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The universality of quadratic L-series for prime discriminants

by

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1. Introduction and statement of results. For an odd prime p let λ_p denote the real Dirichlet character modulo p given by the Legendre symbol $\left(\frac{\cdot}{p}\right)$. Let $L(s,\chi)$ be the Dirichlet L-function associated with a character χ. The value distribution of $L(s,\lambda_p)$ for a complex number s with $\mathrm{Re}\, s > 1/2$ as p varies over the odd primes is investigated e.g. in [El]. The main purpose of the present paper is to study the functional distribution of $L(s,\lambda_p)$ on D, as p varies over the primes in an arithmetic progression; here and henceforth D denotes the strip $\{s \in \mathbb{C} \mid 1/2 < \mathrm{Re}\, s < 1\}$. More precisely, we shall establish the so-called *universality theorem* for $L(s,\lambda_p)$ in the p-aspect.

The universality theorem was first discovered by Voronin ([Vo], [KV]) for the Riemann zeta-function $\zeta(s)$ in the t-aspect; he showed the following.

THEOREM ([Vo]). *Let $0 < r < 1/4$ and $h(s)$ be a continuous function on the disk $|s| \le r$ which is holomorphic and has no zeros in $|s| < r$. Then for any $\varepsilon > 0$ there exists a real number t such that*

$$\max_{|s|\le r} |\zeta(s + 3/4 + it) - h(s)| < \varepsilon.$$

The universality theorem for a Dirichlet L-function $L(s,\chi)$ in the t-aspect was obtained by Bagchi [B1], [B2], Gonek [Go] and Voronin (see [KV, Chapter VII, Section 3]) independently; indeed, the *joint* universality theorem was shown.

Furthermore, Bagchi [B1], Eminyan [Em] and Gonek [Go] independently showed an analogous result for Dirichlet L-functions in another aspect. In fact, they established the universality theorem for the family of $L(s,\chi)$'s as χ varies over the set of characters modulo q with q large.

We denote by $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$ and $\mathbb{N}$ the set of all real numbers, positive real numbers, integers and positive integers, respectively. For a discriminant d, let χ_d denote the real Dirichlet character modulo $|d|$ defined by the Kronecker

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symbol $\left(\frac{d}{\cdot}\right)_{\mathrm{K}}$. Letting γ stand for the plus sign or the minus sign, we define

$$\mathcal{D}^{\gamma} := \begin{cases} \{d > 0 \mid d \text{ is a square-free integer, } d \equiv 1 \bmod 8,\ d \neq 1\} & \text{if } \gamma \text{ is } +, \\ \{d < 0 \mid d \text{ is a square-free integer, } d \equiv 1 \bmod 8\} & \text{if } \gamma \text{ is } -, \end{cases}$$

and

$$\mathcal{D}_X^{\gamma} := \{d \in \mathcal{D}^{\gamma} \mid |d| \leq X\} \quad \text{for } X \in \mathbb{R}^+.$$

The authors [MN1] have recently obtained the following universality theorem, which is an analogue of Bagchi, Eminyan and Gonek's result above for the family $\{L(s,\chi_d) \mid d \in \mathcal{D}^{\gamma}\}$ of L-functions associated with real characters χ_d: *Let Ω, $h(s)$ and K be as in Theorem* 1.1 *below. Then for any $\varepsilon > 0$ we have*

$$\liminf_{X\to\infty} \frac{1}{\#\mathcal{D}_X^{\gamma}} \#\{d \in \mathcal{D}_X^{\gamma} \mid \max_{s\in K} |L(s,\chi_d) - h(s)| < \varepsilon\} > 0. \tag{1.1}$$

In the present paper we investigate the universality theorem for $L(s,\lambda_p)$ in the prime p-aspect, as mentioned above. Noting that $L(s,\lambda_p)$ is equal to $L(s,\chi_q)$ with a certain integer q (see (3.3)), we will deal with $L(s,\chi_q)$ instead of $L(s,\lambda_p)$. Throughout let $\gamma \in \{+,-\}$ and let m and $a = a(\gamma)$ be any fixed positive integers such that $\gcd(m,a) = 1$, $8 \mid m$, $a \equiv 1 \bmod 4$ if γ is $+$ and $a \equiv 3 \bmod 4$ if γ is $-$. We define

$$\mathcal{P}^{\gamma}(m,a) := \begin{cases} \{p \mid p \text{ is a prime, } p \equiv a \bmod m\} & \text{if } \gamma \text{ is } +, \\ \{-p \mid p \text{ is a prime, } p \equiv a \bmod m\} & \text{if } \gamma \text{ is } -, \end{cases}$$

and

$$\mathcal{P}_X^{\gamma}(m,a) := \{q \in \mathcal{P}^{\gamma}(m,a) \mid |q| \leq X\} \quad \text{for } X > 0.$$

By the prime number theorem for arithmetic progressions,

$$\#\mathcal{P}_X^{\gamma}(m,a) \sim \frac{1}{\varphi(m)} \frac{X}{\log X} \quad \text{as } X \to \infty, \tag{1.2}$$

where $\varphi(m)$ denotes the Euler totient function. Every integer q in $\mathcal{P}^{\gamma}(m,a)$ is a *prime discriminant* (for its definition, see e.g. [Ay, p. 310], [Da, p. 41]). In the following, the letter p will stand for a prime number and q for a prime discriminant.

THEOREM 1.1. *Let $\gamma \in \{+,-\}$. Let $m, a \in \mathbb{N}$ be as above. Let Ω be a simply connected region in D which is symmetric with respect to the real axis. Suppose that $h(s)$ is a holomorphic function on Ω which has no zeros on Ω and is $\mathbb{R}^+$-valued on the set $\Omega \cap \mathbb{R}$. Let K be a compact set in Ω, and $\varepsilon > 0$. Then there exist infinitely many $q \in \mathcal{P}^{\gamma}(m,a)$ such that* $\max_{s\in K} |L(s,\chi_q) - h(s)| < \varepsilon$. *More precisely, we have*

$$\liminf_{X\to\infty} \frac{1}{\#\mathcal{P}_X^{\gamma}(m,a)} \#\{q \in \mathcal{P}_X^{\gamma}(m,a) \mid \max_{s\in K} |L(s,\chi_q) - h(s)| < \varepsilon\} > 0. \tag{1.3}$$

It should be noted that the results (1.1) and (1.3) do not directly imply each other, because the *density* of the set $\mathcal{P}^{\gamma}(m,a)$ in $\mathcal{D}^{\gamma}$ is 0 in the sense that $\#\mathcal{P}^{\gamma}_X(m,a)/\#\mathcal{D}^{\gamma}_X \to 0$ as $X \to \infty$ (see [MN1, Lemma 4.1] and (1.2)).

In the same way as in the present paper, we can generalize (1.1) to the result in which d varies over the fundamental discriminants in the arithmetic progression $\{km+a \mid k \in \mathbb{Z}\}$, where $m, a \in \mathbb{N}$ are as in Theorem 1.1.

Theorem 1.1 yields the following corollaries, for example. First we get a denseness result on values of $L(s,\chi_q)$'s for fixed $s \in D$ and variable $q \in \mathcal{P}^{\gamma}(m,a)$. This is analogous to Bohr–Courant's result [BC] on values of the Riemann zeta-function $\zeta(s)$.

Corollary 1.2.

(1) *Let any* $s_0 \in D$ *with* $\operatorname{Im} s_0 \neq 0$ *be fixed. Then the set* $\{L(s_0,\chi_q) \mid q \in \mathcal{P}^{\gamma}(m,a)\}$ *is dense in* $\mathbb{C}$. *More precisely, for any* $z_0 \in \mathbb{C}$ *and* $\varepsilon > 0$ *we have*

$$\liminf_{X\to\infty} \frac{1}{\#\mathcal{P}^{\gamma}_X(m,a)} \#\{q \in \mathcal{P}^{\gamma}_X(m,a) \mid |L(s_0,\chi_q) - z_0| < \varepsilon\} > 0. \tag{1.4}$$

(2) *Let* $1/2 < \sigma_0 < 1$ *be fixed. Then the set* $\{L(\sigma_0,\chi_q) \mid q \in \mathcal{P}^{\gamma}(m,a)\}$ *is dense in* $\mathbb{R}^+$. *More precisely, for any* $x_0 \in \mathbb{R}^+$ *and* $\varepsilon > 0$ *we have*

$$\liminf_{X\to\infty} \frac{1}{\#\mathcal{P}^{\gamma}_X(m,a)} \#\{q \in \mathcal{P}^{\gamma}_X(m,a) \mid |L(\sigma_0,\chi_q) - x_0| < \varepsilon\} > 0.$$

Next we have a non-vanishing result for $L(s,\chi_q)$'s on D, and the following stronger result.

Corollary 1.3. *Let* α, β *be any positive real numbers with* $\alpha < \beta$. *Let* K *be a compact set in* D. *Then*

$$\liminf_{X\to\infty} \frac{1}{\#\mathcal{P}^{\gamma}_X(m,a)} \#\{q \in \mathcal{P}^{\gamma}_X(m,a) \mid \alpha < |L(s,\chi_q)| < \beta \text{ uniformly for } s \in K\} > 0.$$

Noting that $L(s,\chi_q)$ is $\mathbb{R}$-valued on the real segment $(1/2,1)$, we can obtain a result concerning the horizontal distribution of zeros of the derivatives $L^{(r)}(s,\chi_q)$ on $(1/2,1)$ in the q-aspect.

Corollary 1.4. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $1/2 < \alpha < \beta < 1$ *and* $r', N \in \mathbb{N}$. *Then there exist infinitely many* $q \in \mathcal{P}^{\gamma}(m,a)$ *such that for every integer* r *with* $1 \le r \le r'$ *the* r*th derivative* $L^{(r)}(s,\chi_q)$ *has at least* N *zeros on the interval* $[\alpha,\beta] \subset \mathbb{R}$. *More precisely*,

$$\liminf_{X\to\infty} \frac{1}{\#\mathcal{P}^{\gamma}_X(m,a)} \#\{q \in \mathcal{P}^{\gamma}_X(m,a) \mid L^{(r)}(s,\chi_q) \text{ has at least } N \text{ zeros on } [\alpha,\beta] \text{ for every } r = 1,\ldots,r'\} > 0.$$

We shall also study the denseness result on values of $L(s,\chi_q)$ for a fixed complex number $s \neq 1$ with $\mathrm{Re}\, s = 1$ and variable $q \in \mathcal{P}^\gamma(m,a)$.

THEOREM 1.5. *Let $t \in \mathbb{R} - \{0\}$ be fixed. Then the set $\{L(1+it,\chi_q) \mid q \in \mathcal{P}^\gamma(m,a)\}$ is dense in $\mathbb{C}$. More precisely, for any $z_0 \in \mathbb{C}$ and $\varepsilon > 0$ we have*

$$\liminf_{X\to\infty} \frac{1}{\#\mathcal{P}_X^\gamma(m,a)} \#\{q \in \mathcal{P}_X^\gamma(m,a) \mid |L(1+it,\chi_q) - z_0| < \varepsilon\} > 0.$$

In [MN2] the authors showed an analogue of Theorem 1.5 for $L(1,\lambda_p)$ and deduced from it a quantitative result for a problem of Ayoub–Chowla–Walum on certain character sums.

2. Denseness lemma. The purpose of this section is to show Proposition 2.3 below. For $s \in \mathbb{C}$ we write $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. The next lemma is proved in [MN1, Proposition 2.4].

LEMMA 2.1. *Let Ω be a simply connected region in D symmetric with respect to the real axis, as in Theorem 1.1. Let U be a bounded, simply connected region in Ω which is symmetric with respect to the real axis and which satisfies $\overline{U} \subset \Omega$, where $\overline{U}$ denotes the closure of U. Suppose that $g(s)$ is a holomorphic function on Ω which is $\mathbb{R}$-valued on the interval $\Omega \cap \mathbb{R}$. Let $y > 0$ be fixed. Then for any $\varepsilon > 0$ there exist $\nu \in \mathbb{R}^+$ and $c_p \in \{1,-1\}$, for each prime p with $y \le p \le \nu$, such that*

$$\int_U \Big| g(s) - \sum_{y \le p \le \nu} \frac{c_p}{p^s} \Big|^2 \, d\sigma\, dt < \varepsilon.$$

The next lemma is a generalization of [Ti, p. 303, Lemma] and was obtained in [MN1, Lemma 2.5].

LEMMA 2.2. *Let U be a bounded region in $\mathbb{C}$. Let K be a compact subset of $\mathbb{C}$ such that $K \subset U$. Let $B > 0$. Suppose that $f(s)$ is a holomorphic function on U satisfying $\int_U |f(s)|^2\, d\sigma\, dt \le B$. Then $\max_{s\in K} |f(s)| \le b(U,K)\, B^{1/2}$, where $b(U,K)$ is a certain positive constant depending only on U and K.*

PROPOSITION 2.3. *Let Ω be a simply connected region in D symmetric with respect to the real axis. Suppose that $g(s)$ is a holomorphic function on Ω which is $\mathbb{R}$-valued on $\Omega \cap \mathbb{R}$. Let K be a compact set in Ω and $\nu_1 \in \mathbb{R}^+$ with $\nu_1 > m+1$. Let $a_p \in \{1,-1\}$ for each prime p with $p \mid m$. Then for any $\varepsilon > 0$ there exist $\nu > \nu_1$ and $a_p \in \{1,-1\}$, for each prime p with $p \le \nu$ and $p \nmid m$, such that*

$$\max_{s\in K} \Big| g(s) - \log \prod_{p\le\nu} \Big(1 - \frac{a_p}{p^s}\Big)^{-1} \Big| < \varepsilon,$$

where

$$\log \prod_{p\le\nu}\left(1-\frac{a_p}{p^s}\right)^{-1} = -\sum_{p\le\nu}\log\left(1-\frac{a_p}{p^s}\right) = \sum_{p\le\nu}\sum_{n=1}^{\infty}\frac{a_p^n}{np^{ns}}.$$

Proof. Take a bounded, simply connected region U in Ω which is symmetric with respect to the real axis and which satisfies $K\subset U$ and $\overline{U}\subset\Omega$. Set $\sigma_1 := \min\{\operatorname{Re} s \mid s\in\overline{U}\} > 1/2$. Let $\varepsilon>0$ be arbitrary. Fix a real number y satisfying $y>\nu_1$ and $y^{1-2\sigma_1}/(2\sigma_1-1)<\varepsilon$. Then we have

$$\sum_{p\ge y}\sum_{n=2}^{\infty}\frac{1}{np^{n\sigma_1}} \le \sum_{p\ge y}\sum_{n=2}^{\infty}\frac{1}{p^{n\sigma_1}} = \sum_{p\ge y}\frac{p^{-2\sigma_1}}{1-p^{-\sigma_1}} \ll \sum_{n\ge y,\, n\in\mathbb{N}}\frac{1}{n^{2\sigma_1}} \ll \frac{y^{1-2\sigma_1}}{2\sigma_1-1}<\varepsilon. \tag{2.1}$$

Set $a_p=1$ for each prime p with $p<y$ and $p\nmid m$. From Lemma 2.1 it follows that there exist $\nu\ge y$ and $c_p\in\{1,-1\}$, for each prime p with $y\le p\le\nu$, such that

$$\int_U\left|\left(g(s)-\sum_{p<y}\sum_{n=1}^{\infty}\frac{a_p^n}{np^{ns}}\right)-\sum_{y\le p\le\nu}\frac{c_p}{p^s}\right|^2 d\sigma\,dt<\varepsilon^2.$$

This and Lemma 2.2 yield

$$\max_{s\in K}\left|g(s)-\sum_{p<y}\sum_{n=1}^{\infty}\frac{a_p^n}{np^{ns}}-\sum_{y\le p\le\nu}\frac{c_p}{p^s}\right| \ll_{U,K}\varepsilon. \tag{2.2}$$

For each prime p with $y\le p\le\nu$ we set $a_p=c_p$. Then we obtain, by (2.1) and (2.2),

$$\begin{aligned}
&\max_{s\in K}\left|g(s)-\log\prod_{p\le\nu}\left(1-\frac{a_p}{p^s}\right)^{-1}\right| \\
&\quad=\max_{s\in K}\left|g(s)-\sum_{p<y}\sum_{n=1}^{\infty}\frac{a_p^n}{np^{ns}}-\sum_{y\le p\le\nu}\frac{c_p}{p^s}-\sum_{y\le p\le\nu}\sum_{n=2}^{\infty}\frac{c_p^n}{np^{ns}}\right| \\
&\quad\le\max_{s\in K}\left|g(s)-\sum_{p<y}\sum_{n=1}^{\infty}\frac{a_p^n}{np^{ns}}-\sum_{y\le p\le\nu}\frac{c_p}{p^s}\right|+\max_{s\in K}\left|\sum_{y\le p\le\nu}\sum_{n=2}^{\infty}\frac{c_p^n}{np^{ns}}\right| \\
&\quad\ll_{U,K}\varepsilon+\sum_{p\ge y}\sum_{n=2}^{\infty}\frac{1}{np^{n\sigma_1}}\ll\varepsilon,
\end{aligned}$$

which completes the proof. ∎

3. Approximation by finite Euler products. As usual, let $\pi(X)$ denote the number of primes not exceeding $X \in \mathbb{R}^+$. For large $X \in \mathbb{R}^+$, let R_X denote the set

$$\{s = \sigma + it \in \mathbb{C} \mid 1/2 + (\log\log\log X)^{-1/2} \le \sigma \le 5/4, |t| < X^{\frac{1}{13}(2\sigma-1)}\}$$

and put

$$h_X := (\log\log X)^2. \tag{3.1}$$

The next lemma is obtained in [El, Lemma 8].

LEMMA 3.1. *For all large X and uniformly for $s \in R_X$ we have*

$$\sum_{\substack{3\le r\le X \\ r\,:\,\text{prime}}} \left| L(s,\lambda_r) - \prod_{p\le h_X} \left(1 - \frac{\lambda_r(p)}{p^s}\right)^{-1}\right|^2 \ll \pi(X) h_X^{1-2\sigma} (\log h_X)^3 (2\sigma - 1)^{-4}.$$

Recall that for an odd prime r and a positive integer n we have the relation (see e.g. [Ay, p. 290, Lemma 2.2])

$$\left(\frac{n}{r}\right) = \begin{cases} \left(\dfrac{r}{n}\right)_{\mathrm{K}} & \text{if } r \equiv 1 \bmod 4, \\ \left(\dfrac{-r}{n}\right)_{\mathrm{K}} & \text{if } r \equiv 3 \bmod 4, \end{cases} \tag{3.2}$$

and hence

$$L(s,\lambda_r) = \begin{cases} L(s,\chi_r) & \text{if } r \equiv 1 \bmod 4, \\ L(s,\chi_{-r}) & \text{if } r \equiv 3 \bmod 4. \end{cases} \tag{3.3}$$

PROPOSITION 3.2. *Let $\varepsilon > 0$ and K be a compact set in the region $1/2 < \operatorname{Re} s < 5/4$. Define $\mathcal{A}_X^\gamma(m,a) = \mathcal{A}_X^\gamma(m,a,\varepsilon,K)$ by*

$$\mathcal{A}_X^\gamma(m,a) := \left\{ q \in \mathcal{P}_X^\gamma(m,a) \,\middle|\, \max_{s\in K} \left| L(s,\chi_q) - \prod_{p\le h_X}\left(1 - \frac{\chi_q(p)}{p^s}\right)^{-1}\right| < \varepsilon \right\}.$$

Then

$$\frac{\#\mathcal{A}_X^\gamma(m,a)}{\#\mathcal{P}_X^\gamma(m,a)} > 1 - \varepsilon$$

if X is sufficiently large.

Proof. Take an open rectangle U of the form $\{s \in \mathbb{C} \mid \sigma_1 < \operatorname{Re} s < \sigma_2, |\operatorname{Im} s| < A\}$ satisfying $1/2 < \sigma_1 < \min\{\operatorname{Re} s \mid s \in K\} \le \max\{\operatorname{Re} s \mid s \in K\} < \sigma_2 < 5/4$ and $\max\{|\operatorname{Im} s| \mid s \in K\} < A$. Then $K \subset U$. For large $X \in \mathbb{R}^+$ we define $\widetilde{\mathcal{A}}_X^\gamma(m,a)$ to be the set

$$\left\{ q \in \mathcal{P}_X^\gamma(m,a) \,\middle|\, \iint_U \left| L(s,\chi_q) - \prod_{p\le h_X}\left(1 - \frac{\chi_q(p)}{p^s}\right)^{-1}\right|^2 d\sigma\, dt < \frac{\varepsilon^2}{b(U,K)^2} \right\}, \tag{3.4}$$

where $b(U,K)$ is the constant in Lemma 2.2. By Lemma 2.2,

$$\widetilde{\mathcal{A}}_X^\gamma(m,a) \subset \mathcal{A}_X^\gamma(m,a). \tag{3.5}$$

From Lemma 3.1, (3.2), (3.3), the prime number theorem, and (1.2), we infer that for all large X,

$$\begin{aligned}\sum_{q\in\mathcal{P}_X^\gamma(m,a)} \int_U \bigg| L(s,\chi_q) - \prod_{p\le h_X}\bigg(1-\frac{\chi_q(p)}{p^s}\bigg)^{-1}\bigg|^2 d\sigma\, dt \\ \le \sum_{\substack{3\le r\le X\\ r:\,\text{prime}}} \int_U \bigg| L(s,\lambda_r) - \prod_{p\le h_X}\bigg(1-\frac{\lambda_r(p)}{p^s}\bigg)^{-1}\bigg|^2 d\sigma\, dt \\ \ll_U \pi(X) h_X^{1-2\sigma_1}(\log h_X)^3(2\sigma_1-1)^{-4} \\ \ll \varphi(m)\#\mathcal{P}_X^\gamma(m,a) h_X^{1-2\sigma_1}(\log h_X)^3(2\sigma_1-1)^{-4}.\end{aligned} \tag{3.6}$$

Since $h_X^{1-2\sigma_1}(\log h_X)^3 \to 0$ as $X\to\infty$, it follows from (3.6) that there exists a large number $X_0 = X_0(\varepsilon,U,K,m)$ such that for all $X > X_0$,

$$\sum_{q\in\mathcal{P}_X^\gamma(m,a)} \int_U \bigg| L(s,\chi_q) - \prod_{p\le h_X}\bigg(1-\frac{\chi_q(p)}{p^s}\bigg)^{-1}\bigg|^2 d\sigma\, dt < \frac{\varepsilon^3}{b(U,K)^2}\#\mathcal{P}_X^\gamma(m,a). \tag{3.7}$$

Now assume that there exists a real number $X > X_0$ such that $\#(\mathcal{P}_X^\gamma(m,a) - \widetilde{\mathcal{A}}_X^\gamma(m,a)) \ge \varepsilon\#\mathcal{P}_X^\gamma(m,a)$. For this X we have, by (3.4),

$$\begin{aligned}\sum_{q\in\mathcal{P}_X^\gamma(m,a)} \int_U \bigg| L(s,\chi_q) - \prod_{p\le h_X}\bigg(1-\frac{\chi_q(p)}{p^s}\bigg)^{-1}\bigg|^2 d\sigma\, dt \\ \ge \sum_{q\in\mathcal{P}_X^\gamma(m,a)-\widetilde{\mathcal{A}}_X^\gamma(m,a)} \int_U \bigg| L(s,\chi_q) - \prod_{p\le h_X}\bigg(1-\frac{\chi_q(p)}{p^s}\bigg)^{-1}\bigg|^2 d\sigma\, dt \\ \ge \varepsilon\#\mathcal{P}_X^\gamma(m,a)\,\frac{\varepsilon^2}{b(U,K)^2} = \frac{\varepsilon^3}{b(U,K)^2}\#\mathcal{P}_X^\gamma(m,a).\end{aligned}$$

However, this contradicts (3.7). Hence for any $X > X_0$ we have

$$\#(\mathcal{P}_X^\gamma(m,a) - \widetilde{\mathcal{A}}_X^\gamma(m,a)) < \varepsilon\#\mathcal{P}_X^\gamma(m,a),$$

that is, $\#\widetilde{\mathcal{A}}_X^\gamma(m,a)/\#\mathcal{P}_X^\gamma(m,a) > 1-\varepsilon$. This and (3.5) complete the proof. ∎

4. Results on characters χ_q for prime discriminants q. The aim of this section is to obtain Proposition 4.3. As before, the letter γ denotes the plus sign or the minus sign. For $X \in \mathbb{R}^+$ we define I_X to be the interval

$[0, X]$ if γ is $+$, and $[-X, 0]$ if γ is $-$. We define

$$\delta = \delta(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is } +, \\ -1 & \text{if } \gamma \text{ is } -. \end{cases} \tag{4.1}$$

LEMMA 4.1. *Fix a number* $\nu \in \mathbb{R}^+$ *such that* $\pi(\nu) > \pi(m)$. *Let* $a_p \in \{1, -1\}$ *for each prime* p *satisfying* $p \le \nu$ *and* $p \nmid m$. *Define* $\mathcal{P}^\gamma_{X,\nu}(m, a) = \mathcal{P}^\gamma_{X,\nu}(m, a, \{a_p\})$ *to be the set*

$$\{q \in \mathcal{P}^\gamma_X(m, a) \mid \chi_q(p) = a_p \text{ for every prime } p \text{ with } p \le \nu \text{ and } p \nmid m\},$$

and put $C_\nu(m) := \prod_{p \le \nu,\, p\nmid m} \frac{1}{2}$. *Then*

$$\lim_{X\to\infty} \frac{\#\mathcal{P}^\gamma_{X,\nu}(m, a)}{\#\mathcal{P}^\gamma_X(m, a)} = C_\nu(m).$$

Proof. In general, for $n \in \mathbb{N}$ and $b \in \mathbb{Z}$, we denote by $[b]_n$ the set of all integers x such that $x \equiv b \bmod n$, that is, the residue class mod n which b belongs to.

Let p be an odd prime. Let $\mathcal{Q}_p$ be the set of all residue classes $[b]_p$ mod p such that b is a quadratic residue mod p, other than the residue class $[0]_p$, and let $\mathcal{Q}'_p$ be the set of all residue classes $[c]_p$ mod p such that c is a quadratic non-residue mod p. It is well known that

$$\#\mathcal{Q}_p = \#\mathcal{Q}'_p = \frac{p-1}{2}. \tag{4.2}$$

In view of the definitions of Kronecker's symbol and Legendre's symbol, a discriminant q satisfies $\chi_q(p) = a_p$ if and only if q belongs to one of residue classes in $\mathcal{Q}_p$ if $a_p = 1$ and in $\mathcal{Q}'_p$ if $a_p = -1$. From this, (4.2) and the Chinese remainder theorem, it follows, for an integer r such that δr is a prime number, that r satisfies $r \equiv \delta a \bmod m$ (i.e. $r \in \mathcal{P}^\gamma(m, a)$) and $\chi_r(p) = a_p$ for every prime p with $p \le \nu$ and $p \nmid m$ if and only if r belongs to one of exactly $\prod_{p\le\nu,\, p\nmid m}(p-1)/2$ distinct residue classes mod Q, where

$$Q = Q(m, \nu) := m \prod_{p\le\nu,\, p\nmid m} p$$

and δ is as in (4.1). Let $\mathcal{R}^\gamma = \mathcal{R}^\gamma(m, a, \nu)$ denote the set of those residue classes mod Q, so that

$$\#\mathcal{R}^\gamma = \prod_{p\le\nu,\, p\nmid m} \frac{p-1}{2}. \tag{4.3}$$

Thus

$$\begin{aligned}(4.4)\quad \mathcal{P}^{\gamma}_{X,\nu}(m,a) &= \{r\in I_X \mid \delta r \text{ is a prime, } r\equiv \delta a \bmod m,\\ &\qquad \chi_r(p)=a_p \text{ for every prime } p \text{ with } p\le\nu \text{ and } p\nmid m\}\\ &= \bigcup_{[c]_Q\in\mathcal{R}^{\gamma}} \{r\in I_X \mid \delta r \text{ is a prime, } r\equiv c \bmod Q\}.\end{aligned}$$

We note that if $[c]_Q\in\mathcal{R}^{\gamma}$ then

$$\gcd(c,Q)=1, \tag{4.5}$$

since $[0]_p\notin\mathcal{Q}_p$ and $[0]_p\notin\mathcal{Q}'_p$ for all primes p with $p\le\nu$ and $p\nmid m$, and $\gcd(a,m)=1$.

From (4.4) we have

$$\#\mathcal{P}^{\gamma}_{X,\nu}(m,a)=\sum_{[c]_Q\in\mathcal{R}^{\gamma}}\ \sum_{\substack{r\in I_X,\,\delta r:\,\text{prime}\\ r\equiv c \bmod Q}} 1=\sum_{[c]_Q\in\mathcal{R}^{\gamma}}\ \sum_{\substack{p\le X\\ p\equiv\delta c \bmod Q}} 1.$$

By the prime number theorem for arithmetic progressions and (4.5),

$$\sum_{\substack{p\le X\\ p\equiv\delta c \bmod Q}} 1\sim\frac{1}{\varphi(Q)}\frac{X}{\log X}\quad \text{as } X\to\infty. \tag{4.6}$$

Note that the right-hand side of (4.6) is independent of $[c]_Q\in\mathcal{R}^{\gamma}$. Therefore for fixed ν we have

$$\begin{aligned}(4.7)\quad \#\mathcal{P}^{\gamma}_{X,\nu}(m,a) &\sim \frac{\#\mathcal{R}^{\gamma}}{\varphi(Q)}\frac{X}{\log X}\\ &= \Big(\prod_{p\le\nu,\,p\nmid m}\frac{1}{2}\Big)\frac{X}{\varphi(m)\log X}=\frac{C_\nu(m)}{\varphi(m)}\frac{X}{\log X}\end{aligned}$$

as $X\to\infty$, using (4.3) and the fact

$$\varphi(Q)=\varphi(m)\prod_{p\le\nu,\,p\nmid m}\varphi(p)=\varphi(m)\prod_{p\le\nu,\,p\nmid m}(p-1). \tag{4.8}$$

Thus (4.7) and (1.2) give us

$$\frac{\#\mathcal{P}^{\gamma}_{X,\nu}(m,a)}{\#\mathcal{P}^{\gamma}_{X}(m,a)}=\frac{\#\mathcal{P}^{\gamma}_{X,\nu}(m,a)}{\frac{C_\nu(m)}{\varphi(m)}\frac{X}{\log X}}\,\frac{\frac{C_\nu(m)}{\varphi(m)}\frac{X}{\log X}}{\frac{1}{\varphi(m)}\frac{X}{\log X}}\,\frac{\frac{1}{\varphi(m)}\frac{X}{\log X}}{\#\mathcal{P}^{\gamma}_{X}(m,a)}\to C_\nu(m)$$

as $X\to\infty$. This completes the proof. ∎

LEMMA 4.2. *Fix $\nu\in\mathbb{R}^+$ such that $\pi(\nu)>\pi(m)$. Let $a_p\in\{1,-1\}$ for each prime p with $p\le\nu$ and $p\nmid m$. Let $\mathcal{P}^{\gamma}_{X,\nu}(m,a)$ and $C_\nu(m)$ be as in Lemma 4.1, $h_X=(\log\log X)^2$ be as in (3.1), and $\sigma_1>1/2$. Then for all* *large X and uniformly for $s \in \mathbb{C}$ with* $\mathrm{Re}\, s \geq \sigma_1$ *we have*

$$\sum_{q \in \mathcal{P}^{\gamma}_{X,\nu}(m,a)} \Bigg| \sum_{\nu < p \leq h_X} \frac{\chi_q(p)}{p^s} \Bigg|^2 \ll \frac{\nu^{1-2\sigma_1}}{2\sigma_1 - 1} \, C_\nu(m) \, \#\mathcal{P}^{\gamma}_X(m,a). \tag{4.9}$$

Proof. Let Q and $\mathcal{R}^\gamma$ be as in the proof of Lemma 4.1. From (4.4) it follows that

$$\begin{aligned} \sum_{q \in \mathcal{P}^{\gamma}_{X,\nu}(m,a)} \Bigg| \sum_{\nu < p \leq h_X} \frac{\chi_q(p)}{p^s} \Bigg|^2 &= \sum_{[c]_Q \in \mathcal{R}^\gamma} \sum_{\substack{r \in I_X,\, \delta r : \mathrm{prime} \\ r \equiv c \bmod Q}} \Bigg| \sum_{\nu < p \leq h_X} \frac{\chi_r(p)}{p^s} \Bigg|^2 \\ &= \sum_{[c]_Q \in \mathcal{R}^\gamma} \sum_{\substack{u \leq X,\, u : \mathrm{prime} \\ u \equiv \delta c \bmod Q}} \Bigg| \sum_{\nu < p \leq h_X} \frac{\chi_{\delta u}(p)}{p^s} \Bigg|^2. \end{aligned} \tag{4.10}$$

For $[c]_Q \in \mathcal{R}^\gamma$ we have

$$\begin{aligned} &\sum_{\substack{u \leq X,\, u : \mathrm{prime} \\ u \equiv \delta c \bmod Q}} \Bigg| \sum_{\nu < p \leq h_X} \frac{\chi_{\delta u}(p)}{p^s} \Bigg|^2 \\ &= \sum_{\substack{u \leq X,\, u : \mathrm{prime} \\ u \equiv \delta c \bmod Q}} \Bigg(\sum_{\nu < p \leq h_X} \frac{|\chi_{\delta u}(p)|^2}{|p^s|^2} + \sum_{\substack{p_1, p_2 : \mathrm{primes},\, p_1 \neq p_2 \\ \nu < p_1, p_2 \leq h_X}} \frac{\chi_{\delta u}(p_1)\overline{\chi_{\delta u}(p_2)}}{p_1^s \overline{p_2^s}} \Bigg) \\ &= \sum_{\nu < p \leq h_X} \frac{1}{|p^s|^2} \sum_{\substack{u \leq X,\, u : \mathrm{prime} \\ u \equiv \delta c \bmod Q}} |\chi_{\delta u}(p)|^2 \\ &\quad + \sum_{\substack{p_1, p_2 : \mathrm{primes},\, p_1 \neq p_2 \\ \nu < p_1, p_2 \leq h_X}} \frac{1}{p_1^s \overline{p_2^s}} \sum_{\substack{u \leq X,\, u : \mathrm{prime} \\ u \equiv \delta c \bmod Q}} \chi_{\delta u}(p_1)\overline{\chi_{\delta u}(p_2)} \\ &= S_1 + S_2, \quad \text{say.} \end{aligned} \tag{4.11}$$

Using the prime number theorem for arithmetic progressions, we deduce that for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geq \sigma_1$

$$\begin{aligned} |S_1| &\leq \sum_{\nu < p \leq h_X} \frac{1}{p^{2\sigma_1}} \sum_{\substack{u \leq X,\, u : \mathrm{prime} \\ u \equiv \delta c \bmod Q}} 1 \\ &\ll \Bigg(\sum_{n > \nu,\, n \in \mathbb{N}} \frac{1}{n^{2\sigma_1}} \Bigg) \frac{1}{\varphi(Q)} \frac{X}{\log X} \\ &\ll \frac{\nu^{1-2\sigma_1}}{2\sigma_1 - 1} \frac{1}{\varphi(Q)} \frac{X}{\log X}. \end{aligned} \tag{4.12}$$

Next we shall consider the sum S_2. Fix two distinct primes p_1, p_2 satisfying $\nu < p_1 \le h_X$ and $\nu < p_2 \le h_X$. Then by the definition of the Kronecker symbol and the orthogonality relation for Dirichlet characters, we have

$$\begin{aligned}(4.13)\quad \sum_{\substack{u\le X,\, u:\,\mathrm{prime}\\ u\equiv\delta c \bmod Q}} \chi_{\delta u}(p_1)\overline{\chi_{\delta u}(p_2)} &= \sum_{\substack{u\le X,\, u:\,\mathrm{prime}\\ u\equiv\delta c \bmod Q}} \left(\frac{\delta u}{p_1}\right)\left(\frac{\delta u}{p_2}\right)\\ &= \sum_{u\le X,\, u:\,\mathrm{prime}} \left(\frac{\delta u}{p_1}\right)\left(\frac{\delta u}{p_2}\right)\frac{1}{\varphi(Q)}\sum_{\lambda \bmod Q}\lambda(u)\overline{\lambda(\delta c)}\\ &= \frac{1}{\varphi(Q)}\left(\frac{\delta}{p_1}\right)\left(\frac{\delta}{p_2}\right)\sum_{\lambda \bmod Q}\overline{\lambda(\delta c)}\sum_{u\le X,\, u:\,\mathrm{prime}}\left(\frac{u}{p_1}\right)\left(\frac{u}{p_2}\right)\lambda(u),\end{aligned}$$

where $\sum_{\lambda \bmod Q}$ means the sum over all the Dirichlet characters $\lambda \bmod Q$. Since p_1, p_2 and Q are relatively prime in pairs, we find from the Chinese remainder theorem that for any character $\lambda \bmod Q$ the product $\left(\frac{\cdot}{p_1}\right)\left(\frac{\cdot}{p_2}\right)\lambda(\cdot)$ is a non-principal Dirichlet character mod p_1p_2Q. From this, the Siegel–Walfisz theorem (see [Da, p. 132, (3)]) and partial summation, it follows, for fixed ν, that for all large X and all pairs of distinct primes (p_1, p_2) satisfying $\nu < p_1 \le h_X$ and $\nu < p_2 \le h_X$, we have

$$(4.14)\qquad \sum_{u\le X,\, u:\,\mathrm{prime}}\left(\frac{u}{p_1}\right)\left(\frac{u}{p_2}\right)\lambda(u) \ll Xe^{-b\sqrt{\log X}},$$

where b is an absolute positive constant. From this and (4.13) we infer

$$\begin{aligned}(4.15)\quad |S_2| &\le \sum_{\substack{p_1,p_2:\,\mathrm{primes},\, p_1\ne p_2\\ \nu<p_1,p_2\le h_X}} \frac{1}{p_1^{\sigma_1}p_2^{\sigma_1}}\left|\sum_{\substack{u\le X,\, u:\,\mathrm{prime}\\ u\equiv\delta c \bmod Q}} \chi_{\delta u}(p_1)\overline{\chi_{\delta u}(p_2)}\right|\\ &= \sum_{\substack{p_1,p_2:\,\mathrm{primes},\, p_1\ne p_2\\ \nu<p_1,p_2\le h_X}} \frac{1}{p_1^{\sigma_1}p_2^{\sigma_1}}\, O(Xe^{-b\sqrt{\log X}})\\ &\ll \left(\sum_{p\le h_X}\frac{1}{p^{\sigma_1}}\right)^2 O(Xe^{-b\sqrt{\log X}}) \ll h_X^2 Xe^{-b\sqrt{\log X}}\\ &= o\left(\frac{X}{\log X}\right).\end{aligned}$$

Consequently, for fixed ν we find, from (4.11), (4.12) and (4.15), that for all large X and uniformly for $s\in\mathbb{C}$ with $\operatorname{Re} s \ge \sigma_1$,

$$(4.16)\qquad \sum_{\substack{u\le X,\, u:\,\mathrm{prime}\\ u\equiv\delta c \bmod Q}}\left|\sum_{\nu<p\le h_X}\frac{\chi_{\delta u}(p)}{p^s}\right|^2 \ll \frac{\nu^{1-2\sigma_1}}{2\sigma_1-1}\,\frac{1}{\varphi(Q)}\,\frac{X}{\log X}.$$

Note that the right-hand side of (4.16) is independent of $[c]_Q \in \mathcal{R}^\gamma$. Combining (4.16), (4.10), (4.3), (4.8) and (1.2), we conclude that

$$\sum_{q\in\mathcal{P}^\gamma_{X,\nu}(m,a)} \Big|\sum_{\nu<p\le h_X} \frac{\chi_q(p)}{p^s}\Big|^2 \ll \#\mathcal{R}^\gamma \frac{\nu^{1-2\sigma_1}}{2\sigma_1-1}\frac{1}{\varphi(Q)}\frac{X}{\log X}$$

$$\ll \#\mathcal{R}^\gamma \frac{\nu^{1-2\sigma_1}}{2\sigma_1-1}\frac{\varphi(m)}{\varphi(Q)}\#\mathcal{P}^\gamma_X(m,a) \ll \frac{\nu^{1-2\sigma_1}}{2\sigma_1-1} C_\nu(m)\#\mathcal{P}^\gamma_X(m,a).$$

This completes the proof. ∎

To obtain (4.14) we have used the Siegel–Walfisz theorem. We remark that actually, instead of the Siegel–Walfisz theorem, a weaker result (e.g. [Da, p. 123]) is sufficient since $p_1p_2Q \ll_\nu h_X^2 = (\log\log X)^4$.

Proposition 4.3. *Let $\sigma_1 > 1/2$ and K be a compact subset of $\mathbb{C}$ such that $K \subset \{s\in\mathbb{C} \mid \operatorname{Re} s > \sigma_1\}$. Let $\varepsilon > 0$. Then there exists a large real number $\nu_0 = \nu_0(\sigma_1, K, \varepsilon, m)$ depending only on σ_1, K, ε and m, and satisfying $\pi(\nu_0) > \pi(m)$ and the following. Fix any real number $\nu > \nu_0$. Let $a_p \in \{1,-1\}$ for each prime p satisfying $p \le \nu$ and $p\nmid m$. Let $\mathcal{P}^\gamma_{X,\nu}(m,a), C_\nu(m)$ and h_X be as in Lemma 4.2 for large X. Define $\mathcal{B}^\gamma_{X,\nu}(m,a) = \mathcal{B}^\gamma_{X,\nu}(m,a,\varepsilon,\sigma_1,K,\{a_p\})$ by*

$$\mathcal{B}^\gamma_{X,\nu}(m,a) := \Big\{q\in\mathcal{P}^\gamma_{X,\nu}(m,a) \;\Big|\; \max_{s\in K}\Big|\sum_{\nu<p\le h_X}\frac{\chi_q(p)}{p^s}\Big| < \varepsilon\Big\}.$$

Then for all sufficiently large X we have

$$\frac{\#\mathcal{B}^\gamma_{X,\nu}(m,a)}{\#\mathcal{P}^\gamma_X(m,a)} > \frac{1}{2}C_\nu(m).$$

Proof. Set $\sigma_2 = 1+\sup\{\operatorname{Re} s \mid s\in K\}$ and $A = 1+\sup\{|\operatorname{Im} s| \mid s\in K\}$. Let U be the open rectangle $\{s\in\mathbb{C} \mid \sigma_1 < \operatorname{Re} s < \sigma_2, |\operatorname{Im} s| < A\}$ in $\mathbb{C}$, and then $U\supset K$. Take a large real number $\nu_0 = \nu_0(\sigma_1,K,\varepsilon,m)$ satisfying $\pi(\nu_0) > \pi(m)$ and

$$\Big(\int_U 1\,d\sigma\,dt\Big) c\,\frac{\nu_0^{1-2\sigma_1}}{2\sigma_1-1} < \frac{\varepsilon^2}{4b(U,K)^2}, \tag{4.17}$$

where c is the absolute constant implied by the symbol $\ll$ in (4.9), and $b(U,K)$ is the constant in Lemma 2.2. Note that ν_0 depends only on σ_1, K, ε and m.

In the following we fix any $\nu > \nu_0$. For large X we define

$$\widetilde{\mathcal{B}}^\gamma_{X,\nu}(m,a) := \Big\{q\in\mathcal{P}^\gamma_{X,\nu}(m,a) \;\Big|\; \int_U\Big|\sum_{\nu<p\le h_X}\frac{\chi_q(p)}{p^s}\Big|^2 d\sigma\,dt < \frac{\varepsilon^2}{b(U,K)^2}\Big\}. \tag{4.18}$$

By Lemma 2.2,

$$\widetilde{\mathcal{B}}^{\gamma}_{X,\nu}(m,a) \subset \mathcal{B}^{\gamma}_{X,\nu}(m,a). \tag{4.19}$$

By Lemma 4.2 and (4.17), we have, for all large X,

$$\begin{aligned}\sum_{q\in\mathcal{P}^{\gamma}_{X,\nu}(m,a)} \int_U \Big| \sum_{\nu<p\le h_X} \frac{\chi_q(p)}{p^s}\Big|^2 d\sigma\, dt &\le \Big(\int_U 1\, d\sigma\, dt\Big) c\, \frac{\nu^{1-2\sigma_1}}{2\sigma_1-1}\, C_\nu(m)\, \#\mathcal{P}^{\gamma}_X(m,a) \\ &< \frac{\varepsilon^2}{4b(U,K)^2}\, C_\nu(m)\, \#\mathcal{P}^{\gamma}_X(m,a).\end{aligned} \tag{4.20}$$

Now we assume that there exists a large number X such that

$$\#(\mathcal{P}^{\gamma}_{X,\nu}(m,a) - \widetilde{\mathcal{B}}^{\gamma}_{X,\nu}(m,a)) \ge \frac{1}{4}\, C_\nu(m)\, \#\mathcal{P}^{\gamma}_X(m,a).$$

Then for this X we have, using (4.18),

$$\begin{aligned}\sum_{q\in\mathcal{P}^{\gamma}_{X,\nu}(m,a)} \int_U \Big| \sum_{\nu<p\le h_X} \frac{\chi_q(p)}{p^s}\Big|^2 d\sigma\, dt &\ge \sum_{q\in\mathcal{P}^{\gamma}_{X,\nu}(m,a)-\widetilde{\mathcal{B}}^{\gamma}_{X,\nu}(m,a)} \int_U \Big| \sum_{\nu<p\le h_X} \frac{\chi_q(p)}{p^s}\Big|^2 d\sigma\, dt \\ &\ge \frac{1}{4}\, C_\nu(m)\, \#\mathcal{P}^{\gamma}_X(m,a)\, \frac{\varepsilon^2}{b(U,K)^2}.\end{aligned}$$

However, this contradicts (4.20). Hence for all large X we have

$$\#(\mathcal{P}^{\gamma}_{X,\nu}(m,a) - \widetilde{\mathcal{B}}^{\gamma}_{X,\nu}(m,a)) < \frac{1}{4}\, C_\nu(m)\, \#\mathcal{P}^{\gamma}_X(m,a),$$

so

$$\frac{\#\widetilde{\mathcal{B}}^{\gamma}_{X,\nu}(m,a)}{\#\mathcal{P}^{\gamma}_X(m,a)} > \frac{\#\mathcal{P}^{\gamma}_{X,\nu}(m,a)}{\#\mathcal{P}^{\gamma}_X(m,a)} - \frac{1}{4}\, C_\nu(m). \tag{4.21}$$

Further, Lemma 4.1 implies that

$$\frac{\#\mathcal{P}^{\gamma}_{X,\nu}(m,a)}{\#\mathcal{P}^{\gamma}_X(m,a)} > \frac{3}{4}\, C_\nu(m) \quad \text{if } X \text{ is large enough.} \tag{4.22}$$

Combining (4.19), (4.21) and (4.22), we conclude that if X is large enough then

$$\frac{\#\mathcal{B}^{\gamma}_{X,\nu}(m,a)}{\#\mathcal{P}^{\gamma}_X(m,a)} \ge \frac{\#\widetilde{\mathcal{B}}^{\gamma}_{X,\nu}(m,a)}{\#\mathcal{P}^{\gamma}_X(m,a)} > \frac{3}{4}\, C_\nu(m) - \frac{1}{4}\, C_\nu(m) = \frac{1}{2}\, C_\nu(m). \ \blacksquare$$

5. Proofs of Theorem 1.1 and its corollaries

Proof of Theorem 1.1. Let $\varepsilon > 0$ be an arbitrary small number. Take a real number $\sigma_1 > 1/2$ such that $K \subset \{s \in \mathbb{C} \mid \mathrm{Re}\, s > \sigma_1\}$. Fix a large positive number ν_1 satisfying $\nu_1 > \nu_0(\sigma_1, K, \varepsilon, m)$ and $\nu_1^{1-2\sigma_1}/(2\sigma_1 - 1) < \varepsilon$, where $\nu_0(\sigma_1, K, \varepsilon, m)$ is the constant in Proposition 4.3. We set a_2 to be 1 if $a \equiv 1$ or $7 \bmod 8$, and -1 if $a \equiv 3$ or $5 \bmod 8$. Further, we set $a_p = \left(\frac{\delta a}{p}\right)$ for each odd prime p with $p \mid m$, where δ is as in (4.1).

As is shown in [MN1], there exists a holomorphic function $g(s)$ on Ω such that $g(x) \in \mathbb{R}$ for any $x \in \Omega \cap \mathbb{R}$ and

$$h(s) = e^{g(s)}. \tag{5.1}$$

Now Proposition 2.3 implies that there exist $\nu > \nu_1$ and $a_p \in \{1, -1\}$, for each prime p with $p \le \nu$ and $p \nmid m$, such that

$$\max_{s \in K} \left| g(s) - \log \prod_{p \le \nu} \left(1 - \frac{a_p}{p^s}\right)^{-1} \right| < \varepsilon. \tag{5.2}$$

For those a_p's, where $p \le \nu$ and $p \nmid m$, we apply Proposition 4.3. Then for the above number ν and all large X, we have

$$\frac{\#\mathcal{B}^{\gamma}_{X,\nu}(m,a)}{\#\mathcal{P}^{\gamma}_{X}(m,a)} > \frac{1}{2} C_\nu(m). \tag{5.3}$$

Since $8 \mid m$, we have $q \equiv \delta a \bmod 8$ and $q \equiv \delta a \bmod p$ for $q \in \mathcal{P}^{\gamma}(m,a)$ and a prime p with $p \mid m$. This and the definition of Kronecker's symbol yield $\chi_q(2) = a_2$ and $\chi_q(p) = \left(\frac{q}{p}\right) = \left(\frac{\delta a}{p}\right) = a_p$ for $q \in \mathcal{P}^{\gamma}(m,a)$ and an odd prime p with $p \mid m$. Hence, from the definition of $\mathcal{B}^{\gamma}_{X,\nu}(m,a)$ we find that for every $q \in \mathcal{B}^{\gamma}_{X,\nu}(m,a)$ and all large X,

$$\begin{aligned} &\max_{s \in K} \left| \log \prod_{p \le \nu} \left(1 - \frac{a_p}{p^s}\right)^{-1} - \log \prod_{p \le h_X} \left(1 - \frac{\chi_q(p)}{p^s}\right)^{-1} \right| \\ &\quad = \max_{s \in K} \left| \sum_{\nu < p \le h_X} \frac{\chi_q(p)}{p^s} + \sum_{\nu < p \le h_X} \sum_{n=2}^{\infty} \frac{\chi_q(p)^n}{np^{ns}} \right| \\ &\quad \le \max_{s \in K} \left| \sum_{\nu < p \le h_X} \frac{\chi_q(p)}{p^s} \right| + \max_{s \in K} \left| \sum_{\nu < p \le h_X} \sum_{n=2}^{\infty} \frac{\chi_q(p)^n}{np^{ns}} \right| \\ &\quad \le \varepsilon + O(\varepsilon) \ll \varepsilon, \end{aligned} \tag{5.4}$$

since

$$\left| \sum_{\nu < p \le h_X} \sum_{n=2}^{\infty} \frac{1}{np^{ns}} \right| \ll \sum_{\nu < p \le h_X} \frac{1}{p^{2\sigma_1}} \ll \frac{\nu^{1-2\sigma_1}}{2\sigma_1 - 1} < \frac{\nu_1^{1-2\sigma_1}}{2\sigma_1 - 1} < \varepsilon.$$

From (5.4) and (5.2) we deduce, for every $q \in \mathcal{B}^{\gamma}_{X,\nu}(m,a)$,

$$\max_{s\in K}\Big|g(s) - \log\prod_{p\le h_X}\Big(1-\frac{\chi_q(p)}{p^s}\Big)^{-1}\Big| \ll \varepsilon$$

and therefore

$$(5.5)\qquad \max_{s\in K}\Big|\prod_{p\le h_X}\Big(1-\frac{\chi_q(p)}{p^s}\Big)^{-1} - h(s)\Big|$$
$$= \max_{s\in K}\Big|h(s)\Big(\frac{\prod_{p\le h_X}(1-\chi_q(p)/p^s)^{-1}}{h(s)} - 1\Big)\Big|$$
$$\le \max_{s\in K}|h(s)|\max_{s\in K}|e^{\log\prod_{p\le h_X}(1-\chi_q(p)/p^s)^{-1}-g(s)} - 1|$$
$$\ll_{K,\,h(s)} \varepsilon,$$

using (5.1) and the fact that $e^z - 1 \ll |z|$ if $|z|$ is small.

Let ε_1 be a small positive number such that

$$\varepsilon_1 < \min\Big\{\varepsilon, \frac{C_\nu(m)}{2}\Big\}.$$

According to Proposition 3.2, if we put

$$(5.6)\qquad \mathcal{A}^{\gamma}_X(m,a) := \Big\{q \in \mathcal{P}^{\gamma}_X(m,a)\ \Big|\ \max_{s\in K}\Big|L(s,\chi_q) - \prod_{p\le h_X}\Big(1-\frac{\chi_q(p)}{p^s}\Big)^{-1}\Big| < \varepsilon_1\Big\},$$

then for all large X,

$$(5.7)\qquad \frac{\#\mathcal{A}^{\gamma}_X(m,a)}{\#\mathcal{P}^{\gamma}_X(m,a)} > 1 - \varepsilon_1.$$

By (5.6) and (5.5), every $q \in \mathcal{A}^{\gamma}_X(m,a) \cap \mathcal{B}^{\gamma}_{X,\nu}(m,a)$ satisfies

$$(5.8)\qquad \max_{s\in K}|L(s,\chi_q) - h(s)| \ll_{K,\,h(s)} \varepsilon.$$

Furthermore, from (5.3) and (5.7) it follows that for the above number ν and all large X,

$$(5.9)\qquad \#(\mathcal{A}^{\gamma}_X(m,a) \cap \mathcal{B}^{\gamma}_{X,\nu}(m,a))$$
$$\ge \#\mathcal{A}^{\gamma}_X(m,a) + \#\mathcal{B}^{\gamma}_{X,\nu}(m,a) - \#\mathcal{P}^{\gamma}_X(m,a)$$
$$\ge \Big(\frac{C_\nu(m)}{2} - \varepsilon_1\Big)\#\mathcal{P}^{\gamma}_X(m,a).$$

Since $C_\nu(m)/2 - \varepsilon_1 > 0$, (5.8) and (5.9) yield (1.3). This completes the proof. ∎

From Theorem 1.1 we can prove Corollaries 1.2–1.4 by the same arguments as in the proofs of Corollaries 1.2–1.4 in [MN1], respectively.

6. On the line $\mathrm{Re}\, s = 1$**.** In this section we prove Theorem 1.5. The next lemma is proved in [MN1].

LEMMA 6.1. *Let $t \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$ be fixed. Then for any $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, there exist $\nu \geq y$ and $c_p \in \{1, -1\}$, for each prime p with $y \leq p \leq \nu$, such that*

$$\Big| z_0 - \sum_{y \leq p \leq \nu} \frac{c_p}{p^{1+it}} \Big| < \varepsilon.$$

PROPOSITION 6.2. *Let $t \in \mathbb{R}^+$ be fixed. Let $z \in \mathbb{C}$ and $\nu_1 \in \mathbb{R}^+$ with $\nu_1 > m + 1$. Let $a_p \in \{1, -1\}$ for each prime p with $p \mid m$. Then for any $\varepsilon > 0$ there exist $\nu > \nu_1$ and $a_p \in \{1, -1\}$, for each prime p with $p \leq \nu$ and $p \nmid m$, such that*

$$\Big| z - \log \prod_{p \leq \nu} \Big(1 - \frac{a_p}{p^{1+it}}\Big)^{-1} \Big| < \varepsilon,$$

where

$$\log \prod_{p \leq \nu} \Big(1 - \frac{a_p}{p^{1+it}}\Big)^{-1} = -\sum_{p \leq \nu} \log\Big(1 - \frac{a_p}{p^{1+it}}\Big)$$
$$= \sum_{p \leq \nu} \sum_{n=1}^{\infty} \frac{a_p^n}{n p^{n(1+it)}}.$$

Proof. The proof is similar to that of Proposition 2.3. Let $\varepsilon > 0$ be arbitrary. Take a large number $y > \nu_1$ such that $1/y < \varepsilon$. Then

$$\sum_{p \geq y} \sum_{n=2}^{\infty} \frac{1}{n p^n} \ll \sum_{p \geq y} \frac{1}{p^2} \ll \frac{1}{y} < \varepsilon. \tag{6.1}$$

Set $a_p = 1$ for each prime p with $p < y$ and $p \nmid m$.

From Lemma 6.1 it follows that there exist $\nu \geq y$ and $c_p \in \{1, -1\}$, for each prime p with $y \leq p \leq \nu$, such that

$$\Big| \Big(z - \sum_{p < y} \sum_{n=1}^{\infty} \frac{a_p^n}{n p^{n(1+it)}} \Big) - \sum_{y \leq p \leq \nu} \frac{c_p}{p^{1+it}} \Big| < \varepsilon. \tag{6.2}$$

For each prime p with $y \leq p \leq \nu$ we set $a_p = c_p$. Then we obtain, by (6.1) and (6.2),

$$\begin{aligned}
&\left| z - \log \prod_{p \le \nu} \left(1 - \frac{a_p}{p^{1+it}}\right)^{-1} \right| \\
&\quad = \left| z - \sum_{p<y} \sum_{n=1}^{\infty} \frac{a_p^n}{np^{n(1+it)}} - \sum_{y \le p \le \nu} \frac{c_p}{p^{1+it}} - \sum_{y \le p \le \nu} \sum_{n=2}^{\infty} \frac{c_p^n}{np^{n(1+it)}} \right| \\
&\quad \le \left| z - \sum_{p<y} \sum_{n=1}^{\infty} \frac{a_p^n}{np^{n(1+it)}} - \sum_{y \le p \le \nu} \frac{c_p}{p^{1+it}} \right| + \left| \sum_{y \le p \le \nu} \sum_{n=2}^{\infty} \frac{c_p^n}{np^{n(1+it)}} \right| \\
&\quad < \varepsilon + \sum_{p \ge y} \sum_{n=2}^{\infty} \frac{1}{np^n} \ll \varepsilon,
\end{aligned}$$

which completes the proof. ∎

Proof of Theorem 1.5. The proof is similar to that of Theorem 1.1 in Section 5. Since $L(1+it,\chi_q) = \overline{L(1-it,\chi_q)}$, it suffices to verify the assertion in the case $t > 0$. Moreover, it suffices to consider the case $z_0 \in \mathbb{C} - \{0\}$, since the set $\mathbb{C} - \{0\}$ is dense in $\mathbb{C}$.

Fix $z_0 \in \mathbb{C} - \{0\}$ and $t > 0$. Take a complex number z such that $z_0 = e^z$. Let $\varepsilon > 0$ be an arbitrary small number. Take $\sigma_1 \in \mathbb{R}$ with $1/2 < \sigma_1 < 1$, and set $K = \{1+it\}$. Take $\nu_1 \in \mathbb{R}^+$ so large that $1/\nu_1 < \varepsilon$ and $\nu_1 > \nu_0(\sigma_1, K, \varepsilon, m)$, where $\nu_0(\sigma_1, K, \varepsilon, m)$ is the constant in Proposition 4.3. We set a_2 to be 1 if $a \equiv 1$ or 7 mod 8, and -1 if $a \equiv 3$ or 5 mod 8. Further, we set $a_p = \left(\frac{\delta a}{p}\right)$ for each odd prime p with $p \mid m$. According to Proposition 6.2, there exist $\nu > \nu_1$ and $a_p \in \{1, -1\}$, for each prime p with $p \le \nu$ and $p \nmid m$, such that

$$\left| z - \log \prod_{p \le \nu} \left(1 - \frac{a_p}{p^{1+it}}\right)^{-1} \right| < \varepsilon. \tag{6.3}$$

For those a_p's, where $p \le \nu$ and $p \nmid m$, we apply Proposition 4.3. Then for the above number ν and all large X, we have

$$\frac{\#\mathcal{B}^{\gamma}_{X,\nu}(m,a)}{\#\mathcal{P}^{\gamma}_{X}(m,a)} > \frac{1}{2}\, C_\nu(m). \tag{6.4}$$

Noting $\chi_q(2) = a_2$ and $\chi_q(p) = a_p$ for $q \in \mathcal{P}^{\gamma}(m,a)$ and an odd prime p with $p \mid m$, we have, for every $q \in \mathcal{B}^{\gamma}_{X,\nu}(m,a)$ and all large X,

$$\begin{aligned}
&\left| \log \prod_{p \le \nu} \left(1 - \frac{a_p}{p^{1+it}}\right)^{-1} - \log \prod_{p \le h_X} \left(1 - \frac{\chi_q(p)}{p^{1+it}}\right)^{-1} \right| \\
&\quad = \left| \sum_{\nu < p \le h_X} \frac{\chi_q(p)}{p^{1+it}} + \sum_{\nu < p \le h_X} \sum_{n=2}^{\infty} \frac{\chi_q(p)^n}{np^{n(1+it)}} \right| \le \varepsilon + O(\varepsilon) \ll \varepsilon,
\end{aligned} \tag{6.5}$$

since

$$\sum_{\nu<p\le h_X}\sum_{n=2}^{\infty}\frac{1}{np^n} \ll \sum_{\nu<p\le h_X}\frac{1}{p^2} \ll \nu^{-1} < \nu_1^{-1} < \varepsilon.$$

By (6.3) and (6.5), every $q \in \mathcal{B}^{\gamma}_{X,\nu}(m,a)$ satisfies

$$\left| z - \log \prod_{p\le h_X}\left(1-\frac{\chi_q(p)}{p^{1+it}}\right)^{-1}\right| \ll \varepsilon$$

and hence

$$(6.6)\quad \left|\prod_{p\le h_X}\left(1-\frac{\chi_q(p)}{p^{1+it}}\right)^{-1} - z_0\right| = \left|z_0\left(\frac{\prod_{p\le h_X}(1-\chi_q(p)/p^{1+it})^{-1}}{z_0}-1\right)\right|$$
$$= |z_0|\,|e^{\log\prod_{p\le h_X}(1-\chi_q(p)/p^{1+it})^{-1}-z}-1|$$
$$\ll_{z_0} \varepsilon.$$

Let ε_1 be a small positive number such that $\varepsilon_1 < \min\{\varepsilon, C_\nu(m)/2\}$. Proposition 3.2 implies that if we put

$$(6.7)\quad \mathcal{A}^{\gamma}_X(m,a) := \left\{q \in \mathcal{P}^{\gamma}_X(m,a) \,\middle|\, \left|L(1+it,\chi_q) - \prod_{p\le h_X}\left(1-\frac{\chi_q(p)}{p^{1+it}}\right)^{-1}\right| < \varepsilon_1\right\}$$

then

$$(6.8)\quad \frac{\#\mathcal{A}^{\gamma}_X(m,a)}{\#\mathcal{P}^{\gamma}_X(m,a)} > 1-\varepsilon_1$$

for all large X. Hence by (6.6) and (6.7) we conclude that every $q \in \mathcal{A}^{\gamma}_X(m,a)\cap\mathcal{B}^{\gamma}_{X,\nu}(m,a)$ satisfies

$$(6.9)\quad |L(1+it,\chi_q) - z_0| \ll_{z_0} \varepsilon.$$

Furthermore, from (6.4) and (6.8) we see that for the above number ν and all X sufficiently large,

$$(6.10)\quad \#(\mathcal{A}^{\gamma}_X(m,a)\cap\mathcal{B}^{\gamma}_{X,\nu}(m,a)) \ge \left(\frac{C_\nu(m)}{2}-\varepsilon_1\right)\#\mathcal{P}^{\gamma}_X(m,a).$$

Since $C_\nu(m)/2-\varepsilon_1 > 0$, (6.9) and (6.10) complete the proof. ∎

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