $S_{n,\alpha'}(\lambda) > 0$.

Direct calculation shows that

$$\begin{aligned} S_{n,\alpha}(\lambda) &= \sigma_{n-1}(\lambda) + \alpha\sigma_n(\lambda) \\ &= \sigma_{n-1}(\lambda) - \frac{\alpha}{\alpha'}\sigma_{n-1}(\lambda) + \frac{\alpha}{\alpha'}\sigma_{n-1}(\lambda) + \alpha\sigma_n(\lambda) \\ &= \frac{\alpha'-\alpha}{\alpha'}\sigma_{n-1}(\lambda) + \frac{\alpha}{\alpha'}(\sigma_{n-1}(\lambda) + \alpha'\sigma_n(\lambda)) \\ &= \frac{\alpha'-\alpha}{\alpha'}\sigma_{n-1}(\lambda) + \frac{\alpha}{\alpha'}S_{n,\alpha'}(\lambda). \end{aligned}$$

By $\lambda \in \Gamma_{n-1}$, we have $S_{n,\alpha}(\lambda) > 0$. This means

$$\widetilde{\Gamma}_{n,\alpha'} \subset \widetilde{\Gamma}_{n,\alpha}.$$

By [LRW19, Theorem 12], we can prove $S_{n,\alpha}$ is elliptic in $\widetilde{\Gamma}_{n,\alpha'}$. Let $\lambda \in \widetilde{\Gamma}_{n,\alpha'}$. A direct calculation shows

$$\begin{aligned} S^{ii}_{n,\alpha}(\lambda) &= \sigma^{ii}_{n-1}(\lambda) + \alpha\sigma^{ii}_n(\lambda) \\ &= \sigma_{n-1}(\lambda)\left(\frac{\sigma^{ii}_{n-1}(\lambda)}{\sigma_{n-1}(\lambda)} + \alpha\frac{\sigma^{ii}_n(\lambda)}{\sigma_{n-1}(\lambda)}\right) \\ &= \sigma_{n-1}(\lambda)\left[\frac{\sigma^{ii}_{n-1}(\lambda)}{\sigma_{n-1}(\lambda)} + \alpha\frac{\sigma_n(\lambda)\sigma^{ii}_{n-1}(\lambda)}{\sigma^2_{n-1}(\lambda)} + \alpha\left(\frac{\sigma_n(\lambda)}{\sigma_{n-1}(\lambda)}\right)^{ii}\right] \\ &= \sigma_{n-1}(\lambda)\left[S_{n,\alpha}(\lambda)\frac{\sigma^{ii}_{n-1}(\lambda)}{\sigma^2_{n-1}(\lambda)} + \alpha\left(\frac{\sigma_n(\lambda)}{\sigma_{n-1}(\lambda)}\right)^{ii}\right]. \end{aligned}$$

By Newton's inequality $\sigma_{n-2}(\lambda)\sigma_n(\lambda) \le \sigma^2_{n-1}(\lambda)$, we have

$$\begin{aligned} \left(\frac{\sigma_n}{\sigma_{n-1}}\right)^{ii}(\lambda) &= \frac{\sigma^{ii}_n(\lambda)\sigma_{n-1}(\lambda) - \sigma_n(\lambda)\sigma^{ii}_{n-1}(\lambda)}{\sigma^2_{n-1}(\lambda)} \\ &= \frac{\lambda_i\sigma_{n-1}(\lambda|i)\sigma_{n-2}(\lambda|i) + \sigma^2_{n-1}(\lambda|i)}{\sigma^2_{n-1}(\lambda)} \\ &\quad - \frac{(\lambda_i\sigma_{n-1}(\lambda|i)\sigma_{n-2}(\lambda|i) - \sigma_n(\lambda|i)\sigma_{n-2}(\lambda|i))}{\sigma^2_{n-1}(\lambda)} \\ &= \frac{\sigma^2_{n-1}(\lambda|i) - \sigma_n(\lambda|i)\sigma_{n-2}(\lambda|i)}{\sigma^2_{n-1}(\lambda)} > 0. \end{aligned}$$

By $\widetilde{\Gamma}_{n,\alpha'} \subset \widetilde{\Gamma}_{n,\alpha}$, we have $S_{n,\alpha}(\lambda) > 0$. Combining $S_{n,\alpha}(\lambda) > 0$ and $(\sigma_n/\sigma_{n-1})^{ii}(\lambda) > 0$, we have $S^{ii}_{n,\alpha}(\lambda) > 0$. This means $S_{n,\alpha}$ is elliptic in $\widetilde{\Gamma}_{n,\alpha'}$. Let $\kappa = (\kappa_1, \dots, \kappa_n)$ be the principal curvatures of Σ. We say that Σ is $\widetilde{\Gamma}_{n,\alpha'}$-*convex* if $\kappa \in \widetilde{\Gamma}_{n,\alpha'}$.

2.3. A weaker condition. Let $\kappa = (\kappa_1, \dots, \kappa_n)$ be the principal curvatures of a hypersurface Σ in $\mathbb{H}^{n+1}$ and denote

$$\mathscr{J}_A = \{\kappa \in \widetilde{\Gamma}_{n,\alpha} \mid \sigma_n(\kappa) > -A,\ A > 0\}.$$

When $A \in \left(\frac{(n-1)\sigma}{\alpha}, +\infty\right)$, we claim that

$$\kappa \in \widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}} \Longrightarrow \kappa \in \mathscr{J}_A,$$
$$\kappa \in \mathscr{J}_A \not\Longrightarrow \kappa \in \widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}}.$$

In fact, since $\kappa \in \widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}}$, we have

$$\sigma_{n-1}(\kappa) + \frac{n\alpha}{n-1}\sigma_n(\kappa) > 0$$

Combining this with equation (1.2), $\sigma_{n-1}(\kappa) + \alpha\sigma_n(\kappa) = \sigma$, we have

$$\sigma - n\alpha\sigma_n(\kappa) + \frac{n\alpha}{n-1}\sigma_n(\kappa) > 0 \implies \sigma_n(\kappa) > -\frac{(n-1)\sigma}{\alpha}$$

Since $\kappa \in \widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}} \subset \widetilde{\Gamma}_{n,\alpha}$, we obtain $\kappa \in \mathscr{J}_A$. To show that the converse inclusion fails, we construct a counterexample by choosing

$$\kappa \in \Gamma_{n-1}$$

such that

$$\kappa \in \mathscr{J}_A, \quad \text{but} \quad \kappa \notin \widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}}.$$

In dimension $n = 2$, for $\epsilon \in (0, A - \sigma/\alpha)$, $A > \sigma/\alpha$, consider the equation

$$t^2 - (2\sigma + \alpha\epsilon)t - (\sigma/\alpha + \epsilon) = 0.$$

Let the principal curvatures κ_1 and κ_2 be the two real roots of the above equation. According to Vieta's theorem, we have

$$\sigma_1(\kappa) = \kappa_1 + \kappa_2 = 2\sigma + \alpha\epsilon > 0,$$
$$\sigma_2(\kappa) = \kappa_1\kappa_2 = -\sigma/\alpha - \epsilon < 0.$$

This shows $\kappa \in \Gamma_1$ and $\sigma_1(\kappa) + \alpha\sigma_2(\kappa) = \sigma > 0$. Therefore, $\kappa \in \widetilde{\Gamma}_{2,\alpha}$. Since

$$\sigma_1(\kappa) + 2\alpha\sigma_2(\kappa) = 2\sigma + \alpha\epsilon - 2\sigma - 2\alpha\epsilon = -\alpha\epsilon < 0.$$

So we have $\kappa \notin \widetilde{\Gamma}_{2,2\alpha}$. By $\epsilon \in (0, A - \sigma/\alpha)$, we obtain

$$\sigma_2(\kappa) = -\sigma/\alpha - \epsilon > -A.$$

This shows

$$\kappa \in \mathscr{J}_A \not\Longrightarrow \kappa \in \widetilde{\Gamma}_{2,2\alpha}.$$

For a general dimension $n \geq 2$, the above reasoning can be naturally generalized. Hence,

$$\kappa \in \mathscr{J}_A \not\Longrightarrow \kappa \in \widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}}.$$

In summary, the condition $\kappa \in \mathscr{J}_A$ is weaker than $\kappa \in \widetilde{\Gamma}_{n,\frac{n\alpha}{n-1}}$.

2.4. Hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$. For computational convenience, we adopt the upper half-space model of hyperbolic space in this paper. More precisely,

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds^2 = \frac{1}{x_{n+1}^2} \sum_{i=1}^{n+1} dx_i^2.$$

The boundary at infinity of the $(n+1)$-dimensional hyperbolic space, $\partial_\infty \mathbb{H}^{n+1}$, is canonically identified with $\mathbb{R}^n$, where $\mathbb{R}^n$ is embedded in $\mathbb{R}^{n+1}$ as the subspace $\mathbb{R}^n \times \{0\}$. We consider a connected, orientable, complete hypersurface Σ embedded in $\mathbb{H}^{n+1}$ whose asymptotic boundary at infinity is compact. Denote by $\mathbf{n}$ the unit normal vector field along Σ oriented toward the noncompact component of $\mathbb{R}^{n+1}_+ \setminus \Sigma$. Let g be the hyperbolic metric on Σ and ∇ be the Levi-Civita connection with respect to g. On the other hand, we may regard Σ as a hypersurface in $\mathbb{R}^{n+1}$. Let $\tilde{g}$ be the induced metric on Σ from $\mathbb{R}^{n+1}$ and let $\tilde{\nabla}$ be the Levi-Civita connection with respect to $\tilde{g}$. Finally, $\overline{\nabla}$ denotes the Levi-Civita connection of $\mathbb{H}^{n+1}$.

Now we regard Σ as a hypersurface in $\mathbb{R}^{n+1}$. We suppose Σ is the vertical graph of a function $x_{n+1} = u(x_1, \ldots, x_n)$. Let X and ν denote the position vector and the outer unit normal vector of Σ in $\mathbb{R}^{n+1}$, respectively. We define

$$\underline{e}u = X \cdot \mathbf{e}, \qquad \nu^{n+1} = \nu \cdot \mathbf{e},$$

where $\mathbf{e}$ is the unit vector field oriented in the positive x_{n+1} direction within $\mathbb{R}^{n+1}$, and $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^{n+1}$. Viewing Σ as a submanifold of $\mathbb{H}^{n+1}$, the first and second fundamental forms on Σ are then given by

$$\begin{aligned} g_{ij} &= \langle X_i, X_j \rangle_{\mathbb{H}^{n+1}} = \left\langle \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial x_{n+1}}, \frac{\partial}{\partial x_j} + u_j \frac{\partial}{\partial x_{n+1}} \right\rangle_{\mathbb{H}^{n+1}} \\ &= \frac{1}{u^2}(\delta_{ij} + u_i u_j) = \frac{\tilde{g}_{ij}}{u^2}. \\ h_{ij} &= \langle \overline{\nabla}_{X_i} X_j, \mathbf{n} \rangle_{\mathbb{H}^{n+1}} = \frac{\tilde{h}_{ij}}{u} + \frac{\nu^{n+1}}{u^2} \tilde{g}_{ij}. \end{aligned}$$

Let $e_1, \ldots, e_n$ be a local orthonormal frame. Then the Gauss and Codazzi equations are given by

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk},$$

$$\nabla_k h_{ij} = \nabla_j h_{ik}.$$

The Ricci identity is given by

$$\begin{aligned}(2.2)\quad \nabla_{ij}h_{kl} = \ & \nabla_{kl}h_{ij} - h_{ml}(h_{im}h_{kj} - h_{ij}h_{mk}) - h_{mj}(h_{mi}h_{kl} - h_{il}h_{mk}) \\ & - h_{ml}(\delta_{ij}\delta_{km} - \delta_{ik}\delta_{jm}) - h_{mj}(\delta_{il}\delta_{km} - \delta_{ik}\delta_{lm}).\end{aligned}$$

The following lemma is repeatedly employed in the proof of the main theorem.

LEMMA 2.4. *In a local orthonormal frame $e_1, \dots, e_n$ on Σ, we have*

$$\sum_i \frac{(\nabla_i u)^2}{u^2} = 1 - (\nu^{n+1})^2 \le 1, \qquad \nabla_i \nu^{n+1} = \frac{\nabla_i u}{u}(\nu^{n+1} - \kappa_i),$$

$$\begin{aligned}\sum_i S^{ii}_{n,\alpha}\nabla_{ii}\nu^{n+1} = \ & 2\sum_i S^{ii}_{n,\alpha}\frac{\nabla_i u}{u}\nabla_i\nu^{n+1} + (n\sigma - \sigma_{n-1})(1 + (\nu^{n+1})^2) \\ & - \nu^{n+1}\Big(\sum_i S^{ii}_{n,\alpha} + \sum_i S^{ii}_{n,\alpha}\kappa_i^2\Big).\end{aligned}$$

Proof. The first equation can be calculated directly from the definition of ν^{n+1}. Indeed, by $g_{ij} = \frac{1}{u^2}\tilde{g}_{ij}$, we have

$$e_i \cdot e_j = u^2\delta_{ij}.$$

Denote

$$\hat{e}_i = \frac{1}{u}e_i, \qquad i = 1, \dots, n.$$

Since

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij},$$

$\hat{e}_1, \dots, \hat{e}_n$ is a Euclidean local orthonormal frame on Σ. In the orthonormal frame $\hat{e}_1, \dots, \hat{e}_n, \nu$ on $\mathbb{R}^{n+1}$, the vector $\underline{e}$ can be decomposed as

$$\underline{e} = \sum_i (\underline{e} \cdot \hat{e}_i)\,\hat{e}_i + (\underline{e} \cdot \nu)\nu.$$

Direct calculation shows

$$\begin{aligned}1 = |\underline{e}|^2_{\mathbb{R}^{n+1}} &= \sum_i (\underline{e} \cdot \hat{e}_i)^2 + (\underline{e} \cdot \nu)^2 \\ &= \sum_i \frac{(\underline{e} \cdot e_i)^2}{u^2} + (\nu^{n+1})^2 \\ &= \sum_i \frac{(\nabla_i u)^2}{u^2} + (\nu^{n+1})^2.\end{aligned}$$

The second equality is given in [GSX14, (3.9)]. For the third equality, by combining [GSX14, (3.9)] with Lemma 2.1(v), we obtain

$$\begin{aligned}\sum_i S^{ii}_{n,\alpha}\nabla_{ii}\nu^{n+1} = \ & 2\sum_i S^{ii}_{n,\alpha}\frac{(\nabla_i u)^2}{u}\nabla_i\nu^{n+1} + (1 + (\nu^{n+1})^2)\sum_i S^{ii}_{n,\alpha}\kappa_i \\ & - \nu^{n+1}\Big(\sum_i S^{ii}_{n,\alpha} + \sum_i S^{ii}_{n,\alpha}\kappa_i^2\Big)\end{aligned}$$

$$= 2\sum_i S^{ii}_{n,\alpha}\frac{(\nabla_i u)^2}{u}\nabla_i \nu^{n+1} + (n\sigma - \sigma_{n-1})(1 + (\nu^{n+1})^2)$$
$$- \nu^{n+1}\Big(\sum_i S^{ii}_{n,\alpha} + \sum_i S^{ii}_{n,\alpha}\kappa_i^2\Big). \ \blacksquare$$

3. Proof of the main theorems. *Proof of Theorem 1.1.* We consider the test function proposed by Lu [L23],

$$Q = \ln \kappa_1 - N \ln \nu^{n+1},$$

where κ_1 is the largest principal curvature and N is a large constant to be determined later.

Suppose that Q attains its maximum at an interior point X_0 of Σ. When the principal curvature κ_1 has multiplicity greater than 1, the function Q fails to remain smooth at X_0. Consequently, we employ the curvature perturbation technique (cf. [C21]) to resolve this issue. We select a local orthonormal frame $e_1, \ldots, e_n$ near X_0 satisfying the following conditions at X_0:

$$\nabla_{e_i} e_j = 0, \quad h_{ij} = \delta_{ij}\kappa_i \quad \text{with} \quad \kappa_1 \geq \cdots \geq \kappa_n.$$

In a neighborhood of X_0, we introduce a new tensor B defined for tangent vectors V_1, V_2 by

$$B(V_1, V_2) = g(V_1, V_2) - g(V_1, e_1)g(V_2, e_2).$$

Denote $B_{ij} = B(e_i, e_j)$; we construct the modified second fundamental form

$$\tilde{h}_{ij} = h_{ij} - B_{ij}.$$

Let $\tilde{\kappa}_1 \geq \cdots \geq \tilde{\kappa}_n$ denote the eigenvalues of $\tilde{h}_{ij}$. Near X_0, we have $\kappa_1 \geq \tilde{\kappa}_1$, and at X_0, the following holds:

$$\tilde{\kappa}_i = \begin{cases} \kappa_1, & i = 1, \\ \kappa_i - 1, & i > 1. \end{cases}$$

We now replace κ_1 with $\tilde{\kappa}_1$ (smooth at X_0) in the function Q and consider the new test function

$$\tilde{Q} = \ln \tilde{\kappa}_1 - N \ln \nu^{n+1}.$$

Note that $\tilde{Q}$ also attains its maximum at X_0. Therefore, by the maximum principle, at X_0, we have

$$0 = \nabla_i \tilde{Q} = \frac{\nabla_i \tilde{\kappa}_1}{\tilde{\kappa}_1} - N\frac{\nabla_i \nu^{n+1}}{\nu^{n+1}}, \tag{3.1}$$

$$0 \geq \nabla_{ii} \tilde{Q} = \frac{\nabla_{ii} \tilde{\kappa}_1}{\tilde{\kappa}_1} - \frac{(\nabla_i \tilde{\kappa}_1)^2}{\tilde{\kappa}_1^2} - N\frac{\nabla_{ii} \nu^{n+1}}{\nu^{n+1}} + N\frac{(\nabla_i \nu^{n+1})^2}{(\nu^{n+1})^2}. \tag{3.2}$$

We first address the fourth-order terms. By the definition of the tensor B and $\nabla_{e_i} e_j = 0$, we have

$$\nabla_p B_{ij} = 0, \quad \nabla_{ii} B_{11} = 0 \text{ at } X_0.$$

It follows that

$$\nabla_p \tilde{h}_{ij} = \nabla_p h_{ij}, \qquad \nabla_{ii}\tilde{h}_{11} = \nabla_{ii} h_{11}.$$

Direct calculation shows that

$$\begin{aligned}
\nabla_i \tilde{\kappa}_1 &= \frac{\partial \tilde{\kappa}_1}{\partial \tilde{h}_{pq}} \nabla_i \tilde{h}_{pq} = \nabla_i \tilde{h}_{11} = \nabla_i h_{11}, \\
\nabla_{ii} \tilde{\kappa}_1 &= \frac{\partial \tilde{\kappa}_1}{\partial \tilde{h}_{pq}} \nabla_{ii} \tilde{h}_{pq} + \frac{\partial^2 \tilde{\kappa}_1}{\partial \tilde{h}_{pq} \partial \tilde{h}_{rs}} \nabla_i \tilde{h}_{pq} \nabla_i \tilde{h}_{rs} \\
&= \nabla_{ii} \tilde{h}_{11} + 2 \sum_{p>1} \frac{(\nabla_i \tilde{h}_{1p})^2}{\kappa_1 - \tilde{\kappa}_p} = \nabla_{ii} h_{11} + 2 \sum_{p>1} \frac{(\nabla_i h_{1p})^2}{\kappa_1 - \tilde{\kappa}_p},
\end{aligned}$$

where we have used the fact $\tilde{\kappa}_1 = \kappa_1$ at X_0. Substituting the last equation above into (3.2),

$$0 \geq \frac{\nabla_{ii} h_{11}}{\kappa_1} + 2 \sum_{p>1} \frac{(\nabla_i h_{1p})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_p)} - \frac{(\nabla_i h_{11})^2}{\kappa_1^2} - N \frac{\nabla_{ii} \nu^{n+1}}{\nu^{n+1}}. \tag{3.3}$$

By the Ricci identity (2.2), we have

$$\nabla_{ii} h_{11} = \nabla_{11} h_{ii} + \kappa_1^2 \kappa_i - \kappa_1 \kappa_i^2 - \kappa_1 + \kappa_i.$$

Combining this with (3.3) yields

$$\begin{aligned}
0 \geq \frac{\nabla_{11} h_{ii}}{\kappa_1} + 2 \sum_{p>1} \frac{(\nabla_i h_{1p})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_p)} - \frac{(\nabla_i h_{11})^2}{\kappa_1^2} \\
- N \frac{\nabla_{ii} \nu^{n+1}}{\nu^{n+1}} + \kappa_1 \kappa_i - \kappa_i^2 - 1 + \frac{\kappa_i}{\kappa_1}.
\end{aligned} \tag{3.4}$$

Contracting (3.4) with $S^{ii}_{n,\alpha}$, we have

$$\begin{aligned}
0 \geq \sum_i \frac{S^{ii}_{n,\alpha} \nabla_{11} h_{ii}}{\kappa_1} + 2 \sum_i \sum_{p>1} \frac{S^{ii}_{n,\alpha} (\nabla_i h_{1p})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_p)} - \sum_i \frac{S^{ii}_{n,\alpha} (\nabla_i h_{11})^2}{\kappa_1^2} \\
- N \sum_i \frac{S^{ii}_{n,\alpha} \nabla_{ii} \nu^{n+1}}{\nu^{n+1}} + \kappa_1 \sum_i S^{ii}_{n,\alpha} \kappa_i - \sum_i S^{ii}_{n,\alpha} \kappa_i^2 \\
- \sum_i S^{ii}_{n,\alpha} + \sum_i \frac{S^{ii}_{n,\alpha} \kappa_i}{\kappa_1}.
\end{aligned} \tag{3.5}$$

Differentiating (1.2) twice, we have

$$\sum_i S^{ii}_{n,\alpha} \nabla_{11} h_{ii} = - \sum_{p,q,r,s} S^{pq,rs}_{n,\alpha} \nabla_1 h_{pq} \nabla_1 h_{rs}. \tag{3.6}$$

Substituting (3.6) into (3.5) yields

$$
\begin{aligned}
0 \geq & -\frac{1}{\kappa_1}\sum_{p,q,r,s} S_{n,\alpha}^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} + 2\sum_i\sum_{p>1}\frac{S_{n,\alpha}^{ii}(\nabla_i h_{1p})^2}{\kappa_1(\kappa_1-\tilde{\kappa}_p)} \\
& - \sum_i \frac{S_{n,\alpha}^{ii}(\nabla_i h_{11})^2}{\kappa_1^2} - N\sum_i \frac{S_{n,\alpha}^{ii}\nabla_{ii}\nu^{n+1}}{\nu^{n+1}} \\
& + \left(\kappa_1+\frac{1}{\kappa_1}\right)\sum_i S_{n,\alpha}^{ii}\kappa_i - \sum_i S_{n,\alpha}^{ii}\kappa_i^2 - \sum_i S_{n,\alpha}^{ii}.
\end{aligned} \tag{3.7}
$$

By Lemma 2.1(v) and since ν^{n+1} has a positive lower bound, we have

$$
\begin{aligned}
0 \geq & -\frac{1}{\kappa_1}\sum_{p,q,r,s} S_{n,\alpha}^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} + 2\sum_i\sum_{p>1}\frac{S_{n,\nabla}^{ii}(\nabla_i h_{1p})^2}{\kappa_1(\kappa_1-\tilde{\kappa}_p)} \\
& - \sum_i \frac{S_{n,\alpha}^{ii}(\nabla_i h_{11})^2}{\kappa_1^2} - 2N\sum_i S_{n,\alpha}^{ii}\frac{\nabla_i u}{u}\frac{\nabla_i \nu^{n+1}}{\nu^{n+1}} \\
& + \left(\kappa_1+\frac{1}{\kappa_1}\right)\sum_i S_{n,\alpha}^{ii}\kappa_i - nN\sigma\frac{1+(\nu^{n+1})^2}{\nu^{n+1}} \\
& + (N-1)\Big(\sum_i S_{n,\alpha}^{ii} + \sum_i S_{n,\alpha}^{ii}\kappa_i^2\Big).
\end{aligned} \tag{3.8}
$$

By the first inequality in Lemma 2.4, we have

$$
\begin{aligned}
0 \geq & -\frac{1}{\kappa_1}\sum_{p,q,r,s} S_{n,\alpha}^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} + 2\sum_i\sum_{p>1}\frac{S_{n,\alpha}^{ii}(\nabla_i h_{1p})^2}{\kappa_1(\kappa_1-\tilde{\kappa}_p)} \\
& - \sum_i \frac{S_{n,\alpha}^{ii}(\nabla_i h_{11})^2}{\kappa_1^2} - 2N\sum_i S_{n,\alpha}^{ii}\frac{\nabla_i u}{u}\frac{\nabla_i \nu^{n+1}}{\nu^{n+1}} \\
& + \left(\kappa_1+\frac{1}{\kappa_1}\right)\sum_i S_{n,\alpha}^{ii}\kappa_i - CN \\
& + (N-1)\Big(\sum_i S_{n,\alpha}^{ii} + \sum_i S_{n,\alpha}^{ii}\kappa_i^2\Big),
\end{aligned} \tag{3.9}
$$

where $C = C(n,\Omega,\sigma,A,\alpha)$ denotes a universal positive constant that may vary from line to line.

By [LR25, Lemma 2.5], we have

$$
\sum_{p,q,r,s} S_{n,\alpha}^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} = \sum_{p\neq q} S_{n,\alpha}^{pp,qq}\nabla_1 h_{pp}\nabla_1 h_{qq} - \sum_{p\neq q} S_{n,\alpha}^{pp,qq}(\nabla_1 h_{pq})^2.
$$

Inserting into (3.9) yields

$$
\begin{aligned}
0 \geq & -\sum_{p\neq q}\frac{S_{n,\alpha}^{pp,qq}\nabla_1 h_{pp}\nabla_1 h_{qq}}{\kappa_1} + \sum_{p\neq q}\frac{S_{n,\alpha}^{pp,qq}(\nabla_1 h_{pq})^2}{\kappa_1} \\
& + 2\sum_i\sum_{p>1}\frac{S_{n,\alpha}^{ii}(\nabla_i h_{1p})^2}{\kappa_1(\kappa_1-\tilde{\kappa}_p)} - \sum_i\frac{S_{n,\alpha}^{ii}(\nabla_i h_{11})^2}{\kappa_1^2}
\end{aligned} \tag{3.10}
$$

$$- 2N \sum_i S^{ii}_{n,\alpha} \frac{\nabla_i u}{u} \frac{\nabla_i \nu^{n+1}}{\nu^{n+1}} + \left(\kappa_1 + \frac{1}{\kappa_1}\right) \sum_i S^{ii}_{n,\alpha} \kappa_i$$
$$+ (N-1)\Big(\sum_i S^{ii}_{n,\alpha} + \sum_i S^{ii}_{n,\alpha} \kappa_i^2\Big) - CN.$$

Assume that κ_1 has multiplicity m. Then

$$\sum_{p \neq q} \frac{S^{pp,qq}_{n,\alpha} (\nabla_1 h_{pq})^2}{\kappa_1} \geq 2 \sum_{p>m} \frac{S^{11,pp}_{n,\alpha} (\nabla_p h_{11})^2}{\kappa_1} \geq 2 \sum_{p>m} \frac{(S^{pp}_{n,\alpha} - S^{11}_{n,\alpha})(\nabla_p h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_p)}.$$

Substituting into (3.10) gives

$$\begin{aligned} 0 \geq & - \sum_{p \neq q} \frac{S^{pp,qq}_{n,\alpha} \nabla_1 h_{pp} \nabla_1 h_{qq}}{\kappa_1} + 2 \sum_{p>m} \frac{(S^{pp}_{n,\alpha} - S^{11}_{n,\alpha})(\nabla_p h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_p)} \\ & + 2 \sum_i \sum_{p>1} \frac{S^{ii}_{n,\alpha} (\nabla_i h_{1p})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_p)} - \sum_i \frac{S^{ii}_{n,\alpha} (\nabla_i h_{11})^2}{\kappa_1^2} \\ & - 2N \sum_i S^{ii}_{n,\alpha} \frac{\nabla_i u}{u} \frac{\nabla_i \nu^{n+1}}{\nu^{n+1}} + \left(\kappa_1 + \frac{1}{\kappa_1}\right) \sum_i S^{ii}_{n,\alpha} \kappa_i \\ & + (N-1)\Big(\sum_i S^{ii}_{n,\alpha} + \sum_i S^{ii}_{n,\alpha} \kappa_i^2\Big) - CN. \end{aligned} \tag{3.11}$$

Discarding certain nonnegative terms, we obtain

$$\begin{aligned} 2 \sum_i \sum_{p \neq 1} \frac{S^{ii}_{n,\alpha} (\nabla_i h_{1p})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_p)} &\geq 2 \sum_{p \neq 1} \frac{S^{pp}_{n,\alpha} (\nabla_p h_{1p}^2)}{\kappa_1(\kappa_1 - \tilde{\kappa}_p)} + 2 \sum_{p \neq 1} \frac{S^{11}_{n,\alpha} (\nabla_1 h_{1p})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_p)} \\ &= 2 \sum_{i \neq 1} \frac{S^{ii}_{n,\alpha} (\nabla_1 h_{ii})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} + 2 \sum_{i \neq 1} \frac{S^{11}_{n,\alpha} (\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)}. \end{aligned}$$

Combining this with (3.11) yields

$$\begin{aligned} 0 \geq & - \sum_{p \neq q} \frac{S^{pp,qq}_{n,\alpha} \nabla_1 h_{pp} \nabla_1 h_{qq}}{\kappa_1} + 2 \sum_{i>m} \frac{(S^{ii}_{n,\alpha} - S^{11}_{n,\alpha})(\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} \\ & + 2 \sum_{i \neq 1} \frac{S^{11}_{n,\alpha} (\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} + 2 \sum_{i \neq 1} \frac{S^{ii}_{n,\alpha} (\nabla_1 h_{ii})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} \\ & - \sum_i \frac{S^{ii}_{n,\alpha} (\nabla_i h_{11})^2}{\kappa_1^2} - 2N \sum_i S^{ii}_{n,\alpha} \frac{\nabla_i u}{u} \frac{\nabla_i \nu^{n+1}}{\nu^{n+1}} \\ & + \left(\kappa_1 + \frac{1}{\kappa_1}\right) \sum_i S^{ii}_{n,\alpha} \kappa_i - CN \\ & + (N-1)\Big(\sum_i S^{ii}_{n,\alpha} + \sum_i S^{ii}_{n,\alpha} \kappa_i^2\Big). \end{aligned} \tag{3.12}$$

By [LR25, Lemma 2.2(v)] and $\kappa \in \widetilde{\Gamma}_n \subset \Gamma_{n-1}$, we have $\kappa_{n-1} + \kappa_n > 0$. We assume that $1 + \alpha\kappa_n \le 0$. Since $\kappa_{n-1} + \kappa_n > 0$, we have $\kappa_{n-1} > 0$. Thus, $\kappa_1 \ge \cdots \ge \kappa_{n-1} > 0$.

Direct calculation shows

$$\begin{aligned} S_{n,\alpha}(\kappa) &= \sigma_{n-1}(\kappa) + \alpha\sigma_n(\kappa) = \sigma_{n-1}(\kappa|n) + \kappa_n S_{n-1}(\kappa|n) \\ &\le \sigma_{n-1}(\kappa|n) - \frac{1}{\alpha} S_{n-1}(\kappa|n) \\ &= \sigma_{n-1}(\kappa|n) - \frac{1}{\alpha}\left(\sigma_{n-2}(\kappa|n) + \alpha\sigma_{n-1}(\kappa|n)\right) \\ &= -\frac{1}{\alpha}\sigma_{n-2}(\kappa|n) < 0. \end{aligned}$$

This contradicts $S_{n,\alpha}(\kappa) > 0$. Thus, we have $1 + \alpha\kappa_n > 0$. This means $\kappa_i > -1/\alpha$, $i = 1, \ldots, n$.

We now address the troublesome third-order terms. We claim that

$$\begin{aligned} 2\sum_{i>m} \frac{(S^{ii}_{n,\alpha} - S^{11}_{n,\alpha})(\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} + 2\sum_{i\ne 1} \frac{S^{11}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} \\ - \sum_{i\ne 1} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} \ge \frac{1}{2}\sum_{i\ne 1} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2}. \end{aligned} \tag{3.13}$$

Indeed, assuming that $\kappa_1 > 3/\alpha + 3$, we have

$$\begin{aligned} &2\sum_{i>m} \frac{(S^{ii}_{n,\alpha} - S^{11}_{n,\alpha})(\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} + 2\sum_{i\ne 1} \frac{S^{11}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} - \sum_{i\ne 1} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} \\ &= 2\sum_{i>m} \frac{(S^{ii}_{n,\alpha} - S^{11}_{n,\alpha})(\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} + 2\sum_{1<i\le m} \frac{S^{11}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} \\ &\quad + 2\sum_{i>m} \frac{S^{11}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} - \sum_{1<i\le m} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} - \sum_{i>m} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} \\ &= \sum_{i>m} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1}\left(\frac{2}{\kappa_1 - \tilde{\kappa}_i} - \frac{1}{\kappa_1}\right) + 2\sum_{1<i\le m} \frac{S^{11}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} \\ &\quad - \sum_{1<i\le m} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} \\ &= \sum_{i>m} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1} \frac{\kappa_1 + \tilde{\kappa}_i}{\kappa_1(\kappa_1 - \tilde{\kappa}_i)} + \sum_{1<i\le m} \frac{S^{11}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1}\left(\frac{2}{\kappa_1 - \tilde{\kappa}_i} - \frac{1}{\kappa_1}\right) \\ &= (2\kappa_1 - 1)\sum_{1<i\le m} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} + \sum_{i>m} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} \frac{\kappa_1 + \tilde{\kappa}_i}{\kappa_1 - \tilde{\kappa}_i} \\ &\ge \frac{1}{2}\sum_{1<i\le m} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} + \sum_{i>m} \frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} \frac{a + b_i - 2}{a - b_i}, \end{aligned}$$

where $a=\kappa_1+1$, $b_i=\kappa_i$. Direct calculation shows

$$\begin{aligned}
&\sum_{i>m}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2}\frac{a+b_i-2}{a-b_i}\\
&\quad=\sum_{i>m}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2}\left(\frac{a+b_i-2}{a-b_i}-\frac{1}{2}\right)+\frac{1}{2}\sum_{i>m}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2}\\
&\quad=\sum_{i>m}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2}\,\frac{a+3b_i-4}{2(a-b_i)}+\frac{1}{2}\sum_{i>m}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2}\\
&\quad>\frac{1}{2}\sum_{i>m}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2},
\end{aligned}$$

where we have assumed $a>3/\alpha+4$ and $a-b_i>0$.

Thus, the claim (3.13) is valid. Inserting (3.13) into (3.12) yields

$$\begin{aligned}
(3.14)\quad 0\geq&-\sum_{p\neq q}\frac{S^{pp,qq}_{n,\alpha}\nabla_1 h_{pp}\nabla_1 h_{qq}}{\kappa_1}+2\sum_{i\neq 1}\frac{S^{ii}_{n,\alpha}(\nabla_1 h_{ii})^2}{\kappa_1(\kappa_1-\tilde{\kappa}_i)}-\frac{S^{11}_{n,\alpha}(\nabla_1 h_{11})^2}{\kappa_1^2}\\
&+\frac{1}{2}\sum_{i\neq 1}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2}-2N\sum_i S^{ii}_{n,\alpha}\frac{\nabla_i u}{u}\,\frac{\nabla_i\nu^{n+1}}{\nu^{n+1}}-CN\\
&+\left(\kappa_1+\frac{1}{\kappa_1}\right)\sum_i S^{ii}_{n,\alpha}\kappa_i+(N-1)\Big(\sum_i S^{ii}_{n,\alpha}+\sum_i S^{ii}_{n,\alpha}\kappa_i^2\Big)
\end{aligned}$$

Assume that $\kappa_1>3/\alpha+3$. Since $\kappa_i>-1/\alpha$, $i=1,\dots,n$, we have

$$\frac{2}{\kappa_1(\kappa_1-\tilde{\kappa}_i)}\geq\frac{2}{\kappa_1(\kappa_1+\frac{1}{\alpha}+1)}=\frac{2}{1+\frac{1+\alpha}{\alpha\kappa_1}}\frac{1}{\kappa_1^2}>\frac{3}{2}\frac{1}{\kappa_1^2}.$$

From (3.14), we obtain

$$\begin{aligned}
(3.15)\quad 0\geq&-\sum_{p\neq q}\frac{S^{pp,qq}_{n,\alpha}\nabla_1 h_{pp}\nabla_1 h_{qq}}{\kappa_1}+\frac{3}{2}\sum_{i\neq 1}\frac{S^{ii}_{n,\alpha}(\nabla_1 h_{ii})^2}{\kappa_1^2}-\frac{S^{11}_{n,\alpha}(\nabla_1 h_{11})^2}{\kappa_1^2}\\
&+\frac{1}{2}\sum_{i\neq 1}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2}-2N\sum_i S^{ii}_{n,\alpha}\frac{\nabla_i u}{u}\,\frac{\nabla_i\nu^{n+1}}{\nu^{n+1}}-CN\\
&+\left(\kappa_1+\frac{1}{\kappa_1}\right)\sum_i S^{ii}_{n,\alpha}\kappa_i+(N-1)\Big(\sum_i S^{ii}_{n,\alpha}+\sum_i S^{ii}_{n,\alpha}\kappa_i^2\Big).
\end{aligned}$$

Applying Lemma 2.3, we have

$$\begin{aligned}
(3.16)\qquad 0\geq&\frac{1}{2}\sum_{i\neq 1}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2}-2N\sum_i S^{ii}_{n,\alpha}\frac{\nabla_i u}{u}\,\frac{\nabla_i\nu^{n+1}}{\nu^{n+1}}\\
&+\left(\kappa_1+\frac{1}{\kappa_1}\right)\sum_i S^{ii}_{n,\alpha}\kappa_i-CN\\
&+(N-1)\Big(\sum_i S^{ii}_{n,\alpha}+\sum_i S^{ii}_{n,\alpha}\kappa_i^2\Big).
\end{aligned}$$

Since $\sigma_n(\kappa) > -A$ and $\kappa \in \Gamma_{n-1}$, we have

$$\left(\kappa_1 + \frac{1}{\kappa_1}\right)\sum_i S^{ii}_{n,\alpha}\kappa_i = \left(\kappa_1 + \frac{1}{\kappa_1}\right)((n-1)\sigma_{n-1} + n\alpha\sigma_n) > -C\kappa_1.$$

Inserting this into (3.16) yields

$$\begin{aligned} 0 \geq \frac{1}{2}\sum_{i\neq 1}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} - 2N\sum_i S^{ii}_{n,\alpha}\frac{\nabla_i u}{u}\frac{\nabla_i \nu^{n+1}}{\nu^{n+1}} \\ + (N-1)\Big(\sum_i S^{ii}_{n,\alpha} + \sum_i S^{ii}_{n,\alpha}\kappa_i^2\Big) - C\kappa_1 - CN. \end{aligned} \tag{3.17}$$

By Lemma 2.4 and for κ_1 sufficiently large, we can discard a positive term

$$-2NS^{11}_{n,\alpha}\frac{\nabla_1 u}{u}\frac{\nabla_1 \nu^{n+1}}{\nu^{n+1}} = -2NS^{11}_{n,\alpha}\frac{(\nabla_1 u)^2}{u^2}\frac{(\nu^{n+1}-\kappa_1)}{\nu^{n+1}}$$

Consequently, (3.17) implies

$$\begin{aligned} 0 \geq \frac{1}{2}\sum_{i\neq 1}\frac{S^{ii}_{n,\alpha}(\nabla_i h_{11})^2}{\kappa_1^2} - 2N\sum_{i\neq 1} S^{ii}_{n,\alpha}\frac{u_i}{u}\frac{\nabla_i \nu^{n+1}}{\nu^{n+1}} \\ + (N-1)\Big(\sum_i S^{ii}_{n,\alpha} + \sum_i S^{ii}_{n,\alpha}\kappa_i^2\Big) - C\kappa_1 - CN. \end{aligned} \tag{3.18}$$

By the critical equation (3.1), we have

$$\begin{aligned} 0 &\geq \frac{N^2}{2}\sum_{i\neq 1}\frac{S^{ii}_{n,\alpha}(\nabla_i \nu^{n+1})^2}{(\nu^{n+1})^2} - 2N\sum_{i\neq 1} S^{ii}_{n,\alpha}\frac{\nabla_i u}{u}\frac{\nabla_i \nu^{n+1}}{\nu^{n+1}} \\ &\quad + (N-1)\Big(\sum_i S^{ii}_{n,\alpha} + \sum_i S^{ii}_{n,\alpha}\kappa_i^2\Big) - C\kappa_1 - CN \\ &\geq -2\sum_{i\neq 1} S^{ii}_{n,\alpha}\frac{(\nabla_i u)^2}{u^2} + (N-1)\Big(\sum_i S^{ii}_{n,\alpha} + \sum_i S^{ii}_{n,\alpha}\kappa_i^2\Big) \\ &\quad - C\kappa_1 - CN. \end{aligned} \tag{3.19}$$

Choosing N sufficiently large, we have

$$0 \geq (N-1)\sum_i S^{ii}_{n,\alpha}\kappa_i^2 - C\kappa_1 - CN. \tag{3.20}$$

By Lemma 2.2, we have

$$\begin{aligned} 0 &\geq (N-1)\sum_i S^{ii}_{n,\alpha}\kappa_i^2 - C\kappa_1 - CN \\ &\geq (N-1)S^{11}_{n,\alpha}\kappa_1^2 - C\kappa_1 - CN \\ &\geq (N-1)\theta\sigma\kappa_1 - C\kappa_1 - CN. \end{aligned} \tag{3.21}$$

For sufficiently large N, (3.21) implies $\kappa_1 \leq C$. ∎

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Jianbo Yang, Yueming Lu
College of Science
Harbin University of Science and Technology
Harbin, 150080 Heilongjiang Province, P. R. China
E-mail: jianbo151@gmail.com
yueminglu@hrbust.edu.cn