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On exact additive complements

by

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Abstract. Two infinite sets A, B of positive integers are called *additive complements* if all sufficiently large integers can be expressed as $a + b$ with $a \in A$ and $b \in B$. Such A, B are further termed *exact additive complements* if $A(x)B(x) = x + o(x)$, where $A(x)$ (resp. $B(x)$) denotes the number of integers in A (resp. B) not exceeding x. In 2001, Ruzsa [Combinatorica 21 (2001), 279–291] initially showed that a class of growing rapid sets has exact additive complements, that is, if $A = \{a_1, a_2, \dots\}$ is a set of positive integers such that $a_{n+1}/(na_n) \to +\infty$ as $n \to +\infty$, then A has an exact additive complement B. In the present paper, we give a qualitative version of Ruzsa's result and prove that if $A = \{a_1, a_2, \dots\}$ is a set of positive integers with $1 \le a_1 < a_2 < \cdots$ and $a_{i+1} \ge ia_i$ for all sufficiently large i, then there exists an additive complement B of A such that

$$A(x)B(x) = x + \frac{8xh(A(x))}{A(x)} + O\Big(\frac{x}{A(x)}\Big),$$

where $h(t)$ is the integer with

$$(h(t)-1)\rho(h(t)-1) \le t^2 < h(t)\rho(h(t))$$

and $\rho(i) = \inf_{j \ge i}(a_{j+1}/a_j)$. Furthermore, if $a_{i+1}/(ia_i) \to +\infty$ as $i \to +\infty$, then $h(A(x))/A(x) \to 0$ as $x \to +\infty$ and so B is an exact additive complement of A.

1. Introduction.

Two infinite sets A, B of positive integers are called *additive complements* if all sufficiently large integers can be expressed as $a + b$ with $a \in A$ and $b \in B$. Such A, B are further termed *exact additive complements* if

$$A(x)B(x) = x + o(x), \tag{1.1}$$

where $A(x)$ (resp. $B(x)$) denotes the number of integers in A (resp. B) not exceeding x.

In 1994, by confirming a conjecture posed by Danzer [4], Sárközy and Szemerédi [14] proved that, for additive complements A and B with (1.1),

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we have

$$A(x)B(x) - x \to +\infty \quad \text{as } x \to +\infty$$

(see also[5], [9, p. 75] and [11]). They conjectured that there exist additive complements A and B such that $A(x)B(x) - x = O(\min\{A(x), B(x)\})$.

In 2015, Chen and Fang [2] disproved the above conjecture of Sárközy and Szemerédi by proving the following result.

THEOREM A. *For additive complements A and B with* (1.1) *and any number M, we have*

$$A(x)B(x) - x \geq \min\{A(x)^M, B(x)^M\}$$

for all sufficiently large x.

Narkiewicz [11] showed that if A, B are additive complements with (1.1), then

$$\lim_{x\to+\infty} \frac{A(2x)}{A(x)} = 1 \tag{1.2}$$

or

$$\lim_{x\to+\infty} \frac{B(2x)}{B(x)} = 1.$$

For any x with $A\cap[1,x] \neq \emptyset$, let $a^*(x)$ be the largest element in $A\cap[1,x]$ and let $r(x)$ be the number of positive integers $n \leq x$ which cannot be represented as $n = a + b$ with $a \in A$ and $b \in B$. It is clear that A and B are additive complements if and only if $r(x) = O(1)$. If (1.2) holds, then for any $\varepsilon > 0$, $A(x) \leq x^\varepsilon$ for all sufficiently large x. It follows that $A(x) = A(a^*(x)) \leq a^*(x)^\varepsilon$ for all sufficiently large x. This implies that for any real number M, $a^*(x) > A(x)^M$ for all sufficiently large x. So the following Theorem B is an improvement of the above Theorem A.

In 2017, Ruzsa [13] relaxed the condition $r(x) = O(1)$ and improved Theorem A:

THEOREM B. *For two sets A and B satisfying* (1.1) *and* (1.2)*, if $r(x) = o(a^*(x)/A(x))$, then*

$$A(x)B(x) - x \geq (1 - o(1))\frac{a^*(x)}{A(x)}.$$

In 2019, Chen and Fang [3] removed condition (1.1) and showed the following:

THEOREM C. *For two infinite sequences A and B with Narkiewicz's condition* (1.2) *and $r(x) = o(a^*(x))$, we have*

$$A(x)B(x) - x + r(x) \geq (1 - o(1))\frac{a^*(x)}{A(x)}.$$

THEOREM D. *There exist two infinite sequences A and B with Narkiewicz's condition* (1.2) *and $r(x) = O(1)$ such that*

$$A(x)B(x) - x + r(x) \le (1 + o(1))\frac{a^*(x)}{A(x)}$$

for infinitely many positive integers x.

For exact additive complements, Ruzsa [12] initially showed that a class of growing rapid sets has exact additive complements.

THEOREM E. *Let $A = \{a_1, a_2, \dots\}$ be a set of positive integers such that*

$$\frac{a_{n+1}}{na_n} \to +\infty \quad \text{as } n \to +\infty.$$

Then A has an exact additive complement B, that is, $A(x)B(x) = x + o(x)$.

For a set $A = \{a_1, a_2, \dots\}$ of positive integers with $a_1 < a_2 < \cdots$ and any positive integer n, let

$$\rho_A(n) = \inf_{i \ge n} \frac{a_{i+1}}{a_i}.$$

It is clear that

$$\rho_A(n) \le \rho_A(n+1), \quad \rho_A(n) \le \frac{a_{n+1}}{a_n}.$$

It follows that $k\rho(k)$ $(k = 1, 2, \dots)$ is strictly increasing. For any real number t, let $h_A(t)$ be the integer h such that

$$(h-1)\rho_A(h-1) \le t^2 < h\rho_A(h).$$

For convenience, we will omit the index A in $\rho_A(n)$ and $h_A(t)$. It is clear that $h(t) \to +\infty$ as $t \to +\infty$. The function $h(t)$ is not easy to interpret. Lemmas 2.3 and 2.4 give some of its properties. If $a_{n+1} = na_n$ for all sufficiently large n, then one may deduce that $h(t) = t + 1$ for all sufficiently large t and so $h(A(x)) = A(x) + 1$ for all sufficiently large x. If $a_{n+1}/(na_n) \to +\infty$ as $n \to +\infty$, then $\rho(n)/n \to +\infty$ as $n \to +\infty$ and so $h(t)/t \to 0$ as $t \to +\infty$. In particular, $h(A(x)) = o(A(x))$.

In this paper, the following results are proved.

THEOREM 1.1. *Let $A = \{a_1, a_2, \dots\}$ be a set of positive integers with $1 \le a_1 < a_2 < \cdots$ such that $a_{i+1} \ge ia_i$ for all sufficiently large integers i. Then there exists an additive complement B of A such that*

$$A(x)B(x) = x + \frac{8xh(A(x))}{A(x)} + O\left(\frac{x}{A(x)}\right).$$

REMARK 1.2. If $a_{i+1}/(ia_i) \to +\infty$ as $i \to +\infty$, then $h(A(x))/A(x) \to 0$ as $x \to +\infty$ and so B in Theorem 1.1 is an exact additive complement of A.

COROLLARY 1.3. *Let $A = \{a_1, a_2, \dots\}$ be a set of positive integers with $1 \le a_1 < a_2 < \cdots$ and $\gamma \ge 1$ a real number. Suppose that $a_{n+1} \ge n^\gamma a_n$*

for all sufficiently large n. Then there exists an additive complement B of A such that for all sufficiently large x,

$$x + \frac{7xh(A(x))}{A(x)} \le A(x)B(x) \le x + \frac{9x}{A(x)^{(\gamma-1)/(\gamma+1)}}.$$

COROLLARY 1.4. *Let $A = \{(2n)!\}$. Then there exists an additive complement B of A such that*

$$A(x)B(x) = x + (2^{8/3} + o(1))x\left(\frac{\log\log x}{\log x}\right)^{1/3}.$$

COROLLARY 1.5. *Let $A = \{a_n\}$, where a_n is the product of the first n primes. Then there exists an additive complement B of A such that*

$$A(x)B(x) = x + (8 + o(1))\frac{x}{\sqrt{\log\log x}}.$$

COROLLARY 1.6. *Let $A = \{2^{n^2}\}$. Then there exists an additive complement B of A such that*

$$A(x)B(x) = x + \left(\frac{4}{\sqrt{\log 2}} + o(1)\right)\frac{x\log\log x}{\sqrt{\log x}}.$$

Comparing with Theorem E, we introduce

$$\eta_A(i) = \inf_{j\ge i}\frac{a_{j+1}}{ja_j}.$$

By using Theorem 1.1, one can prove the following result.

COROLLARY 1.7. *Let $A = \{a_1, a_2, \dots\}$ be a set of positive integers with $1 \le a_1 < a_2 < \cdots$ such that $a_{i+1} \ge ia_i$ for all sufficiently large integers i. Then there exists an additive complement B of A such that for all sufficiently large x,*

$$x + \frac{7xh(A(x))}{A(x)} \le A(x)B(x) \le x + \frac{9x}{\sqrt{\eta_A(h(A(x))-1)}}.$$

It is clear that if $\eta_A(i) \to +\infty$ as $i \to +\infty$, then B in Corollary 1.7 or in Theorem 1.1 is an exact additive complement of A. We pose a problem here for further research.

PROBLEM 1.8. *Let $A = \{(2n)!\}$ or $A = \{2^{n^2}\}$ or $A = \{a_n\}$, where a_n is the product of the first n primes. Does there exist $0 < \delta < 1$ for which there is an additive complement B of A such that for all sufficiently large x, $A(x)B(x) < x + x^{\delta}$?*

For related results, one may refer to [1, 6–8, 10].

2. Proofs of Theorem 1.1 and its corollaries. We follow the proof of [12, Theorem 2]. We always assume that $A = \{a_1, a_2, \dots\}$ is a set of positive integers with $1 \le a_1 < a_2 < \cdots$. We also use the same definition of $L(m)$ as in [12, Theorem 2]: Let m be an integer with $m \ge a_1$. Define $L(m)$ as the smallest positive integer l for which there exist integers $w_1, \dots, w_l$ such that the integers $a_i + w_j$ $(1 \le i \le A(m),\ 1 \le j \le l)$ cover every residue modulo m. For two sets S and T of integers, let

$$S + T = \{s + t : s \in S,\ t \in T\}$$

and let $|S|$ denote the cardinality of S. For any real number y, let $\lfloor y \rfloor$ denote the largest integer not exceeding y, $\lceil y \rceil$ the least integer no less than y, and let $\{y\} = y - \lfloor y \rfloor$.

Now we sketch the proof of Theorem 1.1. In Lemmas 2.1 and 2.2, we will show that if a set $\{b_0, b_1, \dots, b_{n-1}\}$ of positive integers is rapidly growing and m is large, then there exists a 'small' set W of integers such that $\{b_0, b_1, \dots, b_{n-1}\} + W$ contains every residue modulo m. By using the properties of $h(t)$ obtained in Lemmas 2.3 and 2.4, we will show in Lemma 2.5 that if

$$L(a_i) \le \frac{a_i}{i} + \frac{8a_i h(i)}{i^2}$$

and $a_{i+1} \ge i a_i$ for all large i, then there exists an additive complement B of A such that $A(x)B(x)$ has the desired asymptotic formula.

LEMMA 2.1. *Let $b_0, b_1, \dots, b_{n-1}$ be positive integers and $\Gamma \ge n+1$ a real number such that*

$$\frac{b_{i+1}}{b_i} \ge \Gamma \quad \text{for all } i = 0, 1, \dots, n-2. \tag{2.1}$$

Then there exists a real number $\alpha \in (0,1)$ such that

$$\frac{i}{n} \le \{\alpha b_i\} < \frac{i}{n} + \frac{1}{\Gamma - 1} \quad \text{for all } i = 0, 1, \dots, n-1. \tag{2.2}$$

Proof. We try to find $\alpha \in (0,1)$ such that for every integer i with $0 \le i \le n-1$, there exists an integer r_i satisfying

$$r_i + \frac{i}{n} \le \alpha b_i < r_i + \frac{i}{n} + \frac{1}{\Gamma - 1}, \tag{2.3}$$

that is,

$$\frac{r_i + \frac{i}{n}}{b_i} \le \alpha < \frac{r_i + \frac{i}{n} + \frac{1}{\Gamma - 1}}{b_i}.$$

Write

$$I_i = \left[\frac{r_i + \frac{i}{n}}{b_i}, \frac{r_i + \frac{i}{n} + \frac{1}{\Gamma - 1}}{b_i} \right).$$

We will choose integers $r_0, r_1, \dots, r_{n-1}$ such that

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{n-1}. \tag{2.4}$$

For $i = 0$, choose $r_0 = 0$. Assume that for some positive integer k with $1 \le k \le n-1$, we have chosen $I_0, \dots, I_{k-1}$ satisfying (2.4). We will try to find an integer r_k such that $I_{k-1} \supseteq I_k$, that is,

$$\frac{r_k + \frac{k}{n}}{b_k} \ge \frac{r_{k-1} + \frac{k-1}{n}}{b_{k-1}} \quad \text{and} \quad \frac{r_k + \frac{k}{n} + \frac{1}{\Gamma-1}}{b_k} \le \frac{r_{k-1} + \frac{k-1}{n} + \frac{1}{\Gamma-1}}{b_{k-1}}.$$

So we only need to find an integer r_k satisfying

$$\frac{b_k}{b_{k-1}}\left(r_{k-1} + \frac{k-1}{n}\right) - \frac{k}{n} \le r_k \le \frac{b_k}{b_{k-1}}\left(r_{k-1} + \frac{k-1}{n} + \frac{1}{\Gamma-1}\right) - \frac{k}{n} - \frac{1}{\Gamma-1}.$$

By (2.1) we know that the length of the above interval is greater than or equal to 1. So, the existence of the integer r_k follows. Thus, we can choose $r_0, r_1, \dots, r_{n-1}$ such that (2.4) holds. Noting that $I_0 \subseteq [0,1)$, we may choose a real number $\alpha \in I_{n-1} \cap (0,1)$. This α satisfies (2.3) for all i. Noting that for $0 \le i \le n-1$,

$$\frac{i}{n} + \frac{1}{\Gamma-1} \le \frac{n-1}{n} + \frac{1}{n} = 1,$$

by (2.3) we have $\lfloor \alpha b_i \rfloor = r_i$. Now (2.2) follows from (2.3). ∎

LEMMA 2.2. *Let $m, b_0, b_1, \dots, b_{n-1}$ $(n \ge 2)$ be positive integers and Γ a real number with $\Gamma \ge 2n+1$ such that $m \ge (\Gamma-1)b_{n-1}$ and $b_{i+1} \ge \Gamma b_i$ for all $0 \le i \le n-2$. Then there exists a set W of integers such that $\{b_0, b_1, \dots, b_{n-1}\} + W$ contains every residue modulo m and*

$$|W| \le \frac{m}{n} + \frac{2m}{\Gamma-1}.$$

Proof. By Lemma 2.1, there exists $\alpha \in (0,1)$ such that (2.2) holds. Let r be the integer such that

$$\frac{r-1}{m} < \alpha \le \frac{r}{m}.$$

Then $1 \le r \le m$ and

$$0 \le \frac{r}{m}b_i - \alpha b_i < \frac{1}{m}b_i \le \frac{1}{m}b_{n-1} \le \frac{1}{\Gamma-1}.$$

Since

$$\left\{\frac{rb_i}{m}\right\} = \left\{\frac{rb_i}{m} - \alpha b_i + \alpha b_i\right\} = \left\{\frac{rb_i}{m} - \alpha b_i + \{\alpha b_i\}\right\}$$

and by (2.2),

$$\frac{i}{n} \le \frac{rb_i}{m} - \alpha b_i + \{\alpha b_i\} < \frac{1}{\Gamma-1} + \frac{i}{n} + \frac{1}{\Gamma-1} \le 1,$$

it follows that

$$\frac{i}{n} \le \left\{\frac{rb_i}{m}\right\} < \frac{i}{n} + \frac{2}{\Gamma - 1} \quad \text{for all } i = 0, 1, \dots, n-1. \tag{2.5}$$

Thus, $1 \le r \le m-1$. Let r'/m' be the reduced form of r/m and $d_i \equiv r'b_i \pmod{m'}$ with $0 \le d_i < m'$. Thus, (2.5) is equivalent to

$$\frac{i}{n} \le \frac{d_i}{m'} < \frac{i}{n} + \frac{2}{\Gamma - 1} \quad \text{for all } i = 0, 1, \dots, n-1. \tag{2.6}$$

Write $d_n = d_0 + m'$. Then (2.6) also holds for $i = n$ and

$$d_0 < d_1 < \cdots < d_{n-1} < d_n = d_0 + m'.$$

It is clear that

$$\{d_0, d_1, \dots, d_{n-1}\} + \Big\{0, 1, \dots, \max_{0 \le i \le n-1}\{d_{i+1} - d_i\} - 1\Big\} \supseteq \{d_0, d_0 + 1, \dots, d_0 + m' - 1\}. \tag{2.7}$$

Let r^* be an integer with $r'r^* \equiv 1 \pmod{m'}$ and let

$$W' = \Big\{r^*h : h = 0, 1, \dots, \max_{0 \le i \le n-1}\{d_{i+1} - d_i\} - 1\Big\}.$$

Then $r^*d_i \equiv b_i \pmod{m'}$ for all i. By (2.7), $\{r^*d_0, r^*d_1, \dots, r^*d_{n-1}\} + W'$ contains every residue modulo m'. It follows that $\{b_0, b_1, \dots, b_{n-1}\} + W'$ contains every residue modulo m'. In view of (2.6),

$$|W'| = \max_{0 \le i \le n-1}\{d_{i+1} - d_i\} < \left(\frac{1}{n} + \frac{2}{\Gamma - 1}\right)m'.$$

Let

$$W = W' + \left\{0, m', 2m', \dots, \left(\frac{m}{m'} - 1\right)m'\right\}.$$

Then $\{b_0, \dots, b_{n-1}\} + W$ contains every residue modulo m and

$$|W| = \frac{m}{m'}|W'| \le \frac{m}{n} + \frac{2m}{\Gamma - 1}. \ \blacksquare$$

LEMMA 2.3. *If $\rho(n) \ge n$ for all sufficiently large n, then for all sufficiently large t, we have $h(t+1) - 4 \le h(t) \le t + 1$.*

Proof. Let t be a sufficiently large real number. Since $h(t) \to +\infty$ as $t \to +\infty$, it follows that

$$t^2 \ge (h(t) - 1)\rho(h(t) - 1) \ge (h(t) - 1)^2.$$

So $h(t) \le t + 1$. Suppose that $h(t) \le h(t+1) - 5$. By the definition, $\rho(n)$ is increasing. Thus

$$\begin{aligned} t^2 &< h(t)\rho(h(t)) \le (h(t+1) - 5)\rho(h(t+1) - 5) \\ &\le (h(t+1) - 5)\rho(h(t+1) - 1) \le (t+1)^2 - 4\rho(h(t+1) - 1). \end{aligned}$$

So $\rho(h(t+1)-1) < \frac{1}{4}(2t+1)$. It follows that

$$t^2 < h(t)\rho(h(t)) \le h(t)\rho(h(t+1)-1) < \tfrac{1}{4}(2t+1)h(t) \le \tfrac{1}{4}(2t+1)(t+1),$$

a contradiction. Hence, $h(t) \ge h(t+1) - 4$. ∎

LEMMA 2.4. *The function $h(t)$ is increasing on $(0, +\infty)$.*

Proof. Let $t_1, t_2 \in (0, +\infty)$ with $t_1 < t_2$. Since $x\rho(x)$ is increasing and

$$h(t_2)\rho(h(t_2)) > t_2^2 > t_1^2 \ge (h(t_1)-1)\rho(h(t_1)-1),$$

it follows that $h(t_2) > h(t_1) - 1$. Thus, $h(t_2) \ge h(t_1)$. ∎

LEMMA 2.5. *If*

$$L(a_i) \le \frac{a_i}{i} + \frac{8a_i h(i)}{i^2} \tag{2.8}$$

and $a_{i+1} \ge ia_i$ for all sufficiently large i, then there exists an additive complement B of A such that

$$B(x) = \frac{x}{A(x)} + \frac{8xh(A(x))}{A(x)^2} + O\bigg(\frac{x}{A(x)^2}\bigg).$$

Proof. Let

$$q_i = \left\lceil \frac{a_{i+1}}{a_i} \right\rceil, \quad i = 1, 2, \ldots.$$

Then $q_i \ge 2$ for all $i \ge 1$ and

$$q_i \ge \frac{a_{i+1}}{a_i} \ge i \tag{2.9}$$

for all sufficiently large i. It follows that $\rho(n) \ge n$ for all sufficiently large n. By Lemma 2.3, $h(t) \le t+1$ for all sufficiently large t. Hence

$$\frac{a_i}{i} + \frac{8a_i h(i)}{i^2} < a_i \tag{2.10}$$

for all sufficiently large i. Let $A_i = A \cap [1, a_i]$. In view of (2.8) and (2.10), for every sufficiently large i, there exists $W_i \subseteq [1, a_i]$ such that

$$|W_i| = \left\lfloor \frac{a_i}{i} + \frac{8a_i h(i)}{i^2} \right\rfloor$$

and $A_i + W_i$ contains every residue modulo a_i. Let

$$B_i = W_i + \left\{ (q_i - 1)a_i, q_i a_i, (q_i+1)a_i, \ldots, \left\lfloor \frac{q_{i+1}a_{i+1}}{a_i} \right\rfloor a_i \right\}. \tag{2.11}$$

We follow the arguments in [12] to prove that

$$B = \bigcup_{i=1}^{\infty} B_i$$

is an additive complement of A.

For any given integer n ($> q_1a_1$), there exists a positive integer j such that

$$q_ja_j < n \le q_{j+1}a_{j+1}.$$

By the definitions of A_j and W_j, there exist $a \in A_j$ and $w \in W_j$ such that

$$n \equiv a + w \pmod{a_j}.$$

Thus, $(n-a-w)/a_j$ is an integer. It follows from (2.11) and the inequalities

$$\frac{n-a-w}{a_j} < \frac{q_{j+1}a_{j+1}}{a_j}, \qquad \frac{n-a-w}{a_j} > \frac{q_ja_j - 2a_j}{a_j} = q_j - 2$$

that

$$w + \frac{n-a-w}{a_j}a_j \in B_j.$$

Thus,

$$n = a + \left(w + \frac{n-a-w}{a_j}a_j\right) \in A + B.$$

Hence, $A + B$ contains all large integers.

Now we estimate $B(x)$. Let x ($> q_1a_1$) be a large real number. Then there exists a positive integer j such that $q_ja_j < x \le q_{j+1}a_{j+1}$. Since $q_ja_j \ge a_{j+1}$ and

$$q_{j+1}a_{j+1} = (q_{j+1}-1)a_{j+1} + a_{j+1} < a_{j+2} + a_{j+1} < a_{j+3},$$

it follows that $a_{j+1} < x < a_{j+3}$. So $A(x)$ is $j+1$ or $j+2$. By (2.11), the definition of B and $q_i \ge 2$, we have

$$\begin{aligned} B(x) &\le \sum_{i=1}^{j-1} |W_i|\left(\left\lfloor \frac{q_{i+1}a_{i+1}}{a_i} \right\rfloor - q_i + 2\right) \\ &\quad + |W_j|\left(\left\lfloor \frac{x}{a_j} \right\rfloor - q_j + 2\right) + \delta|W_{j+1}| \\ &\le \sum_{i=1}^{j-1} |W_i|\frac{q_{i+1}a_{i+1}}{a_i} + |W_j|\left(\frac{x}{a_j} - q_j + 2\right) + \delta|W_{j+1}|, \end{aligned} \tag{2.12}$$

where

$$\delta = \begin{cases} 0 & \text{if } x < (q_{j+1}-1)a_{j+1}, \\ 1 & \text{if } (q_{j+1}-1)a_{j+1} \le x \le q_{j+1}a_{j+1}. \end{cases} \tag{2.13}$$

By Lemma 2.3, $h(i) \le i+1$ for all sufficiently large i. Hence, there exists an integer $i_0 \ge 5$ such that for all $i \ge i_0$,

$$|W_i| \le \frac{a_i}{i} + \frac{8a_ih(i)}{i^2} < \frac{10a_i}{i}. \tag{2.14}$$

Let

$$S_1 = 2|W_j| + \sum_{i=i_0}^{j-2} |W_i| \frac{q_{i+1}a_{i+1}}{a_i}, \qquad S_2 = |W_{j-1}| \frac{q_j a_j}{a_{j-1}} + |W_j| \left(\frac{x}{a_j} - q_j \right),$$

$$S_3 = \delta |W_{j+1}|.$$

Now we estimate S_1, S_2 and S_3 separately.

For S_1, by (2.14) and

$$q_{i+1}a_{i+1} = (q_{i+1} - 1)a_{i+1} + a_{i+1} < a_{i+2} + a_{i+1} < 2a_{i+2},$$

we have

$$S_1 < \frac{20a_j}{j} + 10 \sum_{i=i_0}^{j-2} \frac{q_{i+1}a_{i+1}}{i} < \frac{20a_j}{j} + 20 \sum_{i=i_0}^{j-2} \frac{a_{i+2}}{i}.$$

Since

$$\frac{a_{i+2}/i}{a_{i+1}/(i-1)} = \frac{i-1}{i} \frac{a_{i+2}}{a_{i+1}} \geq \frac{i-1}{i}(i+1) > i - 1 \geq 4, \qquad i \geq i_0,$$

it follows that

$$\begin{aligned} S_1 &\leq \frac{20a_j}{j} + 20 \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots \right) \frac{a_j}{j-2} \\ &< \frac{20a_j}{j} + \frac{80}{3} \frac{a_j}{j-2} < \frac{47a_j}{j} = \frac{47q_j a_j}{jq_j} \leq \frac{47x}{j^2} < \frac{48x}{(A(x))^2}. \end{aligned} \tag{2.15}$$

By Lemma 2.4 and $A(x) > j$, we have $h(j-1) \leq h(j) \leq h(A(x))$. In view of the definition of $h(j)$, we deduce that $h(j) \to +\infty$ as $j \to +\infty$. For S_2, by (2.14) and Lemma 2.3 we have

$$\begin{aligned} S_2 &\leq \left(\frac{a_{j-1}}{j-1} + \frac{8a_{j-1}h(j-1)}{(j-1)^2} \right) \frac{q_j a_j}{a_{j-1}} \\ &\quad + \left(\frac{a_j}{j} + \frac{8a_j h(j)}{j^2} \right) \left(\frac{x}{a_j} - q_j \right) \\ &\leq \frac{q_j a_j}{j-1} + \frac{8q_j a_j h(j)}{(j-1)^2} + \frac{x}{j} - \frac{q_j a_j}{j} + \frac{8xh(j)}{j^2} - \frac{8q_j a_j h(j)}{j^2} \\ &= \frac{x}{j+2} + \frac{8xh(j)}{(j+2)^2} + \frac{2x}{j(j+2)} + \frac{8xh(j)(4j+4)}{j^2(j+2)^2} \\ &\quad + \frac{q_j a_j}{(j-1)j} + \frac{8q_j a_j h(j)(2j-1)}{j^2(j-1)^2} \\ &< \frac{x}{j+2} + \frac{8xh(j)}{(j+2)^2} + \frac{52x}{(j+2)^2} \\ &\leq \frac{x}{A(x)} + \frac{8xh(A(x))}{(A(x))^2} + \frac{52x}{(A(x))^2}. \end{aligned} \tag{2.16}$$

Now we give an upper bound of S_3. By (2.13) we only need to consider $\delta = 1$. Then $x \geq (q_{j+1}-1)a_{j+1}$. It follows from (2.14) and $q_{j+1} \geq j+1$ that

$$S_3 \leq |W_{j+1}| \leq \frac{10a_{j+1}}{j+1} \leq \frac{11}{q_{j+1}} \cdot \frac{(q_{j+1}-1)a_{j+1}}{j+2} \leq \frac{11}{j+1} \cdot \frac{x}{j+2} \leq \frac{12x}{(j+2)^2} \leq \frac{12x}{(A(x))^2}. \tag{2.17}$$

Since $a_{i+1} \geq ia_i$ for all sufficiently large i, we have $a_{i+1} \geq i(i-1)$ for all sufficiently large i. Thus,

$$\frac{x}{A(x)^2} \geq \frac{a_j q_j}{(j+2)^2} \geq \frac{(j-1)(j-2)j}{(j+2)^2} \to +\infty.$$

It follows from (2.15)–(2.17) that

$$\begin{aligned} B(x) &< \sum_{i=1}^{i_0-1} |W_i| \frac{q_{i+1}a_{i+1}}{a_i} + \frac{x}{A(x)} + \frac{8xh(A(x))}{(A(x))^2} + \frac{112x}{(A(x))^2} \\ &\leq \frac{x}{A(x)} + \frac{8xh(A(x))}{(A(x))^2} + \frac{113x}{(A(x))^2}. \end{aligned}$$

Finally, we give a lower bound of $B(x)$. Let

$$B'_{j-1} = W_{j-1} + \left\{(q_{j-1}-1)a_{j-1}, q_{j-1}a_{j-1}, \ldots, \left(\left\lfloor \frac{q_j a_j}{a_{j-1}} \right\rfloor - 1\right)a_{j-1}\right\},$$

$$B'_j = W_j + \left\{q_j a_j, \ldots, \left\lfloor \frac{q_{j+1}a_{j+1}}{a_j} \right\rfloor a_j\right\}.$$

Then $B'_{j-1} \subseteq B_{j-1}$, $B'_j \subseteq B_j$,

$$\max B'_{j-1} \leq a_{j-1} + \left(\left\lfloor \frac{q_j a_j}{a_{j-1}} \right\rfloor - 1\right)a_{j-1} \leq q_j a_j < x$$

and

$$\min B'_j \geq 1 + q_j a_j > \max B'_{j-1}.$$

Hence

$$\begin{aligned} B(x) &\geq B'_j(x) + |B'_{j-1}| \\ &\geq |W_j|\left(\left\lfloor \frac{x-a_j}{a_j} \right\rfloor - q_j + 1\right) + |W_{j-1}|\left(\left\lfloor \frac{q_j a_j}{a_{j-1}} \right\rfloor - q_{j-1} + 1\right) \\ &= \left\lfloor \frac{a_j}{j} + \frac{8a_j h(j)}{j^2} \right\rfloor \left(\left\lfloor \frac{x-a_j}{a_j} \right\rfloor - q_j + 1\right) \\ &\quad + \left\lfloor \frac{a_{j-1}}{j-1} + \frac{8a_{j-1}h(j-1)}{(j-1)^2} \right\rfloor \left(\left\lfloor \frac{q_j a_j}{a_{j-1}} \right\rfloor - q_{j-1} + 1\right). \end{aligned}$$

By Lemmas 2.3 and 2.4,

$$\left\lfloor \frac{a_j}{j} + \frac{8a_j h(j)}{j^2} \right\rfloor = \frac{a_j}{j} + \frac{8a_j h(j)}{j^2} + O(1) = \frac{a_j}{j} + \frac{8a_j h(j+2)}{j^2} + O\left(\frac{a_j}{j^2}\right)$$

and

$$\begin{aligned}
&\left\lfloor \frac{a_{j-1}}{j-1} + \frac{8a_{j-1}h(j-1)}{(j-1)^2} \right\rfloor \\
&= \frac{a_{j-1}}{j-1} + \frac{8a_{j-1}h(j-1)}{(j-1)^2} + O(1) = \frac{a_{j-1}}{j} + \frac{8a_{j-1}h(j+2)}{(j-1)^2} + O\left(\frac{a_{j-1}}{(j-1)^2}\right) \\
&= \frac{a_{j-1}}{j} + \frac{8a_{j-1}h(j+2)}{j^2} + \frac{8a_{j-1}h(j+2)(2j-1)}{j^2(j-1)^2} + O\left(\frac{a_{j-1}}{(j-1)^2}\right) \\
&= \frac{a_{j-1}}{j} + \frac{8a_{j-1}h(j+2)}{j^2} + O\left(\frac{a_{j-1}}{j^2}\right).
\end{aligned}$$

It is clear that

$$\left\lfloor \frac{x-a_j}{a_j} \right\rfloor - q_j + 1 = \frac{x}{a_j} - q_j + O(1)$$

and

$$\left\lfloor \frac{q_j a_j}{a_{j-1}} \right\rfloor - q_{j-1} + 1 = \frac{q_j a_j}{a_{j-1}} - q_{j-1} + O(1).$$

Noting that $a_j q_j < x$ and $q_j \geq j$, by Lemmas 2.3 and 2.4 we have

$$\begin{aligned}
&\left\lfloor \frac{a_j}{j} + \frac{8a_j h(j)}{j^2} \right\rfloor \left(\left\lfloor \frac{x-a_j}{a_j} \right\rfloor - q_j + 1 \right) \\
&\qquad = \left(\frac{a_j}{j} + \frac{8a_j h(j+2)}{j^2} + O\left(\frac{a_j}{j^2}\right) \right) \left(\frac{x}{a_j} - q_j + O(1) \right) \\
&\qquad = \frac{x}{j} + \frac{8xh(j+2)}{j^2} - \frac{a_j q_j}{j} - \frac{8a_j q_j h(j+2)}{j^2} \\
&\qquad\quad + O\left(\frac{x}{j^2}\right) + O\left(\frac{a_j q_j}{j^2}\right) + O\left(\frac{a_j}{j}\right) + O\left(\frac{8a_j h(j+2)}{j^2}\right) \\
&\qquad = \frac{x}{j} + \frac{8xh(j+2)}{j^2} - \frac{a_j q_j}{j} - \frac{8a_j q_j h(j+2)}{j^2} + O\left(\frac{x}{j^2}\right).
\end{aligned}$$

Since

$$\frac{a_{j-1}q_{j-1}}{j} = O\left(\frac{a_j}{j}\right) = O\left(\frac{a_j q_j}{j q_j}\right) = O\left(\frac{a_j q_j}{j^2}\right) = O\left(\frac{x}{j^2}\right)$$

and

$$\frac{8a_{j-1}q_{j-1}h(j+2)}{j^2} = O\left(\frac{8a_{j-1}q_{j-1}(j+3)}{j^2}\right) = O\left(\frac{a_{j-1}q_{j-1}}{j}\right) = O\left(\frac{x}{j^2}\right),$$

it follows that

$$\begin{aligned}
&\left\lfloor \frac{a_{j-1}}{j-1} + \frac{8a_{j-1}h(j-1)}{(j-1)^2} \right\rfloor \left(\left\lfloor \frac{q_j a_j}{a_{j-1}} \right\rfloor - q_{j-1} + 1 \right) \\
&\quad = \left(\frac{a_{j-1}}{j} + \frac{8a_{j-1}h(j+2)}{j^2} + O\left(\frac{a_{j-1}}{j^2} \right) \right) \left(\frac{q_j a_j}{a_{j-1}} - q_{j-1} + O(1) \right) \\
&\quad = \frac{q_j a_j}{j} + \frac{8q_j a_j h(j+2)}{j^2} - \frac{a_{j-1}q_{j-1}}{j} - \frac{8a_{j-1}q_{j-1}h(j+2)}{j^2} \\
&\qquad + O\left(\frac{a_j q_j}{j^2} \right) + O\left(\frac{a_{j-1}q_{j-1}}{j^2} \right) + O\left(\frac{a_{j-1}}{j} \right) + O\left(\frac{8a_{j-1}h(j+2)}{j^2} \right) \\
&\quad = \frac{q_j a_j}{j} + \frac{8q_j a_j h(j+2)}{j^2} + O\left(\frac{x}{j^2} \right).
\end{aligned}$$

Thus, by Lemma 2.4 and $j < A(x) \le j+2$, we have

$$B(x) \ge \frac{x}{j} + \frac{8xh(j+2)}{j^2} + O\left(\frac{x}{j^2} \right) \ge \frac{x}{A(x)} + \frac{8xh(A(x))}{A(x)^2} + O\left(\frac{x}{A(x)^2} \right). \blacksquare$$

Proof of Theorem 1.1. Since $a_{i+1} \ge ia_i$ for all sufficiently large i, it follows that $\rho(i) \ge i$ for all sufficiently large i. Let k be a large integer and $u = h(0.75k)$. By Lemma 2.3, $u < k-2$. We will use Lemma 2.2 with $\Gamma = \rho(u)$, $n = k-u$, $m = a_k$ and

$$\{b_0, b_1, \dots, b_{n-1}\} = \{a_u, a_{u+1}, \dots, a_{k-1}\}.$$

By the definition of $\rho(u)$, $a_{i+1} \ge \rho(u)a_i$ for all $i \ge u$. In particular, $a_k \ge \rho(u)a_{k-1}$. In view of $0.75^2k^2 < u\rho(u)$, we have

$$\rho(u) > \frac{0.75^2k^2}{u}.$$

Since

$$2u + \frac{0.75^2k^2}{u} \ge 2\sqrt{2u \cdot \frac{0.75^2k^2}{u}} = 2\sqrt{2 \times 0.75^2}k > 2k+1,$$

it follows that

$$\rho(u) > \frac{0.75^2k^2}{u} > 2(k-u)+1 = 2n+1.$$

By Lemma 2.2, there is a set W of integers such that $\{a_u, a_{u+1}, \dots, a_{k-1}\}+W$ contains every residue modulo a_k and

$$|W| \le \frac{a_k}{n} + \frac{2a_k}{\rho(u)-1} < \frac{a_k}{k-u} + \frac{2.001a_k}{\rho(u)}.$$

By Lemmas 2.3 and 2.4, $u \le 0.75k+1$ and $u \le h(k)$. It follows from

$0.75^2k^2 < u\rho(u)$ that

$$|W| \le \frac{a_k}{k-u} + \frac{2.001a_k}{\rho(u)} \le \frac{a_k}{k}\left(1 + \frac{1.001u}{(1-0.75)k}\right) + \frac{2.001a_k}{\rho(u)}$$
$$< \frac{a_k}{k} + \frac{8a_ku}{k^2} \le \frac{a_k}{k} + \frac{8a_kh(k)}{k^2}.$$

Since $\{a_1,\dots,a_k\}+W$ contains every residue modulo a_k, it follows from the definition of $L(a_k)$ that for all sufficiently large k,

$$L(a_k) \le |W| < \frac{a_k}{k} + \frac{8a_kh(k)}{k^2}.$$

By Lemma 2.5, there exists an additive complement B of A such that

$$B(x) = \frac{x}{A(x)} + \frac{8xh(A(x))}{A(x)^2} + O\left(\frac{x}{A(x)^2}\right).$$

That is,

$$A(x)B(x) = x + \frac{8xh(A(x))}{A(x)} + O\left(\frac{x}{A(x)}\right). \blacksquare$$

Proof of Corollary 1.3. Since $a_{n+1} \ge n^\gamma a_n$ for all sufficiently large n, it follows that $\rho(n) \ge n^\gamma$ for all sufficiently large n. By $(h(n)-1)\rho(h(n)-1) \le n^2$, we have $(h(n)-1)^{\gamma+1} \le n^2$. It follows that for all sufficiently large n,

$$h(n) \le n^{2/(\gamma+1)} + 1. \tag{2.18}$$

By Theorem 1.1 and (2.18), there exists an additive complement B of A such that for all sufficiently large x,

$$x + \frac{7xh(A(x))}{A(x)} \le A(x)B(x) \le x + \frac{8.1xh(A(x))}{A(x)}$$
$$< x + \frac{9x(A(x))^{2/(\gamma+1)}}{A(x)} = x + \frac{9x}{A(x)^{(\gamma-1)/(\gamma+1)}}. \blacksquare$$

Proof of Corollary 1.4. Since $(2n+2)! > 4n^2(2n)!$ for all integers $n \ge 1$, it follows from Theorem 1.1 that there exists an additive complement B of A such that

$$A(x)B(x) = x + \frac{8xh(A(x))}{A(x)} + O\left(\frac{x}{A(x)}\right).$$

Noting that $A = \{(2n)!\}$, we have

$$\rho(n) = (2n+2)(2n+1), \qquad h(n) = (1+o(1))\left(\frac{n}{2}\right)^{2/3},$$
$$A(x) = (1+o(1))\frac{\log x}{2\log\log x}.$$

Thus

$$\frac{h(A(x))}{A(x)} = (1+o(1))2^{-2/3}(A(x))^{-1/3} = (1+o(1))2^{-1/3}\left(\frac{\log\log x}{\log x}\right)^{1/3}.$$

It follows that

$$A(x)B(x) = x + (2^{8/3}+o(1))x\left(\frac{\log\log x}{\log x}\right)^{1/3}. \ \blacksquare$$

Proof of Corollary 1.5. Let p_n be the n-th prime. Then $a_{n+1} = p_{n+1}a_n$. It follows that $\rho(n) = p_{n+1} = (1+o(1))(n+1)\log(n+1)$ by the Prime Number Theorem. By the definition of $h(n)$, we have

$$h(n) = (1+o(1))\frac{n}{\sqrt{\log n}}.$$

Noting that $A = \{p_1 \cdots p_n\}$, we have

$$A(x) = (1+o(1))\frac{\log x}{\log\log x}.$$

Thus

$$\frac{h(A(x))}{A(x)} = (1+o(1))\frac{1}{\sqrt{\log A(x)}} = (1+o(1))\frac{1}{\sqrt{\log\log x}}.$$

It follows from Theorem 1.1 that there exists an additive complement B of A such that

$$\begin{aligned} A(x)B(x) &= x + \frac{8xh(A(x))}{A(x)} + O\left(\frac{x}{A(x)}\right) \\ &= x + (8+o(1))\frac{x}{\sqrt{\log\log x}}. \ \blacksquare \end{aligned}$$

Proof of Corollary 1.6. Since $2^{(n+1)^2} = 2^{2n+1}2^{n^2}$, it follows that $\rho(n) = 2^{2n+1}$. By the definition of $h(n)$, we have

$$h(n) = (1+o(1))\frac{\log n}{\log 2}.$$

Noting that $A = \{2^{n^2}\}$, we have $A(x) = (1+o(1))\sqrt{\log x/\log 2}$. Thus

$$\frac{h(A(x))}{A(x)} = (1+o(1))\frac{\log A(x)}{A(x)\log 2} = (1+o(1))\frac{1}{2\sqrt{\log 2}}\,\frac{\log\log x}{\sqrt{\log x}}.$$

It follows from Theorem 1.1 that there exists an additive complement B of A such that

$$\begin{aligned} A(x)B(x) &= x + \frac{8xh(A(x))}{A(x)} + O\left(\frac{x}{A(x)}\right) \\ &= x + \left(\frac{4}{\sqrt{\log 2}} + o(1)\right)\frac{x\log\log x}{\sqrt{\log x}}. \ \blacksquare \end{aligned}$$

Proof of Corollary 1.7. Since

$$\eta_A(i) = \inf_{j\ge i} \frac{a_{j+1}}{ja_j} \le \frac{1}{i} \inf_{j\ge i} \frac{a_{j+1}}{a_j} = \frac{\rho(i)}{i},$$

it follows that $\rho(i) \ge i\eta_A(i) \ge i$. Hence

$$(h(n)-1)^2\eta_A(h(n)-1) \le (h(n)-1)\rho(h(n)-1) \le n^2. \tag{2.19}$$

Noting that $h(n) \to +\infty$ as $n \to +\infty$, by (2.19), for all sufficiently large integers n, we have

$$\frac{h(n)}{n} < \frac{1.1}{\sqrt{\eta_A(h(n)-1)}}. \tag{2.20}$$

It follows from Theorem 1.1 that there exists an additive complement B of A such that for all sufficiently large x,

$$x + \frac{7xh(A(x))}{A(x)} \le A(x)B(x) \le x + \frac{9x}{\sqrt{\eta_A(h(A(x))-1)}}. \ \blacksquare$$

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