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Ninth degree analogue of Ramanujan's septic theta function identity

by

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Dedicated to George E. Andrews and Bruce C. Berndt for their 85th birthdays

Abstract. On page 206 in his lost notebook, Ramanujan recorded a seventh degree identity for his theta function $\varphi(q)$. We give an analogous ninth degree identity. We also provide an application of an entry from his second notebook on a cubic equation and an interpretation with theta functions for some of his trigonometric identities. Lastly, we calculate five examples for $\varphi(e^{-\pi\sqrt{n}})$.

1. Introduction and the septic identity. Ramanujan's general theta function $f(a,b)$ is defined by ([16, p. 197], [2, p. 34])

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{1}$$

The Jacobi triple product identity ([13, pp. 176–183], [16, p. 197], [2, p. 35]) states that

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty, \quad |ab| < 1, \tag{2}$$

where

$$(a;q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| < 1.$$

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In Ramanujan's notation, the theta function $\varphi(q)$ is defined by ([16, p. 197], [2, p. 36])

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \quad |q|<1, \tag{3}$$

where the series and product representations follow from (1) and (2), respectively. After Ramanujan, we set ([16, p. 197], [2, p. 37])

$$\chi(q) := (-q;q^2)_\infty, \quad |q|<1. \tag{4}$$

If $q = \exp(-\pi\sqrt{n})$, for a positive rational number n, the class invariant G_n is defined by ([14], [15, pp. 23–39], [4, p. 183])

$$G_n := 2^{-1/4} q^{-1/24} \chi(q). \tag{5}$$

Using the theory of modular equations, one can find values of $\varphi(q)$ of the form $\varphi(e^{-\pi\sqrt{n}})$ for a positive rational number n. For an overview of the theory of modular equations, see [2, pp. 213–214], [6], [4, p. 185]. The values obtained by Ramanujan are proved in [6], [4, pp. 323–351]. We also recommend the survey [8] in the subject. For calculating special values of $\varphi(q)$, there are other particular methods, as we shall see in Entry 1.1.

On page 206 in his lost notebook [17], Ramanujan gives the following septic identities.

Entry 1.1. Let

$$\frac{\varphi(q^{1/7})}{\varphi(q^7)} = 1 + u + v + w. \tag{i}$$

Then

$$p := uvw = \frac{8q^2(-q;q^2)_\infty}{(-q^7;q^{14})_\infty^7} \tag{ii}$$

and

$$\frac{\varphi^8(q)}{\varphi^8(q^7)} - (2+5p)\frac{\varphi^4(q)}{\varphi^4(q^7)} + (1-p)^3 = 0. \tag{iii}$$

Furthermore,

$$u = \left(\frac{\alpha^2 p}{\beta}\right)^{1/7}, \quad v = \left(\frac{\beta^2 p}{\gamma}\right)^{1/7}, \quad \text{and} \quad w = \left(\frac{\gamma^2 p}{\alpha}\right)^{1/7}, \tag{iv}$$

where α, β, and γ are the roots of the equation

$$r(\xi) := \xi^3 + 2\xi^2\left(1 + 3p - \frac{\varphi^4(q)}{\varphi^4(q^7)}\right) + \xi p^2(p+4) - p^4 = 0. \tag{v}$$

For example,

$$\varphi(e^{-7\pi\sqrt{7}}) = 7^{-3/4}\varphi(e^{-\pi\sqrt{7}})\{1 + (\quad)^{2/7} + (\quad)^{2/7} + (\quad)^{2/7}\}. \tag{vi}$$

We remark that $0 < |q| < 1$, and u, v, and w depend on the order of α, β, and γ. Parts (i)–(v) were proved by Seung Hwan Son ([19], [21, pp. 198–200], [1, pp. 180–194]). Part (vi) was completed by the second author in [18, Theorem 4.1], by finding the three missing terms given as follows.

THEOREM 1.2. *We have*

$$\varphi(e^{-7\pi\sqrt{7}})$$
$$= 7^{-3/4}\varphi(e^{-\pi\sqrt{7}})\left\{1+\left(\frac{\cos\frac{\pi}{7}}{2\cos^2\frac{2\pi}{7}}\right)^{2/7}+\left(\frac{\cos\frac{2\pi}{7}}{2\cos^2\frac{3\pi}{7}}\right)^{2/7}+\left(\frac{\cos\frac{3\pi}{7}}{2\cos^2\frac{\pi}{7}}\right)^{2/7}\right\}.$$

Berndt and the second author [9] discussed cubic and quintic analogues of Entry 1.1 and Theorem 1.2, with an extensive list of further examples. The aim of the present paper is to propose a ninth degree analogue.

In Section 2, we give our ninth degree analogue of Entry 1.1. In Section 3, we discuss an entry from Ramanujan's second notebook, and we offer an alternative formulation of our main result. In Section 4, we give examples, such as the analogous example for Theorem 1.2.

For a positive rational n, we evaluate values of $\varphi(e^{-\pi\sqrt{n}})$ in terms of trigonometric function values. The first four out of our five examples are given in [9] in an alternative formulation. We list them in the next theorem.

THEOREM 1.3. *We have*

(i) $$\frac{\varphi(e^{-27\pi})}{\varphi(e^{-3\pi})} = \frac{1}{3}\left(1+(\sqrt{3}-1)\left(\frac{(2(\sqrt{3}+1))^{1/3}+1}{(2(\sqrt{3}-1))^{1/3}-1}\right)^{1/3}\right),$$

(ii) $$\frac{\varphi(e^{-9\pi\sqrt{3}})}{\varphi(e^{-3\pi\sqrt{3}})} = \frac{1}{3^{1/4}(1+2^{1/3})}\left(1+\left(\frac{2}{2^{1/3}-1}\right)^{1/3}\right),$$

(iii) $$\frac{\varphi(e^{-27\pi\sqrt{3}})}{\varphi(e^{-\pi\sqrt{3}})} = \frac{1+2^{1/3}}{3^{7/4}}\left(1+2^{1/3}(2^{1/3}-1)\frac{2^{1/3}+2^{2/3}+3^{1/3}}{(9-2\cdot 3^{4/3})^{1/3}}\right),$$

and

(iv) $$\frac{\varphi(e^{-81\pi})}{\varphi(e^{-\pi})} = \frac{1+(2(\sqrt{3}+1))^{1/3}}{9}\times$$
$$\left(1+2^{1/3}\left(\frac{(2(\sqrt{3}-1))^{1/3}-1}{(2(\sqrt{3}+1))^{1/3}+1}\right)^{8/9}\left(\frac{3(\sqrt{3}+1)}{\sqrt{3}+1-\left(\frac{(2(\sqrt{3}+1))^{1/3}+1}{(2(\sqrt{3}-1))^{1/3}-1}\right)^{1/3}}-2\right)^{1/3}\right).$$

Theorem 1.3(i) was first proved in print by Berndt and Chan [6]. It can be obtained by combining [8, Corollaries 3.12 and 3.5]. Likewise, to obtain Theorem 1.3(ii), combine [8, Corollaries 3.7 and 3.4]. Finally, the values in Theorem 1.3(iii),(iv) are given in [8, Corollary 3.27] and [8, Corollary 3.28], respectively. All of these values can be expressed in terms of gamma functions. From [23, pp. 524–525], [2, p. 103], [6], [4, p. 325], and from [25], [11],

[10, p. 298],

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)}{\sqrt{2}\pi^{3/4}} \quad \text{and} \quad \varphi(e^{-\pi\sqrt{3}}) = \frac{3^{1/8}\Gamma^{3/2}\left(\frac{1}{3}\right)}{2^{2/3}\pi},$$

respectively. The values of $\varphi(e^{-3\pi})$ and $\varphi(e^{-3\pi\sqrt{3}})$ follow from [8, Corollaries 3.5 and 3.4].

2. A ninth degree analogue of Entry 1.1

THEOREM 2.1. *For $|q| < 1$, let*

$$u_k := u_k(q) := \frac{2q^{k^2/9} f(q^{9+2k}, q^{9-2k})}{\varphi(q^9)}, \quad k = 1, \dots, 4. \tag{0}$$

Then

$$\frac{\varphi(q^{1/9})}{\varphi(q^9)} = 1 + u_1 + u_2 + u_3 + u_4, \tag{i}$$

$$p := u_1 u_2 u_4 = \frac{8q^{7/3}\chi(q)}{\chi^6(q^9)\chi(q^3)}, \tag{ii}$$

$$u_3 = \frac{\varphi(q)}{\varphi(q^9)} - 1. \tag{iii}$$

Furthermore, for $0 < |q| < 1$,

$$u_1 = \left(\frac{\beta\, p}{\alpha}\right)^{1/3}, \quad u_2 = \left(\frac{\gamma\, p}{\beta}\right)^{1/3}, \quad \textit{and} \quad u_4 = \left(\frac{\alpha\, p}{\gamma}\right)^{1/3}, \tag{iv}$$

where α, β, and γ are the roots of the cubic equation

$$r(\xi) := \xi^3 - u_3\left(\frac{\varphi^2(q)}{\varphi^2(q^9)} + 3\right)(\xi^2 - 2u_3^2\xi) - p^3 = 0. \tag{v}$$

Throughout the paper, we use the notation introduced in Theorem 2.1. Note that

$$u_1 = \frac{2q^{1/9} f(q^{11}, q^7)}{\varphi(q^9)}, \quad u_2 = \frac{2q^{4/9} f(q^{13}, q^5)}{\varphi(q^9)},$$
$$u_3 = \frac{2q f(q^{15}, q^3)}{\varphi(q^9)}, \quad u_4 = \frac{2q^{16/9} f(q^{17}, q)}{\varphi(q^9)}.$$

Also note that, by Theorem 2.1(iii), we can express r in terms of u_3 and p as

$$r(\xi) = \xi^3 - u_3(u_3^2 + 2u_3 + 4)\xi^2 + 2u_3^3(u_3^2 + 2u_3 + 4)\xi - p^3,$$

and as we shall see in Lemma 2.4(iv), p^3 is a rational function in u_3.

There are some differences compared to Entry 1.1. Here, u_3 is not obtained from the roots of r, and in Theorem 2.1(iv), the roots are employed in a different form than in Entry 1.1(iv). The order of the roots α, β, and γ

in Theorem 2.1(iv) is not provided. We shall give a solution for this in Lemma 4.8. Without loss of generality, one can assume that

$$\alpha := u_2u_4^2, \quad \beta := u_4u_1^2, \quad \text{and} \quad \gamma := u_1u_2^2. \tag{6}$$

In order to establish Theorem 2.1, we shall need some lemmas. We use Entry 31 of Chapter 16 of Ramanujan's second notebook ([16, p. 200], [2, pp. 48–49]).

LEMMA 2.2. *Let* $\mathcal{U}_k = a^{k(k+1)/2}b^{k(k-1)/2}$ *and* $\mathcal{V}_k = a^{k(k-1)/2}b^{k(k+1)/2}$ *for each nonnegative integer* k*. Then, for each positive integer* n*,*

$$f(\mathcal{U}_1, \mathcal{V}_1) = \sum_{k=0}^{n-1} \mathcal{U}_k f\left(\frac{\mathcal{U}_{n+k}}{\mathcal{U}_k}, \frac{\mathcal{V}_{n-k}}{\mathcal{U}_k}\right), \qquad |ab| < 1,\ ab \neq 0.$$

Proof. See Berndt's book [2, pp. 48–49, Entry 31]. ∎

Theorem 2.1(i) holds in the following more general form.

LEMMA 2.3. *For any positive odd integer* n *and* $|q| < 1$*,*

$$\frac{\varphi(q^{1/n})}{\varphi(q^n)} = 1 + \sum_{k=1}^{(n-1)/2} \frac{2q^{k^2/n} f(q^{n+2k}, q^{n-2k})}{\varphi(q^n)}.$$

Proof. The case $q = 0$ is trivial. Otherwise, from Entry 18(i),(iv) in [16, p. 197], [2, pp. 34–35], we obtain the identity

$$f(a, b) = af\big(\tfrac{1}{a}, a^2b\big), \qquad |ab| < 1,\ a \neq 0. \tag{7}$$

Now, for a positive integer n and $0 < |q| < 1$, let

$$\widetilde{u}_k := q^{k^2/n} f(q^{n+2k}, q^{n-2k}), \qquad k = 0, \ldots, n-1. \tag{8}$$

Because of (7), we have $\widetilde{u}_k = \widetilde{u}_{n-k}$, for $k = 1, \ldots, n-1$. By applying Lemma 2.2 for $(a, b) = (q^{1/n}, q^{1/n})$, with the aid of (3), (7), and (8), for each odd n, we find that

$$\varphi(q^{1/n}) = \sum_{k=0}^{n-1} \widetilde{u}_k = \varphi(q^n) + \sum_{k=1}^{n-1} \widetilde{u}_k = \varphi(q^n) + \sum_{k=1}^{(n-1)/2} 2\widetilde{u}_k.$$

After rearrangement, the proof is complete. ∎

Next, we propose some useful lemmas. Lemma 2.4 gives different representations for u_3, and a connection between $u_3(q^{1/3})$ and $u_3(q)$. Lemma 2.5 gives expressions for p.

LEMMA 2.4. *For* $|q| < 1$,

(i) $$u_3(q) = \frac{\varphi(q)}{\varphi(q^9)} - 1,$$

(ii) $$u_3(q) = \left(\frac{\varphi^4(q^3)}{\varphi^4(q^9)} - 1\right)^{1/3},$$

(iii) $$u_3(q) = \frac{2q\chi(q^3)}{\chi^3(q^9)},$$

and

(iv) $$u_3(q^{1/3}) = \left(\frac{(u_3(q)+1)^4}{u_3^3(q)+1} - 1\right)^{1/3}.$$

Proof. Part (i) follows from the first equation of [2, p. 49, Corollary (i)]. Then, (ii) follows from the first equation of [2, pp. 345–346, Entry 1(iii)], by setting $q \mapsto q^3$. Now, (iii) is obtained by [4, p. 330, (4.6)], with $q \mapsto q^3$. From (i) and (ii), we find

$$(u_3(q)+1)^4 = \frac{\varphi^4(q)}{\varphi^4(q^9)} \quad \text{and} \quad u_3^3(q)+1 = \frac{\varphi^4(q^3)}{\varphi^4(q^9)}, \tag{9}$$

respectively. Set $q \mapsto q^{1/3}$ in (ii) and combine it with (9) to conclude (iv). ∎

LEMMA 2.5. *For* $|q| < 1$,

(i) $$p = \frac{8q^{7/3}\chi(q)}{\chi^6(q^9)\chi(q^3)},$$

(ii) $$p = u_3^2(q)u_3(q^{1/3}),$$

(iii) $$p = \left(\frac{\varphi(q)}{\varphi(q^9)} - 1\right)^2 \left(\frac{\varphi^4(q)}{\varphi^4(q^3)} - 1\right)^{1/3},$$

(iv) $$p^3(u_3^2 - u_3 + 1) = u_3^7(u_3^2 + 2u_3 + 4).$$

Proof. To prove (i), we use a standard argument as in [19, Lemma 4.1] or [1, p. 183, Lemma 8.3.5]. The case $q = 0$ is trivial. Otherwise, apply the Jacobi triple product identity (2) to establish the representations

$$f(q^{9+2k}, q^{9-2k}) = (-q^{9+2k}; q^{18})_\infty (-q^{9-2k}; q^{18})_\infty (q^{18}; q^{18})_\infty, \quad k = 1, \ldots, 4,$$

and

$$\varphi(q^9) = f(q^9, q^9) = (-q^9; q^{18})_\infty^2 (q^{18}; q^{18})_\infty.$$

Thus,

$$u_1u_2u_3u_4 = \frac{16q^{10/3}}{\varphi^4(q^9)}\prod_{k=1}^{4} f(q^{9+2k}, q^{9-2k})$$
$$= \frac{16q^{10/3}}{(-q^9;q^{18})_\infty^8}\prod_{k=1}^{4}(-q^{9+2k};q^{18})_\infty(-q^{9-2k};q^{18})_\infty.$$

Since

$$(-q;q^2)_\infty = \prod_{k=1}^{9}(-q^{2k-1};q^{18})_\infty,$$

and since $9\pm 2k$ for $k=1,\dots,4$ covers $\{1,3,5,\dots,17\}\setminus\{9\}$, and $2k-1$ for $k=1,\dots,9$ covers $\{1,3,5,\dots,17\}$, we find that

$$u_1u_2u_3u_4 = \frac{16q^{10/3}(-q;q^2)_\infty}{(-q^9;q^{18})_\infty^9} = \frac{16q^{10/3}\chi(q)}{\chi^9(q^9)},$$

where $\chi(q)$ is defined in (4). Using Lemma 2.4(iii), we conclude that

$$p = u_1u_2u_4 = \frac{16q^{10/3}\chi(q)}{\chi^9(q^9)}\cdot\frac{\chi^3(q^9)}{2q\chi(q^3)} = \frac{8q^{7/3}\chi(q)}{\chi^6(q^9)\chi(q^3)}.$$

Now, to prove (ii), by Lemma 2.4(iii), observe that

$$p = \frac{8q^{7/3}\chi(q)}{\chi^6(q^9)\chi(q^3)} = \frac{4q^2\chi^2(q^3)}{\chi^6(q^9)}\cdot\frac{2q^{1/3}\chi(q)}{\chi^3(q^3)} = u_3^2(q)u_3(q^{1/3}).$$

Then (iii) follows directly by Lemma 2.4(i),(ii). Lastly, (iv) can be deduced from (ii) and Lemma 2.4(iv), as follows:

$$p^3 = u_3^6(q)u_3^3(q^{1/3}) = u_3^6\cdot\left(\frac{(u_3+1)^4}{u_3^3+1}-1\right) = \frac{u_3^7(u_3^2+2u_3+4)}{u_3^2-u_3+1}.\ \blacksquare$$

In the following lemma, we express $u_1+u_2+u_4$ in two ways, first with the aid of $\varphi^4(q^{1/3})$, and second in terms of u_3 and p.

LEMMA 2.6. *For $|q|<1$,*

$$\text{(i)}\qquad (u_1+u_2+u_4)^3 = \frac{\varphi^3(q)}{\varphi^3(q^9)}\left(\frac{\varphi^4(q^{1/3})}{\varphi^4(q)}-1\right),$$

and for $0<|q|<1$,

$$\text{(ii)}\qquad (u_1+u_2+u_4)^3 = (u_3+1)^3\left(\frac{\left(\frac{p}{u_3^2}+1\right)^4}{\left(\frac{p}{u_3^2}\right)^3+1}-1\right).$$

Proof. By Lemma 2.3 for $n = 9$ and Lemma 2.4(i), or from [2, pp. 349–352, Entry 2(ix)],

$$\frac{\varphi(q^{1/9})}{\varphi(q^9)} = \frac{\varphi(q)}{\varphi(q^9)} + u_1 + u_2 + u_4.$$

After rearrangement, by Lemma 2.4(i),(ii), we deduce that

$$u_1 + u_2 + u_4 = \frac{\varphi(q)}{\varphi(q^9)}\left(\frac{\varphi(q^{1/9})}{\varphi(q)} - 1\right) = \frac{\varphi(q)}{\varphi(q^9)}u_3(q^{1/9}) = \frac{\varphi(q)}{\varphi(q^9)}\left(\frac{\varphi^4(q^{1/3})}{\varphi^4(q)} - 1\right)^{1/3}, \tag{10}$$

and by Lemmas 2.4(i),(iv), and 2.5(ii), we find that

$$\begin{aligned} &u_1 + u_2 + u_4 \\ &= \frac{\varphi(q)}{\varphi(q^9)}\left(\frac{\varphi(q^{1/9})}{\varphi(q)} - 1\right) = (u_3(q) + 1)u_3(q^{1/9}) \\ &= (u_3(q) + 1)\left(\frac{(u_3(q^{1/3}) + 1)^4}{u_3^3(q^{1/3}) + 1} - 1\right)^{1/3} = (u_3 + 1)\left(\frac{\left(\frac{p}{u_3^2} + 1\right)^4}{\left(\frac{p}{u_3^2}\right)^3 + 1} - 1\right)^{1/3}. \end{aligned} \tag{11}$$

Cubing both sides of (10) and (11), we obtain (i) and (ii), respectively. ∎

LEMMA 2.7. *For* $0 < |q| < 1$,

$$\frac{p}{u_3^2}(u_3^3 + 4) + 6p + 3u_3(u_3^2 + 2u_3 + 4)\left(\frac{2u_3^2}{p} + 1\right) = (u_3 + 1)^3\left(\frac{\left(\frac{p}{u_3^2} + 1\right)^4}{\left(\frac{p}{u_3^2}\right)^3 + 1} - 1\right).$$

Proof. After rearrangement, by Lemma 2.5(iv), we find that

$$\begin{aligned} &\frac{p}{u_3^2}(u_3^3 + 4) + 6p + 3u_3(u_3^2 + 2u_3 + 4)\left(\frac{2u_3^2}{p} + 1\right) - (u_3 + 1)^3\left(\frac{\left(\frac{p}{u_3^2} + 1\right)^4}{\left(\frac{p}{u_3^2}\right)^3 + 1} - 1\right) \\ &= \frac{3(p - 2u_3^2)}{pu_3^2(p^2 - pu_3^2 + u_3^4)}(p^3(u_3^2 - u_3 + 1) - u_3^7(u_3^2 + 2u_3 + 4)) = 0. \end{aligned}$$ ∎

For now, let $\{x\} \in [0, 1)$ denote the fractional part of a real number x. For a real number $\alpha \in [0, 1)$, an enumerable set S of real numbers, and a power series $\sum_{v\in S} a_v q^v$, we define the operator $\mathcal{M}_\alpha$ by

$$\mathcal{M}_\alpha\Big(\sum_{v\in S} a_v q^v\Big) := \sum_{\substack{w\in S \\ \{w\}=\alpha}} a_w q^w. \tag{12}$$

Note that $\mathcal{M}_\alpha$ collects the terms of a power series whose exponents have fractional part α. Also note that since power series representations are unique, by applying $\mathcal{M}_\alpha$ to an equation, we obtain an identity. A similar

definition is introduced by Son [19], [1, pp. 180–194], with a more general concept appearing in [20, Definition 2.8].

LEMMA 2.8. *For* $|q| < 1$,

$$u_1u_2^2 + u_2u_4^2 + u_4u_1^2 = u_3\left(\frac{\varphi^2(q)}{\varphi^2(q^9)} + 3\right), \tag{i}$$

$$p(u_1^2u_2 + u_2^2u_4 + u_4^2u_1) = 2u_3^3\left(\frac{\varphi^2(q)}{\varphi^2(q^9)} + 3\right), \tag{ii}$$

and for $0 < |q| < 1$,

$$u_1^3 + u_2^3 + u_4^3 = \frac{p}{u_3^2}(u_3^3 + 4). \tag{iii}$$

Proof. The case $q = 0$ is trivial. Otherwise, combining Lemmas 2.7 and 2.6(ii), we have

$$\frac{p}{u_3^2}(u_3^3 + 4) + 6p + 3u_3(u_3^2 + 2u_3 + 4)\left(\frac{2u_3^2}{p} + 1\right) = (u_1 + u_2 + u_4)^3. \tag{13}$$

Expanding the right-hand side of (13) and using Lemma 2.4(i), after simplification, we obtain

$$\begin{aligned}\frac{p}{u_3^2}(u_3^3 + 4) + 3u_3\left(\frac{\varphi^2(q)}{\varphi^2(q^9)} + 3\right)\left(\frac{2u_3^2}{p} + 1\right)\\ = u_1^3 + u_2^3 + u_4^3 + 3(u_1u_2^2 + u_2u_4^2 + u_4u_1^2 + u_1^2u_2 + u_2^2u_4 + u_4^2u_1).\end{aligned} \tag{14}$$

Now, using the series representations of u_1, u_2, u_3, and u_4, and applying $\mathcal{M}_0$, defined in (12), to both sides of (14), i.e., collecting the terms with integer powers of q, we obtain part (i). Similarly, applying $\mathcal{M}_{2/3}$ and $\mathcal{M}_{1/3}$ to (14), we arrive at (ii) and (iii), respectively. ∎

Now, we summarize our proof of Theorem 2.1.

Proof of Theorem 2.1. Part (i) is obtained by Lemma 2.3 with $n = 9$. Then, (ii) and (iii) are given in Lemmas 2.5(i) and 2.4(i), respectively. Assuming (6), the construction in (iv) holds, since the coefficients of the quadratic, linear, and constant terms of r in (v) follow by Lemma 2.8(i),(ii) and (6), respectively. These are

$$\alpha + \beta + \gamma = u_1u_2^2 + u_2u_4^2 + u_4u_1^2 = u_3\left(\frac{\varphi^2(q)}{\varphi^2(q^9)} + 3\right),$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = p(u_1^2u_2 + u_2^2u_4 + u_4^2u_1) = 2u_3^3\left(\frac{\varphi^2(q)}{\varphi^2(q^9)} + 3\right),$$

$$\alpha\beta\gamma = (u_1u_2u_4)^3 = p^3.$$

Thus, α, β, and γ are the roots of (v). ∎

COROLLARY 2.9. *For* $0 < |q| < 1$,

$$\frac{1}{u_1^3} + \frac{1}{u_2^3} + \frac{1}{u_4^3} = \frac{1}{pu_3^3}(u_3^4 - u_3^3 + 6u_3^2 - 8u_3 + 8).$$

Proof. First, by multiplying the left-hand sides of Lemma 2.8(i),(ii), after rearrangement, we find that

$$\frac{1}{u_1^3} + \frac{1}{u_2^3} + \frac{1}{u_4^3} = \frac{1}{p^3}(u_1u_2^2 + u_2u_4^2 + u_4u_1^2)(u_1^2u_2 + u_2^2u_4 + u_4^2u_1) - \frac{1}{p^2}(u_1^3 + u_2^3 + u_4^3) - \frac{3}{p}.$$

Then, by Lemmas 2.8(i)–(iii), 2.4(i), and 2.5(iv), we deduce that

$$\begin{aligned}\frac{1}{u_1^3} + \frac{1}{u_2^3} + \frac{1}{u_4^3} &= \frac{2u_3^4}{p^4}(u_3^2 + 2u_3 + 4)^2 - \frac{1}{u_3^2p}(u_3^3 + 4) - \frac{3}{p}\\ &= \frac{1}{pu_3^3}\left(\frac{2u_3^7}{p^3}(u_3^2 + 2u_3 + 4)^2 - u_3(u_3^3 + 4) - 3u_3^3\right)\\ &= \frac{1}{pu_3^3}\left(2\frac{u_3^2 - u_3 + 1}{u_3^2 + 2u_3 + 4}(u_3^2 + 2u_3 + 4)^2 - u_3(u_3^3 + 4) - 3u_3^3\right)\\ &= \frac{1}{pu_3^3}(u_3^4 - u_3^3 + 6u_3^2 - 8u_3 + 8). \ \blacksquare\end{aligned}$$

3. Ramanujan's entry on a cubic equation. On page 356 in his second notebook [16], Ramanujan states the following algebraic identity.

ENTRY 3.1. If α, β, and γ are the roots of

$$x^3 - ax^2 + bx - 1 = 0,$$

then

$$(\alpha/\beta)^{1/3} + (\beta/\gamma)^{1/3} + (\gamma/\alpha)^{1/3} \quad \text{and} \quad (\beta/\alpha)^{1/3} + (\gamma/\beta)^{1/3} + (\alpha/\gamma)^{1/3} \tag{15}$$

are the roots of

$$z^2 - tz + a + b + 3 = 0,$$

where

$$t^3 - 3(a + b + 3)t - (ab + 6(a + b) + 9) = 0.$$

The quantities (15) are also roots of

$$y^6 - (ab + 6(a + b) + 9)y^3 + (a + b + 3)^3 = 0. \tag{16}$$

Lastly,

$$\alpha^{1/3}+\beta^{1/3}+\gamma^{1/3}$$
$$=\left(a+6+3\left\{\frac{ab+9}{2}+3(a+b)+\left(\left(\frac{ab+9}{2}\right)^2-a^3-b^3-27\right)^{1/2}\right\}^{1/3}\right.$$
$$\left.+3\left\{\frac{ab+9}{2}+3(a+b)-\left(\left(\frac{ab+9}{2}\right)^2-a^3-b^3-27\right)^{1/2}\right\}^{1/3}\right)^{1/3}.$$

Entry 3.1 is a sequel to Ramanujan's earlier result ([3, pp. 22–23, Entry 10], [5, pp. 652–653]).

Proof. See Berndt's book [3, pp. 40–41, Entry 30]. ∎

For our purposes, we restate some parts of Entry 3.1.

LEMMA 3.2. *If α, β, and γ are the roots of*

$$x^3-ax^2+bx-1=0,$$

and if μ and ν are the roots of

$$y^2-(ab+6(a+b)+9)y+(a+b+3)^3=0,$$

then

$$\mu^{1/3}=(\alpha/\beta)^{1/3}+(\beta/\gamma)^{1/3}+(\gamma/\alpha)^{1/3},$$
$$\nu^{1/3}=(\beta/\alpha)^{1/3}+(\gamma/\beta)^{1/3}+(\alpha/\gamma)^{1/3}.$$

Proof. We utilize (15) and (16) of Entry 3.1. Replace y by $y^{1/3}$ in (16) to obtain the result. ∎

Note the symmetry in Lemma 3.2. Interchanging a and b produces the same case.

THEOREM 3.3. *Let u_3 and p be as defined in Theorem* 2.1. *Furthermore, for $0<|q|<1$, let*

$$a:=\frac{u_3}{p}\left(\frac{\varphi^2(q)}{\varphi^2(q^9)}+3\right)\quad\text{and}\quad b:=\frac{2u_3^3}{p^2}\left(\frac{\varphi^2(q)}{\varphi^2(q^9)}+3\right)=\frac{2u_3^2a}{p}.$$

Then

$$\frac{\varphi(q^{1/9})}{\varphi(q^9)}=1+u_3+(py)^{1/3},$$

where

$$y^2-(ab+6(a+b)+9)y+(a+b+3)^3=0.$$

Theorem 3.3 is a corollary of Theorem 2.1 and Lemma 3.2.

Proof. From Theorem 2.1(v), consider the polynomial

$$r(\xi)=\xi^3-ap\xi^2+bp^2\xi-p^3=0. \tag{17}$$

By using an appropriate polynomial transformation, we find that α, β, and γ are the roots of (17) exactly if α/p, β/p, and γ/p are the roots of

$$\frac{r(xp)}{p^3} = x^3 - ax^2 + bx - 1 = 0.$$

The rest of the proof follows directly by Theorem 2.1(i),(iv) and Lemma 3.2. ∎

4. Examples for ninth degree identities. In this section, we present five ninth degree examples. To obtain examples for Theorem 2.1, we need the values of triplets of class invariants G_n, G_{9n}, and G_{81n}, for certain positive rational numbers n. Ramanujan [4, pp. 189–199] calculated the class invariant G_n for a total of 78 values of n. In our first four examples, we use the values for $n = 1, 3, 9, 27$, and 81. In our fifth example, we use G_{243}, given by Watson [22]. We give our values of Theorem 2.1(i) in trigonometric form. As we mentioned in the introduction, alternative expressions for Theorems 4.9–4.12 are given in terms of radicals in Theorem 1.3(i)–(iv). Theorem 4.13 appears to be new. Our examples are summarized in Table 1.

Table 1. Overview of ninth degree examples

n	$9n$	$81n$	Value	Ex. for Thm. 2.1	Other representation
1/9	1	9	$\varphi(e^{-27\pi})/\varphi(e^{-3\pi})$	Theorem 4.9	Theorem 1.3(i)
1/3	3	27	$\varphi(e^{-9\pi\sqrt{3}})/\varphi(e^{-3\pi\sqrt{3}})$	Theorem 4.10	Theorem 1.3(ii)
1/27	1/3	3	$\varphi(e^{-27\pi\sqrt{3}})/\varphi(e^{-\pi\sqrt{3}})$	Theorem 4.11	Theorem 1.3(iii)
1/81	1/9	1	$\varphi(e^{-81\pi})/\varphi(e^{-\pi})$	Theorem 4.12	Theorem 1.3(iv)
1/243	1/27	1/3	$\varphi(e^{-81\pi\sqrt{3}})/\varphi(e^{-\pi\sqrt{3}})$	Theorem 4.13	—

First, we shall need some lemmas. Lemmas 4.1–4.4 are classical. Lemmas 4.5, 4.6, and 4.8 are analogous to the septic case lemmas in [18, Lemmas 2.3, 3.3, and 3.7], respectively.

LEMMA 4.1. *If n is a positive rational number, then*

$$\varphi(e^{-\pi/\sqrt{n}}) = n^{1/4}\varphi(e^{-\pi\sqrt{n}}).$$

Proof. The transformation formula for $\varphi(q)$ states that if $\mathrm{Re}(\alpha^2), \mathrm{Re}(\beta^2) > 0$ and $\alpha\beta = \pi$, then ([16, p. 199], [2, p. 43])

$$\sqrt{\alpha}\varphi(e^{-\alpha^2}) = \sqrt{\beta}\varphi(e^{-\beta^2}).$$

The lemma is the special case for $\alpha^2 = \pi/\sqrt{n}$. ∎

LEMMA 4.2. *If n is a positive rational number, then $G_n = G_{1/n}$.*

Proof. See Ramanujan's paper [14], [15, pp. 23–39] or Yi's thesis [24, pp. 18–19]. ∎

LEMMA 4.3. *If n is a positive rational number, then*

$$\frac{\varphi(e^{-9\pi\sqrt{n}})}{\varphi(e^{-\pi\sqrt{n}})} = \frac{1}{3}\left(1 + \frac{\sqrt{2}G_{9n}}{G_n^3}\right).$$

Proof. See [10, p. 145], [4, p. 334, (5.7)], [6, (3.30)], or [9, Theorem 3.3]. ∎

Berndt [4, pp. 205–207, Theorem 3.1] expresses G_{9n} in terms of G_n. The Borwein brothers [10, p. 145, (4.7.12)] connect G_{81n} with G_{9n} and G_n, which we provide in our next lemma.

LEMMA 4.4. *If n is a positive rational number, then*

$$G_{81n}^3 = G_{9n}\frac{\sqrt{2}G_{9n} + G_n^3}{\sqrt{2}G_n^3 - G_{9n}}.$$

Proof. See the Borweins' book [10, p. 145, (4.7.12)]. ∎

To find their values, in our next lemma, we express p and u_3 in terms of G_n.

LEMMA 4.5. *If $q = \exp(-\pi\sqrt{n})$ for a positive rational number n, then*

(i) $$p = \frac{2\sqrt{2}G_n}{G_{9n}G_{81n}^6},$$

(ii) $$u_3 = \frac{\sqrt{2}G_{9n}}{G_{81n}^3}.$$

Proof. To prove (i), from Lemma 2.5(i) and (5) we have

$$p = \frac{8q^{7/3}\chi(q)}{\chi^6(q^9)\chi(q^3)} = \frac{2\sqrt{2}G_n}{G_{81n}^6 G_{9n}}.$$

Similarly, to obtain (ii), from Lemma 2.4(iii) and (5) we deduce

$$u_3 = \frac{2q\chi(q^3)}{\chi^3(q^9)} = \frac{\sqrt{2}G_{9n}}{G_{81n}^3}.$$ ∎

Next, we prove a monotonicity property of the values u_k defined in Theorem 2.1(0).

LEMMA 4.6. *For $0 < q < 1$,*

$$2 > u_1 > u_2 > u_3 > u_4 > 0.$$

Proof. From [18, Lemma 3.2], if n is a positive integer and $0 < q < 1$, then $\widetilde{u}_k$ defined in (8) is positive and strictly monotonically decreasing for $k = 0, \ldots, n/2$ when n is even, and for $k = 0, \ldots, (n-1)/2$ when n is odd. The lemma follows from this result with $n = 9$. ∎

We shall need the following inequality.

LEMMA 4.7. *For* $0 < q < 1$,

$$\frac{p}{u_1^3} + \frac{p}{u_2^3} + \frac{p}{u_4^3} > \frac{u_1^3}{p} + \frac{u_2^3}{p} + \frac{u_4^3}{p}.$$

Proof. By Corollary 2.9 and Lemma 2.8(iii), we have

$$\left(\frac{p}{u_1^3} + \frac{p}{u_2^3} + \frac{p}{u_4^3}\right) - \left(\frac{u_1^3}{p} + \frac{u_2^3}{p} + \frac{u_4^3}{p}\right) = \frac{u_3^4 - u_3^3 + 6u_3^2 - 8u_3 + 8}{u_3^3} - \frac{u_3^3 + 4}{u_3^2}$$
$$= \left(\frac{2 - u_3}{u_3}\right)^3 > 0,$$

where the inequality holds by Lemma 4.6. ∎

The next lemma provides the correct order of the roots of r in the case of $0 < q < 1$.

LEMMA 4.8. *For* $0 < q < 1$, *suppose that the roots of* r *are given in order* (α, β, γ) *such that they satisfy the following two conditions:*

(i) $$(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) > 0,$$

(ii) $$\frac{\beta}{\alpha} > \frac{\gamma}{\beta} > \frac{\alpha}{\gamma}.$$

Then

$$u_1 = \left(\frac{\beta\, p}{\alpha}\right)^{1/3}, \quad u_2 = \left(\frac{\gamma\, p}{\beta}\right)^{1/3}, \quad \textit{and} \quad u_4 = \left(\frac{\alpha\, p}{\gamma}\right)^{1/3}.$$

Condition (i) guarantees the correct order of α, β, and γ in Theorem 2.1(iv), so that Theorem 2.1(i) holds. Condition (ii) is needed to fulfill Theorem 2.1(0).

Proof. For each possible order of α, β, and γ, consider the set of possible values of u_1, u_2, and u_4 given in Theorem 2.1(iv). For $(\alpha, \beta, \gamma), (\beta, \gamma, \alpha)$, (γ, α, β) and for $(\gamma, \beta, \alpha), (\beta, \alpha, \gamma), (\alpha, \gamma, \beta)$, we have

(18)
$$\left\{\left(\frac{\beta p}{\alpha}\right)^{1/3}, \left(\frac{\gamma p}{\beta}\right)^{1/3}, \left(\frac{\alpha p}{\gamma}\right)^{1/3}\right\} \quad \text{and} \quad \left\{\left(\frac{\alpha p}{\beta}\right)^{1/3}, \left(\frac{\beta p}{\gamma}\right)^{1/3}, \left(\frac{\gamma p}{\alpha}\right)^{1/3}\right\},$$

respectively. Thus, it is enough to consider the order (α, β, γ) and its reverse (γ, β, α). By Lemma 4.6, p is positive. By condition (i), since $\alpha\beta\gamma = p^3 > 0$, we find that

$$\frac{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)}{\alpha\beta\gamma} > 0,$$

and therefore, after rearrangement,

(19) $$\frac{\alpha}{\beta} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha} > \frac{\beta}{\alpha} + \frac{\gamma}{\beta} + \frac{\alpha}{\gamma}.$$

In view of Theorem 2.1(iv) and Lemma 4.7, by Lemma 2.8(iii) and Corollary 2.9, we obtain

$$\frac{\beta}{\alpha}+\frac{\gamma}{\beta}+\frac{\alpha}{\gamma}=\frac{u_3^3+4}{u_3^2} \quad \text{and} \quad \frac{\alpha}{\beta}+\frac{\beta}{\gamma}+\frac{\gamma}{\alpha}=\frac{u_3^4-u_3^3+6u_3^2-8u_3+8}{u_3^3}, \tag{20}$$

respectively. Thus, we find that the first set in (18) is the correct choice.

Since $p>0$, the condition (ii) guarantees that $u_1>u_2>u_4$, which holds by Lemma 4.6. ∎

Because of Corollary 2.9, the condition in Lemma 4.8(i) is simpler than in [18, Lemma 3.7(i)], but one could replace it with one of the equations in (20). In some cases, it might be useful to rewrite Lemma 4.8(i) into the form of (19). As in [18, Lemma 3.6], one can show that for $0<q<1$, r has three distinct positive roots.

Now, in Theorems 4.9–4.13, we provide five examples for Theorem 2.1(i).

THEOREM 4.9. *We have*

$$\frac{\varphi(e^{-27\pi})}{\varphi(e^{-3\pi})}=\frac{1}{3}\Bigg\{1+\frac{(16(11\sqrt{3}-19))^{1/9}}{\sqrt{3}} \times\left(\left(1+\frac{\sqrt{3}}{2\cos\frac{4\pi}{9}}\right)^{1/3}+\left(1+\frac{\sqrt{3}}{2\cos\frac{2\pi}{9}}\right)^{1/3}+\left(1-\frac{\sqrt{3}}{2\cos\frac{\pi}{9}}\right)^{1/3}\right)\Bigg\}.$$

Note that Theorem 4.9 is the ninth degree analogue of Theorem 1.2.

First proof of Theorem 4.9. We use Theorem 2.1 with $q=\exp(-\pi/3)$. First, by using Lemma 4.1 with $n=729=27^2$, we rewrite Theorem 2.1(i) as

$$\frac{\varphi(e^{-27\pi})}{\varphi(e^{-3\pi})}=\frac{1}{3\sqrt{3}}(1+u_1+u_2+u_3+u_4). \tag{21}$$

From [4, p. 189], $G_1=1$, and from [14], [15, p. 24], [4, p. 189], with the use of Lemma 4.2,

$$G_{1/9}=G_9=\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3}=(2+\sqrt{3})^{1/6}. \tag{22}$$

By Lemma 4.5(i) with $n=1/9$, for Theorem 2.1(ii), we have

$$\begin{aligned} p&=\frac{2\sqrt{2}G_{1/9}}{G_1G_9^6}=\frac{2\sqrt{2}}{G_9^5}=\frac{4\cdot 2^{1/3}}{(1+\sqrt{3})^{5/3}}\\ &=(4(\sqrt{3}-1)^5)^{1/3}=(16(11\sqrt{3}-19))^{1/3}. \end{aligned} \tag{23}$$

By Theorem 2.1(iii) and Lemma 4.1 with $n=9$, or by Lemma 4.5(ii), we find that

$$u_3=\frac{\varphi(q)}{\varphi(q^9)}-1=\frac{\varphi(e^{-\pi/3})}{\varphi(e^{-3\pi})}-1=\sqrt{3}-1. \tag{24}$$

Now, we have all the coefficients of r given in Theorem 2.1(v). We have to determine the zeros of

$$r(\xi) = \xi^3 - 6(\sqrt{3}-1)\xi^2 + 12(\sqrt{3}-1)^3\xi - 4(\sqrt{3}-1)^5. \tag{25}$$

By a proper polynomial transformation, we find that

$$\frac{r(2(\sqrt{3}-1)(1-\sqrt{2}(\sqrt{3}-1)\xi))}{8\sqrt{2}(\sqrt{3}-1)^4(\sqrt{3}-2)} = 4\xi^3 - 3\xi + \frac{\sqrt{2-\sqrt{3}}}{2}. \tag{26}$$

Set $\theta = 7\pi/36, 17\pi/36$, and $31\pi/36$ in the power-reduction trigonometric identity ([12, p. 32], [2, pp. 347–348], [3, p. 39])

$$4\cos^3\theta = 3\cos\theta + \cos 3\theta. \tag{27}$$

By (25)–(27), with the values of $\cos(3\theta) = -(2-\sqrt{3})^{1/2}/2$, we find that the roots of r are

$$2(\sqrt{3}-1)\left(1-\sqrt{2}(\sqrt{3}-1)\cos\left(\frac{k\pi}{36}\right)\right), \qquad k = 7, 17, 31.$$

In the case of $k = 7$, by angle sum and difference identities [12, p. 29], observe that

$$\begin{aligned}\cos\left(\frac{7\pi}{36}\right) &= \cos\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{9}\right) - \sin\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{9}\right)\\ &= \frac{\sqrt{2+\sqrt{3}}}{2}\cos\left(\frac{\pi}{9}\right) - \frac{\sqrt{2-\sqrt{3}}}{2}\sin\left(\frac{\pi}{9}\right)\\ &= \frac{1}{\sqrt{2}(\sqrt{3}-1)}\cos\left(\frac{\pi}{9}\right) - \frac{2-\sqrt{3}}{\sqrt{2}(\sqrt{3}-1)}\sin\left(\frac{\pi}{9}\right).\end{aligned} \tag{28}$$

Taking similar steps in the other two cases as well, we find that

$$\alpha = 2(\sqrt{3}-1)\left(1-\cos\left(\frac{\pi}{9}\right) + (2-\sqrt{3})\sin\left(\frac{\pi}{9}\right)\right), \tag{29}$$

$$\beta = 2(\sqrt{3}-1)\left(1+(2-\sqrt{3})\cos\left(\frac{\pi}{9}\right) - \sin\left(\frac{\pi}{9}\right)\right), \tag{30}$$

$$\gamma = 2(\sqrt{3}-1)\left(1+(\sqrt{3}-1)\left(\cos\left(\frac{\pi}{9}\right) + \sin\left(\frac{\pi}{9}\right)\right)\right). \tag{31}$$

By rearrangement and product-to-sum identities [12, p. 29], with $\cos(5\pi/9) = -\cos(4\pi/9)$ and $\sin(5\pi/9) = \sin(4\pi/9)$, and with the values $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$, we find that

$$\begin{aligned}\frac{\beta}{\alpha} &= \frac{1+(2-\sqrt{3})\cos(\frac{\pi}{9}) - \sin(\frac{\pi}{9})}{1-\cos(\frac{\pi}{9}) + (2-\sqrt{3})\sin(\frac{\pi}{9})}\\ &= 1 + \sqrt{3}\frac{(\sqrt{3}-1)(\cos(\frac{\pi}{9}) - \sin(\frac{\pi}{9}))}{1-\cos(\frac{\pi}{9}) + (2-\sqrt{3})\sin(\frac{\pi}{9})}\end{aligned} \tag{32}$$

$$= 1 + \frac{\sqrt{3}}{2\cos(\frac{4\pi}{9})} \cdot \frac{(\sqrt{3}-1)(\frac{1}{2} - \cos(\frac{4\pi}{9}) + \frac{\sqrt{3}}{2} - \sin(\frac{4\pi}{9}))}{1 - \cos(\frac{\pi}{9}) + (2-\sqrt{3})\sin(\frac{\pi}{9})}$$
$$= 1 + \frac{\sqrt{3}}{2\cos(\frac{4\pi}{9})} \cdot \frac{1 - (\sqrt{3}-1)(\sin(\frac{4\pi}{9}) + \cos(\frac{4\pi}{9}))}{1 - (\cos(\frac{\pi}{9}) - (2-\sqrt{3})\sin(\frac{\pi}{9}))}.$$

The last fraction in (32) is equal to 1. To see this, take similar steps to those in (28), i.e.,

$$\frac{1}{\sqrt{2}}\sin\left(\frac{4\pi}{9}\right) + \frac{1}{\sqrt{2}}\cos\left(\frac{4\pi}{9}\right)$$
$$= \frac{1}{\sqrt{2}(\sqrt{3}-1)}\cos\left(\frac{\pi}{9}\right) - \frac{2-\sqrt{3}}{\sqrt{2}(\sqrt{3}-1)}\sin\left(\frac{\pi}{9}\right) = \cos\left(\frac{7\pi}{36}\right).$$

Thus, with Theorem 2.1(iv) in mind, for the other two cases in a similar way we obtain

$$\frac{\beta}{\alpha} = 1 + \frac{\sqrt{3}}{2\cos\frac{4\pi}{9}}, \quad \frac{\gamma}{\beta} = 1 + \frac{\sqrt{3}}{2\cos\frac{2\pi}{9}}, \quad \text{and} \quad \frac{\alpha}{\gamma} = 1 - \frac{\sqrt{3}}{2\cos\frac{\pi}{9}}. \tag{33}$$

Lastly, we show that our choice of the order (α, β, γ) is correct. By (29)–(31), and by the inequalities $\sqrt{3} > 3/2$ and $\cos(\pi/9) > \sin(\pi/9)$, we have $\alpha - \beta < 0$, $\beta - \gamma < 0$, and $\gamma - \alpha > 0$, therefore the condition in Lemma 4.8(i) holds. The condition in Lemma 4.8(ii) is satisfied by (33) together with the inequalities $0 < \cos(4\pi/9) < \cos(2\pi/9) < \cos(\pi/9) < 1$. After recalling that p and u_3 are obtained in (23) and (24), respectively, by Theorem 2.1(iv), we conclude that

$$u_1 = \left\{p\left(1 + \frac{\sqrt{3}}{2\cos\frac{4\pi}{9}}\right)\right\}^{1/3}, \; u_2 = \left\{p\left(1 + \frac{\sqrt{3}}{2\cos\frac{2\pi}{9}}\right)\right\}^{1/3}, \tag{34}$$
$$u_4 = \left\{p\left(1 - \frac{\sqrt{3}}{2\cos\frac{\pi}{9}}\right)\right\}^{1/3}.$$

We complete the proof by rearranging (21). ∎

Chapter 20 of Ramanujan's second notebook [16, pp. 241–251], [2, pp. 325–453] contains modular equations of higher and composite degrees. In the first entry of this chapter [16, p. 241], among theta function identities, there are the following two trigonometric identities:

$$\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = \cos\left(\frac{\pi}{9}\right), \tag{35}$$

$$\frac{1}{\cos\frac{4\pi}{9}} + \frac{1}{\cos\frac{2\pi}{9}} = \frac{1}{\cos\frac{\pi}{9}} + 6. \tag{36}$$

Ramanujan gave them in degrees instead of radians. In Berndt's book [2, pp. 345–349], these are the third and fourth identities of Entry 1(iii), and

are proved by the identity (27). Berndt mentions that this proof does not relate these identities to the rest of the material in Entry 1, but Ramanujan clearly had a connection in mind.

From the first proof of Theorem 4.9, we see a connection with theta functions. By using Lemma 2.8(iii) with $q = \exp(-\pi/3)$, with the values of p and u_3 given in (23) and (24), respectively, and with the values of u_1, u_2, and u_4 obtained in (34), we find (36). Similarly, by trigonometric transformations, we find that (34) can be written in the form

$$u_1 = \left\{ p\left(1 + \sqrt{3} + 2\sqrt{3}\cos\left(\frac{\pi}{9}\right)\right)\right\}^{1/3},$$
$$u_2 = \left\{ p\left(1 + \sqrt{3} - 2\sqrt{3}\cos\left(\frac{4\pi}{9}\right)\right)\right\}^{1/3},$$
$$u_4 = \left\{ p\left(1 + \sqrt{3} - 2\sqrt{3}\cos\left(\frac{2\pi}{9}\right)\right)\right\}^{1/3}.$$

Therefore, using these forms, we can deduce (35) in the same way.

Our second proof relies on the entry of Ramanujan from his second notebook, discussed in Section 3. In the following, the notations α, β, γ, and r have different meanings than those introduced in Theorem 2.1 and used elsewhere.

Second proof of Theorem 4.9. By Lemma 4.3, for $n = 9$, we find that

$$\frac{\varphi(e^{-27\pi})}{\varphi(e^{-3\pi})} = \frac{1}{3}\left(1 + \frac{\sqrt{2}G_{81}}{G_9^3}\right), \tag{37}$$

with the value G_9 from (22), and by Lemma 4.4, for $n = 1$, with $G_1 = 1$ [4, p. 189],

$$G_{81}^3 = G_9\frac{\sqrt{2}G_9 + 1}{\sqrt{2} - G_9}. \tag{38}$$

Note by (22) that

$$(16(11\sqrt{3} - 19))^{1/9} = \sqrt{2}G_9^{-5/3} \quad \text{and} \quad \sqrt{3} - 1 = \sqrt{2}G_9^{-3}. \tag{39}$$

Let α, β, and γ denote the expressions under the cube roots in the statement, i.e.,

$$\alpha := 1 + \frac{\sqrt{3}}{2\cos\frac{2\pi}{9}}, \quad \beta := 1 + \frac{\sqrt{3}}{2\cos\frac{4\pi}{9}}, \quad \gamma := 1 - \frac{\sqrt{3}}{2\cos\frac{\pi}{9}}. \tag{40}$$

Then, by using (37), (39), and (40), we can restate the theorem as

$$\frac{\sqrt{3}G_{81}}{G_9^{4/3}} = \alpha^{1/3} + \beta^{1/3} + \gamma^{1/3}. \tag{41}$$

By setting $\theta = 2\pi/9, 4\pi/9$, and $8\pi/9$ in the power-reduction formula (27), we find that $2\cos(2\pi/9), 2\cos(4\pi/9)$, and $2\cos(8\pi/9) = -2\cos(\pi/9)$ are the roots of

$$r(x) := x^3 - 3x + 1.$$

Thus, by a proper polynomial transformation, α, β, and γ are the roots of the polynomial

$$\widetilde{r}(x) := (x-1)^3 r\left(\frac{\sqrt{3}}{x-1}\right) = x^3 - 3(1+\sqrt{3})x^2 + 3(1+2\sqrt{3})x - 1. \tag{42}$$

By Entry 3.1, we know that if α, β, and γ are the roots of

$$x^3 - ax^2 + bx - 1 = 0,$$

then

$$\alpha^{1/3} + \beta^{1/3} + \gamma^{1/3} = \left(a + 6 + 3\{A+B\}^{1/3} + 3\{A-B\}^{1/3}\right)^{1/3}$$

with

$$A = \frac{ab+9}{2} + 3a + 3b \quad \text{and} \quad B = \left(\frac{(ab+9)^2}{4} - a^3 - b^3 - 27\right)^{1/2}.$$

We apply this theorem with $a = 3(1+\sqrt{3})$ and $b = 3(1+2\sqrt{3})$. Since

$$a + 6 = 3\sqrt{3}(1+\sqrt{3}), \quad A + B = 54(1+\sqrt{3}), \quad A - B = \tfrac{27}{2}(1+\sqrt{3})^2,$$

and knowing that α, β, and γ in (40) are the roots of $\widetilde{r}$ in (42), with (22) in mind, we find that

$$\begin{aligned} \alpha^{1/3} + \beta^{1/3} + \gamma^{1/3} &= (3\sqrt{3}(1+\sqrt{3}) + 9\sqrt[3]{2}(1+\sqrt{3})^{1/3} + \frac{9}{\sqrt[3]{2}}(1+\sqrt{3})^{2/3})^{1/3} \\ &= (3\sqrt{6}G_9^3 + 9\sqrt{2}G_9 + 9G_9^2)^{1/3}. \end{aligned} \tag{43}$$

Now, combining (41) and (43), we find that we are required to prove that

$$\frac{3\sqrt{3}G_{81}^3}{G_9^4} = 3\sqrt{6}G_9^3 + 9\sqrt{2}G_9 + 9G_9^2. \tag{44}$$

Substituting (38) into (44) yields

$$\frac{\sqrt{2}G_9 + 1}{\sqrt{2} - G_9} = \sqrt{2}G_9^6 + \sqrt{6}G_9^4 + \sqrt{3}G_9^5.$$

After rearrangement, we may recast (44) in the form

$$\sqrt{2}G_9^7 - (2-\sqrt{3})G_9^6 - 2\sqrt{3}G_9^4 + \sqrt{2}G_9 + 1 = 0. \tag{45}$$

By factoring (45), we see that it remains to show that

$$\frac{1}{4\sqrt{2}}\big(2G_9^3-\sqrt{2}G_9^2+(\sqrt{3}-1)G_9-\sqrt{2}\big) \times\big(2G_9^3-\sqrt{6}-\sqrt{2}\big)\big(2G_9+\sqrt{6}-\sqrt{2}\big)=0. \tag{46}$$

From (22), we know that $2G_9^3-\sqrt{6}-\sqrt{2}=0$, and thus (46) holds to complete the proof. ∎

THEOREM 4.10. *We have*

$$\frac{\varphi(e^{-9\pi\sqrt{3}})}{\varphi(e^{-3\pi\sqrt{3}})}=3^{-5/4}\bigg(1+\bigg(p\frac{1-a\sin\frac{\pi}{9}}{1-a\sin\frac{2\pi}{9}}\bigg)^{1/3}+\bigg(p\frac{1+a\sin\frac{4\pi}{9}}{1-a\sin\frac{\pi}{9}}\bigg)^{1/3} +(2^{2/3}-2^{1/3})+\bigg(p\frac{1-a\sin\frac{2\pi}{9}}{1+a\sin\frac{4\pi}{9}}\bigg)^{1/3}\bigg),$$

where

$$p=2(2^{1/3}-1)^2,$$
$$a=2(2^{2/3}-1)^{1/2}.$$

Proof. We apply Theorem 2.1 with $q=\exp(-\pi/\sqrt{3})$. By using Lemma 4.1 with $n=(9\sqrt{3})^2$, we rewrite Theorem 2.1(i) as

$$\frac{\varphi(e^{-9\pi\sqrt{3}})}{\varphi(e^{-3\pi\sqrt{3}})}=3^{-5/4}(1+u_1+u_2+u_3+u_4). \tag{47}$$

From [4, pp. 189–190], by Lemma 4.2,

$$G_{1/3}=G_3=2^{1/12}\quad\text{and}\quad G_{27}=2^{1/12}(2^{1/3}-1)^{-1/3}. \tag{48}$$

By Lemma 4.5(i),(ii) with $n=1/3$, for Theorem 2.1(ii),(iii), we find

$$p=\frac{2\sqrt{2}}{G_{27}^6}=2(2^{1/3}-1)^2\quad\text{and}\quad u_3=\frac{\sqrt{2}G_3}{G_{27}^3}=2^{1/3}(2^{1/3}-1). \tag{49}$$

We have to determine the zeros of r in Theorem 2.1(v), i.e., the zeros of

$$r(\xi)=\xi^3-6(2^{1/3}-1)\xi^2+12(2^{1/3}-1)^2(2-2^{2/3})\xi-8(2^{1/3}-1)^6=0. \tag{50}$$

By a polynomial transformation, we find that

$$-\frac{\sqrt{2}}{a^3p^{3/2}}r((2p)^{1/2}(1-a\xi))=4\xi^3-3\xi+\frac{\sqrt{3}}{2}. \tag{51}$$

Set $\theta=\pi/9, 2\pi/9$, and $-4\pi/9$ in the power-reduction trigonometric identity [12, p. 32]

$$4\sin^3\theta=3\sin\theta-\sin 3\theta. \tag{52}$$

By (50)–(52), with the values of $\sin(3\theta) = \sqrt{3}/2$, since sine is odd, the roots of r are

$$\alpha = (2p)^{1/2}\left(1 - a\sin\left(\frac{2\pi}{9}\right)\right), \quad \beta = (2p)^{1/2}\left(1 - a\sin\left(\frac{\pi}{9}\right)\right),$$
$$\gamma = (2p)^{1/2}\left(1 + a\sin\left(\frac{4\pi}{9}\right)\right).$$

Mathematica confirms that both conditions in Lemma 4.8(i),(ii) hold. This can be done directly, or by using numerical evaluations to exclude all possible orders of α, β, and γ, which do not meet the conditions. Thus (α, β, γ) is the correct order of the roots. With (47) in mind, we recall that p and u_3 are given in (49), and by Theorem 2.1(iv), we find that

$$u_1 = \left(p\frac{1 - a\sin\frac{\pi}{9}}{1 - a\sin\frac{2\pi}{9}}\right)^{1/3}, \quad u_2 = \left(p\frac{1 + a\sin\frac{4\pi}{9}}{1 - a\sin\frac{\pi}{9}}\right)^{1/3},$$
$$u_4 = \left(p\frac{1 - a\sin\frac{2\pi}{9}}{1 + a\sin\frac{4\pi}{9}}\right)^{1/3},$$

which completes the proof. ∎

THEOREM 4.11. *We have*

$$\frac{\varphi(e^{-27\pi\sqrt{3}})}{\varphi(e^{-\pi\sqrt{3}})} = 3^{-7/4}\left(1 + \left(p\frac{1 + a\cos\frac{4\pi}{9}}{1 - a\cos\frac{\pi}{9}}\right)^{1/3} + \left(p\frac{1 + a\cos\frac{2\pi}{9}}{1 + a\cos\frac{4\pi}{9}}\right)^{1/3} + 2^{1/3} + \left(p\frac{1 - a\cos\frac{\pi}{9}}{1 + a\cos\frac{2\pi}{9}}\right)^{1/3}\right),$$

where

$$p = 2(1 + 2^{1/3} + 2^{2/3})^{1/3},$$
$$a = 16p^{-3} = 2(2^{1/3} - 1).$$

Proof. We use Theorem 2.1 with $q = \exp(-\pi/\sqrt{27})$. By applying Lemma 4.1 for $n = (27\sqrt{3})^2$, we transform Theorem 2.1(i) into

$$\frac{\varphi(e^{-27\pi\sqrt{3}})}{\varphi(e^{-\pi\sqrt{3}})} = 3^{-7/4}(1 + u_1 + u_2 + u_3 + u_4). \tag{53}$$

Utilize the values of G_3 and G_{27} in (48), with Lemma 4.2. By Lemma 4.5(i),(ii) with $n = 1/27$, for Theorem 2.1(ii),(iii), we obtain

$$p = \frac{2\sqrt{2}G_{27}}{G_3^7} = \frac{2}{(2^{1/3} - 1)^{1/3}} = 2(1 + 2^{1/3} + 2^{2/3})^{1/3}, \tag{54}$$
$$u_3 = \frac{\sqrt{2}}{G_3^2} = 2^{1/3}.$$

We have to determine the roots of r in Theorem 2.1(v), i.e., the roots of

$$r(\xi)=\xi^3-2(1+2^{1/3})^2\xi^2+4(1+2^{1/3})(2+2^{2/3})\xi-8(1+2^{1/3}+2^{2/3})=0. \tag{55}$$

By a polynomial transformation, we arrive at

$$\frac{3}{8a}r\left(\frac{2}{3}(1+2^{1/3})^2(1+a\xi)\right)=4\xi^3-3\xi+\frac{1}{2}. \tag{56}$$

By (55), (56), and by setting $\theta = 2\pi/9, 4\pi/9$, and $8\pi/9$ in the power-reduction identity (27), with the values of $\cos(3\theta) = -1/2$, we find that the roots of r are

$$\alpha=b\left(1-a\cos\left(\frac{\pi}{9}\right)\right),\ \beta=b\left(1+a\cos\left(\frac{4\pi}{9}\right)\right),\ \gamma=b\left(1+a\cos\left(\frac{2\pi}{9}\right)\right),$$

where $b=2(1+2^{1/3})^2/3$, and we used the fact that $\cos(8\pi/9)=-\cos(\pi/9)$.

As at the end of the proof of Theorem 4.10, we verify with *Mathematica* that the conditions in Lemma 4.8(i),(ii) hold, thus (α,β,γ) is the correct order of the roots. In view of (53), we recall that p and u_3 are given in (54), and by Theorem 2.1(iv) we conclude that

$$u_1=\left(p\frac{1+a\cos\frac{4\pi}{9}}{1-a\cos\frac{\pi}{9}}\right)^{1/3},\quad u_2=\left(p\frac{1+a\cos\frac{2\pi}{9}}{1+a\cos\frac{4\pi}{9}}\right)^{1/3},$$
$$u_4=\left(p\frac{1-a\cos\frac{\pi}{9}}{1+a\cos\frac{2\pi}{9}}\right)^{1/3}. \ \blacksquare$$

By combining the results of Theorems 4.10 and 4.11 with Lemma 2.8(iii), after simplification, we obtain the trigonometric identities

$$\frac{1-a\sin\frac{\pi}{9}}{1-a\sin\frac{2\pi}{9}}+\frac{1+a\sin\frac{4\pi}{9}}{1-a\sin\frac{\pi}{9}}+\frac{1-a\sin\frac{2\pi}{9}}{1+a\sin\frac{4\pi}{9}}=(1+2^{1/3})^3(2+2^{1/3}),$$
$$a=2(2^{2/3}-1)^{1/2},$$

and

$$\frac{1+a\cos\frac{4\pi}{9}}{1-a\cos\frac{\pi}{9}}+\frac{1+a\cos\frac{2\pi}{9}}{1+a\cos\frac{4\pi}{9}}+\frac{1-a\cos\frac{\pi}{9}}{1+a\cos\frac{2\pi}{9}}=3\cdot 2^{1/3},\qquad a=2(2^{1/3}-1).$$

On the one hand, these identities are analogous to Ramanujan's identities (36) and (35), and on the other hand, to an identity for the septic case given in [18, Lemma 3.8], [7, Example 3.3].

Another family of related identities can be obtained by Corollary 2.9. With $q=\exp(-\pi/3)$, with the values of p and u_3 from (23) and (24), respectively, and with the values of u_1, u_2, and u_4 in (34), after rearrangement, we find that

$$\frac{1}{2\cos(\frac{2\pi}{9})+\sqrt{3}}+\frac{1}{2\cos(\frac{4\pi}{9})+\sqrt{3}}=\frac{1}{2\cos(\frac{\pi}{9})-\sqrt{3}}-6,$$

which is similar to Ramanujan's identity (36). By combining the findings of Theorems 4.10 and 4.11 with Corollary 2.9, and simplifying, we arrive at

$$\frac{1-a\sin\frac{2\pi}{9}}{1-a\sin\frac{\pi}{9}}+\frac{1-a\sin\frac{\pi}{9}}{1+a\sin\frac{4\pi}{9}}+\frac{1+a\sin\frac{4\pi}{9}}{1-a\sin\frac{2\pi}{9}}=\frac{3}{(2^{1/3}-1)^3},$$
$$a=2(2^{2/3}-1)^{1/2},$$

and

$$\frac{1-a\cos\frac{\pi}{9}}{1+a\cos\frac{4\pi}{9}}+\frac{1+a\cos\frac{4\pi}{9}}{1+a\cos\frac{2\pi}{9}}+\frac{1+a\cos\frac{2\pi}{9}}{1-a\cos\frac{\pi}{9}}=\frac{9}{2^{1/3}+1},\qquad a=2(2^{1/3}-1).$$

Lastly, a third family of trigonometric identities can be found by comparing the values of Theorems 4.9, 4.10, and 4.11 with the values in Theorem 1.3(i), (ii), and (iii), respectively. After simplification, we obtain the identities

$$\left(1+\frac{\sqrt{3}}{2\cos\frac{4\pi}{9}}\right)^{1/3}+\left(1+\frac{\sqrt{3}}{2\cos\frac{2\pi}{9}}\right)^{1/3}+\left(1-\frac{\sqrt{3}}{2\cos\frac{\pi}{9}}\right)^{1/3}$$
$$=\sqrt{3}(7-4\sqrt{3})^{1/9}\left(\frac{(2(\sqrt{3}+1))^{1/3}+1}{(2(\sqrt{3}-1))^{1/3}-1}\right)^{1/3},$$

$$\left(\frac{1-a\sin\frac{\pi}{9}}{1-a\sin\frac{2\pi}{9}}\right)^{1/3}+\left(\frac{1+a\sin\frac{4\pi}{9}}{1-a\sin\frac{\pi}{9}}\right)^{1/3}+\left(\frac{1-a\sin\frac{2\pi}{9}}{1+a\sin\frac{4\pi}{9}}\right)^{1/3}=(2^{1/3}+1)^2,$$
$$a=2(2^{2/3}-1)^{1/2},$$

and

$$\left(\frac{1+a\cos\frac{4\pi}{9}}{1-a\cos\frac{\pi}{9}}\right)^{1/3}+\left(\frac{1+a\cos\frac{2\pi}{9}}{1+a\cos\frac{4\pi}{9}}\right)^{1/3}+\left(\frac{1-a\cos\frac{\pi}{9}}{1+a\cos\frac{2\pi}{9}}\right)^{1/3}$$
$$=\frac{3^{1/3}(2^{1/3}-1)^{4/9}(2^{1/3}+(2^{1/3}-1)^{1/3})}{(3-2\cdot3^{1/3})^{1/3}},\quad a=2(2^{1/3}-1),$$

respectively. Further investigation of these identities may be of interest.

The forms of our last two examples are more complicated.

THEOREM 4.12. *We have*

$$\frac{\varphi(e^{-81\pi})}{\varphi(e^{-\pi})}=\frac{1}{9}(1+u_1+u_2+u_3+u_4),\tag{57}$$

where

$$u_1=\left(\frac{\beta p}{\alpha}\right)^{1/3},\quad u_2=\left(\frac{\gamma p}{\beta}\right)^{1/3},\quad u_3=(2(\sqrt{3}+1))^{1/3},$$
$$u_4=\left(\frac{\alpha p}{\gamma}\right)^{1/3},$$

with

$$\alpha = \frac{2}{3}\bigg(a - b\cos\Big(\frac{\pi}{9}\Big) - c\cos\Big(\frac{2\pi}{9}\Big)\bigg),$$
$$\beta = \frac{2}{3}\bigg(a + d\cos\Big(\frac{\pi}{9}\Big) - b\cos\Big(\frac{2\pi}{9}\Big)\bigg),$$
$$\gamma = \frac{2}{3}\bigg(a - c\cos\Big(\frac{\pi}{9}\Big) + d\cos\Big(\frac{2\pi}{9}\Big)\bigg),$$

and

$$p = 2\bigg(2(\sqrt{3}-1)\bigg(\frac{(2(\sqrt{3}+1))^{1/3}+1}{(2(\sqrt{3}-1))^{1/3}-1}\bigg)\bigg)^{1/3},$$
$$a = 2(2(\sqrt{3}+1))^{1/3} + (2(\sqrt{3}+1))^{2/3} + \sqrt{3} + 1,$$
$$b = 2\big((2(\sqrt{3}+1))^{1/3} - (2(\sqrt{3}-1))^{1/3}\big),$$
$$c = 2\big((2(\sqrt{3}-1))^{1/3} - (2(\sqrt{3}+1))^{1/3} + \sqrt{3} - 1\big),$$
$$d = 2(\sqrt{3}-1).$$

Proof. We appeal to Theorem 2.1 with $q = \exp(-\pi/9)$. By using Lemma 4.1 with $n = 81^2$, we rewrite Theorem 2.1(i) as in (57). Recall the value $G_1 = 1$ and the value of G_9 from (22). The value of G_{81} can be obtained by (38), or from [4, p. 193], by Lemma 4.2,

$$G_{1/81} = G_{81} = \bigg(\frac{(2(\sqrt{3}+1))^{1/3}+1}{(2(\sqrt{3}-1))^{1/3}-1}\bigg)^{1/3}.$$

By Lemma 4.5(i),(ii) with $n = 1/81$, and by Lemma 4.2, for Theorem 2.1(ii),(iii), we find

$$p = \frac{2\sqrt{2}G_{81}}{G_9} \quad \text{and} \quad u_3 = \sqrt{2}G_9.$$

We verify with *Mathematica* that α, β, and γ are the roots of r in Theorem 2.1(v), and that the conditions in Lemma 4.8(i),(ii) hold; therefore (α, β, γ) is the correct order of the roots, and by Theorem 2.1(iv), the proof is complete. ∎

THEOREM 4.13. *We have*

$$\frac{\varphi(e^{-81\pi\sqrt{3}})}{\varphi(e^{-\pi\sqrt{3}})} = \frac{1}{9}(1 + u_1 + u_2 + u_3 + u_4), \tag{58}$$

where

$$u_1 = \left(\frac{\beta p}{\alpha}\right)^{1/3}, \qquad u_2 = \left(\frac{\gamma p}{\beta}\right)^{1/3},$$
$$u_3 = (2(1+2^{1/3}+2^{2/3}))^{1/3}, \quad u_4 = \left(\frac{\alpha p}{\gamma}\right)^{1/3},$$

with

$$\alpha = \frac{1}{9}\left(a - b\cos\left(\frac{\pi}{9}\right) - c\cos\left(\frac{2\pi}{9}\right)\right),$$
$$\beta = \frac{1}{9}\left(a + d\cos\left(\frac{\pi}{9}\right) - b\cos\left(\frac{2\pi}{9}\right)\right),$$
$$\gamma = \frac{1}{9}\left(a - c\cos\left(\frac{\pi}{9}\right) + d\cos\left(\frac{2\pi}{9}\right)\right),$$

and

$$p = \frac{2(2^{1/3}+2^{2/3}+3^{1/3})(2^{1/3}-1)^{1/3}}{(9-2\cdot 3^{4/3})^{1/3}},$$
$$a = 4\cdot 2^{1/3}3^{2/3}(1+2^{1/3}) + 2\cdot 3^{1/3}(2^{1/3}+2^{2/3})^2 + 6(1+2^{1/3}+2^{2/3}),$$
$$b = 4\big(1+3^{1/3}(3^{1/3}+1) - 2\cdot 2^{1/3}(3^{1/3}+2^{1/3}-1)\big),$$
$$c = 4\big(1-2^{1/3}(3^{1/3}-1)(3^{1/3}+2^{1/3}-1)\big),$$
$$d = 4\big((2-2^{1/3})(1+2^{1/3}) - 3^{1/3}(2^{1/3}-1)(1+2^{1/3}+3^{1/3})\big).$$

Proof. We invoke Theorem 2.1 with $q = \exp(-\pi/\sqrt{243})$. Applying Lemma 4.1 twice with $n = (81\sqrt{3})^2$ and $n = 3$ for Theorem 2.1(i), we find the representation in (58). With Lemma 4.2 in mind, we utilize the values of G_3 and G_{27} from (48). From Watson's paper [22, pp. 100–105], with Lemma 4.2, we also know that

$$G_{1/243} = G_{243} = \frac{2^{1/12}(2^{1/3}+2^{2/3}+3^{1/3})}{(9-2\cdot 3^{4/3})^{1/3}}.$$

By Lemma 4.5(i),(ii) with $n = 1/243$, for Theorem 2.1(ii),(iii), we obtain the values

$$p = \frac{2\sqrt{2}G_{243}}{G_{27}G_3^6} \quad \text{and} \quad u_3 = \frac{\sqrt{2}G_{27}}{G_3^3}.$$

Mathematica confirms that α, β, and γ are the roots of r in Theorem 2.1(v), and that the conditions in Lemma 4.8(i),(ii) hold. Thus, by Theorem 2.1(iv), we finish the proof. ∎

From [9], we know the value of G_{729}, and this allows us to evaluate $\varphi(e^{-243\pi})$, but the solutions of the corresponding cubic equation seem much more complicated.

In the articles of Son [19] and the second author [18], Ramanujan's septic theta function identity was examined. Analogous cubic and quintic identities are the topic of the paper by Berndt and the second author [9]. In this paper, we obtained a ninth degree theta function identity, and calculated five examples of this kind. We believe that these results are special cases of a grand theory that Ramanujan envisioned at the end of Section 12 of Chapter 20 in his second notebook [16, p. 247], [2, p. 400].

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References

[1] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part II*, Springer, New York, 2009.
[2] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer, New York, 1991.
[3] B. C. Berndt, *Ramanujan's Notebooks, Part IV*, Springer, New York, 1994.
[4] B. C. Berndt, *Ramanujan's Notebooks, Part V*, Springer, New York, 1998.
[5] B. C. Berndt and S. Bhargava, *Ramanujan—for lowbrows*, Amer. Math. Monthly 100 (1993), 644–656.
[6] B. C. Berndt and H. H. Chan, *Ramanujan's explicit values for the classical theta function*, Mathematika 42 (1995), 278–294.
[7] B. C. Berndt, S. Kim, and A. Zaharescu, *Exact evaluations and reciprocity theorems for finite trigonometric sums*, Res. Math. Sci. 10 (2023), art. 40, 70 pp.
[8] B. C. Berndt and Ö. Rebák, *Explicit values for Ramanujan's theta function* $\varphi(q)$, Hardy–Ramanujan J. 44 (2021), 41–50.
[9] B. C. Berndt and Ö. Rebák, *Cubic and quintic analogues of Ramanujan's septic theta function identity*, Ramanujan J. 66 (2025), art. 66, 40 pp.
[10] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, Wiley, New York, 1987.
[11] J. M. Borwein and I. J. Zucker, *Fast evaluation of the gamma function for small rational fractions using complete elliptic integrals of the first kind*, IMA J. Numer. Anal. 12 (1992), 519–526.
[12] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., Academic Press, New York, 2007.
[13] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Sumtibus fratrum Borntræger, Regiomonti, 1829.
[14] S. Ramanujan, *Modular equations and approximations to* π, Quart. J. Math. 45 (1914), 350–372.
[15] S. Ramanujan, *Collected Papers of Srinivasa Ramanujan*, Cambridge Univ. Press, Cambridge, 1927.
[16] S. Ramanujan, *Notebooks of Srinivasa Ramanujan, Volume II*, Tata Institute of Fundamental Research, Bombay, 1957.

[17] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
[18] Ö. Rebák, *The three missing terms in Ramanujan's septic theta function identity*, Ramanujan J. 60 (2023), 885–911.
[19] S. H. Son, *Septic theta function identities in Ramanujan's lost notebook*, Acta Arith. 98 (2001), 361–374.
[20] S. H. Son, *Circular summations of theta functions in Ramanujan's lost notebook*, Ramanujan J. 8 (2004), 235–272.
[21] S. H. Son, *Ramanujan's symmetric theta functions in his lost notebook*, in: Special Functions and Orthogonal Polynomials, Contemp. Math. 471, Amer. Math. Soc., Providence, RI, 2008, 187–202.
[22] G. N. Watson, *Singular moduli (3)*, Proc. London Math. Soc. 40 (1936), 83–142.
[23] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge Univ. Press, Cambridge, 1950.
[24] J. Yi, *The Construction and Applications of Modular Equations*, Ph.D. thesis, Univ. of Illinois at Urbana-Champaign, 2001.
[25] I. J. Zucker, *The evaluation in terms of Γ-functions of the periods of elliptic curves admitting complex multiplication*, Math. Proc. Cambridge Philos. Soc. 82 (1977), 111–118.

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