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The Yang–Hua theorems in Several Complex Variables

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Abstract. In this paper, we investigate meromorphic solutions in $\mathbb{C}^m$ of the nonlinear differential equation

$$f^n \partial_u(f) g^n \partial_u(g) = 1,$$

where $\partial_u(f) = \sum_{j=1}^m u_j \partial_j(f)$ and $\sum_{j=1}^m u_j \neq 0$. Our results extend those of C. C. Yang and X. H. Hua [Ann. Acad. Sci. Fenn. Math. 22 (1997), 395–406] to the framework of several complex variables. Moreover, we establish new uniqueness theorems that further generalize their conclusions to higher dimensions. As applications, explicit solutions of certain nonlinear partial differential equations in several variables are derived.

1. Introduction and historical background. Solving nonlinear differential equations remains one of the most challenging and stimulating problems in mathematical analysis. Among the earliest contributions to this field, Yang and Hua [37] were the first to investigate the nonlinear differential equation

$$f^n f^{(1)} g^n g^{(1)} = 1, \tag{1.1}$$

and to determine possible functional forms for the solutions f and g. Their pioneering work laid a foundation that has inspired extensive developments in the study of meromorphic functions and nonlinear differential equations.

Historical results. In 1997, Yang and Hua [37] presented a complete characterization of the nonconstant entire and meromorphic solutions of equation (1.1). Their main results can be summarized as follows:

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THEOREM A ([37], Theorem 3). *Let f and g be two nonconstant entire functions in $\mathbb{C}$, and let n be a positive integer. If*

$$f^n f^{(1)} g^n g^{(1)} = 1,$$

then

$$f(z) = c_1 \exp(-cz) \quad \text{and} \quad g(z) = c_2 \exp(cz),$$

where c, c_1, and c_2 are nonzero constants satisfying $c^2(c_1c_2)^{n+1} = -1$.

THEOREM B ([37], Theorem 2). *Let f and g be two nonconstant meromorphic functions in $\mathbb{C}$, and let $n \geq 6$ be an integer. If $f^n f^{(1)} g^n g^{(1)} = 1$, then*

$$f(z) = c_1 \exp(-cz) \quad \text{and} \quad g(z) = c_2 \exp(cz),$$

where c, c_1, and c_2 are nonzero constants satisfying $c^2(c_1c_2)^{n+1} = -1$.

Uniqueness of meromorphic functions. These results have significantly influenced the study of uniqueness problems for meromorphic functions, particularly in the context of nonlinear differential polynomials sharing a common value. In the same work, Yang and Hua also explored the uniqueness of meromorphic functions when two linear differential polynomials share a common a-point, for $a \in \mathbb{C} \setminus \{0\}$. The following theorem captures their key result:

THEOREM C ([37], Theorem 1). *Let f and g be two nonconstant meromorphic functions in $\mathbb{C}$ and $n \geq 11$ be an integer. If $f^n f^{(1)}$ and $g^n g^{(1)}$ share the value $a \in \mathbb{C}$ counting multiplicities (CM), then either $f = dg$, where $d^{n+1} = 1$ or*

$$g(z) = c_1 \exp(cz) \quad \text{and} \quad f(z) = c_2 \exp(-cz),$$

where c, c_1, and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -a^2$.

REMARK A. If f and g are entire, it suffices to assume $n \geq 7$ in Theorem C.

Modern context and multi-variable generalizations. Nevanlinna theory, originally formulated for meromorphic functions of a single complex variable, has been extended to several complex variables to study the value distribution of meromorphic maps from $\mathbb{C}^n$ into complex projective spaces or complex manifolds. It provides profound insights into growth and value distribution of meromorphic mappings; uniqueness and rigidity phenomena for solutions to linear and nonlinear partial differential equations (PDEs); and compactness and convergence criteria in families of meromorphic maps. Its applications span complex analysis, partial differential equations, complex geometry, and mathematical physics, offering both powerful techniques and elegant results. References [1]–[12], [15]–[29], [31], [33]–[36] provide a strong

foundation for understanding the current state of research in the Nevanlinna value distribution theory in several complex variables.

We define $\mathbb{Z}_+ = \mathbb{Z}[0,+\infty) = \{n \in \mathbb{Z} : 0 \leq n < +\infty\}$ and $\mathbb{Z}^+ = \mathbb{Z}(0,+\infty) = \{n \in \mathbb{Z} : 0 < n < +\infty\}$. For any set S, $\#(S)$ denotes the *cardinality* (number of elements) of S.

On $\mathbb{C}^m$, we define the following operators

$$\partial_{z_i} = \frac{\partial}{\partial z_i}, \partial^2_{z_j z_i} = \frac{\partial^2}{\partial z_j z_i}, \ldots, \partial^{l_i}_{z_i} = \frac{\partial^{l_i}}{\partial z_i^{l_i}} \quad \text{and} \quad \partial^I = \frac{\partial^{|I|}}{\partial z_1^{i_1} \cdots \partial z_m^{i_m}}$$

where $l_i \in \mathbb{Z}^+$ $(i = 1, \ldots, m)$ and $I = (i_1, \ldots, i_m) \in \mathbb{Z}_+^m$ is a multi-index such that $|I| = \sum_{j=1}^m i_j$.

Let $S_m = \{(u_1, \ldots, u_m) \in \mathbb{C}^m : u_1 + \cdots + u_m \neq 0\}$. For $u \in S_m$, we define

$$\partial_u(f) = \sum_{j=1}^m u_j \frac{\partial f}{\partial z_j} \not\equiv 0.$$

The k-th order derivative $\partial_u^k(f)$ of a function f is defined inductively by

$$\partial_u^k(f) = \partial_u(\partial_u^{k-1}(f)).$$

Problem statement. In this paper, we focus on the following nonlinear partial differential equation:

$$f^n \partial_u(f)\, g^n \partial_u(g) = 1, \tag{1.2}$$

where f and g are nonconstant meromorphic functions in $\mathbb{C}^m$ and n is a positive integer.

Furthermore, we investigate the uniqueness problem for meromorphic functions in $\mathbb{C}^m$ with respect to nonlinear partial differential polynomials generated by them that share a common value.

Brief discussion of the theory of several complex variables. We begin by recalling some basic notions and definitions from the theory of several complex variables (see, for example, [14, 32]). The purpose of this section is to introduce the notation and fundamental concepts that will be used throughout the paper, particularly those relevant to Nevanlinna theory in higher dimensions. For clarity and completeness, we also include a concise conceptual breakdown to help illuminate the central ideas and their interrelations within this framework.

On $\mathbb{C}^m$, the exterior derivative d splits as $d = \partial + \bar{\partial}$ and twists to $d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$; clearly $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$. For $z \in \mathbb{C}^m$, we define $\tau_m(z) = ||z||^2$. The standard Kähler metric on $\mathbb{C}^m$ is given by $\upsilon_m = dd^c\tau_m > 0$. On $\mathbb{C}^m \setminus \{0\}$, we define $\omega_m = dd^c \log \tau_m \geq 0$ and $\sigma_m = d^c \log \tau_m \wedge \omega_m^{m-1}$. For any $S \subseteq \mathbb{C}^m$, we define $S[r] = \{z \in S : ||z|| \leq r\}$, $S(r) = \{z \in S : ||z|| < r\}$ and $S\langle r\rangle = \{z \in S : ||z|| = r\}$, where $r > 0$.

The *zero multiplicity* $\mu_f^0(a)$ of an entire function f in $\mathbb{C}^m$ at a point $a \in \mathbb{C}^m$ is defined to be the order of vanishing of f at a. If $a = (a_1, \ldots, a_m)$, then $\partial^I(f(a)) = 0$, where $|I| \leq \mu_f^0(a) - 1$. In other words, we can write $f(z) = \sum_{i=0}^{\infty} P_i(z-a)$, where the term $P_i(z-a)$ is either identically zero or a homogeneous polynomial of degree i. Clearly $\mu_f^0(a) = \min\{i : P_i(z-a) \not\equiv 0\}$.

Let f be a meromorphic function in $\mathbb{C}^m$. Then there exist holomorphic functions g and h such that $hf = g$ on $\mathbb{C}^m$ and $\dim_z(h^{-1}(\{0\}) \cap g^{-1}(\{0\})) \leq m-2$. Therefore the c-multiplicity of f is just $\mu_f^c = \mu_{g-ch}^0$ if $c \in \mathbb{C}$ and $\mu_f^c = \mu_h^0$ if $c = \infty$. The function $\mu_f^c : \mathbb{C}^m \to \mathbb{Z}$ is nonnegative and is called the *c-divisor* of f. If $f \not\equiv 0$, then $\nu = \mu_f = \mu_f^0 - \mu_f^\infty$ is called the *divisor* of f. For $t > 0$, the *counting function* n_ν is defined by

$$n_\nu(t) = t^{-2(m-1)} \int\limits_{A[t]} \nu \upsilon_m^{m-1},$$

where $A = \operatorname{supp} \nu = \overline{\{z \in G : \nu_f(z) \neq 0\}}$.

The *valence function* of ν is defined by

$$N_\nu(r) = N_\nu(r, r_0) = \int\limits_{r_0}^{r} n_\nu(t) \frac{dt}{t} \qquad (r \geq r_0).$$

Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. We write $N_{\mu_f^a}(r) = N(r, a; f)$ if $a \in \mathbb{C}$ and $N_{\mu_f^a}(r) = N(r, f)$ if $a = \infty$. For $k \in \mathbb{N}$, define the *truncated multiplicity functions* on $\mathbb{C}^m$ by $\mu_{f,k}^a(z) = \min\{\mu_f^a(z), k\}$, $\mu_{f(k}^a(z) = \mu_f^a(z)$ if $\mu_f^a(z) \geq k$ and $\mu_{f(k}^a(z) = 0$ if $\mu_f^a(z) < k$. Also we define the *truncated valence functions* $N_\nu(t) = \overline{N}(t, a; f)$ if $\nu = \mu_{f,1}^a$ and $N_\nu(t) = N_{(k}(t, a; f)$, if $\nu = \mu_{f(k}^a$.

With the help of the positive logarithm function, we define the *proximity function* of f by

$$m(r, f) = \int\limits_{\mathbb{C}^m\langle r \rangle} \log^+ |f| \, \sigma_m \geq 0.$$

The *characteristic function* of f is defined by $T(r, f) = m(r, f) + N(r, f)$. We define $m(r, a; f) = m(r, f)$ if $a = \infty$ and $m(r, a; f) = m(r, 1/(f-a))$ if $a \in \mathbb{C}$. Now if $a \in \mathbb{C}$, then the first main theorem becomes

$$m(r, a; f) + N(r, a; f) = T(r, f) + O(1),$$

where $O(1)$ denotes a bounded function when r is sufficiently large.

A property is said to hold *generically* if it holds everywhere except on an analytic subset of strictly smaller dimension. For instance, if f is a meromorphic function on $\mathbb{C}^m$, then there exist holomorphic functions g and h such that $fh = g$ and $\dim(g^{-1}(\{0\}) \cap h^{-1}(\{0\})) \leq m-2$. Consequently, the common zero factor of g and h can be cancelled *generically* on its support (see [14, p. 103]).

Let f, g, and a be meromorphic functions on $\mathbb{C}^n$. Then one can find three pairs of entire functions f_1 and f_2, g_1 and g_2, and a_1 and a_2, in which each pair is coprime at each point in $\mathbb{C}^n$ such that $f = f_2/f_1$, $g = g_2/g_1$ and $a = a_2/a_1$. We say that f and g share *a counting multiplicities* (CM) if

$$\mu^0_{a_1f_2-a_2f_1} = \mu^0_{a_1g_2-a_2g_1} \ (a \not\equiv \infty) \quad \text{and} \quad \mu^0_{f_1} = \mu^0_{g_1} \ (a = \infty).$$

Again we say that f and g share *a ignoring multiplicities* (IM) if

$$\mu^0_{a_1f_2-a_2f_1,1} = \mu^0_{a_1g_2-a_2g_1,1} \ (a \not\equiv \infty) \quad \text{and} \quad \mu^0_{f_1,1} = \mu^0_{g_1,1} \ (a = \infty).$$

A closed set $A \subset \mathbb{C}^m$ is said to be *analytic* if for each $a \in A$, there are a finite number of holomorphic functions $f_1, \dots, f_l$ defined in a neighbourhood $N(a)$ of a such that (see [30, p. 42])

$$A \cap N(a) = \{z \in N(a) : f_1(z) = \cdots = f_l(z) = 0\}.$$

If $f_j(z)$, $1 \le j \le l$, can be taken so that their differentials at $a \in A$, $df_1(a), \dots, df_l(a)$ are linearly independent, then a is called a *regular point* of A. Any point in A that is not regular is *singular*. The subset of all singular points of A is denoted by A_S. If $A_S = \varnothing$, then A is said to be *regular*.

The remainder of the paper is organized as follows: in Section 2, we determine the possible solutions of the nonlinear partial differential equation (1.2) for the functions f and g and in Section 3, we consider the uniqueness problem for meromorphic functions in $\mathbb{C}^m$ concerning nonlinear partial differential polynomials generated by them that share a common value.

2. Solutions of $f^n\partial_u(f)g^n\partial_u(g) = 1$. We investigate the nonlinear partial differential equation

$$f^n\,\partial_u(f)\,g^n\,\partial_u(g) = 1,$$

where f and g are nonconstant entire or meromorphic functions of several complex variables, and ∂_u denotes the directional derivative along a fixed nonzero vector u.

This equation establishes a direct relationship between the two functions through their respective powers and directional derivatives, thereby imposing simultaneous constraints on their growth and analytic properties. The presence of the directional derivative further reflects the multivariable nature of the problem and introduces additional structural considerations.

The aim of this section is to obtain a complete characterization of all nonconstant entire and meromorphic pairs (f, g) satisfying the above equation. The following theorems establish the precise forms of such solutions.

THEOREM 2.1. *Let $f : \mathbb{C}^m \to \mathbb{C}$ and $g : \mathbb{C}^m \to \mathbb{C}$ be two nonconstant entire functions and let $n \ge 1$ be an integer. If*

$$(f^n\partial_u f)(g^n\partial_u g) \equiv 1,$$

then $f = \exp(\alpha)$ *and* $g = \exp(\beta)$, *where* α *and* β *are nonconstant entire functions in* $\mathbb{C}^m$ *such that* $\partial_u(\alpha) = c$, $\partial_u(\beta) = -c$ *and* $c^2 e^{(n+1)(\alpha+\beta)} = -1$.

THEOREM 2.2. *Let* $f : \mathbb{C}^m \to \mathbb{P}^1$ *and* $g : \mathbb{C}^m \to \mathbb{P}^1$ *be two nonconstant meromorphic functions and let* $n \geq 6$ *be an integer. If*

$$f^n \partial_u(f) g^n \partial_u(g) \equiv 1,$$

then the conclusions of Theorem 2.1 *hold.*

Let $u = (u_1, \ldots, u_m)$ such that $u_j = 1$ and $u_i = 0$ for $i \neq j$. Clearly $\partial_u(f) = \partial_j(f)$ for $m > 1$ and $\partial_u(f) = f^{(1)}$ for $m = 1$ and $u_1 = 1$. Then from Theorem 2.2, we get the following.

COROLLARY 2.1. *Let* $f : \mathbb{C}^m \to \mathbb{P}^1$ *and* $g : \mathbb{C}^m \to \mathbb{P}^1$ *be two nonconstant meromorphic functions and let* $n \geq 6$ *be an integer. If*

$$f^n \partial_j(f) g^n \partial_j(g) \equiv 1,$$

where $j \in \mathbb{Z}[1, m]$, *then* $f = \exp(\alpha)$ *and* $g = \exp(\beta)$, *where* α *and* β *are nonconstant entire functions in* $\mathbb{C}^m$ *such that* $\partial_j(\alpha) = c$, $\partial_j(\beta) = -c$ *and* $c^2 e^{(n+1)(\alpha+\beta)} = -1$.

COROLLARY 2.2. *Let* $f : \mathbb{C}^m \to \mathbb{P}^1$ *and* $g : \mathbb{C}^m \to \mathbb{P}^1$ *be two nonconstant meromorphic functions and let* $n \geq 6$ *be an integer. If*

$$f^n \partial_j(f) g^n \partial_j(g) \equiv 1,$$

for all $j \in \mathbb{Z}[1, m]$, *then*

$$f(z) = c_1 \exp(c(z_1 + \ldots + z_m)) \quad \text{and} \quad g(z) = c_2 \exp(-c(z_1 + \ldots + z_m)),$$

where c, c_1 *and* c_2 *are nonzero constants such that* $c^2 (c_1 c_2)^{n+1} = -1$.

Table 1. Comparison of Theorems 2.1, 2.2 and Corollaries 2.1, 2.2

Result	Type of functions	n	Equation	Conclusions
Theorem 2.1	Nonconstant entire functions $f, g : \mathbb{C}^m \to \mathbb{C}$	$n \geq 1$	$(f^n \partial_u f)(g^n \partial_u g) \equiv 1$	$f = e^\alpha$, $g = e^\beta$, where $\partial_u \alpha = c$, $\partial_u \beta = -c$, and $c^2 e^{(n+1)(\alpha+\beta)} = -1$
Theorem 2.2	Nonconstant entire functions $f, g : \mathbb{C}^m \to \mathbb{P}^1$	$n \geq 6$	$f^n \partial_u(f) g^n \partial_u(g) \equiv 1$	Same conclusions as in Th. 2.1.
Corollary 2.1	Nonconstant meromorphic functions $f, g : \mathbb{C}^m \to \mathbb{P}^1$	$n \geq 6$	$f^n \partial_j(f) g^n \partial_j(g) \equiv 1$ for a fixed $j \in \mathbb{Z}[1, m]$	$f = e^\alpha$, $g = e^\beta$, where $\partial_j \alpha = c$, $\partial_j \beta = -c$, and $c^2 e^{(n+1)(\alpha+\beta)} = -1$
Corollary 2.2	Nonconstant meromorphic functions $f, g : \mathbb{C}^m \to \mathbb{P}^1$	$n \geq 6$	$f^n \partial_j(f) g^n \partial_j(g) \equiv 1$ for all $j \in \mathbb{Z}[1, m]$	$f(z) = c_1 e^{c(z_1 + \cdots + z_m)}$, $g(z) = c_2 e^{-c(z_1 + \cdots + z_m)}$, where $c, c_1, c_2 \neq 0$ and $c^2 (c_1 c_2)^{n+1} = -1$

2.1. Auxiliary lemmas. In the proof of Theorems 2.1–2.2, we use the following key lemmas.

LEMMA 2.1 ([14, Lemma 1.37]). *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function and let $I = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$. Then for any $\varepsilon > 0$, we have*

$$\| \quad m(r, \partial^I(f)/f) \le |I| \log^+ T(r,f) + |I|(1+\varepsilon) \log^+ \log T(r,f) + O(1),$$

where $\|$ indicates that the inequality holds only outside a set of finite measure on $\mathbb{R}^+$.

LEMMA 2.2 ([14, Lemma 1.68]). *Let $f_1 : \mathbb{C}^m \to \mathbb{P}^1$ and $f_2 : \mathbb{C}^m \to \mathbb{P}^1$ be two nonconstant meromorphic functions. Then for $r > 0$ we have*

$$\| \quad N(r,0;f_1f_2) - N(r,f_1f_2) = N(r,0;f_1) + N(r,0;f_2) - N(r,f_1) - N(r,f_2).$$

2.2. Proof of Theorem 2.2. We begin with the partial differential equation

$$f^n \partial_u(f)\, g^n \partial_u(g) \equiv 1. \tag{2.1}$$

STEP 1: *Preliminary setup.* Define

$$I = I_f \cup I_g \quad \text{and} \quad S = \bigcup_{a \in \{0,\infty\}} \big(\operatorname{supp} \mu_f^a\big)_s \cup \big(\operatorname{supp} \mu_g^a\big)_s,$$

where A_s denotes the set of singular points of the analytic set A. Note that $\dim(I \cup S) \le m-2$. Choose $z_0 \in \mathbb{C}^m \setminus (I \cup S)$.

If possible, suppose $\mu_f^0(z_0) = l > 0$. Then by (2.1), we have $\mu_g^\infty(z_0) = k > 0$. Since $z_0 \notin S$, there exists a holomorphic coordinate system

$$(U; \varphi_1, \dots, \varphi_m)$$

around z_0 in $\mathbb{C}^m \setminus (I \cup S)$ such that

$$U \cap \operatorname{supp} \mu_g^\infty = \{z \in U \mid \varphi_1(z) = 0\}, \quad \varphi_j(z_0) = 0,\ j = 1, \dots, m$$

(see [21, Lemma 2.3]).

Thus, there exists a biholomorphic transformation $z_j = z_j(\varphi_1, \dots, \varphi_m)$, $j = 1, \dots, m$, near 0 such that $z_0 = z(0) = (z_1(0), \dots, z_m(0))$. Thus, we can write

$$f = \varphi_1^l \hat{f}(\varphi_1, \dots, \varphi_m) \quad \text{and} \quad g = \varphi_1^{-k} \hat{g}(\varphi_1, \dots, \varphi_m),$$

where $\hat{f}$ and $\hat{g}$ are holomorphic and nonvanishing on $\operatorname{supp} \mu_g^\infty$.

STEP 2: *Computing directional derivatives.* For $i \in \{1, \dots, m\}$, we have

$$\frac{\partial f}{\partial z_i} = \sum_{j=1}^m \frac{\partial f}{\partial \varphi_j} \frac{\partial \varphi_j}{\partial z_i} = l\varphi_1^{l-1} \hat{f} \frac{\partial \varphi_1}{\partial z_i} + \varphi_1^l \sum_{j=1}^m \frac{\partial \hat{f}}{\partial \varphi_j} \frac{\partial \varphi_j}{\partial z_i}.$$

Hence,

$$\partial_u(f) = l\varphi_1^{l-1}\hat{f}\partial_u(\varphi_1) + \varphi_1^l \sum_{j=1}^{m} \frac{\partial \hat{f}}{\partial \varphi_j}\partial_u(\varphi_j). \tag{2.2}$$

Similarly,

$$\partial_u(g) = -\frac{k}{\varphi_1^{k+1}}\hat{g}\partial_u(\varphi_1) + \frac{1}{\varphi_1^k} \sum_{j=1}^{m} \frac{\partial \hat{g}}{\partial \varphi_j}\partial_u(\varphi_j). \tag{2.3}$$

Step 3: *Expanding the main identity.* It follows that

$$f^n\partial_u(f) = l\varphi_1^{(n+1)l-1}\hat{f}^{n+1}\partial_u(\varphi_1) + \varphi_1^{(n+1)l}\hat{f}^n \sum_{j=1}^{m} \frac{\partial \hat{f}}{\partial \varphi_j}\partial_u(\varphi_j), \tag{2.4}$$

$$g^n\partial_u(g) = -\frac{k}{\varphi_1^{(n+1)k+1}}\hat{g}^{n+1}\partial_u(\varphi_1) + \frac{1}{\varphi_1^{(n+1)k}}\hat{g}^n \sum_{j=1}^{m} \frac{\partial \hat{g}}{\partial \varphi_j}\partial_u(\varphi_j). \tag{2.5}$$

Now from (2.1) and (2.4)–(2.5), it follows that $(n+1)(l-k) = 2$, a contradiction. Hence $l = \mu_f^0(z_0) = 0$ and so $f \neq 0$. Similarly, $g \neq 0$. From (2.3), we get

$$\| \quad N(r, \partial_u(g)) \leq N(r, g) + \overline{N}(r, g),$$

where $\|$ is defined as in Lemma 2.1. Similarly

$$\| \quad N(r, \partial_u(f)) \leq N(r, f) + \overline{N}(r, f).$$

Step 4: *Setting up the auxiliary function.* Let

$$h = \frac{1}{fg}.$$

Then h is an entire function. We consider two cases.

Case 1: *h is nonconstant.* Since $f \neq 0$, from (2.1) we see that $\mu^{\infty}_{g^n\partial_u g} \leq \mu^0_{\partial_u f}$ holds on $\operatorname{supp}\mu_f^{\infty}$ generically, and so (2.5) gives

$$(n+1)\mu_g^{\infty}(z_0) + \mu_{g,1}^{\infty}(z_0) \leq \mu^0_{\partial_u f}(z_0).$$

Therefore,

$$\| \quad (n+1)N(r, g) + \overline{N}(r, g) \leq N(r, 0; \partial_u(f)). \tag{2.6}$$

Now by Lemma 2.2, we have

$$\begin{aligned} \| \quad & N(r, f/\partial_u(f)) - N(r, \partial_u(f)/f) \\ & = N(r, f) + N(r, 0; \partial_u(f)) - N(r, \partial_u(f)) - N(r, 0; f) \\ & = N(r, 0; \partial_u(f)) + N(r, f) - N(r, \partial_u(f)). \end{aligned} \tag{2.7}$$

Using the first main theorem, we get

$$\| \quad N(r, f/\partial_u(f)) - N(r, \partial_u(f)/f) = m(r, \partial_u(f)/f) - m(r, f/\partial_u(f)) + O(1),$$

and so from (2.7) we have

$$\| \quad N(r,0;\partial_u(f)) = N(r,\partial_u(f)) - N(r,f) + m(r,\partial_u(f)/f) - m(r,f/\partial_u(f)) + O(1). \tag{2.8}$$

Again from (2.1) we have $\partial_u g/g = (f/\partial_u(f))h^{n+1}$, and so

$$\| \quad m(r,\partial_u(g)/g) \le m(r,f/\partial_u(f)) + (n+1)m(r,h),$$

i.e.,

$$\| \quad m(r,f/\partial_u(f)) \ge m(r,\partial_u(g)/g) - (n+1)m(r,h) - O(1). \tag{2.9}$$

Note that $\| \ N(r,\partial_u(f)) \le N(r,f)+\overline{N}(r,f)$, and so from (2.6) and (2.8)–(2.9) we get

$$\| \quad (n+1)N(r,g) + \overline{N}(r,g) \le \overline{N}(r,f) + m(r,\partial_u(f)/f) - m(r,\partial_u(g)/g) + (n+1)m(r,h) + O(1). \tag{2.10}$$

Similarly, we have

$$\| \quad (n+1)N(r,f) + \overline{N}(r,f) \le \overline{N}(r,g) + m(r,\partial_u(g)/g) - m(r,\partial_u(f)/f) + (n+1)m(r,h) + O(1). \tag{2.11}$$

Now adding (2.10) and (2.11) gives

$$\| \quad N(r,f) + N(r,g) \le 2m(r,h) + O(1). \tag{2.12}$$

Since $f = 1/(gh)$, we get

$$\partial_u(f) = -\big(g\partial_u(h) + h\partial_u(g)\big)/g^2h^2,$$

and so from (2.1) we get

$$\xi^2 = -h^{n+1}(\partial_u(h)/h)^2/4, \tag{2.13}$$

where

$$\xi = \partial_u(g)/g + (\partial_u(h)/h)/2. \tag{2.14}$$

First suppose $\xi \equiv 0$. Then from (2.13), we get $h^{n+1} = (\partial_u(h)/h)^2/4$. Now by Lemma 2.1, we have $\| \ (n+1)T(r,h) = (n+1)m(r,h) = o(T(r,h))$, which is absurd.

Next suppose $\xi \not\equiv 0$. Clearly, (2.13) yields

$$2\xi^2(\partial_u(\xi)/\xi) = 2\xi\partial_u(\xi) = (1/2)(\partial_u(h)/h)\partial_u(\partial_u(h)/h) - (n+1)h^n\partial_u(h), \tag{2.15}$$

and so from (2.13) we get

$$h^{n+1}H_1 = \partial_u(h)(\partial_u(\partial_u(h)/h) - \partial_u(h)/h)/2h, \tag{2.16}$$

where

$$H_1 = (n+1)\partial_u(h)/h - (2\partial_u(\xi)/\xi).$$

Further manipulations involving ξ defined by

$$\xi = \frac{\partial_u(g)}{g} + \frac{1}{2}\frac{\partial_u(h)}{h}$$

lead to the key relation

$$\xi^2 = -\frac{h^{n+1}}{4}\left(\frac{\partial_u(h)}{h}\right)^2. \tag{2.17}$$

Now we consider the following two subcases.

SUBCASE 1.1: $H_1 \equiv 0$. Then $(n+1)\partial_u(h)/h - (2\partial_u(\xi)/\xi) \equiv 0$, i.e,

$$(n+1)\partial_u(h)h \equiv 2\partial_u(\xi)/\xi. \tag{2.18}$$

If $\dim(\mathbb{C}^m) = 1$, then from (2.18), we get $\xi^2 \equiv c_1 h^{n+1}$ and so (2.17) gives

$$(c_1+1)h^{n+1} \equiv (1/4)(h^{(1)}/h)^2.$$

Since $h^{(1)} \not\equiv 0$ and $c_1 \neq -1$, by Lemma 2.1, we get $\| \ (n+1)T(r,h) = (n+1)m(r,h) = o(T(r,h))$, which is absurd. Hence $\dim(\mathbb{C}^m) \geq 2$. From (2.16) and (2.18), we get

$$\partial_u(\partial_u(h)/h) \equiv \partial_u(h)/h.$$

Again from (2.15), we get

$$(n+1)\partial_u(h)/h\xi^2 = (1/2)(\partial_u(h)/h)^2 - (n+1)h^{n+1}\partial_u(h)/h,$$

and so from (2.17) we have

$$((n+1)/4)(\partial_u(h)/h)^3 = (\partial_u(h)/h)^2/2. \tag{2.19}$$

First suppose $\partial_u(h) \not\equiv 0$. Then (2.19) gives $\partial_u(h)/h = 2/(n+1)$. Since $\partial_u(\partial_u(h)/h) \equiv \partial_u(h)/h$, we get a contradiction.

Next suppose $\partial_u(h) \equiv 0$. Clearly from (2.13) and (2.14), we have $\xi^2 = -h^{n+1}$ and $\xi = \partial_u(g)/g$ respectively and so $(\partial_u(g)/g)^2 = -h^{n+1}$. Since h is an entire function and $g \neq 0$, we see that g is also an entire function. Again, $h = 1/(fg)$ gives

$$\partial_u(h)/h = -(\partial_u(f)/f) - (\partial_u(g)/g)$$

and so $\partial_u(f)/f = -\partial_u(g)/g$. This shows that f is also an entire function. Let us take $f = \exp(\alpha)$ and $g = \exp(\beta)$, where α and β are nonconstant entire functions in $\mathbb{C}^m$. Clearly $\partial_u(\alpha) \equiv -\partial_u(\beta)$. Since $(\partial_u(g)/g)^2 = -h^{n+1}$, we have $(\partial_u(\beta))^2 \exp(-(n+1)(\alpha+\beta)) = -1$. Consequently,

$$2\partial_u^2(\beta) - (n+1)\partial_u(\beta)(\partial_u(\alpha) + \partial_u(\beta)) \equiv 0,$$

and so $\partial_u^2(\beta) \equiv 0$. Finally, we have

$$f = \exp(\alpha) \quad \text{and} \quad g = \exp(\beta),$$

where α and β are nonconstant entire functions in $\mathbb{C}^m$ such that $\partial_u(\alpha) = c$, $\partial_u(\beta) = -c$ and $c^2 e^{-(n+1)(\alpha+\beta)} = -1$.

SUBCASE 1.2: $H_1 \not\equiv 0$. Since $\partial_u(h)/h = -(\partial_u(f)/f) - (\partial_u(g)/g)$, from (2.14) we get

$$\xi = (\partial_u(g)/g - \partial_u(f)/f)/2,$$

and so

$$\partial_u(\xi) = (\partial_u(\partial_u(g)/g) - \partial_u(\partial_u(f)/f))/2.$$

Therefore,

$$H_1 = -(n+1)(\partial_u(f)/f + \partial_u(g)/g) - \frac{\partial_u(\partial_u(g)/g) - \partial_u(\partial_u(f)/f)}{\xi}.$$

Since $f \neq 0$ and $g \neq 0$, it is easy to verify that $\mu^{\infty}_{H_1} \leq \mu^{\infty}_{f,1} + \mu^{\infty}_{g,1} + \mu^{0}_{\xi,1}$ holds over $\operatorname{supp} \mu^{\infty}_{H_1}$ generically, and so

$$\| \quad N(r, H_1) \leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}(r, 0; \xi). \tag{2.20}$$

On the other hand applying the first main theorem and Lemma 2.1 to (2.13), we have

$$\begin{aligned} \| \quad 2T(r,\xi) &\leq 2T(r, \partial_u(h)/h) + (n+1)T(r,h) \\ &\leq 2N(r, \partial_u(h)/h) + (n+1)T(r,h) \\ &\leq 2\overline{N}(r,0;h) + (n+1)T(r,h) \\ &\leq (n+3)T(r,h) + o(T(r,h)). \end{aligned} \tag{2.21}$$

Again, using the first main theorem, Lemma 2.1, (2.12) and (2.16)–(2.21), we get

$$\begin{aligned} \| \quad (n+1)T(r,h) &\leq m(r, h^{n+1}H_1) + m(r,0;H_1) + O(1) \\ &\leq m\big(r, \tfrac{1}{2}(\partial_u(h)/h)(\partial_u(\partial_u(h)/h)\big) - \partial_u(h)/h)) + T(r, H_1) \\ &\leq N(r, H_1) + o(T(r,h)) + o(T(r,\xi)) \\ &\leq \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}(r,0;\xi) + o(T(r,h)) + o(T(r,\xi)) \\ &\leq \tfrac{n+7}{2} T(r,h) + o(T(r,h)), \end{aligned}$$

i.e., $\| \ (1/2)(n-5)T(r,h) \leq o(T(r,h))$, which is impossible.

CASE 2: *h is constant.* Since $f \neq 0$ and $g \neq 0$, both f and g are nonconstant entire functions. Note that $f = 1/(gh)$ and so $\partial_u(f) = -\partial_u(g)/g^2 h$. Consequently,

$$\frac{\partial_u(f)}{f} = -\frac{\partial_u(g)}{g},$$

and so from (2.1) we get $\partial_u(g) = cg$ and $\partial_u(f) = -cf$, where $c = ih^{(n+1)/2}$. Hence, $f = e^{\alpha}$ and $g = e^{\beta}$, where α, β are nonconstant entire functions in $\mathbb{C}^m$ satisfying

$$\partial_u(\alpha) = c, \quad \partial_u(\beta) = c, \quad c^2 e^{-(n+1)(\alpha+\beta)} = -1.$$

Conclusion. In all cases, the only nontrivial admissible solutions are exponential type functions: $f = e^{\alpha}$ and $g = e^{\beta}$, with $\partial_u(\alpha)$ and $\partial_u(\beta)$ satisfying the above relations. ∎

2.3. Proof of Theorem 2.1. The proof of Theorem 2.1 follows directly from the proof of Theorem 2.2, so we omit the details.

Proof of Corollary 2.2. By Theorem 2.2, we have $f = \exp(\alpha)$ and $g = \exp(\beta)$, where α and β are nonconstant entire functions on $\mathbb{C}^m$ satisfying $\partial_j(\alpha) = c$, $\partial_j(\beta) = c$ for all $j \in \mathbb{Z}[1, m]$ and $4c^2e^{-(n+1)(\alpha+\beta)} = -1$. In view of (1.2), we may assume that $\alpha(z) = c(z_1 + \ldots + z_m) + d_1$ and $\beta(z) = -c(z_1 + \cdots + z_m) + d_2$, where d_1 and d_2 are constants. Therefore $f(z) = c_1 \exp(c(z_1 + \cdots + z_m))$ and $g(z) = c_2 \exp(-c(z_1 + \cdots + z_m))$, where c, c_1, c_2 are nonzero constants satisfying $c^2(c_1c_2)^{n+1} = -1$. ∎

3. Uniqueness problems. In this section, our main objective is to consider the uniqueness problem for meromorphic functions in $\mathbb{C}^m$ such that nonlinear differential polynomials generated by them share one value. Moreover, our result shows that Theorem C remains valid in several complex variables. Now we state our main results.

Theorem 3.1. *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ and $g : \mathbb{C}^m \to \mathbb{P}^1$ be two nonconstant meromorphic functions and let $n \geq 11$ be an integer. If $f^n\partial_u(f)$ and $g^n\partial_u(g)$ share a CM, where $a \in \mathbb{C} \setminus \{0\}$, then one of the following cases holds:*

(i) *$f \equiv dg$, where $d^{n+1} = 1$;*
(ii) *$f = \exp(\alpha)$ and $g = \exp(\beta)$, where $c \in \mathbb{C} \setminus \{0\}$, α and β are nonconstant entire functions in $\mathbb{C}^m$ such that $\partial_u(\alpha) = c$, $\partial_u(\beta) = -c$ and $c^2e^{(n+1)(\alpha+\beta)} = -a^2$.*

Theorem 3.2. *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ and $g : \mathbb{C}^m \to \mathbb{P}^1$ be two nonconstant meromorphic functions and let $n \geq 11$ be an integer. If $f^n\partial_j(f)$ and $g^n\partial_j(g)$ share a CM, where $j \in \mathbb{Z}[1, m]$ and $a \in \mathbb{C} \setminus \{0\}$, then one of the following cases holds:*

(i) *$f \equiv dg$, where $d^{n+1} = 1$;*
(ii) *$f = \exp(\alpha)$ and $g = \exp(\beta)$, where $c \in \mathbb{C} \setminus \{0\}$, α and β are nonconstant entire functions in $\mathbb{C}^m$ such that $\partial_j(\alpha) = c$, $\partial_j(\beta) = -c$ and $c^2e^{(n+1)(\alpha+\beta)} = -a^2$.*

For nonconstant entire functions, we get the following corollary from Theorem 3.1.

Corollary 3.1. *Let $f : \mathbb{C}^m \to \mathbb{C}$ and $g : \mathbb{C}^m \to \mathbb{C}$ be two nonconstant entire functions such that $\partial_u(f) \not\equiv 0$ and $\partial_u(g) \not\equiv 0$ and let $n \geq 7$ be an integer. If $f^n\partial_u(f)$ and $g^n\partial_u(g)$ share a CM, where $a \in \mathbb{C} \setminus \{0\}$, then the conclusions of Theorem* 3.1 *hold.*

The following example ensures the necessity of the condition $a \in \mathbb{C} \setminus \{0\}$ in Corollary 3.1.

EXAMPLE 3.1. Let $f(z) = \exp(\exp(z_1 + \ldots + z_m))$, $g(z) = \exp(z_1 + \ldots + z_m)$ and $a = 0$. Note that $f^n \partial_u(f) = \exp((n+1)\exp(z_1 + \ldots + z_m))\exp(z_1 + \ldots + z_m)\sum_{j=1}^m u_j$ and $g^n \partial_u(g) = \exp((n+1)(z_1 + \ldots + z_m))\sum_{j=1}^m u_j$. Clearly $f^n \partial_u(f)$ and $g^n \partial_u(g)$ share a CM, but f and g do not satisfy the conclusion of Corollary 3.1.

REMARK 3.1. The lower bounds on n in Theorems 3.1 and 3.2, namely $n \geq 11$, and in Corollary 3.1, namely $n \geq 7$, arise from the estimates used in the proofs. At present, we do not claim that these bounds are optimal. In particular, it remains an open question whether the conclusions of these results continue to hold for smaller values of n. Determining the smallest possible lower bound for n would require a separate analysis of the estimates involved in the proofs and is left for future investigation.

3.1. Auxiliary lemmas. In order to prove Theorems 3.1–3.2, we require the following lemmas.

LEMMA 3.1 ([14, Corollary 1.40]). *Let $f_0, \ldots, f_n$ be linearly independent meromorphic functions in $\mathbb{C}^m$. Write $f = (f_0, \ldots, f_n)$. Then there exist multi-indices $\nu_i \in \mathbb{Z}_+^m$ with $0 < |\nu_i| \leq i$ $(i = 1, \ldots, n)$ such that f, $\partial^{\nu_1}(f), \ldots, \partial^{\nu_n}(f)$ are linearly independent over $\mathbb{C}^m$ and there exists a partition*

$$\{\nu_1, \ldots, \nu_n\} = \bigcup_{i=1}^{s} I_i \qquad (1 \leq s \leq n)$$

such that $I_k \subset \{\nu \in \mathbb{Z}_+^m \mid |\nu| = k\}$, $k = 1, \ldots, s$ and when $1 \leq k < s$ each element in

$$\{\partial^\nu f \mid \nu \in \mathbb{Z}_+^m,\ |\nu| = k,\ \nu \notin I_k\}$$

can be expressed as a linear combination of the family

$$\Big\{ f, \partial^\nu(f) \ \Big|\ \nu \in \bigcup_{i=1}^{k} I_i \Big\}.$$

REMARK 3.2. By using [14, Lemma 1.41], we make a remark on Lemma 3.1. If $\#(I_s) = 1$ in Lemma 3.1, say, $\nu_n \in I_s$, then [14, Lemma 1.41] shows that we may choose ν_n such that $\nu_n = \nu + \imath_j$ for some $\nu \in I_{s-1}$, $j \in \mathbb{Z}[1, m]$, where $\imath_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}_+^m$ in which 1 is the j-th component of $\imath_j$.

LEMMA 3.2 ([13, Lemma 1.2]). *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function and let $a_1, a_2, \ldots, a_q$ be different points in $\mathbb{P}^1$. Then*

$$\Big\| \quad (q-2)T(r,f) \leq \sum_{j=1}^{q} \overline{N}(r, a_j; f) + O(\log(rT(r,f))).$$

LEMMA 3.3 ([14, Theorem 1.26]). *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function. If$R(z,w) = A(z,w)/B(z,w)$, then*

$$\| \quad T(r,R_f) = \max\{p,q\}T(r,f) + O\Big(\sum_{j=0}^{p} T(r,a_j) + \sum_{j=0}^{q} T(r,b_j)\Big),$$

where $R_f(z) = R(z,f(z))$ and two coprime polynomials $A(z,w)$ and $B(z,w)$ are given by $A(z,w) = \sum_{j=0}^{p} a_j(z)w^j$ and $B(z,w) = \sum_{j=0}^{q} b_j(z)w^j$ respectively.

LEMMA 3.4 ([14, Theorem 1.101]). *Suppose that $f_1,\dots,f_n$ are linearly independent meromorphic functions in $\mathbb{C}^m$ such that $\sum_{i=1}^{n} f_i = 1$. Then for $1 \le j \le n$ and $R > \rho > r > r_0$,*

$$\begin{aligned}\| \quad T(r,f_j) \le N(r,f_j) + \sum_{j=1}^{n}\{N(r,0;f_j) - N(r,f_j)\} + N(r,\mathbf{W})\\ - N(r,0;\mathbf{W}) + l_1\log\{(\rho/r)^{2m-1}T(R)/(\rho-r)\} + O(1),\end{aligned}$$

where $T(R) = \max_{1\le j\le n} T(r,f_j)$ and $\mathbf{W} = \mathbf{W}_{\nu_1\dots\nu_{n-1}}(f_1,\dots,f_n) \not\equiv 0$ is the Wronskian determinant and $n-1 \le l_1 = |\nu_1| + \cdots + |\nu_{n-1}| \le n(n-1)/2$.

LEMMA 3.5. *Let $f : \mathbb{C}^m \to \mathbb{P}^1$ and $g : \mathbb{C}^m \to \mathbb{P}^1$ be two nonconstant meromorphic functions. If f and g share one CM, then one of the following three cases holds:*

(i) $\|\ T(r,f) \le N_2(r,f) + N_2(r,g) + N_2(r,0;f) + N_2(r,0;g) + o(T(r))$, *where $T(r) = \max\{T(r,f),T(r,g)\}$ and the same inequality holding for $T(r,g)$;*
(ii) *$f \equiv g$;*
(iii) *$fg \equiv 1$.*

Proof. We consider the following two cases.

CASE 1: *$f-1$ and $g-1$ are linearly dependent.* Since f and g are nonconstant, there exist $c_1 \ne 0$ and $c_2 \ne 0$ such that $c_1(f-1)+c_2(g-1) = 0$. Then

$$g - 1 = d(f-1), \tag{3.1}$$

where $d = -c_1/c_2$. If $d = 1$, then $f \equiv g$ and so we get (ii). Next let $d \ne 1$. Now (3.1) gives $\|\ \overline{N}(r,0;g) = \overline{N}(r,(d-1)/d;f)$, and so by Lemma 3.2 we get

$$\begin{aligned}\| \quad T(r,f) &\le \overline{N}(r,0;f) + \overline{N}(r,f) + \overline{N}(r,(d-1)/d;f) + o(T(r,f))\\ &\le N_2(r,f) + N_2(r,g) + N_2(r,0;f) + N_2(r,0;g) + o(T(r,f)).\end{aligned}$$

Therefore we obtain (i).

CASE 2: *Let $f-1$ and $g-1$ are linearly independent.* Suppose

$$F_1 = f,\ F_2 = (f-1)/(g-1) \quad \text{and} \quad F_3 = -gF_2. \tag{3.2}$$

Now we consider the following two subcases.

SUBCASE 2.1: *F_1, F_2 and F_3 are linearly independent.* Clearly (3.2) gives $F_1 + F_2 + F_3 = 1$, and so by Lemma 3.4 we get

$$\begin{aligned} \| \quad T(r,F) \le N(r,F_1) + \sum_{k=1}^{3}\{N(r,0;F_k) - N(r,F_k)\} + N(r,\mathbf{W}) \\ - N(r,0;\mathbf{W}) + l_1 \log\{(\rho/r)^{2m-1}T_1(R)/(\rho - r)\} + O(1), \end{aligned} \tag{3.3}$$

where $T_1(r) = \max\{T(r,F_1), T(r,F_2), T(r,F_3)\}$, $2 \le l = |\nu_1| + |\nu_2| \le 3$ and

$$\mathbf{W} = \begin{vmatrix} F_1 & F_2 & F_3 \\ \partial^{\nu_1}F_1 & \partial^{\nu_1}F_2 & \partial^{\nu_1}F_3 \\ \partial^{\nu_2}F_1 & \partial^{\nu_2}F_2 & \partial^{\nu_2}F_3 \end{vmatrix}.$$

Since $F_1 + F_2 + F_3 = 1$, we have

$$\mathbf{W} = \partial^{\nu_1}F_1\partial^{\nu_2}F_2 - \partial^{\nu_1}F_2\partial^{\nu_2}F_1, \tag{3.4}$$

$$\mathbf{W} = \partial^{\nu_1}F_3\partial^{\nu_2}F_1 - \partial^{\nu_1}F_1\partial^{\nu_2}F_3. \tag{3.5}$$

We now prove the following inequality:

$$\mu := \sum_{k=1}^{3}\mu^0_{F_k} - \mu^\infty_{F_2} - \mu^\infty_{F_3} + \mu^\infty_{\mathbf{W}} - \mu^0_{\mathbf{W}} \le \mu^0_{f,2} + \mu^0_{g,2} + \mu^\infty_{f,2} + \mu^\infty_{g,2} =: \nu. \tag{3.6}$$

Note that generically, we have

$$\mu^0_{F_2} = \max\{\mu^\infty_g - \mu^\infty_f, 0\} \quad \text{and} \quad \mu^\infty_{F_2} = \max\{\mu^\infty_f - \mu^\infty_g, 0\}. \tag{3.7}$$

Define

$$I = I_f \cup I_g \cup I_{F_2} \cup I_{F_3} \quad \text{and} \quad S = \bigcup_{a\in\{0,1,\infty\}} (\operatorname{supp}\mu^a_f)_s \cup (\operatorname{supp}\mu^a_g)_s,$$

where A_s denotes the set of singular points of the analytic set A. It suffices to prove (3.6) on $\mathbb{C}^m - (I \cup S)$, since $\dim(I \cup S) \le m-2$. Take $z_0 \in \mathbb{C}^m - (I \cup S)$.

First, we assume $\mu^1_f(z_0) > 0$. Since $\operatorname{supp}\mu^1_f = \operatorname{supp}\mu^1_g$, we have $\mu^1_g(z_0) > 0$. By the given condition we get $\mu^1_f(z_0) = \mu^1_g(z_0)$. Clearly $\mu(z_0) \le 0$ and so the inequality (3.6) holds.

Next, we assume $\mu^1_f(z_0) = 0$. Then $\mu^1_g(z_0) = 0$. Now we will prove the inequality (3.6) by distinguishing following four subcases.

SUBCASE 2.1.1: *$\mu^\infty_f(z_0) = 0$ and $\mu^\infty_g(z_0) = 0$.* In this case from (3.7), we have

$$\mu(z_0) = \mu^0_f(z_0) + \mu^0_g(z_0) - \mu^0_{\mathbf{W}}(z_0).$$

Also from (3.5), we find

$$\begin{aligned}\mu^0_{\mathbf{W}}(z_0) &\geq \mu^0_{F_1}(z_0) - \mu^0_{F_1,2}(z_0) + \mu^0_{F_3}(z_0) - \mu^0_{F_3,2}(z_0)\\ &= \mu^0_f(z_0) - \mu^0_{f,2}(z_0) + \mu^0_g(z_0) - \mu^0_{g,2}(z_0).\end{aligned}$$

Therefore

$$\mu(z_0) \leq \mu^0_{f,2}(z_0) + \mu^0_{g,2}(z_0) = \nu(z_0)$$

and so inequality (3.6) holds.

Subcase 2.1.2: $\mu^\infty_f(z_0) = 0$ *and* $\mu^\infty_g(z_0) > 0$. In this case from (3.7) we have $\mu(z_0) = \mu^0_f(z_0) + \mu^\infty_g(z_0) - \mu^0_{\mathbf{W}}(z_0)$. Now from (3.4), we get

$$\begin{aligned}\mu^0_{\mathbf{W}}(z_0) &\geq \mu^0_{F_1}(z_0) - \mu^0_{F_1,2}(z_0) + \mu^0_{F_2}(z_0) - \mu^0_{F_2,2}(z_0)\\ &= \mu^0_f(z_0) - \mu^0_{f,2}(z_0) + \mu^\infty_g(z_0) - \mu^\infty_{g,2}(z_0).\end{aligned}$$

Consequently,

$$\mu(z_0) \leq \mu^0_{f,2}(z_0) + \mu^\infty_{g,2}(z_0) = \nu(z_0),$$

and so the inequality (3.6) holds.

Subcase 2.1.3: $\mu^\infty_f(z_0) > 0$ *and* $\mu^\infty_g(z_0) = 0$. Clearly,

$$\mu(z_0) = \mu^0_{F_3}(z_0) - \mu^\infty_{F_2}(z_0) - \mu^\infty_{F_3}(z_0) + \mu^\infty_{\mathbf{W}}(z_0) - \mu^0_{\mathbf{W}}(z_0). \tag{3.8}$$

Let $l = \mu^\infty_f(z_0)$ and $k = \mu^\infty_f(z_0) - \mu^0_g(z_0)$.

If $k = 0$, i.e., if $\mu^0_g(z_0) = \mu^\infty_f(z_0) > 0$, then from (3.5) we have

$$\mu^\infty_{\mathbf{W}}(z_0) \leq \mu^\infty_{F_1}(z_0) + 2,$$

and so by (3.8) we get

$$\mu(z_0) \leq -\mu^\infty_{F_2}(z_0) + \mu^\infty_{\mathbf{W}}(z_0) \leq \mu^\infty_{F_1}(z_0) - \mu^\infty_{F_2}(z_0) + 2 = 2 \leq \nu(z_0).$$

Consequently, the inequality (3.6) holds.

Next, we suppose $k \neq 0$. Since $2 \leq |\nu_1| + |\nu_2| \leq 3$, for the sake of simplicity, we may assume that $\partial^{\nu_1} = \partial_{z_i}$ for some $i \in \mathbb{Z}[1, m]$. Since $z_0 \notin S$, there is a holomorphic coordinate system $(U; \varphi_1, \ldots, \varphi_m)$ of z_0 in $\mathbb{C}^m \setminus (I \cup S)$ such that $U \cap \operatorname{supp} \mu^\infty_f = \{z \in U \mid \varphi_1(z) = 0\}$ and $\varphi_j(z_0) = 0$ for $j = 1, \ldots, m$. Hence, there exists a biholomorphic coordinate transformation $z_j = z_j(\varphi_1, \ldots, \varphi_m)$ near 0 such that $z_0 = z(0) = (z_1(0), \ldots, z_m(0))$ such that

$$F_1 = \frac{\hat{F}_1(\varphi_1, \ldots, \varphi_m)}{\varphi_1^l},\ F_2 = \frac{\hat{F}_2(\varphi_1, \ldots, \varphi_m)}{\varphi_1^l},\ F_3 = \frac{\hat{F}_3(\varphi_1, \ldots, \varphi_m)}{\varphi_1^k}, \tag{3.9}$$

where $\hat{F}_i$ $(i = 1, 2, 3)$ are holomorphic functions near 0, which do not vanish on $\operatorname{supp} \mu^\infty_f$. We consider the following two subcases.

SUBCASE 2.1.3.1: $|\nu_2|=1$. Suppose $\partial^{\nu_2}=\partial_{z_j}$ for some $j\in\mathbb{Z}[1,m]-\{i\}$. Now applying (3.9) to (3.5), we get

$$\begin{aligned}(3.10)\quad \mathbf{W}&=(-k\varphi_1^{-k-1}\hat{F}_3\partial_{z_i}\varphi_1+\varphi_1^{-k}\partial_{z_i}\hat{F}_3)(-l\varphi_1^{-l-1}\hat{F}_1\partial_{z_j}\varphi_1+\varphi_1^{-l}\partial_{z_j}\hat{F}_1)\\&\quad-(-l\varphi_1^{-l-1}\hat{F}_1\partial_{z_i}\varphi_1+\varphi_1^{-l}\partial_{z_i}\hat{F}_1)(-k\varphi_1^{-k-1}\hat{F}_3\partial_{z_j}\varphi_1+\varphi_1^{-k}\partial_{z_j}\hat{F}_3)\\&=\varphi_1^{-l-k-1}(l\hat{F}_1\partial_{z_i}\varphi_1\partial_{z_j}\hat{F}_3+k\hat{F}_3\partial_{z_i}\hat{F}_1\partial_{z_j}\varphi_1-l\hat{F}_1\partial_{z_i}\hat{F}_3\partial_{z_j}\varphi_1\\&\quad-k\hat{F}_3\partial_{z_i}\varphi_1\partial_{z_j}\hat{F}_1)+\varphi_1^{-l-k}(\partial_{z_i}\hat{F}_3\partial_{z_j}\hat{F}_1-\partial_{z_i}\hat{F}_1\partial_{z_j}\hat{F}_3).\end{aligned}$$

Let $-l-k>0$. Then from (3.10), we get $\mu^0_{\mathbf{W}}(z_0)\geq -l-k-1=-2\mu^\infty_f(z_0)+\mu^0_g(z_0)-1$, and so by (3.8) we find

$$\mu(z_0)=\mu^0_g(z_0)-2\mu^\infty_f(z_0)-\mu^0_{\mathbf{W}}(z_0)\leq 1\leq\mu^\infty_{f,2}(z_0)\leq\nu(z_0),$$

which shows that the inequality (3.6) holds.

Let $-l-k\leq 0$. Then (3.10) gives

$$\mu^\infty_{\mathbf{W}}(z_0)\leq l+k+1=2\mu^\infty_f(z_0)-\mu^0_g(z_0)+1,$$

and so from (3.8) we get

$$\mu(z_0)\leq\mu^0_g(z_0)-2\mu^\infty_f(z_0)+\mu^\infty_{\mathbf{W}}(z_0)\leq 1\leq\mu^\infty_{f,2}(z_0)\leq\nu(z_0),$$

which shows that the inequality (3.6) holds.

SUBCASE 2.1.3.2: $|\nu_2|=2$. Now by Remark 3.2, there exists some $j\in\mathbb{Z}[1,m]$ such that $\partial^{\nu_2}=\partial_{z_j}\partial_{z_i}$, and so from (3.5) we get

$$\begin{aligned}(3.11)\quad \mathbf{W}&=\partial_{z_i}F_3\partial_{z_j}(\partial_{z_i}F_1)-\partial_{z_i}F_1\partial_{z_j}(\partial_{z_i}F_3)\\&=\varphi_1^{-l-k-3}(-k\hat{F}_3\partial_{z_i}\varphi_1+\varphi_1\partial_{z_i}\hat{F}_3)\times(l(l+1)\hat{F}_1\partial_{z_i}\varphi_1\partial_{z_j}\varphi_1\\&\quad-l\varphi_1\partial_{z_i}\hat{F}_1\partial_{z_j}\varphi_1-l\varphi_1\hat{F}_1\partial_{z_i}\partial_{z_j}\varphi_1-l\varphi_1\partial_{z_i}\varphi_1\partial_{z_j}\hat{F}_1+\varphi_1^2\partial_{z_i}\partial_{z_j}\hat{F}_1)\\&\quad-\varphi_1^{-l-k-3}(-l\hat{F}_1\partial_{z_i}\varphi_1+\varphi_1\partial_{z_i}\hat{F}_1)\times(k(k+1)\hat{F}_3\partial_{z_i}\varphi_1\partial_{z_j}\varphi_1\\&\quad-k\varphi_1\partial_{z_i}\hat{F}_3\partial_{z_j}\varphi_1-k\varphi_1\hat{F}_3\partial_{z_i}\partial_{z_j}\varphi_1-k\varphi_1\partial_{z_i}\varphi_1\partial_{z_j}\hat{F}_3\\&\quad+\varphi_1^2\partial_{z_i}\partial_{z_j}\hat{F}_3).\end{aligned}$$

Also by Lemma 3.1, we get

$$(3.12)\qquad \mathbf{W}_1=\begin{vmatrix}F_1&F_2&F_3\\\partial_{z_i}F_1&\partial_{z_i}F_2&\partial_{z_i}F_3\\\partial_{z_j}F_1&\partial_{z_j}F_2&\partial_{z_j}F_3\end{vmatrix}\equiv 0.$$

First, we suppose $k=l$. Then $\mu^0_g(z_0)=0$, and so from (3.11) we get

$$\begin{aligned}(3.13)\qquad \mathbf{W}&=\varphi_1^{-2l-2}l\partial_{z_i}\varphi_1(l\partial_{z_i}\varphi_1(\hat{F}_3\partial_{z_j}\hat{F}_1-\hat{F}_1\partial_{z_j}\hat{F}_3)\\&\quad-\partial_{z_j}\varphi_1(\hat{F}_3\partial_{z_i}\hat{F}_1-\hat{F}_1\partial_{z_i}\hat{F}_3))+\cdots.\end{aligned}$$

If $l\geq 2$, then from (3.13) we get

$$\mu(z_0)\leq-2\mu^\infty_f(z_0)+\mu^\infty_{\mathbf{W}}(z_0)\leq-2\mu^\infty_f(z_0)+2l+2=2=\mu^\infty_{f,2}(z_0)\leq\nu(z_0).$$

Consequently, the inequality (3.6) holds. Next let $k = l = 1$. It is clear from (3.9) that $\hat{F}_1 + \hat{F}_2 + \hat{F}_3 = u_1$ on U. From (3.12), we see that

$$\mathbf{W}_1 = \varphi_1^{-3} \begin{vmatrix} \hat{F}_1 & \varphi_1 & \hat{F}_3 \\ \partial_{z_i}\hat{F}_1 & \partial_{z_i}\varphi_1 & \partial_{z_i}\hat{F}_3 \\ \partial_{z_j}\hat{F}_1 & \partial_{z_j}\varphi_1 & \partial_{z_j}\hat{F}_3 \end{vmatrix} \equiv 0,$$

and so

$$\partial_{z_i}\varphi_1(\hat{F}_3\partial_{z_j}\hat{F}_1 - \hat{F}_1\partial_{z_j}\hat{F}_3) - \partial_{z_j}\varphi_1(\hat{F}_3\partial_{z_i}\hat{F}_1 - \hat{F}_1\partial_{z_i}\hat{F}_3) = \varphi_1(\partial_{z_i}\hat{F}_3\partial_{z_j}\hat{F}_1 - \partial_{z_i}\hat{F}_1\partial_{z_j}\hat{F}_3).$$

Therefore, from (3.13) we get $\mu^0_{\mathbf{W}}(z_0) = 0$ and $\mu^\infty_{\mathbf{W}}(z_0) \le 3$. Consequently,

$$\mu(z_0) \le -2\mu^\infty_f(z_0) + \mu^\infty_{\mathbf{W}}(z_0) = 1 = \mu^\infty_{f,2}(z_0) \le \nu(z_0),$$

which shows that the inequality (3.6) holds.

Next we suppose $k \ne l$. Let $-l-k \ge 3$. Clearly $\mu^0_g(z_0) \ge 3+2\mu^\infty_f(z_0) \ge 5$. Then from (3.11) we get $\mu^\infty_{\mathbf{W}}(z_0) = 0$ and $\mu^0_{\mathbf{W}}(z_0) \ge -l-k-3 = -2\mu^\infty_f(z_0) + \mu^0_g(z_0) - 3$. Consequently,

$$\mu(z_0) = \mu^0_g(z_0) - 2\mu^\infty_f(z_0) - \mu^0_{\mathbf{W}}(z_0) \le 3 \le \mu^\infty_{f,2}(z_0) + \mu^0_{g,2}(z_0) \le v(z_0),$$

which shows that the inequality (3.6) holds. Let $-l-k < 3$. Then (3.11) gives $\mu^0_{\mathbf{W}}(z_0) = 0$ and

$$\begin{aligned}\mu(z_0) &\le \mu^0_g(z_0) - 2\mu^\infty_f(z_0) + \mu^\infty_{\mathbf{W}}(z_0) \\ &\le \mu^0_g(z_0) - 2\mu^\infty_f(z_0) + l + k + 3 = 3 \le \mu^\infty_{f,2}(z_0) + \mu^0_{g,2}(z_0) \le \nu(z_0),\end{aligned}$$

which shows that the inequality (3.6) holds.

Subcase 2.1.4: $\mu^\infty_f(z_0) > 0$ *and* $\mu^\infty_g(z_0) > 0$. In this case, we have

$$\mu(z_0) = \mu^0_{F_2}(z_0) - \mu^\infty_{F_2}(z_0) - \mu^\infty_{F_3}(z_0) + \mu^\infty_{\mathbf{W}}(z_0) - \mu^0_{\mathbf{W}}(z_0). \tag{3.14}$$

Let $\mu^\infty_f(z_0) = \mu^\infty_g(z_0)$. From (3.2), we have $\mu^0_{F_2}(z_0) = \mu^\infty_{F_2}(z_0) = 0$. Then from (3.4) we get $\mu^0_{\mathbf{W}}(z_0) = 0$ and $\mu^\infty_{\mathbf{W}}(z_0) \le \mu^\infty_f(z_0) + 2$. Since $\mu^\infty_{F_3}(z_0) = \mu^\infty_f(z_0)$, from (3.14) we have

$$\mu(z_0) \le -\mu^\infty_f(z_0) + \mu^\infty_{\mathbf{W}}(z_0) \le 2 \le \mu^\infty_{f,2}(z_0) + \mu^\infty_{g,2}(z_0) \le \nu(z_0),$$

which shows that the inequality (3.6) holds.

Let $\mu^\infty_f(z_0) < \mu^\infty_g(z_0)$. Then from (3.4) we have either

$$\mu^\infty_{\mathbf{W}}(z_0) \le 2\mu^\infty_f(z_0) - \mu^\infty_g(z_0) + 3 \text{ if } \mu^0_{\mathbf{W}}(z_0) = 0$$

or

$$\mu^0_{\mathbf{W}}(z_0) \ge \mu^\infty_g(z_0) - 2\mu^\infty_f(z_0) - 3 \text{ if } \mu^\infty_{\mathbf{W}}(z_0) = 0.$$

Therefore, in either case, we must have

$$\begin{aligned}\mu(z_0) &= \mu_g^{\infty}(z_0) - 2\mu_f^{\infty}(z_0) + \mu_{\mathbf{W}}^{\infty}(z_0) - \mu_{\mathbf{W}}^{0}(z_0) \le 3\\ &\le \mu_{f,2}^{\infty}(z_0) + \mu_{g,2}^{\infty}(z_0) \le \nu(z_0),\end{aligned}$$

which shows that the inequality (3.6) holds.

Let $\mu_f^{\infty}(z_0) > \mu_g^{\infty}(z_0)$. Then from (3.4), $\mu_{\mathbf{W}}^{\infty}(z_0) \le 2\mu_f^{\infty}(z_0) - \mu_g^{\infty}(z_0) + 3$ and hence

$$\mu(z_0) = \mu_g^{\infty}(z_0) - 2\mu_f^{\infty}(z_0) + \mu_{\mathbf{W}}^{\infty}(z_0) \le 3 \le \mu_{f,2}^{\infty}(z_0) + \mu_{g,2}^{\infty}(z_0) \le \nu(z_0),$$

which shows that the inequality (3.6) holds.

Therefore, all the foregoing discussion shows that the inequality (3.6) holds and so we get

$$\begin{aligned}(3.15)\qquad \| \quad N(r,F_1) + \sum_{k=1}^{3}\{N(r,0;F_k) - N(r,F_k)\} + N(r,\mathbf{W}) - N(r,0;\mathbf{W})\\ \le N_2(r,0;f) + N_2(r,0;g) + N_2(r,f) + N_2(r,g).\end{aligned}$$

On the other hand, from (3.2), we get $\| \ T_1(r) \le O(T(r))$. Hence, (3.3) and (3.15) give

$$\| \quad T(r,f) \le N_2(r,f) + N_2(r,g) + N_2(r,0;f) + N_2(r,0;g) + o(T(r)),$$

and so we obtain (i).

Subcase 2.2: *F_1, F_2 and F_3 are linearly dependent.* Then there exists $(c_1,c_2,c_3) \in \mathbb{C}^3 - \{0\}$ such that $c_1F_1 + c_2F_2 + c_3F_3 = 0$. Let $c_1 - c_2 = A$, $c_2 = B$, $c_1 - c_3 = C$ and $c_3 = D$. Then

$$(3.16)\qquad g \equiv \frac{Af+B}{Cf+D}.$$

Clearly $AD - BC \neq 0$. Now applying Lemma 3.2 to (3.16), we obtain

$$(3.17)\qquad \| \quad T(r,g) = T(r,f) + o(T(r,f)).$$

Next we consider the following subcases.

Subcase 2.2.1: $AC \neq 0$. Then from (3.16), we get $g - \frac{A}{C} = \frac{B - AD/C}{Cf+D}$. Clearly,

$$\| \quad \overline{N}(r,A/C;g) = \overline{N}(r,f),$$

and so using Lemma 2.1 and (3.17) we get

$$\begin{aligned}\| \quad T(r,f) &= T(r,g) + o(T(r,f))\\ &\le \overline{N}(r,g) + \overline{N}(r,A/C;g) + \overline{N}(r,0;g) + o(T(r,g))\\ &= \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}(r,0;g) + o(T(r,g)).\end{aligned}$$

Therefore, we get (i).

Subcase 2.2.2: *$A \neq 0$ and $C = 0$.* Then from (3.16), we have $g \equiv (Af + B)/D$.

Let $B \neq 0$. Clearly $\| \overline{N}(r, B/D; g) = \overline{N}(r, 0; f)$, and so using Lemma 2.1 and (3.17) we get

$$\begin{aligned} \| \quad T(r,f) &= T(r,g) + o(T(r,g)) \\ &\leq \overline{N}(r,g) + \overline{N}(r,B/D;g) + \overline{N}(r,0;g) + o(T(r,f)) \\ &= \overline{N}(r,g) + \overline{N}(r,0;f) + \overline{N}(r,0;g) + o(T(r,f)) \end{aligned}$$

and so we obtain (i).

Let $B = 0$. Then $g \equiv Af/D$. Let 1 not be a Picard exceptional value of f. Since f and g share 1 CM, there exists $\hat{z} \in \mathbb{C}^m$ such that $f(\hat{z}) = 1$ and $g(\hat{z}) = 1$. Since $g \equiv Af/D$, we get $A/D = 1$ and so $f \equiv g$, which is (ii). Next let 1 be a Picard exceptional value of f. Then by Lemma 2.1, we get $\| T(r,f) \leq \overline{N}(r,f) + \overline{N}(r,0;f) + o(T(r,f))$, and so (i) follows.

Subcase 2.2.3: *$A = 0$ and $C \neq 0$.* Then $g \equiv B/(Cf + D)$.

Let $D \neq 0$. Clearly $\| \overline{N}(r, B/D; g) = \overline{N}(r, 0; f)$ and so using Lemma 2.1 and (3.17), we get

$$\begin{aligned} \| \quad T(r,f) &= T(r,g) + o(T(r,g)) \\ &\leq \overline{N}(r,g) + \overline{N}(r,B/D;g) + \overline{N}(r,0;g) + o(T(r,f)) \\ &= \overline{N}(r,g) + \overline{N}(r,0;f) + \overline{N}(r,0;g) + o(T(r,f)). \end{aligned}$$

Thus we get (i).

Let $D = 0$. Then $g = B/Cf$. Let 1 not be a Picard exceptional value of f. Then there exists $\hat{z} \in \mathbb{C}^m$ such that $f(\hat{z}) = 1$ and $g(\hat{z}) = 1$. Since $g = B/Cf$, we get $B/C = 1$ and so $fg \equiv 1$, which is (iii). Next let 1 be a Picard exceptional value of f. Then by Lemma 2.1, we get

$$\| \quad T(r,f) \leq \overline{N}(r,f) + \overline{N}(r,0;f) + o(T(r,f)),$$

and (i) follows. ∎

We define the following divisor

$$\mu^0_{\partial_u f | f \neq 0}(z) = \begin{cases} \mu^0_{\partial_u f}(z) & \text{if } \mu^0_{\partial_u f}(z) > 0 \text{ and } \mu^0_f(z) = 0 \\ 0 & \text{or else.} \end{cases}$$

We denote by $N(r, 0; \partial_u(f) \,|\, f \neq 0)$ the counting function of the divisor $\mu^0_{\partial_u f | f \neq 0}(z)$.

Lemma 3.6. *Let $g : \mathbb{C}^m \to \mathbb{P}^1$ be a nonconstant meromorphic function and let $G = g^n \partial_u(g)$, where n is a positive integer. Then*

(i) $\| \ (n+1)T(r,g) \leq T(r,G) + N(r,0;g) - N(r,0;\partial_u(g)) + o(T(r,g))$,
(ii) $\| \ N(r,0;\partial_u(g) \,|\, g \neq 0) \leq \overline{N}(r,0;g) + \overline{N}(r,g) + o(T(r,g))$,
(iii) $\| \ N(r,0;\partial_u(g)) \leq N(r,0;g) + \overline{N}(r,g) + o(T(r,g))$.

Proof. (i) Using the first main theorem and Lemma 2.1, we get

$$\begin{aligned}\| \quad (n+1)T(r,g) &= T(r,g^{n+1})\\ &= m(r,0,g^{n+1}) + N(r,0;g^{n+1}) + O(1)\\ &\le m(r,G/g^{n+1}) + m(r,0;G) + N(r,0;g^{n+1}) + O(1)\\ &= m(r,0;G) + N(r,0,g^{n+1}) + o(T(r,g))\\ &= T(r,G) - N(r,0;G) + N(r,0;g^{n+1}) + o(T(r,g))\\ &\le T(r,G) + N(r,0;g) - N(r,0;\partial_u(g)) + o(T(r,g)).\end{aligned}$$

(ii) Using the first main theorem and Lemma 2.1, we get

$$\begin{aligned}\| \quad N(r,0;\partial_u(g)\big|g\neq 0) \le N(r,g/\partial_u(g)) &\le T(r,g/\partial_u(g))\\ &= T(r,\partial_u(g)/g) + O(1)\\ &\le N(r,\partial_u(g)/g) + o(T(r,g)),\end{aligned}$$

i.e.,

$$\| \quad N(r,0;\partial_u(g)\,|\,g\neq 0) \le N(r,H) + o(T(r,g)), \tag{3.18}$$

where $H = \partial_u(g)/g$.

We now prove $\mu_H^\infty \le \mu_{g,1}^0 + \mu_{g,1}^\infty$. Define $S = \bigcup_{a\in\{0,\infty\}}(\operatorname{supp}\mu_g^a)_s$, where A_s denotes the set of singular points of the analytic set A. Take $z_0 \in \mathbb{C}^m \setminus S$.

First we assume that $l = \mu_g^\infty(z_0) > 0$. Then from (2.3), we obtain

$$\| \quad N(r,\partial_u(g)) \le N(r,g) + \overline{N}(r,g)$$

and

$$H = \frac{\partial_u(g)}{g} = -\frac{l}{\varphi_1}\partial_u(\varphi_1) + \frac{1}{\hat g}\sum_{j=1}^m \frac{\partial \hat g}{\partial \varphi_j}\partial_u(\varphi_j).$$

It is easy to verify that $\mu_H^\infty(z_0) \le \mu_{g,1}^\infty(z_0)$ and so $\mu_H^\infty(z_0) \le \mu_{g,1}^0(z_0) + \mu_{g,1}^\infty(z_0)$.

Next we assume that $l = \mu_g^0(z_0) > 0$. Proceeding in the same way as above, we can write $g = \varphi_1^l \hat g(\varphi_1,\dots,\varphi_m)$, where $\hat g$ is holomorphic function near 0, which does not vanish on $\operatorname{supp}\mu_g^0$. Obviously

$$\partial_u(g) = l\varphi_1^{l-1}\hat g\partial_u(\varphi_1) + \varphi_1^l\sum_{j=1}^m \frac{\partial \hat g}{\partial \varphi_j}\partial_u(\varphi_j),$$

and so

$$H = \frac{\partial_u(g)}{g} = \frac{l}{\varphi_1}\partial_u(\varphi_1) + \frac{1}{\hat g}\sum_{j=1}^m \frac{\partial \hat g}{\partial \varphi_j}\partial_u(\varphi_j),$$

which means that $\mu_H^\infty(z_0) \le \mu_{g,1}^0(z_0)$ and so $\mu_H^\infty(z_0) \le \mu_{g,1}^0(z_0) + \mu_{g,1}^\infty(z_0)$.

Therefore

$$\| \quad N(r,\partial_u(g)/g) \le \overline{N}(r,0;g) + \overline{N}(r,g),$$

and so from (3.18) we have

$$\| \quad N(r,0;\partial_u(g)\,|\,g\neq 0) \le \overline{N}(r,0;g)+\overline{N}(r,g)+o(T(r,g)).$$

(iii) Using the first main theorem and Lemma 2.1, we get

$$\| \quad m(r,0,g) \le m(r,0,\partial_u(g))+m(r,\partial_u(g)/g) = m(r,0;\partial_u(g))+o(T(r,g)),$$

i.e.,

$$\begin{aligned}\| \quad N(r,0;\partial_u(g)) &\le T(r,\partial_u(g))-T(r,g)+N(r,0,g)+o(T(r,g))\\ &\le N(r,\partial_u(g))+m(r,g)-T(r,g)+N(r,0;g)+o(T(r,g))\\ &\le \overline{N}(r,g)+N(r,0;g)+o(T(r,g)).\ \blacksquare\end{aligned}$$

3.2. Proof of Theorem 3.1. Let $F=f^n\partial_u(f)/a$ and $G=g^n\partial_u(g)/a$. Clearly F and G share 1 CM. Now by Lemma 3.5, we now consider the following cases.

Case 1: Let

$$\| \quad T(r,F)\le N_2(r,F)+N_2(r,G)+N_2(r,0;F)+N_2(r,0;G)+o(T_2(r)), \tag{3.19}$$

where $T_2(r)=\max\{T(r,F),T(r,G)\}$.

Since $\| \ N(r,\partial_u(f))\le N(r,f)+\overline{N}(r,f)$, by Lemma 2.1 we get $\| \ T(r,F)\le (n+2)T(r,f)+o(T(r,f))$. Similarly, we have $\| \ T(r,G)\le (n+2)T(r,g)+o(T(r,g))$.

Therefore if we take $T(r)=\max\{T(r,f),T(r,g)\}$, then $o(T_2(r))$ can be replaced by $o(T(r))$. Now in view of (3.19) and using Lemma 3.6, we get

$$\begin{aligned}\| \quad (n+1)T(r,f) &\le N_2(r,F)+N_2(r,G)+N_2(r,0,F)+N_2(r,0,G)\\ &\quad +N(r,0;f)-N(r,0;\partial_u(f))+o(T(r))\\ &\le 2[\overline{N}(r,F)+\overline{N}(r,G)]+3N(r,0;f)+2N(r,0;g)\\ &\quad +N(r,0;\partial_u(g)\big|g\neq 0)+o(T(r))\\ &\le 2\overline{N}(r,f)+3\overline{N}(r,g)+3N(r,0;f)+3N(r,0;g)+o(T(r))\\ &\le 11T(r)+o(T(r)).\end{aligned} \tag{3.20}$$

Similarly, we have $\| \ (n+1)T(r,g)\le 11T(r)+o(T(r))$, and so from (3.20) we get $\| \ (n+1)T(r)\le 11T(r)+o(T(r))$, which is impossible.

Case 2: $FG\equiv 1$. Then $f^n\partial_u(f)g^n\partial_u(g)\equiv a^2$, and so by Theorem 2.2 we get $f=e^{\alpha}$ and $g=e^{\beta}$, where α and β are nonconstant entire functions in $\mathbb{C}^m$ such that $\partial_u(\alpha)\equiv -\partial_u(\beta)$, $\partial_u^2(\alpha)\equiv 0$ and $(\partial_u(\alpha))^2e^{(n+1)(\alpha+\beta)}\equiv -a^2$.

Case 3: $F\equiv G$. Then $\partial_u(f^{n+1})\equiv\partial_u(g^{n+1})$. Let $f=hg$. Then

$$\partial_u(g^{n+1}(h^{n+1}-1))\equiv 0. \tag{3.21}$$

First suppose $h^{n+1}\equiv 1$. Then $f=cg$, where $c^{n+1}=1$.

Next we suppose $h^{n+1} \not\equiv 1$. If h is a constant, then from (3.21) we get $\partial_u(g) \equiv 0$, which is impossible. Hence h is nonconstant. From (3.21), we have

$$(n+1)g^n\partial_u(g) \equiv -\frac{\partial_u(h^{n+1}-1)}{h^{n+1}-1} = -(n+1)\frac{h^n\partial_u(h)}{h^{n+1}-1}. \tag{3.22}$$

Let $h_1 = h^{n+1} - 1 = (h-\alpha_1)\dots(h-\alpha_{n+1})$, where $\alpha_1, \dots, \alpha_{n+1}$ are distinct $(n+1)$-th roots of unity. Suppose $\mu^0_{h_1}(z_0) > 0$ for some $z_0 \in \mathbb{C}^m$. Clearly $\mu^0_{\partial_u(h)}(z_0) \le \mu^0_{h_1}(z_0)$ holds generically. If $\mu^0_{\partial_u(h)}(z_0) = 0$, then from (3.22) we have generically $\mu^0_{h_1}(z_0) \ge 2$ over $\operatorname{supp}\mu^0_{h_1}$ and so

$$\| \quad \overline{N}(r,\alpha_j,h) \le (1/2)N(r,\alpha_j;h) \le (1/2)T(r,h) + O(1),$$

where $j = 1, \dots, n+1$. If $\mu^0_{\partial_u(h)}(z_0) > 0$, then by Lemma 3.6 we get

$$\| \quad \overline{N}(r,1;h^{n+1}) \le \overline{N}(r,0;\partial_u(h)) \le N(r,0;h) + \overline{N}(r,h) + o(T(r,h)).$$

Therefore, in either case,

$$\begin{aligned}\| \quad \sum_{j=1}^{n+1}\overline{N}(r,\alpha_j;h) &\le ((n+1)/2)T(r,h) + N(r,0;h) + \overline{N}(r,h) + o(T(r,h)) \\ &\le ((n+5)/2)T(r,h) + o(T(r,h)),\end{aligned}$$

and so by Lemma 3.2 we get

$$\begin{aligned}\| \quad (n-1)T(r,h) &\le \sum_{j=1}^{n+1} N(r,\alpha_j;h) + o(T(r,h)) \\ &\le ((n+5)/2)T(r,h) + o(T(r,h)),\end{aligned}$$

which is impossible since $n \ge 11$. ∎

3.3. Proof of Theorem 3.2. Using Corollary 2.1 instead of Theorem 2.2, we get Theorem 3.2 directly from the proof of Theorem 3.1. Hence we omit the details.

4. Applications. As an application of Corollary 3.1, we present the derivations of entire solutions in $\mathbb{C}^m$ of the nonlinear partial differential equations

$$y^n\partial_u(y) - 1 = e^{\alpha}(e^{\beta} - 1), \tag{4.1}$$

where $n \ge 7$ is an integer, α and β are entire functions in $\mathbb{C}^m$.

If we take $e^{\beta} = g^n\partial_u(g)$ for some entire function g in $\mathbb{C}^m$, then g has no zeros. Therefore we may assume that $g = e^{\gamma}$, where γ is an entire function in $\mathbb{C}^m$. Since $e^{\beta} = g^n\partial_u(g)$, we have

$$e^{\beta-(n+1)\gamma} = \partial_u(\gamma), \tag{4.2}$$

which shows that $\partial_u(\gamma)$ has no zeros and so there exists an entire function δ in $\mathbb{C}^m$ such that $\partial_u(\gamma) = e^{\delta}$. From (4.2), we deduce that $\beta - (n+1)\gamma - \delta$ is a constant, and so

$$\partial_u(\beta) = (n+1)e^{\delta} + \partial_u(\delta) = (n+1)\partial_u(\gamma) + \partial_u^2(\gamma)/\partial_u(\gamma) \tag{4.3}$$

for every $u \in S_m$. Now by Corollary 3.1, we see that one of the following cases holds:

(i) $y \equiv dg$, where $d^{n+1} = 1$;
(ii) $y = e^{\tilde{\alpha}}$ and $g = e^{\gamma}$, where c is a nonzero constant in $\mathbb{C}$, $\tilde{\alpha}$ and γ are nonconstant entire functions in $\mathbb{C}^m$ such that $\partial_u(\tilde{\alpha}) = -c$, $\partial_u(\gamma) = c$ and $c^2 e^{(n+1)(\tilde{\alpha}+\gamma)} = -1$.

For (ii), we have $\partial_u(\gamma) = c$. Since $\partial_u(\gamma) = e^{\delta}$, it follows that $\delta \in \mathbb{C}$. Clearly $e^{\delta} = c$ and so $\partial_u(\delta) = 0$. Therefore, from (4.3) we get $\partial_u(\beta) = (n+1)c$ for all $u \in S_m$. Consequently $y = e^{\tilde{\alpha}}$, where $\tilde{\alpha}$ is a nonconstant entire function in $\mathbb{C}^m$ such that $\partial_u(\tilde{\alpha}) = -c \in \mathbb{C} \setminus \{0\}$ and $(n+1)\partial_u(\tilde{\alpha}) = -\partial_u(\beta)$.

If β is a polynomial, then from the proof of Corollary 2.2 we see that $\deg(\beta) = 1$, and so by (4.3), δ is a constant. Since $\partial_u(\gamma) = e^{\delta}$ for all $u \in S_m$, it follows that $\partial_u(\gamma)$ for all $u \in S_m$, and so from the proof of Corollary 2.2 we see that $\deg(\gamma) = 1$.

Finally, we get the following results.

THEOREM 4.1. *Let $n \geq 7$ be an integer and let α and β be entire functions in $\mathbb{C}^m$. Then every solution of (4.1) is of the form $y = ce^{\gamma}$, where γ is an entire function in $\mathbb{C}^m$ such that $\partial_u(\gamma)$ has no zeros and*

$$\partial_u(\beta)\partial_u(\gamma) = (n+1)(\partial_u(\gamma))^2 + \partial_u^2(\gamma)$$

for all $u \in S_m$ or $y = c_1 e^{\tilde{\alpha}}$, where $\tilde{\alpha}$ is a nonconstant entire function in $\mathbb{C}^m$ such that $\partial_u(\tilde{\alpha}) = -c$, a nonzero constant in $\mathbb{C}$ and $(n+1)\partial_u(\tilde{\alpha}) = -\partial_u(\beta)$ for all $u \in S_m$.

THEOREM 4.2. *Let $n \geq 7$ be an integer and α be an entire function in $\mathbb{C}^m$ and let β be a polynomial in $\mathbb{C}^m$. Then β reduces to a polynomial of degree one and every solution of (4.1) is of the form $y = ce^{\gamma}$, where γ is a polynomial in $\mathbb{C}^m$ such that $\deg(\gamma) = 1$ and $(n+1)\partial_u(\gamma) = \partial_u(\beta)$ for all $u \in S_m$, or $y = c_1 e^{\tilde{\alpha}}$, where $\tilde{\alpha}$ is a polynomial in $\mathbb{C}^m$ of degree one such that $(n+1)\partial_u(\tilde{\alpha}) = -\partial_u(\beta)$ for all $u \in S_m$.*

EXAMPLE 4.1. Let $\dim(\mathbb{C}^m) = 1$ and $n \geq 7$. Consider the nonlinear differential equation

$$y^n y' - 1 = e^{\alpha(z)}(e^{\beta(z)} - 1),$$

where $\alpha(z)$ and $\beta(z)$ are entire functions. Suppose $\alpha \equiv 0$ and $\beta(z) = az + b$ with constants $a, b \in \mathbb{C}$, $a \neq 0$. Then by Theorem 4.2, every solution satisfies $y(z) = ce^{\gamma(z)}$, $(n+1)\gamma'(z) = \beta'(z) = a$, so that $\gamma(z) = \frac{az}{n+1} + c_1$, where c_1 is

a constant. Hence $y(z) = Ce^{\frac{az}{n+1}}$,where $C = ce^{c_1}$. Thus, the equation admits an exponential-type entire solution whose growth rate is determined by the slope of β.

EXAMPLE 4.2. Let $\dim(\mathbb{C}^m) = 2$ and $u = (1,1)$ so that $\partial_u = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$. Consider $\alpha \equiv 0$ and $\beta(z_1, z_2) = a_1 z_1 + a_2 z_2 + b$, where $a_1, a_2, b \in \mathbb{C}$. Then by Theorem 4.2, any entire solution of $y^n \partial_u(y) - 1 = e^{\beta(z_1,z_2)} - 1$ is of the form $y(z_1, z_2) = c_1 e^{\gamma(z_1,z_2)}$, $(n+1)\partial_u(\gamma) = \partial_u(\beta)$. Since $\partial_u(\beta) = a_1 + a_2$, we obtain $\partial_u(\gamma) = \frac{a_1+a_2}{n+1}$. Integration gives $\gamma(z_1, z_2) = \frac{a_1 z_1 + a_2 z_2}{n+1} + c_2$ and hence

$$y(z_1, z_2) = Ce^{\frac{a_1 z_1 + a_2 z_2}{n+1}},$$

where $C = c_1 e^{c_2}$. This shows that the solution is an entire exponential surface depending linearly on both variables.

EXAMPLE 4.3. Let $\dim(\mathbb{C}^m) \geq 1$, $u = (1, 1, \ldots, 1)$, and assume $\alpha \equiv 0$, $\beta(z) = \sum_{j=1}^m a_j z_j + b$ with constants $a_j, b \in \mathbb{C}$. Then, from Theorem 4.2, each solution of $y^n \partial_u(y) - 1 = e^{\alpha(z)}(e^{\beta(z)} - 1)$ satisfies

$$(n+1)\partial_u(\gamma) = \partial_u(\beta) = \sum_{j=1}^m a_j.$$

Consequently, $\gamma(z) = \frac{\sum_{j=1}^m a_j z_j}{n+1} + c_3$, so that

$$y(z) = C \exp\left(\frac{\sum_{j=1}^m a_j z_j}{n+1}\right),$$

where $C = ce^{c_3}$. Thus, the solution represents an entire exponential hypersurface in $\mathbb{C}^m$, determined by the direction vector $(a_1, \ldots, a_m)$.

EXAMPLE 4.4. Let $\dim(\mathbb{C}^m) = 2$, $u = (1,1)$, $\alpha \equiv 2\pi i$ and $\beta(z_1, z_2) = a(z_1 + z_2) + b(z_1 - z_2) + c$, where $a, b, c \in \mathbb{C}$ and $b \neq 0$. Then $\partial_u(\beta) = \frac{\partial \beta}{\partial z_1} + \frac{\partial \beta}{\partial z_2} = (a+b) + (a-b) = 2a$. The equation

$$y^n \left(\frac{\partial y}{\partial z_1} + \frac{\partial y}{\partial z_2}\right) - 1 = e^{\alpha(z_1,z_2)}(e^{\beta(z_1,z_2)} - 1)$$

admits the entire solution

$$y(z_1, z_2) = Ce^{\frac{a+b}{n+1} z_1 + \frac{a-b}{n+1} z_2}, \qquad C \in \mathbb{C} \setminus \{0\}.$$

EXAMPLE 4.5. Let $\dim(\mathbb{C}^m) = 3$, $u = (1, 0, 1)$, $\alpha \equiv 2\pi i$ and $\beta(z_1, z_2, z_3)$ $= a_1 z_1 + a_2 z_2 + a_3 z_3 + b\, z_1 z_3 + c$, with $a_j, b, c \in \mathbb{C}$ and b small. Then $\partial_u \beta = \frac{\partial}{\partial z_1}\beta + \frac{\partial}{\partial z_3}\beta = a_1 + bz_3 + a_3 + bz_1 = (a_1 + a_3) + b(z_1 + z_3)$. For $b = 0$, the equation

$$y^n \left(\frac{\partial y}{\partial z_1} + \frac{\partial y}{\partial z_3}\right) - 1 = e^{\alpha(z_1,z_2,z_3)}(e^{\beta(z_1,z_2,z_3)} - 1)$$

has the solution

$$y(z_1, z_2, z_3) = C \exp\left(\frac{a_1 z_1 + a_2 z_2 + a_3 z_3}{n+1}\right),$$

where $C \in \mathbb{C} \setminus \{0\}$. For small $b \neq 0$, the solution can be adjusted by perturbation theory.

EXAMPLE 4.6. Let $\dim(\mathbb{C}^m) = 2$, $u = (1,1)$, $\alpha \equiv 2\pi i$ and $\beta(z_1, z_2) = a(z_1 + \lambda z_2) + b\cos(z_1 - z_2) + c$, with $a, b, c, \lambda \in \mathbb{C}$, $|\lambda| \neq 1$, $b \neq 0$. Then $\partial_u \beta = a(1+\lambda)$. The equation

$$y^n \left(\frac{\partial y}{\partial z_1} + \frac{\partial y}{\partial z_2}\right) - 1 = e^{\alpha(z_1,z_2)}(e^{\beta(z_1,z_2)} - 1)$$

admits the explicit entire solution

$$y(z_1, z_2) = C \exp\left(\frac{a}{n+1}(z_1 + \lambda z_2) + \frac{b}{n+1}\cos(z_1 - z_2)\right), \quad C \in \mathbb{C} \setminus \{0\}.$$

EXAMPLE 4.7. Let $\dim(\mathbb{C}^m) = 2$, $u = (1,0)$ so that $\partial_u = \frac{\partial}{\partial z_1}$. Take $\beta(z_1, z_2) = az_1^2 + bz_2 + c$, $\alpha \equiv 0$, where $a, b, c \in \mathbb{C}$ and $a \neq 0$. Then $\partial_u(\beta) = \frac{\partial \beta}{\partial z_1} = 2az_1$, which is nonconstant and depends on z_1. Consider the nonlinear partial differential equation

$$y^n \frac{\partial y}{\partial z_1} - 1 = e^{\beta(z_1,z_2)} - 1.$$

To satisfy the compatibility relation $(n+1)\partial_u(\gamma) = \partial_u(\beta)$, we would need $\partial_u(\gamma) = \frac{2az_1}{n+1}$, which clearly vanishes at $z_1 = 0$ and hence $\partial_u(\gamma)$ has zeros. Consequently, $\partial_u(\gamma)$ cannot be entire and zero-free. Moreover, $\partial_u^2(\gamma) = \frac{2a}{n+1}$ does not satisfy the nonlinear condition

$$\partial_u(\beta)\partial_u(\gamma) = (n+1)(\partial_u(\gamma))^2 + \partial_u^2(\gamma)$$

for all $z_1 \in \mathbb{C}$. Hence, in this case there exists no entire solution of exponential type. This example demonstrates that any quadratic dependence of β on the direction variable u destroys the exponential-type entire solution structure.

EXAMPLE 4.8. Let $\dim(\mathbb{C}^m) = 2$, $u = (1,1)$ and take a mixed linear-periodic function $\beta(z_1, z_2) = a(z_1 + z_2) + d\sin(z_1 + 2z_2) + c$, where $a, d, c \in \mathbb{C}$. Then $\partial_u(\beta) = \frac{\partial \beta}{\partial z_1} + \frac{\partial \beta}{\partial z_2} = 2a + d\cos(z_1 + 2z_2)(1+2) = 2a + 3d\cos(z_1 + 2z_2)$, which is a nonconstant oscillatory function. Substituting this into the compatibility condition

$$\partial_u(\beta)\partial_u(\gamma) = (n+1)(\partial_u(\gamma))^2 + \partial_u^2(\gamma),$$

we obtain a differential equation involving γ. If we let $\gamma(z_1, z_2) = \Gamma(s)$, where $s = z_1 + 2z_2$, then $\partial_u(\gamma) = \frac{\partial \gamma}{\partial z_1} + \frac{\partial \gamma}{\partial z_2} = 3\Gamma'(s)$. The above compatibility condition then becomes the following Riccati-type equation:

$$(2a + 3d\cos s)\Gamma(s) = (n+1)\,3\Gamma(s)^2 + \Gamma'(s).$$

In general, this equation does not admit an entire zero-free solution $G(s)$ unless $d = 0$, in which case β becomes linear and Theorem 4.2 applies. Therefore, for $d \neq 0$, there is no entire exponential-type solution of (4.1). This example further confirms that the linearity of β in the direction of u is essential for the existence of entire exponential solutions.

REMARK 4.1. Examples 4.7 and 4.8 emphasize the necessity of the linearity condition on β. When β involves nonlinear or oscillatory terms (such as quadratic or trigonometric components) along the direction of differentiation ∂_u, the functional relation

$$\partial_u(\beta)\partial_u(\gamma) = (n+1)(\partial_u(\gamma))^2 + \partial_u^2(\gamma)$$

fails to possess zero-free entire solutions γ. Hence, entire exponential-type solutions arise only when β is a polynomial of degree one in the variables along u, fully confirming the restriction imposed in Theorem 4.2.

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