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Zones of large and small values for Dedekind sums

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1. Introduction and main results. Throughout this paper let m and N be integers, $N \neq 0$, with $(m, N) = 1$. The classical Dedekind sum $s(m, N)$ is defined by

$$s(m,N) = \sum_{k=1}^{|N|} ((k/N))((mk/N))$$

where $((\ldots))$ denotes the usual sawtooth-function (cf., e.g., [2]). Because of

$$s(m,-N) = s(m,N) \quad \text{and} \quad s(m+N,N) = s(m,N),$$

it suffices to consider $N \geq 1$ and m in the range $0 \leq m < N$. The general definition, however, will be needed below. In the present context it is more natural to work with

$$S(m,N) = 12s(m,N).$$

This paper deals with *zones* of *large* and *small* values of $|S(m,N)|$ for m in the aforesaid range. To this end we observe, first,

$$(1) \qquad |S(m,N)| < N$$

for all possible integers m (cf., e.g., [5, (14)]). Our distinction between "large" and "small" is oriented towards the *quadratic mean value* of $S(m,N)$. It is known that

$$(2) \qquad \left(\frac{1}{N} \sum_{0 \leq m < N} |S(m,N)|^2\right)^{1/2} \asymp N^{1/2}$$

for N tending to infinity (more precisely, the asymptotic main term of (2) lies between $2\sqrt{N}$ and $5\sqrt{N}$, cf. [9]). Having (2) in mind we say that $S(m,N)$ is *small* if $S(m,N) \ll \sqrt{N}$ and *large* if $\sqrt{N} = o(S(m,N))$ as $N \to \infty$. It has been observed by various authors (cf. [2], [4], [5]) that $S(m,N)$ becomes large for arguments m lying near points $N \cdot c/d$, where d is a small natural number and $(c,d) = 1$. In [5] we conjectured a sort of converse, namely, that $S(m,N)$ is small (in the above sense) if m is outside a certain union

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of intervals with mid-points $N \cdot c/d$, $1 \le d \le \sqrt{N}$. In this paper we prove a stronger version of this conjecture (cf. Theorem 1). Indeed, the intervals considered here are smaller than those of [5] (cf. the remark at the end of Section 2 below), and their definition is simpler.

The following terminology will be used: A *Farey point* (or simply an *F-point*) has the shape $N \cdot c/d$, $1 \le d \le \sqrt{N}$, $0 \le c \le d$, $(c,d) = 1$. The denominator d is called the *order* of the F-point. Further, we fix an arbitrary constant $C > 0$. The interval

$$I_{c/d} = \{x : 0 \le x \le N,\ |x - N \cdot c/d| \le C\sqrt{N}/d^2\} \tag{3}$$

is called the *F-neighbourhood* of the point $N \cdot c/d$. We write

$$\mathcal{F}_d = \bigcup_{\substack{0\le c\le d\\ (c,d)=1}} I_{c/d} \tag{4}$$

for the union of all neighbourhoods belonging to F-points of a fixed order d. Further,

$$\mathcal{F} = \bigcup_{1\le d\le\sqrt{N}} \mathcal{F}_d. \tag{5}$$

The integers m (relatively prime to N, as always) lying in $\mathcal{F}$ are called *F-neighbours*. More precisely, m is an F-neighbour of order d if $m \in \mathcal{F}_d$, and it is an F-neighbour of $N \cdot c/d$ if it lies in $I_{c/d}$. An integer m, $0 \le m < N$, which is not in $\mathcal{F}$ is called an *ordinary integer*.

THEOREM 1. *Let $N \ge 15$ and m be an ordinary integer. Then*

$$|S(m,N)| \le (2 + 1/C)\sqrt{N} + 5.$$

It is not hard to see that the set $\mathcal{F}$ is small in terms of its Lebesgue measure: By (3) and (4), the measure of $\mathcal{F}_d$ is $\le 2C\varphi(d)\sqrt{N}/d^2$; accordingly, the measure of $\mathcal{F}$ is

$$\le 2C\sqrt{N} \sum_{1\le d\le\sqrt{N}} \varphi(d)/d^2 = \frac{6C}{\pi^2}\sqrt{N}(\log N + O(1))$$

for large numbers N (cf. [1, p. 71]). Nevertheless, the number of F-points might be large, since $\mathcal{F}$ is the union of many intervals—their number amounts to $\asymp N$. The following theorem says that this is not the case. In particular, the number of ordinary integers exceeds that of F-neighbours by far, which justifies our choice of names.

THEOREM 2. *For each $N \ge 17$ the number of F-neighbours is*

$$\le C\sqrt{N}(\log N + 2\log 2).$$

Let us look briefly at the graph

$$G = \{(m, S(m,N)) : 0 \le m < N\}$$

of the function $m \mapsto S(m, N)$. Theorems 1 and 2 say that G consists of a *very large flat* zone outside of $\mathcal{F}$. This is illustrated by Diagram 1, which represents all 1772 pairs $(m, S(m, N))$ with $m \notin \mathcal{F}$ for $N = 2997 = 3^4 \cdot 37$ and $C = 1$. But the said flat zone is interrupted by many potential *zones of disturbance*, namely, the F-neighbourhoods. Since we used small circles to represent points $(m, S(m, N))$, only the largest of these neighbourhoods become visible in this diagram, the smaller ones disappear in the cluster of circles.

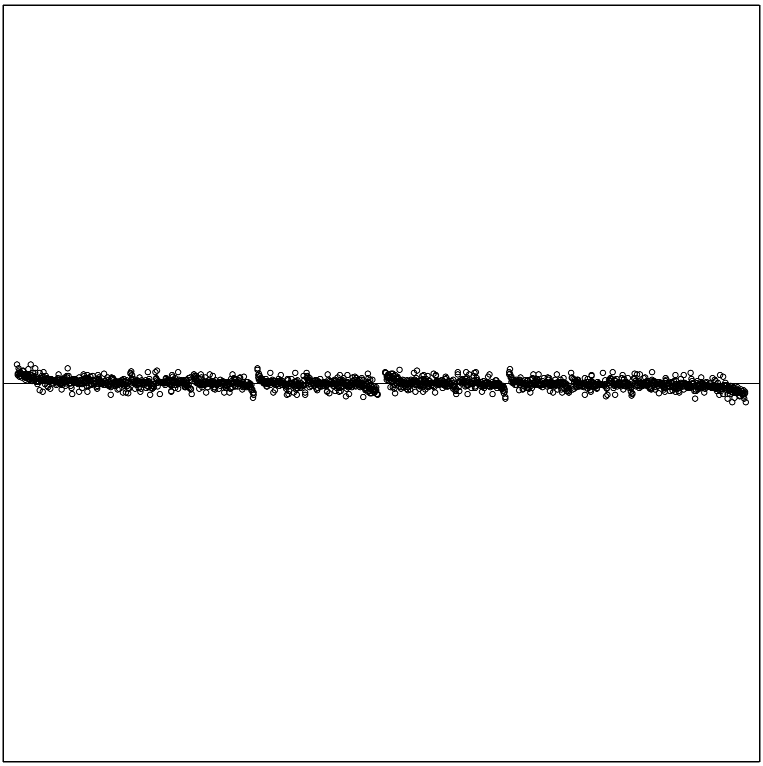

Diagram 1. The graph G outside of $\mathcal{F}$ for $N = 2997$, $C = 1$

Are the F-neighbourhoods really zones of disturbance for the graph G? The answer is of an asymptotic nature, of course. Hence suppose that N runs through a sequence of natural numbers tending to infinity. The number $d \le \sqrt{N}$ need not be constant but may also tend to infinity, and the same is true of the number m, $0 \le m < N$. Suppose that m remains an F-neighbour of order d while N grows. This means that the abscissa

$$x_m = m - N \cdot c/d \tag{6}$$

of m relative to the corresponding F-point fulfils $x_m \ll \sqrt{N}/d^2$. We say m is a *distant* F-neighbour (*close* F-neighbour, respectively) of order d if

$|x_m| \asymp \sqrt{N}/d^2$ ($x_m = o(\sqrt{N}/d^2)$, respectively). In accordance with this notion we call an interval I with mid-point $N\cdot c/d$ a *close* F-neighbourhood of order d if its length is $o(\sqrt{N}/d^2)$ for N tending to infinity.

Theorem 3. (a) *If, in the above setting, m is a distant F-neighbour, then $S(m,N)$ is small, i.e., $S(m,N) \ll \sqrt{N}$.*

(b) *If, on the other hand, m is a close F-neighbour of order d, then $S(m,N)$ is large. More precisely,*

$$S(m,N) = \frac{N}{d^2 x_m} + o(\sqrt{N}) \quad \textit{and} \quad \sqrt{N} = o\left(\frac{N}{d^2 x_m}\right).$$

In view of Theorems 1 and 2, assertion (a) is not surprising. Indeed, if we change the constant C, an ordinary integer m becomes an F-neighbour and conversely—so the asymptotic behaviour of distant F-neighbours and ordinary integers should be much the same. However, we combine this assertion with another observation: Since an F-neighbour m of order d is an integer, we have $|x_m| \geq 1/d$, by (6). Suppose that $d \asymp \sqrt{N}$ as N tends to infinity. Then $1/d \asymp \sqrt{N}/d^2$, so m remains distant in the above sense. Hence we obtain

Corollary 1. *If m is an F-neighbour of order $d \asymp \sqrt{N}$, then $S(m,N) \ll \sqrt{N}$ for N tending to infinity.*

The corollary says that F-neighbourhoods of an order $d \asymp \sqrt{N}$ are not really zones of disturbance for the graph G. In fact, $|S(m,N)|$ may be considerably larger than $\sqrt{N}$ only if both $d = o(\sqrt{N})$ and m is a close F-neighbour of order d. In view of (6), assertion (b) of Theorem 3 may be stated as follows:

Corollary 2. *Let N tend to infinity and $d = o(\sqrt{N})$. Let I run through a sequence of close F-neighbourhoods of order d. Then the points $(m, S(m,N))$, $m \in I$, of the above graph G tend to the corresponding points (x,y), $x = m$, on the hyperbola*

$$(x - N\cdot c/d)\cdot y = N/d^2.$$

The hyperbolic nature of the graph G in the vicinity of F-points of small order has been observed in the literature (cf. [2], [4], [5]). The hyperbola of Corollary 2 is equilateral, its mid-point is the F-point $N\cdot c/d$, its asymptotes are given by $x = N\cdot c/d$ and $y = 0$, and its parameter is $\sqrt{2N}/d$. One should, however, not think that the part $\{(m, S(m,N)) : m \in I\}$ of the graph has a *symmetric* shape relative to the mid-point of the hyperbola, since the distribution of right and left F-neighbours m (i.e., those with $x_m > 0$ and $x_m < 0$, respectively) around $N\cdot c/d$ is in general not symmetric. In Section 3 we shall discuss some more details of this kind.

Diagram 2 displays almost all pairs $(m, S(m,N))$ with $m \in \mathcal{F}$ for the same $N = 2997$ and $C = 1$. There are only two exceptions: the values $m = 1$ and $m = N - 1$ have been omitted for reasons of space, since $S(m,N)$ is close to $\pm N$ in these cases, whereas all other values do not exceed $\pm N/2$.

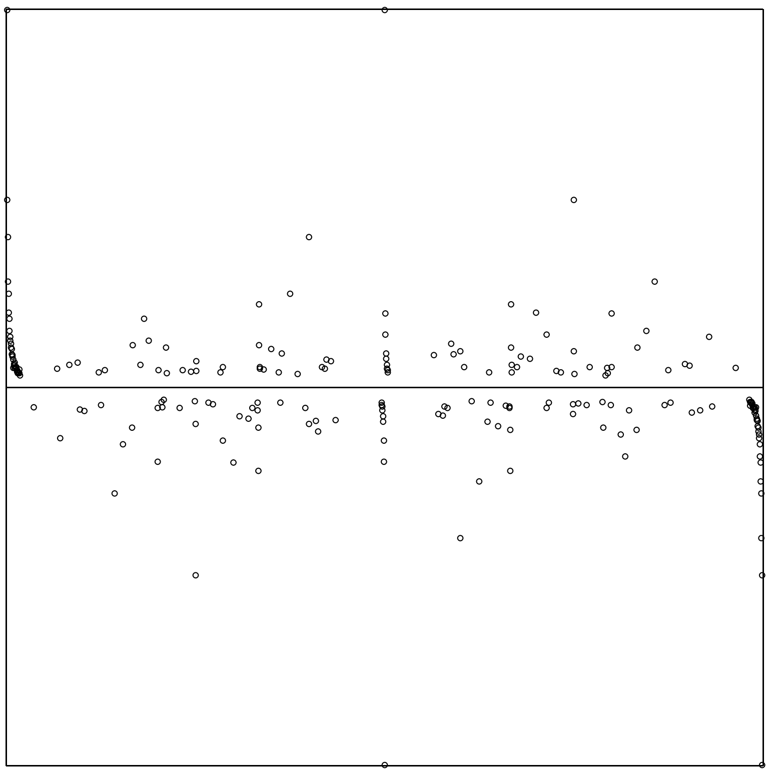

Diagram 2. The graph G restricted to $\mathcal{F}$ for $N = 2997$, $C = 1$

Our understanding of "large" and "small" comes from a quadratic mean value (cf. (2)) which favours large numbers, of course. From this point of view the majority of Dedekind sums is not only small but even "microscopic". Indeed,

$$\frac{1}{N} \sum_{0 \le m < N} |S(m,N)| \ll \log^2 N \tag{7}$$

(cf. [4, Lemma 6], cf. also [8]). The microscopic size of most Dedekind sums has some influence on $S(m,N)$ for *distant* F-neighbours m; namely, the hyperbolic shape of the above graph G in general extends over those m, too—up to few exceptions, cf. the example and the remark at the end of Section 3.

The distribution of Dedekind sums has attracted a great deal of interest (cf. [8], [2], [3], [4], [9]). The results of this paper belong to the easier ones. In our opinion they contribute something to the understanding of pictures of the graph G.

2. Farey neighbours and ordinary integers. In this section we prove Theorems 1 and 2. As above, let N and d be natural numbers with $d < N$, and m and c integers with $(m,N) = (c,d) = 1$. We put

$$q = md - Nc, \tag{8}$$

which means $m - N \cdot c/d = q/d$. Then $q \neq 0$, for otherwise $m/N = c/d$, which is impossible since $d < N$ and both fractions are reduced. A crucial ingredient of all proofs is the *generalized reciprocity law* for Dedekind sums, which we state as follows (cf., e.g., [5, Lemma 1]): For some integer r with $(r,q) = 1$,

$$S(m,N) = S(c,d) \pm S(r,q) + \frac{N^2 + d^2 + q^2}{Ndq} \pm 3, \tag{9}$$

where the $\pm$ sign is the sign of q in both cases. Combined with (1), the reciprocity law gives

$$|S(m,N)| \leq d + |q| + \frac{N}{d|q|} + \frac{d}{N|q|} + \frac{|q|}{Nd} + 3 \tag{10}$$

for all m, N, c, d, and q as above.

Proof of Theorem 1. Suppose now, in addition, that c/d is an F-point and $0 \leq m < N$. Because of $d < N$, we have $N \geq 2$ and $m > 0$. Further,

$$d \leq \sqrt{N} \quad \text{and} \quad \frac{d}{N|q|} \leq \frac{d}{N} \leq \frac{1}{\sqrt{N}}. \tag{11}$$

By [6, p. 127, Theorem 10.5], there is always an F-point c/d such that

$$\left|\frac{m}{N} - \frac{c}{d}\right| < \frac{1}{d\lfloor\sqrt{N}\rfloor}.$$

We fix such an F-point and obtain, from (8),

$$|q| \leq \frac{N}{\sqrt{N} - 1} = \sqrt{N} + 1 + \frac{1}{\sqrt{N} - 1} \tag{12}$$

and

$$\frac{|q|}{dN} \leq \frac{|q|}{N} \leq \frac{1}{\sqrt{N}} + \frac{2}{N}, \tag{13}$$

provided that $N \geq 4$. Finally, suppose that m is an ordinary integer. So its distance to the above F-point satisfies $|m - N \cdot c/d| > C\sqrt{N}/d^2$, which means $|q| \ge C\sqrt{N}/d$ and

$$\frac{N}{d|q|} \le \sqrt{N}/C. \tag{14}$$

On inserting the estimates (11)–(14) into (10), we obtain

$$|S(m,N)| \le \left(2+\frac{1}{C}\right)\sqrt{N} + 4 + \frac{1}{\sqrt{N}-1} + \frac{2}{\sqrt{N}} + \frac{2}{N},$$

which is $\le (2+1/C)\sqrt{N}+5$ for $N \ge 15$. ∎

The proof of Theorem 2 is based on the following

LEMMA 1. *Let $1 \le d < N$. Then*

$$\sharp(\mathcal{F}_d \cap \{m \in \mathbb{Z} : (m,N)=1\}) \le 2C\sqrt{N}/d.$$

Proof. Put $\delta = (N,d)$, so $N = \delta\cdot N'$, $d = \delta \cdot d'$ with $(N',d')=1$. Consider an element m of the set in question. Since m is in $\mathcal{F}_d$, there is an integer q with $|q| \le C\sqrt{N}/d$ such that $md \equiv q \bmod N$ (cf. (8)). Then δ divides q, so $q = \delta \cdot q'$ for some integer q' with $|q'| \le C\sqrt{N}/(\delta d)$. Further, $md' \equiv q' \bmod N'$ and $q' \ne 0$ since, otherwise, $q=0$ and $N \mid d$, which is impossible. Let m' be the unique solution of the congruence $md' \equiv q' \bmod N'$ that lies in $\{0,1,\ldots,N'-1\}$. Then m has the shape

$$m = m' + l\cdot N'$$

for some $l \in \mathbb{Z}$, $0 \le l < \delta$. Since we have at most $2C\sqrt{N}/(\delta d)$ possibilities for q', the assertion follows. ∎

Proof of Theorem 2. The set of F-neighbours is

$$\mathcal{F} \cap \{m \in \mathbb{Z} : (m,N)=1\}.$$

By (5) and Lemma 1, its cardinality is bounded by

$$\sum_{1\le d\le\sqrt{N}} 2C\sqrt{N}/d = 2C\sqrt{N}\sum_{1\le d\le\sqrt{N}} 1/d.$$

However, one readily infers from [7, p. 6, Theorem 5] that the sum on the right hand side is $\le \log(2\sqrt{N})$ whenever $N \ge 17$. ∎

REMARK. In [5] we considered intervals around the F-points which were larger than our F-neighbourhoods when the order $d \le \sqrt{N}$ was large, their size being (roughly) $\sqrt{N/d^3}$. Altogether, those intervals contained $\asymp N^{2/3}$ integers, in contrast with the situation of Theorem 2.

3. The behaviour of Farey neighbours. In what follows let m be an F-neighbour of order d, so $d \le \sqrt{N}$ and $q = md - Nc$ fulfils $|q| \le C\sqrt{N}/d$ for the corresponding F-point $N\cdot c/d$. Then (9) gives

$$S(m,N) = \frac{N}{dq} + E(d+|q|+4) \tag{15}$$

if N is sufficiently large, where $E(x)$ denotes an error term of absolute value $\leq x$.

Proof of Theorem 3. From (15) we clearly obtain

$$S(m,N)=\frac{N}{dq}+E((1+C)\sqrt{N}+4) \tag{16}$$

for large numbers N. If m remains *distant* while N tends to infinity, then $|q|\asymp\sqrt{N}/d$ and $N/(d|q|)\asymp\sqrt{N}$, so (16) shows $S(m,N)\ll\sqrt{N}$, which is assertion (a) of Theorem 3.

As to assertion (b), suppose that m remains a *close* F-neighbour. Then $x_m=o(\sqrt{N}/d^2)$ and $1=o(\sqrt{N}/(d^2x_m))$. So the main term $N/(dq)=N/(d^2x_m)$ of (15) satisfies $\sqrt{N}=o(N/(d^2x_m))$. Further, $q=o(\sqrt{N}/d)$, so both $q=o(\sqrt{N})$ and $d=o(\sqrt{N})$. Altogether, the error term in (15) is $o(\sqrt{N})$. ∎

As in Section 1, let us have a look at the graph G of the function $m\mapsto S(m,N)$. We concentrate upon one particular order $d\geq 2$ with $d=o(\sqrt{N})$, which means that close F-neighbours m of order d are possible. The corresponding set $\mathcal{F}_d$ consists of $\varphi(d)$ (pairwise disjoint) intervals $I_{c/d}$, and Theorem 3 (b) says that close to the center $N\cdot c/d$ of $I_{c/d}$ the graph becomes similar to the hyperbola

$$y=\frac{N}{d^2(x-N\cdot c/d)}.$$

Accordingly, G has a positive spike on the right and a negative one on the left of $N\cdot c/d$. The possible height (or depth) of these spikes is asymptotically bounded by $N/(d\delta)$ with $\delta=(N,d)$. Indeed, since $|q|=|md-Nc|\geq\delta$, the asymptotic value of $|S(m,N)|$ is $|N/(dq)|\leq N/(d\delta)$. If, for instance, $\delta=1$, then the whole set $\mathcal{F}_d$ contains exactly one m with $q=1$, so the (asymptotically) maximal height N/d is taken for *exactly one* integer $m\in\mathcal{F}_d$; the same holds for the depth $-N/d$. The case $d=1$ is exceptional inasmuch as we have *two* intervals I_0 and I_1 instead, each of which defines only *one* branch of the respective hyperbola.

However, the similarity of G with the said hyperbolas is restricted by the fact that the distribution of numbers m, $(m,N)=1$, in the set $\mathcal{F}_d$ may not be uniform. If the order d grows, the F-neighbourhoods $I_{c/d}$ contain fewer of these integers and become empty with increasing frequency: In fact, $\mathcal{F}_d$ contains at most $2C\sqrt{N}/d$ such integers, by Lemma 1. So each F-neighbourhood $I_{c/d}$ contains

$$\leq\frac{2C\sqrt{N}}{d\varphi(d)}$$

numbers m on average. This mean value tends to zero if $d\gg N^{1/4+\varepsilon}$, say.

Moreover, the integers m are not symmetric about the center $N \cdot c/d$ of $I_{c/d}$ in general, as the following trivial example shows: If $I_{c/d}$ contains *both* a right F-neighbour m and a left F-neighbour m', then

$$1 \le m - m' \le 2C\sqrt{N}/d^2,$$

which requires $d \ll N^{1/4}$. For the same reason, m' is relatively far away from the center $N \cdot c/d$ if m is close to it. Thus, if $x_m = 1/d$, say, then $|x_{m'}| \ge 1 - 1/d$.

EXAMPLE. Let $N = 2 \cdot 10^6 + 3$ (a prime number), $d = 11$, $c = 7$, so the corresponding F-point is ≈ 1272729.182. Choosing $C = 1$, we have $C\sqrt{N}/d^2 \approx 11.688$. Hence $I_{c/d}$ contains 23 integers m with $(m, N) = 1$ (viz., all of $1272718, \ldots, 1272740$). For these numbers m, $|q| \le \lfloor \sqrt{N}/11 \rfloor = 128$, so (15) shows that $S(m,N) = N/(dq) + E(143)$; and going back to (9) one even obtains $E(142.001)$. Because of $N/(d|q|) \ge 1420.45$, $S(m,N)$ must be equal to $N/(dq)$ up to a relative error of less than 10 percent. Thus, the main term of Theorem 3 (b) is essentially the correct value of $S(m,N)$ for *all* 22 F-neighbours of $N \cdot 7/11$. On computing the exact values of $S(m,N)$ one sees that the error is even much smaller, the largest value of $|N/(dq) - S(m,N)|$ being ≈ 29.897 for $m = 1272736$. This is due to the fact that the terms q and d in the E-term of (15) come from the Dedekind sums $S(r,q)$ and $S(c,d)$ of (9), which are expected to be much smaller than d and q themselves (cf. (7)). In the following table we list the closest F-neighbours m of $N \cdot c/d$ together with $S(m,N)$ and $\Delta = N/(dq) - S(m,N)$. The table shows that the size of the positive spike of the graph G is considerably smaller than that of the negative one here, which is a consequence of $x_m = -2/11$ for $m = 1272729$ but $x_m = 9/11$ for $m = 1272730$.

m	$S(m,N)$	Δ	m	$S(m,N)$	Δ
1272725	−3954.081	1.506	1272730	20199.192	2.859
1272726	−5184.021	−10.792	1272731	9090.786	0.136
1272727	−7553.322	−22.447	1272732	5861.443	3.669
1272728	−13985.594	−0.441	1272733	4328.184	0.827
1272729	−90907.864	−1.364	1272734	3422.278	8.259

REMARK. Although it often happens, it is not always true that $N/(dq)$ is a reasonable approximation of $S(m,N)$ for *distant* F-neighbours, especially if d is small. So take $N = 1009$, $C = 1.2$, and $d = 1$. Then $m = 36$ lies in I_0; however, $S(m,N) \approx -7.992$, whereas $N/(dq) = N/m \approx 28.028$. If, on the other hand, C has been chosen small enough (say $C = 1/5$), then $N/(dq)$ clearly dominates the error term of (16) for *all* F-neighbours.

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