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On the higher mean over arithmetic progressions of Fourier coefficients of cusp forms

by

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1. Introduction. Throughout this paper, we consider the holomorphic forms or Maass forms for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ which are eigenfunctions of all the Hecke operators T_n. Let k be an even integer and let H_k denote the set of normalized primitive holomorphic cusp forms of even integral weight k. Recall that $f(z) \in H_k$ has Fourier expansion at the cusp ∞ given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz), \tag{1.1}$$

where $e(x) := \exp(2\pi i x)$ is an additive character, and the coefficients $\lambda_f(n) \in \mathbb{R}$ are eigenvalues of T_n. Deligne's bound [7] asserts that

$$|\lambda_f(n)| \le d(n) \tag{1.2}$$

for all $n \ge 1$, where $d(n)$ is the divisor function. Similarly, let S_r be the set of normalized primitive Maass cusp forms of eigenvalue $\lambda = 1/4 + r^2$. Then $f(z) \in S_r$ has Fourier expansion

$$f(z) = \sum_{n \ne 0} \lambda_f(n) \sqrt{y}\, K_{ir}(2\pi |n| y) e(nx), \tag{1.3}$$

where K_{ir} is the K-Bessel function and $\lambda_f(n) \in \mathbb{R}$ are eigenvalues of T_n. The current best estimate is due to Kim and Sarnak [15],

$$|\lambda_f(n)| \le n^{7/64} d(n) \tag{1.4}$$

It is an interesting problem to study the $2j$th power sum of $|\lambda_f(n)|$,

$$\sum_{n \le x} |\lambda_f(n)|^{2j}.$$

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For $j = 1$, the well-known Rankin–Selberg method (see [8, Theorem A]) yields

$$\sum_{n\le x} |\lambda_f(n)|^2 = C_f x + O_f(x^{3/5}).$$

For $j = 2, 3, 4$, Lau, Lü and Wu showed that

$$\sum_{n\le x} |\lambda_f(n)|^{2j} = xP_{f,2j}(\log x) + O_{f,\varepsilon}(x^{c_{2j}+\varepsilon}) \tag{1.5}$$

for any $\varepsilon > 0$, where P_{2j} are polynomials with $\deg P_4 = 1, \deg P_6 = 4$ and $\deg P_8 = 13$, with the constants $c_4 = \frac{151}{175}, c_6 = \frac{175}{181}$ and $c_8 = \frac{2933}{2957}$ if $f \in H_k$ (in [19]), and $c_4 = \frac{15}{17}, c_6 = \frac{63}{65}$ and $c_8 = \frac{255}{257}$ if $f \in S_r$ (in [18]). For higher moments with $j \ge 5$, assuming the Generalized Ramanujan Conjecture and that $L(\mathrm{sym}^r \phi)$, $r = 1, \dots, j$, are all automorphic cuspidal, Lau and Lü [18] recently proved

$$\sum_{n\le x} |\lambda_f(n)|^{2j} = xP_{f,2j}(\log x) + O_{f,\varepsilon}(x^{\eta_{2j}+\varepsilon}),$$

where P_{2j} is a polynomial of degree $(2j)!/(j!(j + 1)!) - 1$. For the general case $j > 1$, $j \in \mathbb{R}$, Rankin [27] showed that, if $f \in H_k$, then

$$\sum_{n\le x} |\lambda_f(n)|^{2j} = O_f(x \log^{2^{2(j-1)}-1} x).$$

Let $l, q \in \mathbb{Z}$ with $0 \le l < q$ and $(q, l) = 1$. In this paper, we will consider the $2j$th power sum of $|\lambda_f(n)|$ over arithmetic progressions,

$$\sum_{\substack{n\le x\\ n\equiv l \,(\mathrm{mod}\, q)}} |\lambda_f(n)|^{2j} \tag{1.6}$$

as $x \to \infty$, in which the parameter q can grow with x in a definite way.

Andrianov and Fomenko [2] firstly treated the second power sum of $|\lambda_f(n)|$ over arithmetic progressions for holomorphic cusp forms. Later Akbarov [1] improved the error term. The result was further strengthened when q is a prime or a power of an odd prime by Ichihara [12], [13]:

$$\begin{aligned}\sum_{\substack{n\le x\\ n\equiv l \,(\mathrm{mod}\, q)}} |\lambda_f(n)|^2 = \frac{c}{\varphi(q)} \prod_{p\mid q} &(1-\alpha(p)^2p^{-1})(1-p^{-1})(1-\beta(p)^2p^{-1}) \\ &\times (1+p^{-1})^{-1}x + O_{f,\varepsilon}(x^{3/5}q^{4/5+\varepsilon}).\end{aligned} \tag{1.7}$$

If $x \ll q^2$, the error term was estimated as $O_{f,\varepsilon}(x^{3/5}q^{4/5})$, where c is a constant only depending on f, and $\alpha(p), \beta(p)$ are the parameters in (2.1).

Our aim here is to investigate the fourth, sixth and eighth moments of Fourier coefficients over arithmetic progressions. The main results are the following.

THEOREM 1.1. *Let $f \in H_k$ and let q be a prime with $(q,l)=1$. For any $\varepsilon>0$ and $j=2,3,4$, if $q \le x^{\theta_{2j}}$, then*

$$\sum_{\substack{n\le x\\ n\equiv l\,(\mathrm{mod}\,q)}} |\lambda_f(n)|^{2j} = \frac{1}{\varphi(q)}R_{2j}(x,q) + O_{f,\varepsilon}(qx^{1-\frac{3}{2}\theta_{2j}+\varepsilon}), \tag{1.8}$$

where $\theta_4 = 2/23, \theta_6 = 4/187$, $\theta_8 = 4/755$, $\varphi(q)$ is the Euler function, and

$$\begin{aligned} R_{2j}(x,q) = \sum_{k=0}^{n_{2j}} {\sum_{d=0}^{k}}' \sum_{\Omega(d)} \frac{dL_{2j,q}^{-1^{(m)}}(q^{-1})}{m!r_0!\cdots r_{2j}!} B_{k,d}\biggl(\frac{-\log q}{q},\ldots,\frac{(-\log q)^{k-d+1}}{q}\biggr) \\ \times \prod_{i=0}^{2j} \frac{C_{2j}^i!((-\alpha(q))^{2(j-i)})^{r_i}}{(C_{2j}^i - r_i)!}\biggl(1-\frac{\alpha(q)^{2(j-i)}}{q}\biggr)^{C_{2j}^i - r_i} xP_{n_{2j}-k}(\log x). \end{aligned}$$

Here $\sum'$ means that the $d=0$ term is absent if $k\ge 1$, $\Omega(d)$ denotes summing over $m, r_0, \ldots, r_{2j}$ with $m + r_0 + \cdots + r_{2j} = d$, $g^{(m)}$ means the mth derivative of g, $B_{k,d}$ is the Bell polynomial given by (3.10), *C_{2j}^i are binomial coefficients, $L_{2j,q}(T)$ are polynomials with* $\deg L_{4,q} = 14$, $\deg L_{6,q} = 62$ *and* $\deg L_{8,q} = 254$ *as in Lemma* 2.5, *P_k is a polynomial of degree k, and $n_{2j} = 1, 4, 13$ respectively.*

The proof of Theorem 1.1 is based on Deligne's bound (1.2). Although the Ramanujan Conjecture is not available for Maass cusp forms, we have

THEOREM 1.2. *Let $f \in S_r$ and let q be a prime with $(q,l)=1$. For any $\varepsilon>0$ and $j=2,3,4$, we have*

$$\sum_{\substack{n\le x\\ n\equiv l\,(\mathrm{mod}\,q)}} |\lambda_f(n)|^{2j} = \frac{1}{\varphi(q)}R_{2j}(x,q) + O_{f,\varepsilon}\bigl(q^{\frac{4^j}{4^j+1}}x^{\frac{4^j-1}{4^j+1}+\varepsilon}\bigr), \tag{1.9}$$

where $R_{2j}(x,q)$ is as in Theorem 1.1.

REMARK 1.1. Applying the method of Theorem 1.2 for the decomposition equation (2.14), we can obtain the result on Maass cusp forms for $j=1$, which coincides with Ichihara's result (1.7) on holomorphic cusp forms.

2. Preliminaries. In this section we will briefly recall some fundamental facts about cusp forms and their L-functions. For f in H_k or S_r, the associated L-function is given by

$$L(s,f) = \sum_{n\ge 1}\frac{\lambda_f(n)}{n^s} = \prod_p \biggl(1-\frac{\alpha(p)}{p^s}\biggr)^{-1}\biggl(1-\frac{\beta(p)}{p^s}\biggr)^{-1}, \tag{2.1}$$

which converges absolutely for $\sigma = \Re s > 1$. The local parameters $\alpha(p)$ and $\beta(p)$ are related to the normalized Fourier coefficients in the following way:

$$\alpha(p) + \beta(p) = \lambda_f(p), \qquad \alpha(p)\beta(p) = 1.$$

Now let χ be a Dirichlet character modulo q. Then we define the twisted mth *symmetric power L-function* to be the degree $m+1$ Euler product

$$L(s, \operatorname{sym}^m f \otimes \chi) = \prod_p \prod_{0\le j\le m} \left(1 - \frac{\alpha(p)^{m-2j}\chi(p)}{p^s}\right)^{-1}, \tag{2.2}$$

and the *Rankin–Selberg convolution* of $\operatorname{sym}^m f$ and $\operatorname{sym}^n f \otimes \chi$ via the degree $m+n+2$ Euler product

$$\begin{aligned} L(s, \operatorname{sym}^m f \times \operatorname{sym}^n f \otimes \chi) \\ = \prod_p \prod_{0\le j\le m} \prod_{0\le i\le n} \left(1 - \frac{\alpha(p)^{m-2j}\alpha(p)^{n-2i}\chi(p)}{p^s}\right)^{-1}. \end{aligned} \tag{2.3}$$

It is easy to see that

$$\begin{cases} L(s, \operatorname{sym}^0 f \otimes \chi) = L(s, \chi), \\ L(s, \operatorname{sym}^1 f \otimes \chi) = L(s, f \otimes \chi), \\ L(s, \operatorname{sym}^m f \times \operatorname{sym}^0 f \otimes \chi) = L(s, \operatorname{sym}^m f \otimes \chi). \end{cases}$$

LEMMA 2.1. *Let f in H_k or S_r be a Hecke eigencuspform and χ be a primitive character modulo a prime q. The completed L-function defined as*

$$\begin{aligned} \Lambda(s, \operatorname{sym}^m f \times \operatorname{sym}^n f \otimes \chi)) \\ = q^{(m+1)(n+1)s/2}\gamma(s) L(s, \operatorname{sym}^m f \times \operatorname{sym}^n f \otimes \chi) \end{aligned} \tag{2.4}$$

is an entire function and satisfies a functional equation

$$\Lambda(s, \operatorname{sym}^m f \times \operatorname{sym}^n f \otimes \chi)) = \epsilon(f,\chi)\Lambda(1-s, \operatorname{sym}^m f \times \operatorname{sym}^n f \otimes \overline{\chi})), \tag{2.5}$$

where $1 \le m \le 4$ and $0 \le n \le m$, $|\epsilon(f,\chi)| = 1$ and $\gamma(s)$ is essentially a product of some gamma functions $\Gamma((s+\kappa_i)/2)$, $i = 1, \ldots, (m+1)(n+1)$, with κ_i depending on the weight or spectrum of f and the parity of the character χ and $\Re\kappa_i \ge 0$.

Proof. If $m = 0, 1$ or 2, $n = 0$, this follows from the classical results of [4, Theorem 1.1.1, equation 5.9] and Li [20]. For the general case, the proof is a little different for χ odd and for χ even. A primitive character χ corresponds to a Hecke character of the idele class group $\mathbb{A}^\times/\mathbb{Q}^\times$ trivial on $R_+^\times$, so χ is of the form $\chi = \bigotimes_{p\le\infty} \chi_p$. It is known that there exists an automorphic cuspidal self-dual representation $\operatorname{sym}^k \pi = \bigotimes_{p\le\infty} \operatorname{sym}^k \pi_p$ associated with $\operatorname{sym}^k f$ for k up to 4, from the work of Gelbart and Jacquet [9] for $k = 2$ and the works of Kim and Shahidi [16, 17] and Kim [15] when $k = 3, 4$.

If χ is even, that is, $\chi(-1) = 1$, which is equivalent to the local factor at infinity satisfying $\chi_\infty \equiv 1$, by the method of Luo, Rudnick and Sarnak [22] we can show

$$\epsilon(f,\chi) = \epsilon(\mathrm{sym}^m \pi \times \mathrm{sym}^n \pi \otimes \chi) = (\tau(\chi)/\sqrt{q})^{(m+1)(n+1)}$$

and

$$\begin{aligned}\gamma(s) &= L(s, \mathrm{sym}^m \pi_\infty \times \mathrm{sym}^n \pi_\infty \otimes \chi_\infty) = L(s, \mathrm{sym}^m \pi_\infty \times \mathrm{sym}^n \pi_\infty)\\ &= \prod_{r=0}^{n} L(s, \mathrm{sym}^{m+n-2r} \pi_\infty).\end{aligned}$$

The last step is obtained due to $\mathrm{sym}^m \pi \otimes \mathrm{sym}^n \pi = \bigoplus_{0\le r\le n} \mathrm{sym}^{m+n-2r} \pi$. Further, $\gamma(s)$ is essentially a product of some gamma functions $\Gamma((s+\kappa_i)/2)$ with $\Re\kappa_i \ge 0$, because of the explicit formula for $L(s, \mathrm{sym}^k \pi_\infty)$ of Cogdell and Michel [6] if $f \in H_k$ and of Murty [26] if $f \in S_r$.

If χ is odd, similar results can be obtained. We know from [10, Remark 10.8.7] that the root number $\epsilon(f,\chi)$ and the gamma factor $\gamma(s)$ in the functional equation change a little, namely

$$\epsilon(f,\chi) = \epsilon(\mathrm{sym}^m \pi \times \mathrm{sym}^n \pi \otimes \chi) = \left(\frac{\tau(\chi)}{i\sqrt{q}}\right)^{(m+1)(n+1)}$$

and

$$\begin{aligned}\gamma(s) &= L(s, \mathrm{sym}^m \pi_\infty \times \mathrm{sym}^n \pi_\infty \otimes \chi_\infty) = L(s+1, \mathrm{sym}^m \pi_\infty \times \mathrm{sym}^n \pi_\infty)\\ &= \prod_{r=0}^{n} L(s+1, \mathrm{sym}^{m+n-2r} \pi_\infty).\end{aligned}$$

Hence Lemma 2.1 is proved. ∎

2.1. Mean values and subconvexity bounds

Lemma 2.2. *Let $f(s)$ be an analytic function of s, real for real s, regular for $\sigma \ge \alpha$ except possibly for a pole at $s = s_0$, and $O(e^{\varepsilon|t|})$ as $|t| \to \infty$ for every positive ε and $\sigma \ge \alpha$. Let $\alpha < \sigma < \beta$.*

(1) *Assume that for all $T > 0$,*

$$\int_0^T |f(\alpha+it)|^2\,dt \le C(T^a+1), \qquad \int_0^T |f(\beta+it)|^2\,dt \le C'(T^b+1),$$

where $a, b \ge 0$, and C, C' depend on $f(s)$. Then for $T \ge 2$,

$$\int_{\frac{1}{2}T}^{T} |f(\sigma+it)|^2\,dt \le K(CT^a)^{(\beta-\sigma)/(\beta-\alpha)}(C'T^b)^{(\sigma-\alpha)/(\beta-\alpha)}, \tag{2.6}$$

where K depends on a, b, α, β only, and is bounded if these are bounded.

(2) *Assume that*

$$|f(\alpha+it)| \le C(|t|+1)^a, \qquad |f(\beta+it)| \le C'(|t|+1)^b,$$

for $t \in \mathbb{R}$. *Then*

$$|f(\sigma+it)| \le (C(|t|+1)^a)^{(\beta-\sigma)/(\beta-\alpha)}(C'(|t|+1)^b)^{(\sigma-\alpha)/(\beta-\alpha)}. \tag{2.7}$$

The result (1) comes from a convexity theorem of Titchmarsh [29, p. 149], and (2) is the Phragmén–Lindelöf principle for a strip [14, Theorem 5.53]. We shall estimate the mean values and hybrid bounds of Dirichlet L-functions in the strip by using Lemma 2.2.

LEMMA 2.3. *Let* χ *be a primitive character modulo* q. *Then, for any* $\varepsilon > 0$ *and* $T \ge 1$ *with* $q \ll T^2$,

$$L(\sigma+iT,\chi) \ll_\varepsilon (q(|T|+1))^{\max\{\frac{1}{3}(1-\sigma),0\}+\varepsilon}, \tag{2.8}$$

and if further q *is a prime*,

$$\int_0^T |L(\sigma+it,\chi)|^{12}\,dt \ll_\varepsilon q^{4(1-\sigma)}T^{3-2\sigma+\varepsilon}. \tag{2.9}$$

Proof. The corresponding results on the critical line $s = 1/2+it$ were stated by Heath-Brown [11] and Motohashi [25]. Since the Dirichlet L-function $L(s,\chi)$ converges absolutely for $\sigma = \Re s > 1$, the claim follows from Lemma 2.2. ∎

2.2. Mean values and convexity bounds for higher rank L-functions. We shall use the notation of [19]. For $\mathbf{d} := \{d_1,\dots,d_J\}, \mathbf{m} := \{m_1,\dots,m_J\}, \mathbf{n} := \{n_1,\dots,n_J\}$ with $d_j \in \mathbb{N}$, $1 \le m_j \le 4$ and $0 \le n_j \le m_j$, define

$$\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(s,\chi) := \prod_{j=1}^{J} L(s, \mathrm{sym}^{m_j} f \times \mathrm{sym}^{n_j} f \otimes \chi)^{d_j}. \tag{2.10}$$

By Lemma 2.1, we know that $L(s, \mathrm{sym}^m f \otimes \mathrm{sym}^n f \otimes \chi))$ is Perelli's general L-function as defined in [28]. Then $\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(s,\chi)$ is also a general L-function satisfying the functional equation

$$Q^s \prod_{i=1}^{N} \Gamma(\alpha_i s+\beta_i)\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(s,\chi) = WQ^{1-s}\prod_{i=1}^{N} \Gamma(\alpha_i(1-s)+\beta_i)\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(1-s,\chi)$$

with $|W| = 1$, $\alpha_i = 1/2$, $\beta_i \ge 0$ for all i and

$$N = d_1(m_1+1)(n_1+1)+\cdots+d_J(m_J+1)(n_J+1).$$

Let

$$A = N/2, \qquad B = \sum_{i=1}^{N}\beta_i \ge 0, \qquad q^A \ll Q \ll q^A,$$

$$H = 1+\Re(B/A)-(N-1)/(2A) > 0$$

and

$$G(Q) = Q^H\Big(1 + Q^{-\min \Re(\beta_i/\alpha_i)}\Big(Q^{-1} + Q \max_{\substack{i=1,\dots,N-1\\ z}} |\mathfrak{L}^{\mathbf{d}(i)}_{\mathbf{m},\mathbf{n}}(z,\chi)|\Big)\Big) \ll Q^H,$$

where z runs over the poles of $\prod_{i=1}^N \Gamma(\alpha_i s + \beta_i)\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(s,\chi)$. Then from [28, Theorem 4] and [23, Proposition 1] we deduce the estimate (2.11) below. The estimate (2.12) can be obtained by standard arguments used to establish convexity bounds, applying Lemma 2.1, Lemma 2.2 and the definition of $\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(\sigma + it,\chi)$.

LEMMA 2.4. *Let $f \in H_k$ be a Hecke eigencuspform and χ be a primitive character modulo q. Let $\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(s,\chi)$ be defined as in* (2.10). *Then for any $\varepsilon > 0$, we have*

$$\int_T^{2T} |\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(\sigma + it,\chi)|^2\, dt \ll (qT)^{2A(1-\sigma)+\varepsilon} \tag{2.11}$$

uniformly for $1/2 \le \sigma \le 1$ and $T \ge 1$. Moreover,

$$\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(\sigma + it,\chi) \ll (q(|t|+1))^{\max\{A(1-\sigma),0\}+\varepsilon} \tag{2.12}$$

uniformly for $-\varepsilon \le \sigma \le 1+\varepsilon$.

From (2.4), we know that the mean value and convexity of a product of symmetric L-functions are determined by the parameter A. So we only need to specify the value of A for L-functions in our applications.

2.3. Decomposition of $F_{2j}(s,\chi)$. Let f in H_k or S_r be a Hecke eigencuspform and χ be a Dirichlet character modulo q. Define an L-function as

$$F_{2j}(s,\chi) = \sum_{n=1}^{\infty} |\lambda_f(n)|^{2j}\chi(n)n^{-s}. \tag{2.13}$$

The aim is to decompose it into some functions whose properties are well known. For $j = 1$, we know from [12] that

$$F_2(s,\chi) = L(s,\chi)L(s,\mathrm{sym}^2 f \otimes \chi)L(2s,\chi^2)^{-1}. \tag{2.14}$$

Clearly, $L(2s,\chi^2)^{-1}$ is absolutely convergent and free from zeros for $\Re s \ge 1/2+\varepsilon$. For the higher cases, we have the following.

LEMMA 2.5. *With notation as above and $\Re s > 1$, for any $\varepsilon > 0$ we have*

$$F_{2j}(s,\chi) = G_{2j}(s,\chi)H_{2j}(s,\chi) \tag{2.15}$$

for $j = 2, 3, 4,$ *where*

$$\begin{aligned} G_4(s,\chi) &= L(s,\chi)^2 L(s,\mathrm{sym}^2 f\otimes\chi)^3 L(s,\mathrm{sym}^4 f\otimes\chi),\\ G_6(s,\chi) &= L(s,\chi)^5 L(s,\mathrm{sym}^2 f\otimes\chi)^8\\ &\quad\times L(s,\mathrm{sym}^4 f\otimes\chi)^4 L(s,\mathrm{sym}^4 f\times\mathrm{sym}^2 f\otimes\chi),\\ G_8(s,\chi) &= L(s,\chi)^{13} L(s,\mathrm{sym}^2 f\otimes\chi)^{21} L(s,\mathrm{sym}^4 f\otimes\chi)^{13}\\ &\quad\times L(s,\mathrm{sym}^4 f\times\mathrm{sym}^2 f\otimes\chi)^6 L(s,\mathrm{sym}^4 f\times\mathrm{sym}^4 f\otimes\chi), \end{aligned}$$

and $H_{2j}(s,\chi) := \prod_p L_{2j,p}(\chi(p)p^{-s})$. *Here* $L_{2j,p}$ *are polynomials of degree* 14, 62 *and* 254 *for* $j = 2, 3, 4$ *respectively, whose coefficients of constant, linear and highest terms equal* $1, 0, -1$. *Furthermore, if* $f \in H_k$, *then* $H_{2j}(s,\chi)$ *admits a Dirichlet series absolutely convergent in* $\Re s \geq 1/2+\varepsilon$. *If* $f \in S_r$, *then* $H_4(s,\chi)$ *and* $H_6(s,\chi)$ *converge absolutely in* $\Re s \geq 1/2+\varepsilon$, *and* $H_8(s,\chi)$ *converges absolutely in* $\Re s \geq 23/32+\varepsilon$. *For* f *in* H_k *or* S_r, *the convergence in all cases is uniform in* q *in the respective regions.*

Proof. From the multiplicativity property of $|\lambda_f(n)|^{2j}$, we have the Euler product identity

$$F_{2j}(s,\chi) = \prod_p \Big(\sum_{v=0}^{\infty} |\lambda_f(p^v)|^{2j}\chi^v(p)p^{-vs}\Big).$$

Now we shall take $T = \chi(p)p^{-s}$. For $j = 2$, we can easily derive

$$\sum_{v=0}^{\infty} |\lambda_f(p^v)|^4 T^v = \frac{L_{4,p}(T)}{(1-T)^2 S_2(T)^3 S_4(T)^4},$$

where

$$L_{4,p}(T) = 1 - \big(7 - 12\lambda_f(p)^2 + 6\lambda_f(p)^4\big)T^2 + \cdots - T^{14},$$

and

$$S_{m\times n}(T) = \prod_{0\leq j\leq m}\prod_{0\leq i\leq n}\big(1 - \alpha(p)^{m-2j}\alpha(p)^{n-2i}T\big),$$

in particular $S_m(T) = S_{m\times 0}(T)$, because of the facts from [24, Lemma 2, Remarks], namely

$$\sum_{v=0}^{\infty} |\tau_0(p^v)|^4 T^v = \frac{L'_{4,p}(T)}{(1-T)^2 S'_2(T)^3 S'_4(T)^4}$$

and

$$L'_{4,p}(T) = 1 - (7 - 12\tau_0(n)^2 + 6\tau_0(n)(p)^4)T^2 + \cdots - T^{14},$$

where $\tau_0(n)$ is the nth normalized Ramanujan τ-function, and $L'_{4,p}, S'_2, S'_4$ are functions similar to $L_{4,p}, S_2, S_4$ with local parameters for $\tau_0(n)$.

For $j = 3$, the claim follows if we prove that

$$\sum_{v=0}^{\infty} |\lambda_f(p^v)|^6 T^v = \frac{L_{6,p}(T)}{(1-T)^5 S_2(T)^8 S_4(T)^4 S_{4\times 2}(T)}.$$

It follows from the theory of Hecke operators that

$$\lambda_f(p^v) = \frac{\alpha(p)^{v+1} - \alpha(p)^{-v-1}}{\alpha(p) - \alpha(p)^{-1}}. \tag{2.16}$$

Then the problem is reduced to showing that

$$\sum_{v=0}^{\infty} \left(\frac{\alpha(p)^{v+1} - \alpha(p)^{-v-1}}{\alpha(p) - \alpha(p)^{-1}} \right)^6 T^v = \frac{L_{6,p}(T)}{(1-T)^5 S_2(T)^8 S_4(T)^4 S_{4\times 2}(T)}.$$

By a straightforward calculation one shows that

$$\sum_{v=0}^{\infty} \left(\frac{a^{v+1} - 1}{a - 1} \right)^6 t^v$$
$$= \frac{1 + R_1(a)t + R_2(a)t^2 + R_1(a)a^5 t^3 + R_2(a)a^7 t^4 + a^{15} t^5}{(1-a^6 t)(1-a^5 t)\cdots(1-t)},$$

where

$$R_1(a) = a(5 + 14a + 19a^2 + 14a^3 + 5a^4),$$
$$R_2(a) = a^3(10 + 35a + 66a^2 + 80a^3 + 66a^4 + 35a^5 + 10a^6).$$

Inserting $a = a(p)^2$ and $t = T/a(p)^6$ into this identity, and using the facts that

$$R_1(a)t = \big(1 - 6\lambda_f(p)^2 + 5\lambda_f(p)^4\big)T,$$
$$R_2(a)t^2 = \big(-2 + 16\lambda_f(p)^2 - 25\lambda_f(p)^4 + 10\lambda_f(p)^6\big)T^2,$$

we obtain

$$\sum_{v=0}^{\infty} |\lambda_f(p^v)|^6 T^v = \frac{N_6(T)}{S_6(T)},$$

where

$$\begin{aligned} N_6(T) = 1 &+ (5\lambda_f(p)^4 - 6\lambda_f(p)^2 + 1)T \\ &+ \big(10\lambda_f(p)^6 - 25\lambda_f(p)^4 + 16\lambda_f(p)^2 - 2\big)T^2 \\ &+ (10\lambda_f(p)^6 - 25\lambda_f(p)^4 + 16\lambda_f(p)^2 - 2)T^3 \\ &+ (5\lambda_f(p)^4 - 6\lambda_f(p)^2 + 1)T^4 + T^5. \end{aligned}$$

It is easy to derive

$$S_2(T) = 1 - (\lambda_f(p)^2 - 1)T + (\lambda_f(p) - 1)T^2 - T^3,$$
$$\begin{aligned} S_4(T) = 1 &- (\lambda_f(p)^4 - 3\lambda_f(p)^2 + 1)T + (\lambda_f(p)^6 - 5\lambda_f(p)^4 + 7\lambda_f(p)^2 - 2)T^2 \\ &- (\lambda_f(p)^6 - 5\lambda_f(p)^4 + 7\lambda_f(p)^2 - 2)T^3 + (\lambda_f(p)^4 - 3\lambda_f(p)^2 + 1)T^4 - T^5, \end{aligned}$$

and

$$S_{4\times 2}(T) = S_2(T)S_4(T)S_6(T).$$

Therefore,

$$\begin{aligned}L_{6,p}(T) &= N_6(T)(1-T)^5 S_2(T)^9 S_4(T)^5\\ &= 1-(31-90\lambda_f(p)^2+105\lambda_f(p)^4-60\lambda_f(p)^6+15\lambda_f(p)^8)T^2+\cdots-T^{62}.\end{aligned}$$

For $j=4$, the goal is to show

$$\sum_{v=0}^{\infty}|\lambda_f(p^v)|^8T^v = \frac{L_{8,p}(T)}{(1-T)^{13}S_2(T)^{21}S_4(T)^{13}S_{4\times 2}(T)^6 S_{4\times 4}(T)}.$$

From (2.16) and some elementary calculations we derive that

$$\sum_{v=0}^{\infty}|\lambda_f(p^v)|^8T^v = \sum_{v=0}^{\infty}\left(\frac{\alpha(p)^{v+1}-\alpha(p)^{-v-1}}{\alpha(p)-\alpha(p)^{-1}}\right)^8 T^v = \frac{N_8(T)}{S_8(T)},$$

where

$$\begin{aligned}&N_8(T)\\ &\;= 1+(7\lambda_f(p)^6-15\lambda_f(p)^4+10\lambda_f(p)^2-1)T+(21\lambda_f(p)^{10}-105\lambda_f(p)^8\\ &\quad+183\lambda_f(p)^6-138\lambda_f(p)^4+42\lambda_f(p)^2-3)T^2+(35\lambda_f(p)^{12}-231\lambda_f(p)^{10}\\ &\quad+560\lambda_f(p)^8-610\lambda_f(p)^6+293\lambda_f(p)^4-52\lambda_f(p)^2+3)T^3+(35\lambda_f(p)^{12}\\ &\quad-231\lambda_f(p)^{10}+560\lambda_f(p)^8-610\lambda_f(p)^6+293\lambda_f(p)^4-52\lambda_f(p)^2+3)T^4\\ &\quad+(21\lambda_f(p)^{10}-105\lambda_f(p)^8+183\lambda_f(p)^6-138\lambda_f(p)^4+42\lambda_f(p)^2-3)T^5\\ &\quad+(7\lambda_f(p)^6-15\lambda_f(p)^4+10\lambda_f(p)^2-1)T^6+T^7.\end{aligned}$$

Since

$$S_{4\times 4}(T) = (1-T)S_2(T)S_4(T)S_6(T)S_8(T)$$

and

$$\begin{aligned}S_6(T) = 1&-(\lambda_f(p)^6-5\lambda_f(p)^4+6\lambda_f(p)^2-1)T+(\lambda_f(p)^{10}-9\lambda_f(p)^8\\ &+29\lambda_f(p)^6-40\lambda_f(p)^4+22\lambda_f(p)^2-3)T^2-(\lambda_f(p)^{12}-11\lambda_f(p)^{10}\\ &+46\lambda_f(p)^8-90\lambda_f(p)^6+81\lambda_f(p)^4-28\lambda_f(p)^2+3)T^3+(\lambda_f(p)^{12}\\ &-11\lambda_f(p)^{10}+46\lambda_f(p)^8-90\lambda_f(p)^6+81\lambda_f(p)^4-28\lambda_f(p)^2+3)T^4\\ &-(\lambda_f(p)^{10}-9\lambda_f(p)^8+29\lambda_f(p)^6-40\lambda_f(p)^4+22\lambda_f(p)^2-3)T^5\\ &+(\lambda_f(p)^6-5\lambda_f(p)^4+6\lambda_f(p)^2-1)T^6-T^7.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}L_{8,p}(T) &= N_8(T)(1-T)^{14}S_2(T)^{28}S_4(T)^{20}S_6(T)^7\\ &= 1-(28\lambda_f(p)^{12}-168\lambda_f(p)^{10}+490\lambda_f(p)^8-840\lambda_f(p)^6\\ &\quad+868\lambda_f(p)^4-504\lambda_f(p)^2+127)T^2+\cdots-T^{254}.\end{aligned}$$

When $f \in H_k$, Deligne's bound (1.2) holds. From the convergence of the series $\sum_p p^{-2s}$ for $\Re s > 1/2$ and the expression of $L_{2j,p}(T)$, we know that $H_{2j}(s,\chi)$ converges absolutely in the region $\Re s \ge 1/2+\varepsilon$ for any $\varepsilon > 0$.

When $f \in S_r$, we use an assertion equivalent to (1.4), which states that

$$|\alpha(p)|, |\alpha^{-1}(p)| \le p^{7/64}. \tag{2.17}$$

From the expression of $L_{2j,p}(T)$, we obtain

$$\begin{aligned} H_4(s,\chi) &\ll \prod_p \Big(1 + O\Big(\frac{|\alpha(p)|^4 + |\alpha(p)|^{-4}}{p^{2\sigma}}\Big)\Big),\\ H_6(s,\chi) &\ll \prod_p \Big(1 + O\Big(\frac{|\alpha(p)|^8 + |\alpha(p)|^{-8}}{p^{2\sigma}}\Big)\Big),\\ H_8(s,\chi) &\ll \prod_p \Big(1 + O\Big(\frac{|\alpha(p)|^{12} + |\alpha(p)|^{-12}}{p^{2\sigma}}\Big)\Big). \end{aligned} \tag{2.18}$$

The convexity bound for $L(s, \mathrm{sym}^4 f \times \mathrm{sym}^4 f)$ (see [21]) implies that for any $\sigma > 1+\varepsilon$, $\varepsilon > 0$,

$$\prod_p \Big(1 + \frac{|\alpha(p)|^8 + |\alpha(p)|^{-8}}{p^{\sigma}}\Big) \ll 1. \tag{2.19}$$

Combining (2.17)–(2.19), we deduce that $H_4(s,\chi)$ and $H_6(s,\chi)$ converge absolutely in the region $\Re s \ge 1/2+\varepsilon$, and $H_8(s,\chi)$ converges absolutely in $\Re s \ge 23/32+\varepsilon$. Moreover, since $|\chi(p)| = 1$, the convergence is uniform in q in the respective regions for all cases. This completes the proof of Lemma 2.5. ∎

3. Proof of Theorem 1.1. We shall complete the proof of Theorem 1.1 by using the orthogonality relation of characters. Thus we have to investigate the sum $\sum_{n\le x} |\lambda_f(n)|^{2j}\chi(n)$.

PROPOSITION 3.1. *Let $f \in H_k$ and let χ be a primitive character modulo a prime q. For any $\varepsilon > 0$ and $q \le x^{\theta_{2j}}$, we have*

$$\sum_{n\le x} |\lambda_f(n)|^{2j}\chi(n) \ll_{f,\varepsilon} q x^{1-\frac{3}{2}\theta_{2j}+\varepsilon}. \tag{3.1}$$

where $\theta_4 = 2/23, \theta_6 = 4/187$ and $\theta_8 = 4/755$.

Proof. Applying the Perron formula [14, Proposition 5.54] for the function (2.13), we get, due to (1.2),

$$\sum_{n\le x} |\lambda_f(n)|^{2j}\chi(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} F_{2j}(s,\chi)\frac{x^s}{s}\,ds + O_{f,\varepsilon}(x^{1+\varepsilon}/T),$$

uniformly for $2 \le T \le x$. By Lemmas 2.1 and 2.5, $F_{2j}(s,\chi)$ is free from poles in the rectangle $1/2+\varepsilon \le \sigma \le 1+\varepsilon$. Then we derive

$$\sum_{n\le x} |\lambda_f(n)|^{2j}\chi(n) = -\frac{1}{2\pi i}\int_C F_{2j}(s,\chi)\frac{x^s}{s}\,ds + O_{f,\varepsilon}(x^{1+\varepsilon}/T),$$

where C is the contour joining $1+\varepsilon+iT, 1/2+\varepsilon+iT, 1/2+\varepsilon-iT, 1+\varepsilon-iT$ with straight line segments. Since $H_{2j}(s,\chi)$ converges absolutely and uniformly in q and $\Re s \ge 1/2+\varepsilon$, it follows that

$$\sum_{n\le x} |\lambda_f(n)|^{2j}\chi(n) = O(I_1) + O(I_2) + O_{f,\varepsilon}(x^{1+\varepsilon}/T), \tag{3.2}$$

where

$$\begin{aligned} I_1 &:= \frac{1}{T}\int_{1/2+\varepsilon}^{1+\varepsilon} |G_{2j}(\sigma+iT,\chi)|x^\sigma\,d\sigma, \\ I_2 &:= x^{1/2+\varepsilon}\sup_{0\le t\le 1}|G_{2j}(1/2+\varepsilon+it,\chi)| + x^{1/2+\varepsilon}\int_1^T |G_{2j}(1/2+\varepsilon+it,\chi)|\,\frac{dt}{t} \\ &\ll x^{1/2+\varepsilon}\sup_{0\le t\le 1}|G_{2j}(1/2+\varepsilon+it,\chi)| \\ &\quad + x^{1/2+\varepsilon}\sup_{1\le T_1\le T}\frac{1}{T_1}\int_{T_1}^{2T_1}|G_{2j}(1/2+\varepsilon+it,\chi)|\,dt. \end{aligned}$$

Next we shall only treat the case of $j=2$, since the proofs of other cases have similar steps. Inserting the upper bounds (2.8) and (2.12), namely,

$$\begin{aligned} &L(\sigma+iT,\chi) \ll (q(|T|+1))^{\max\{\frac{1}{3}(1-\sigma),0\}+\varepsilon}, \\ &L(\sigma+iT,\operatorname{sym}^2 f\otimes\chi)^3 L(\sigma+iT,\operatorname{sym}^4 f\otimes\chi) \\ &\qquad\qquad \ll (q(|T|+1))^{\max\{7(1-\sigma),0\}+\varepsilon}, \end{aligned} \tag{3.3}$$

we show, for $q \ll T^2$,

$$\begin{aligned} I_1 &\ll \frac{1}{T}\int_{1/2+\varepsilon}^{1+\varepsilon} (qT)^{\frac{2}{3}(1-\sigma)+\varepsilon}(qT)^{7(1-\sigma)+\varepsilon}x^\sigma\,d\sigma, \\ &\ll q^{23/3+\varepsilon}T^{20/3+\varepsilon}\int_{1/2+\varepsilon}^{1+\varepsilon}\left(\frac{x}{(qT)^{23/3}}\right)^\sigma d\sigma, \\ &\ll q^{23/6+\varepsilon}T^{17/6+\varepsilon}x^{1/2+\varepsilon} + x^{1+\varepsilon}/T. \end{aligned} \tag{3.4}$$

Applying the generalization of Hölder's inequality and (3.3), we obtain

$$\begin{aligned} I_2 &\ll q^{23/6+\varepsilon}x^{1/2+\varepsilon} \\ &\quad + x^{1/2+\varepsilon}\sup_{1\le T_1\le T}\frac{1}{T_1}I_{2,1}(T_1)^{1/6}I_{2,2}(T_1)^{1/3}I_{2,3}(T_1)^{1/2}, \end{aligned} \tag{3.5}$$

where

$$I_{2,1}(T_1) := \int_{T_1}^{2T_1} |L(1/2+\varepsilon+it,\chi)|^{12}\,dt,$$

$$I_{2,2}(T_1) := \int_{T_1}^{2T_1} |L(1/2+\varepsilon+it,\mathrm{sym}^2 f\otimes\chi)|^{6}\,dt,$$

$$I_{2,3}(T_1) := \int_{T_1}^{2T_1} |L(1/2+\varepsilon+it,\mathrm{sym}^2 f\otimes\chi)L(1/2+\varepsilon+it,\mathrm{sym}^4 f\otimes\chi)|^{2}\,dt.$$

Since $L(1/2+\varepsilon+it,\mathrm{sym}^2 f\otimes\chi)^3$ and $L(1/2+\varepsilon+it,\mathrm{sym}^2 f\otimes\chi)L(1/2+\varepsilon+it,\mathrm{sym}^4 f\otimes\chi)$ are general L-functions defined in (2.10) with $A=9/2$ and $A=4$ respectively, we deduce from (2.9) and (2.11) that

$$I_{2,1}(T_1) \ll (qT)^{2+\varepsilon},\quad I_{2,2}(T_1) \ll (qT)^{9/2+\varepsilon},\quad I_{2,3}(T_1) \ll (qT)^{4+\varepsilon}.$$

By inserting these into (3.5) we obtain

$$I_2 \ll q^{23/6+\varepsilon}T^{17/6+\varepsilon}x^{1/2+\varepsilon}. \tag{3.6}$$

Combining (3.4), (3.6) and (3.2), we get (3.1) by choosing $T=x^{3/23}/q$. Since $q\ll T^2$, we have proved (3.1) for $q\le x^{2/23}$. ∎

Proposition 3.2. *Let $f\in H_k$ and let χ_0 be a principal character modulo a prime q. For any $\varepsilon>0$, $q\ll x$ and $j=2,3,4$ we have*

$$\sum_{n\le x}|\lambda_f(n)|^{2j}\chi_0(n) = R_{2j}(x,q)+O_{f,\varepsilon}(x^{c_{2j}+\varepsilon}), \tag{3.7}$$

where $R_{2j}(x,q)$ is given in (3.9), and $c_4=\frac{151}{175}, c_6=\frac{175}{181}, c_8=\frac{2933}{2957}$.

Proof. By Lemma 2.5, it can easily be seen that

$$F_{2j}(s,\chi_0) = L_{2j,q}^{-1}(q^{-s})\prod_{i=0}^{2j}(1-\alpha(q)^{2(j-i)}q^{-s})^{C_{2j}^i}G_{2j}(s)H_{2j}(s), \tag{3.8}$$

where (see [19])

$$G_4(s) = \zeta(s)^2L(s,\mathrm{sym}^2 f)^3L(s,\mathrm{sym}^4 f),$$

$$G_6(s) = \zeta(s)^5L(s,\mathrm{sym}^2 f)^8L(s,\mathrm{sym}^4 f)^4L(s,\mathrm{sym}^4 f\times\mathrm{sym}^2 f),$$

$$\begin{aligned}G_8(s) = \zeta(s)^{13}L(s,\mathrm{sym}^2 f\otimes\chi)^{21}L(s,\mathrm{sym}^4 f)^{13}L(s,\mathrm{sym}^4 f\times\mathrm{sym}^2 f)^6\\ \times L(s,\mathrm{sym}^4 f\times\mathrm{sym}^4 f),\end{aligned}$$

and $H_{2j}(s)$ converges absolutely in $\Re s\ge 1/2+\varepsilon$ and $H_{2j}(1+it)\neq 0$ for

$j = 2, 3, 4$. Applying the Perron formula for this function, we have

$$\sum_{n\le x} |\lambda_f(n)|^{2j}\chi_0(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} F_{2j}(s,\chi_0)\frac{x^s}{s}\,ds + O_{f,\varepsilon}(x^{1+\varepsilon}/T),$$

uniformly for $2 \le T \le x$. From (3.8) and the expressions of $G_{2j}(s)$, the point $s = 1$ is the only possible pole of the integrand in the rectangle $\kappa \le \sigma \le 1+\varepsilon$ and $|\tau| \le T$ for any $\kappa \in [1/2+\varepsilon, 1)$, and it is of order $n_{2j} + 1 = 2, 5, 14$ respectively. Thus, we obtain

$$\sum_{n\le x} |\lambda_f(n)|^{2j}\chi_0(n) = R(x) - \frac{1}{2\pi i}\int_{C} F_{2j}(s,\chi)\frac{x^s}{s}\,ds + O_{f,\varepsilon}(x^{1+\varepsilon}/T),$$

where $R_{2j}(x,q) = \mathrm{res}_{s=1} F_{2j}(s,\chi_0)x^s/s$, and C is the contour joining $1 + \varepsilon + iT, \kappa + iT, \kappa - iT, 1 + \varepsilon - iT$ with straight line segments. By the residue formula, Leibniz's rule and Bruno's formula, we have

(3.9)

$$\begin{aligned}R_{2j}(x,q) &= \sum_{k=0}^{n_{2j}}\Big(L_{2j,q}^{-1}(q^{-s})\prod_{i=0}^{2j}(1-\alpha(q)^{2(j-i)}q^{-s})^{C_{2j}^i}\Big)^{(k)}\Big|_{s=1} xP_{n_{2j}-k}(\log x)\\ &= \sum_{k=0}^{n_{2j}}\sideset{}{'}\sum_{d=0}^{k}\sum_{\Omega(d)}\frac{d!L_{2j,q}^{-1^{(m)}}(q^{-1})}{m!r_0!\cdots r_{2j}!}B_{k,d}\Big(\frac{-\log q}{q},\ldots,\frac{(-\log q)^{k-d+1}}{q}\Big)\\ &\quad\times\prod_{i=0}^{2j}\frac{C_{2j}^i!((-\alpha(q))^{2(j-i)})^{r_i}}{(C_{2j}^i-r_i)!}\Big(1-\frac{\alpha(q)^{2(j-i)}}{q}\Big)^{C_{2j}^i-r_i} xP_{n_{2j}-k}(\log x),\end{aligned}$$

where $\sum'$ means that the $d = 0$ term is absent if $k \ge 1$, $\Omega(d)$ denotes summing over $m, r_0, \ldots, r_{2j}$ with $m + r_0 + \cdots + r_{2j} = d$, and $B_{k,d}$ is the Bell polynomial given by

$$B_{k,d}(x_1,x_2,\ldots,x_{k-d+1}) = \sum_{\substack{j_i\ge 0,\, j_1+j_2+\cdots+j_{k-d+1}=d\\ j_1+2j_2+\cdots+(k-d+1)j_{k-d+1}=k}}\frac{k!}{j_1!\cdots j_{k-d+1}!}\times\Big(\frac{x_1}{1}\Big)^{j_1}\Big(\frac{x_2}{2!}\Big)^{j_2}\cdots\Big(\frac{x_{k-d+1}}{(k-d+1)!}\Big)^{j_{k-d+1}}. \tag{3.10}$$

We also have

$$\prod_{i=0}^{2j}(1-\alpha(q)^{2(j-i)}q^{-s})^{C_{2j}^i} \ll (1+q^{-\sigma})^{2jC_{2j}^i} \ll 1.$$

In addition we know that $H_{2j}(s,\chi)$ converges absolutely and uniformly in q

and $\Re s \geq 1/2 + \varepsilon$. It follows that

$$\sum_{n \leq x} |\lambda_f(n)|^{2j} \chi_0(n) = R_{2j}(x, q) + O(I_1) + O(I_2) + O_{f,\varepsilon}(x^{1+\varepsilon}/T),$$

where

$$\begin{aligned} I_1 &:= \frac{1}{T} \int_{\kappa}^{1+\varepsilon} |G_{2j}(\sigma + iT)| x^{\sigma} \, d\sigma, \\ I_2 &:= x^{\kappa+\varepsilon} \sup_{0 \leq t \leq 1} |G_{2j}(\kappa + it, \chi)| + x^{\kappa+\varepsilon} \int_1^T |G_{2j}(\kappa + it)| \frac{dt}{t} \\ &\ll x^{\kappa+\varepsilon} \sup_{1 \leq T_1 \leq T} \frac{1}{T_1} \int_{T_1}^{2T_1} |G_{2j}(\kappa + it)| \, dt. \end{aligned}$$

Finally, Proposition 3.2 follows from the same procedure as in the proof of [19, Theorem 1]. ∎

Proof of Theorem 1.1. Let χ be a Dirichlet character modulo a prime q. Applying Propositions 3.1 and 3.2, we derive by orthogonality

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \equiv l \,(\mathrm{mod}\, q)}} |\lambda_f(n)|^{2j} &= \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi}(l) \sum_{n \leq x} |\lambda_f(n)|^{2j} \chi(n) \\ &= \frac{1}{\varphi(q)} \sum_{n \leq x} |\lambda_f(n)|^{2j} \chi_0(n) + O\Big(\sum_{n \leq x} |\lambda_f(n)|^{2j} \chi(n)\Big) \\ &= \frac{1}{\varphi(q)} R_{2j}(x, q) + O_{f,\varepsilon}(q x^{1 - \frac{3}{2}\theta_{2j} + \varepsilon}). \end{aligned}$$

Note that $1 - \frac{3}{2}\theta_{2j} > c_{2j}$. This completes the proof. ∎

4. Proof of Theorem 1.2. Chandrasekharan and Narasimhan [5] considered the average order of a class of arithmetical functions. To estimate a special class, we refine their main O-theorem [5, Theorem 4.1] including the conductor. We first give some notation.

Let $\{a_n\}, \{b_n\}$ be two sequences of complex numbers, not all zero. Let $\{\lambda_n\}, \{\mu_n\}$ be two sequences of strictly positive numbers, strictly increasing to infinity. We define Dirichlet series $\varphi(s)$ and $\psi(s)$ by

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \qquad \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s},$$

each of which converges absolutely in some half-plane and satisfies [5, (2.3)]. Let

$$\Delta(s) = \prod_{v=1}^{N} \Gamma\Big(\frac{\alpha_v s + \beta_v}{2}\Big),$$

where $N \geq 1$, β_v is an arbitrary complex number, and $A = \sum_{v=1}^{N} \alpha_v \geq 1$. Suppose that the functional equation

$$q^{As} \Delta(s)\varphi(s) = \epsilon(q) q^{A(\delta-s)} \Delta(\delta - s)\psi(\delta - s)$$

is satisfied with $\delta > 0$, $|\epsilon(q)| = 1$ and that the only singularities of the function φ are poles. Let

$$A_\lambda^0(x) = \sum_{\lambda_n \leq x}{}' a_n,$$

where the dash denotes that the last term has to be multiplied by 1/2 if $x = \lambda_n$. Let

$$Q_\lambda^0(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_0} \varphi(s) \frac{x^s}{s}\, ds$$

where $\mathcal{C}_0$ encloses all the singularities of the integrand.

Lemma 4.1. *We have*

$$A_\lambda^0(x) - Q_\lambda^0(x) = O(q^{A(2\beta-\delta)} y^{-2Au} x^{\frac{\delta}{2} - \frac{1}{4A} + (2A-1)u}) + O(yx^{-1}|Q_\lambda^0(x)|) + O\Big(\sum_{x<\lambda_n \leq x+O(y)} |a_n|\Big),$$

for every $0 < y \ll x$, and $u = \beta - \delta/2 - 1/(4A)$, where β is such that $\sum_{n=1}^{\infty} |b_n| \mu_n^{-\beta} < \infty$.

If in addition $a_n \geq 0$, then

$$A_\lambda^0(x) - Q_\lambda^0(x) = O(q^{A(2\beta-\delta)} y^{-2Au} x^{\frac{\delta}{2} - \frac{1}{4A} + (2A-1)u}) + O(yx^{-1}|Q_\lambda^0(x)|).$$

Proof. The proof is similar to that of Chandrasekharan and Narasimhan. Their results follow immediately by changing $b_n \to \epsilon(q) q^{A\delta} b_n$, $\mu_n \to q^{2A} \mu_n$. ∎

Proposition 4.2. *Let $f \in S_r$ and let χ be a primitive character modulo q. For any $\varepsilon > 0$, $q \ll x$ and $j = 2, 3, 4$ we have*

$$\sum_{n \leq x} |\lambda_f(n)|^{2j} \chi(n) = O_{f,\varepsilon}\big(q^{\frac{4^j}{4^j+1}} x^{\frac{4^j-1}{4^j+1}+\varepsilon}\big). \tag{4.1}$$

Proof. Let $G_{2j}(s,\chi), H_{2j}(s,\chi)$ be expressed as Dirichlet series

$$G_{2j}(s,\chi) = \sum_{n=1}^{\infty} \frac{a_{2j}(n)\chi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{a_{2j}(n,\chi)}{n^s}, \quad H_{2j}(s,\chi) = \sum_{n=1}^{\infty} \frac{b_{2j}(n,\chi)}{n^s},$$

where $a_{2j}(n)$ are the coefficients of $G_{2j}(s)$; it is known that $a_{2j}(n)$ are nonnegative (see [18, Lemma 7.1]). Then by the refinement of Landau's Lemma [3, Theorem 3.2] we derive that

$$\sum_{n \leq x} a_{2j}(n) = x P_{n_{2j}}(\log x) + O\big(x^{\frac{4^j-1}{4^j+1}+\varepsilon}\big). \tag{4.2}$$

For any $x^{\varepsilon} \ll y \ll x$, this yields the upper bound of $a_{2j}(n)$ in short intervals by subtracting the x sum from the $x+y$ sum:

$$\sum_{x<n\leq x+y} a_{2j}(n) = O(y \log^{n_{2j}} x) + O\big(x^{\frac{4^j-1}{4^j+1}+\varepsilon}\big).$$

Clearly, it follows from Lemma 2.1 that

$$q^{4^j s/2} \gamma_{2j}(s) G_{2j}(s,\chi) = \epsilon(f,\chi) q^{4^j(1-s)/2} \gamma_{2j}(1-s) G_{2j}(1-s,\chi).$$

If χ is even, that is, $\chi(-1) = 1$, it is clear that $Q_\lambda^0(0) = G_{2j}(0,\chi) = L(0,\chi) = 0$. Applying Lemma 4.1 with the parameters

$$\delta = 1, \quad A = \frac{4^j}{2}, \quad \beta = 1, \quad u = \frac{1}{2} - \frac{1}{2\cdot 4^j},$$

we deduce from (1.4) that

$$\begin{aligned}
(4.3)\qquad & \sum_{n\leq x} a_{2j}(n,\chi) \\
&= O\big(q^{\frac{4^j}{2}} y^{\frac{-4^j+1}{2}} x^{\frac{4^j-1}{2}}\big) + O\Big(\sum_{x<n\leq x+O(y)} |a_{2j}(n)|\Big) + O\big(x^{\frac{7}{32}j}\big) \\
&= O\big(q^{\frac{4^j}{2}} y^{\frac{-4^j+1}{2}} x^{\frac{4^j-1}{2}}\big) + O(y \log^{\deg P_{2j}} x) + O\big(x^{\frac{4^j-1}{4^j+1}+\varepsilon}\big) \\
&= O_{f,\varepsilon}\big(q^{\frac{4^j}{4^j+1}} x^{\frac{4^j-1}{4^j+1}+\varepsilon}\big),
\end{aligned}$$

which follows by taking $y = q^{\frac{4^j}{4^j+1}} x^{\frac{4^j-1}{4^j+1}+\varepsilon}$.

If χ is odd, that is, $\chi(-1) = -1$, we have $Q_\lambda^0(0) = G_{2j}(0,\chi) \ll q^{4^j/2}$ by the convexity bound (2.12), which may lead to a higher bound than we expect when $q^{\frac{4^j}{2}} \gg q^{\frac{4^j}{4^j+1}} x^{\frac{4^j-1}{4^j+1}+\varepsilon}$, namely, $q \gg x^{\frac{2}{4^j}}$. However, we can treat this case by shifting the variable. A new Dirichlet series is defined by

$$G'_{2j}(s,\chi) = G_{2j}(s-1,\chi) = \sum_{n=1}^{\infty} \frac{n a_{2j}(n,\chi)}{n^s},$$

which satisfies a functional equation

$$q^{4^j s/2} \gamma_{2j}(s) G'_{2j}(s,\chi) = \epsilon(f,\chi) q^{4^j(3-s)/2} \gamma_{2j}(3-s) G'_{2j}(3-s,\chi).$$

Using Lemma 4.1 for $G'_{2j}(s,\chi)$ with the parameters

$$\delta = 3, \quad A = \frac{4^j}{2}, \quad \beta = 2, \quad u = \frac{1}{2} - \frac{1}{2\cdot 4^j}$$

and $Q_\lambda^0(0) = G'_{2j}(0,\chi) = G_{2j}(-1,\chi) = L(-1,\chi) = 0$, one derives by choosing $y = q^{\frac{4^j}{4^j+1}} x^{\frac{4^j-1}{4^j+1}+\varepsilon}$ that

$$\begin{aligned}(4.4)\qquad &\sum_{n\le x} n a_{2j}(n,\chi)\\ &= O\big(q^{\frac{4^j}{2}} y^{\frac{-4^j+1}{2}} x^{\frac{4^j+1}{2}}\big) + O\Big(x \sum_{x<n\le x+O(y)} |a_{2j}(n)|\Big) + O\big(x^{1+\frac{7}{32}j}\big)\\ &= O\big(q^{\frac{4^j}{2}} y^{\frac{-4^j+1}{2}} x^{\frac{4^j+1}{2}}\big) + O\big(yx \log^{\deg P_{2j}} x\big) + O\big(x^{\frac{2\cdot 4^j}{4^j+1}+\varepsilon}\big)\\ &= O_{f,\varepsilon}\big(q^{\frac{4^j}{4^j+1}} x^{\frac{2\cdot 4^j}{4^j+1}+\varepsilon}\big).\end{aligned}$$

Then the estimate (4.3) also holds for odd χ by partial summation. Note that the estimate above is only established for $y = q^{\frac{4^j}{4^j+1}} x^{\frac{4^j-1}{4^j+1}+\varepsilon} \ll x$, that is, $q \ll x^{1/4^j-\varepsilon}$. For $q \gg x^{1/4^j-\varepsilon}$, by (4.2) we obtain

$$\sum_{n\le x} a_{2j}(n,\chi) \ll \sum_{n\le x} a_{2j}(n) \ll x \log^{n_{2j}} x \ll q^{\frac{4^j}{4^j+1}} x^{\frac{4^j-1}{4^j+1}+\varepsilon}.$$

Combining all cases, one can derive that (4.3) is valid for any primitive χ and $q > 0$. Moreover, we know from (2.15) that

$$|\lambda_f(n)|^{2j}\chi(n) = \sum_{n=uv} a_{2j}(u,\chi) b_{2j}(v,\chi). \tag{4.5}$$

By Lemma 2.5, it follows from the absolute and uniform convergence of $H_{2j}(s,\chi)$ in the respective regions of s that

$$\begin{aligned}(4.6)\qquad \sum_{n\le x} |\lambda_f(n)|^{2j}\chi(n) &= \sum_{v\le x} b_{2j}(v,\chi) \sum_{u\le x/v} a_{2j}(u,\chi)\\ &\ll q^{\frac{4^j}{4^j+1}} x^{\frac{4^j-1}{4^j+1}+\varepsilon} \sum_{v=1}^{\infty} \frac{|b_{2j}(v,\chi)|}{v^{\frac{4^j-1}{4^j+1}+\varepsilon}}\\ &\ll q^{\frac{4^j}{4^j+1}} x^{\frac{4^j-1}{4^j+1}+\varepsilon}. \ \blacksquare\end{aligned}$$

PROPOSITION 4.3. *Let $f \in S_r$ and let χ_0 be a principal character modulo a prime q. For any $\varepsilon > 0$, $q \ll x$ and $j = 2, 3, 4$ we have*

$$\sum_{n\le x} |\lambda_f(n)|^{2j}\chi_0(n) = R_{2j}(x,q) + O_{f,\varepsilon}\big(x^{\frac{4^j-1}{4^j+1}+\varepsilon}\big). \tag{4.7}$$

Proof. By Lemma 2.5,

$$\begin{aligned}(4.8)\qquad &F_{2j}(s,\chi_0)\\ &= \prod_{i=0}^{2j} (1-\alpha(q)^{2(j-i)} q^{-s})^{C_{2j}^i} G_{2j}(s) H_{2j}(s,\chi_0) =: G_{2j}(s) H_{2j}(s,q).\end{aligned}$$

Let $b_{2j}(n,q)$ be the coefficients of $H_{2j}(s,q)$. Then

$$H_{2j}(s,q)^{(k)}|_{s=1} = (-1)^k \sum_{n=1} \frac{b_{2j}(n,q)\log^k n}{n^s}, \tag{4.9}$$

and

$$|b_{2j}(n,q)| \le \sum_{i=0}^{\infty} \sum_{q^i m=n} |b_{2j}(m,\chi_0)|\big(|\alpha(q)|^{2j} + |\alpha(q)|^{-2j}\big)^i.$$

Further, from (2.17) and the convergence properties of $H_{2j}(s,\chi_0)$ we have

$$\begin{aligned}\sum_{n\le x} &|b_{2j}(n,q)| \\ &\le \sum_{i\le \log_2 x} \big(|\alpha(q)|^{2j} + |\alpha(q)|^{-2j}\big)^i \sum_{m\le x/q^i} |b_{2j}(m,\chi_0)| \\ &\le x^{\frac{4^j-1}{4^j+1}} \sum_{i\le \log_2 x} \left(\frac{|\alpha(q)|^{2j} + |\alpha(q)|^{-2j}}{q^{\frac{4^j-1}{4^j+1}}}\right)^i \sum_{m\le x/q^i} \frac{|b_{2j}(m,\chi_0)|}{m^{\frac{4^j-1}{4^j+1}}} \\ &\le x^{\frac{4^j-1}{4^j+1}} \sum_{i\le \log_2 x} \left(\frac{|\alpha(q)|^{2j} + |\alpha(q)|^{-2j}}{q^{\frac{4^j-1}{4^j+1}}}\right)^i \\ &\le x^{\frac{4^j-1}{4^j+1}+\varepsilon}.\end{aligned} \tag{4.10}$$

It is easily seen from (4.8) that

$$|\lambda_f(n)|^{2j}\chi_0(n) = \sum_{n=uv} a_{2j}(u) b_{2j}(v,q).$$

Combining this with (4.2), we obtain

$$\begin{aligned}\sum_{n\le x} |\lambda_f(n)|^{2j}\chi_0(n) &= \sum_{v\le x} b_{2j}(v,q) \sum_{u\le x/v} a_{2j}(u) \\ &= x\sum_{v\le x} \frac{b_{2j}(v,q)}{v} P_{n_{2j}}\Big(\log\frac{x}{v}\Big) + O\bigg(x^{\frac{4^j-1}{4^j+1}+\varepsilon} \sum_{v\le x} \frac{|b_{2j}(v,q)|}{v^{\frac{4^j-1}{4^j+1}+\varepsilon}}\bigg).\end{aligned}$$

By partial summation and (4.10), we have

$$\begin{aligned}\sum_{n\le x} |\lambda_f(n)|^{2j}\chi_0(n) &= x\sum_{v=1}^{\infty} \frac{b_{2j}(v,q)}{v} P_{n_{2j}}\Big(\log\frac{x}{v}\Big) + O\big(x^{\frac{4^j-1}{4^j+1}+\varepsilon}\big) \\ &=: M_{2j}(x,q) + O\big(x^{\frac{4^j-1}{4^j+1}+\varepsilon}\big).\end{aligned} \tag{4.11}$$

It remains to compute $M_{2j}(x,q)$. For convenience, suppose P_k is a polynomial of degree k, not necessarily the same at each occurrence. Then from

(4.9), we have

$$\begin{aligned}
M_{2j}(x,q) &= x\sum_{v=1}^{\infty}\frac{b_{2j}(v,q)}{v}P_{n_{2j}}(\log x-\log v)\\
&= \sum_{k=0}^{n_{2j}}\sum_{v=1}^{\infty}\frac{b_{2j}(v,q)\log^k v}{v}xP_{n_{2j}-k}(\log x)\\
&= \sum_{k=0}^{n_{2j}}\Big(L_{2j,q}^{-1}(q^{-s})\prod_{i=0}^{2j}(1-\alpha(q)^{2(j-i)}q^{-s})^{C_{2j}^i}H_{2j}(s)\Big)^{(k)}\Big|_{s=1}\\
&\quad\times xP_{n_{2j}-k}(\log x)\\
&= \sum_{k=0}^{n_{2j}}\Big(L_{2j,q}^{-1}(q^{-s})\prod_{i=0}^{2j}(1-\alpha(q)^{2(j-i)}q^{-s})^{C_{2j}^i}\Big)^{(k)}\Big|_{s=1}xP_{n_{2j}-k}(\log x).
\end{aligned}$$

Then the result can be obtained from (3.9). In fact, a precise calculation shows $M_{2j}(x,q)=R_{2j}(x,q)=\mathrm{res}_{s=1}F_{2j}(s,\chi_0)x^s/s$. ∎

Finally, as in the proof of Theorem 1.1, combining Propositions 4.2 and 4.3, we have

$$\sum_{\substack{n\le x\\ n\equiv l\,(\mathrm{mod}\,q)}}|\lambda_f(n)|^{2j}=\frac{1}{\varphi(q)}R_{2j}(x,q)+O_{f,\varepsilon}\big(q^{\frac{4^j}{4^j+1}}x^{\frac{4^j-1}{4^j+1}+\varepsilon}\big). \tag{4.12}$$

This completes the proof of Theorem 1.2.

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