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Visibility of lattice points

by

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1. Introduction. Two integer points $P(a_1,\dots,a_k)$ and $Q(b_1,\dots,b_k)$ are said to be *visible to each other* if either $P=Q$ or there are no other integer points on the line segment joining P and Q. It is not difficult to verify that if $P\neq Q$, then P and Q are visible to each other if and only if $\gcd(a_1-b_1,\dots,a_k-b_k)=1$. We say that an integer point set A is *visible from an integer point set* B if each point of A is visible from some point of B.

For $k\ge 2$, let

$$\Delta_n^k=\{(x_1,\dots,x_k): x_i \text{ integers and } 1\le x_i\le n\,(1\le i\le n)\}.$$

Define

$$f_k(n)=\min\{|S|: S\subset \mathbb{Z}^k,\ \Delta_n^k \text{ is visible from } S\},$$
$$F_k(n)=\min\{|S|: S\subseteq \Delta_n^k,\ \Delta_n^k \text{ is visible from } S\}.$$

It is clear that $f_k(n)\le F_k(n)$. Erdős, Gruber and Hammer [4] asked for an explicit construction of S such that $S\subset\Delta_n^2$, $|S|=O(\log n)$ and Δ_n^2 is visible from S. A better construction of S was given by Adhikari and Balasubramanian [2]. We have

$$F_2(n)\ge \frac{1}{2}\,\frac{\log n}{\log\log n},\quad n\ge n_0, \qquad \text{(Abbott [1])}$$

$$F_2(n)=O\biggl(\frac{\log n\log\log\log n}{\log\log n}\biggr), \qquad \text{(Adhikari, Balasubramanian [2])}$$

$$F_k(n)=O\biggl(\frac{\log n}{\log\log n}\biggr),\quad k\ge 3, \qquad \text{(Adhikari, Chen [3]).}$$

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In fact, the method in Abbott [1] implies that

$$F_k(n) \geq f_k(n) \geq \frac{1}{2}\,\frac{\log n}{\log\log n}, \quad n \geq n_0, \quad \text{for all } k \geq 2.$$

Thus, for $k \geq 3$, the main orders of $f_k(n)$ and $F_k(n)$ are $\log n/\log\log n$. In this note we are interested in the constant factors. Let

$$\zeta(k) = \sum_{m=1}^{\infty} \frac{1}{m^k} = \prod_p \left(1 - \frac{1}{p^k}\right).$$

The following results are proved.

THEOREM 1. *For $k \geq 2$ we have*

$$f_k(n) \geq \zeta(k)\,\frac{\log n}{\log\log n}\,(1 + o(1)).$$

THEOREM 2. *For $k \geq 3$ we have*

$$F_k(n) \leq \zeta(k-1)\,\frac{\log n}{\log\log n}\,(1 + o(1)).$$

REMARK. The first author conjectures that

$$f_k(n) = \zeta(k)\,\frac{\log n}{\log\log n}\,(1 + o(1)), \qquad F_k(n) = \zeta(k)\,\frac{\log n}{\log\log n}\,(1 + o(1)).$$

By an analogous argument to the proof of Theorem 2, we can prove that the conjecture for $k \geq 3$ follows from the conjecture that for every fixed s,

$$\max_{s<m\leq n} \omega((m-1)(m-2)\dots(m-s)) = (1 + o(1))\,\frac{\log n}{\log\log n},$$

and the conjecture for $k = 2$ follows from the conjecture that

$$\max_{s_n<m\leq n} \omega((m-1)(m-2)\dots(m-s_n)) = (1 + o(1))\,\frac{\log n}{\log\log n},$$

where $s_n = [2\log n/\log\log n]$.

2. Proofs. Let $p_1, p_2, \dots$ be all positive primes in increasing order, that is, $p_1 = 2,\ p_2 = 3, \dots$ As usual, we will use p to denote a prime. For two points P and Q in $\mathbb{Z}^k$ and an integer m, we say that P and Q are *congruent* $\bmod m$ if all coordinates are congruent $\bmod m$.

LEMMA 1. *Let B be a finite subset of $\mathbb{Z}^k$. Then there exist at least $|B|/p^k$ points in B which are congruent mod p.*

Proof. Lemma 1 follows from the fact that points in $\mathbb{Z}^k \bmod p$ has p^k different possible values.

Proof of Theorem 1. Let $0<\varepsilon<1/8$. We take an integer t such that

$$\prod_{i=1}^{t}\left(1-\frac{1}{p_i^k}\right)\le\zeta(k)^{-1}(1+\varepsilon).$$

For $n\ge n_1(\varepsilon,k)$ there exists an integer r such that

$$\zeta(k)\,\frac{\log n}{\log\log n}\,(1-6\varepsilon)\le r\le\zeta(k)\,\frac{\log n}{\log\log n}\,(1-5\varepsilon),\qquad p_1^kp_2^k\ldots p_t^k\mid r.$$

Suppose that $Q_1,\ldots,Q_r$ are r distinct points in $\mathbb{Z}^k$. By Lemma 1 there exists $A_1\subseteq\{Q_1,\ldots,Q_r\}$ such that $|A_1|=r/p_1^k$ and the points in A_1 are congruent $\bmod\,p_1$. Let $B_1=\{Q_1,\ldots,Q_r\}\setminus A_1$. Then $|B_1|=(1-1/p_1^k)r$ is divisible by p_2^k. By Lemma 1 there exists $A_2\subseteq B_1$ such that $|A_2|=|B_1|/p_2^k$ and the points in A_2 are congruent $\bmod\,p_2$. Let $B_2=B_1\setminus A_2$. Then $|B_2|=(1-1/p_2^k)(1-1/p_1^k)r$ is divisible by p_3^k. Similarly, we obtain $A_3,\ldots,A_t$ and $B_3,\ldots,B_t$ such that $|A_i|=|B_{i-1}|/p_i^k$, $B_i=B_{i-1}\setminus A_i$ and the points in A_i are congruent $\bmod\,p_i$ for $3\le i\le t$. Then

$$|B_t|=\left(1-\frac{1}{p_t^k}\right)|B_{t-1}|=\ldots=r\prod_{i=1}^{t}\left(1-\frac{1}{p_i^k}\right)\le\frac{\log n}{\log\log n}\,(1-4\varepsilon).$$

Hence, for $n\ge n_2(\varepsilon,k)$,

$$t+|B_t|\le\frac{\log n}{\log\log n}\,(1-3\varepsilon). \tag{1}$$

Let $s=t+|B_t|$. Let $B_t=A_{t+1}\cup\ldots\cup A_s$ with $|A_i|=1$ and $A_i\cap A_j=\emptyset$ for $t+1\le i,j\le s$ and $i\ne j$. By the Chinese Remainder Theorem, there exists a point Q in $\Delta^k_{(r+1)p_1\ldots p_s}$ which is different from $Q_1,\ldots,Q_r$ and congruent to the points of A_i $\bmod\,p_i$ for each i. For $n\ge n_3(\varepsilon,k)$ by (1) we have

$$\begin{aligned}\log((r+1)p_1\ldots p_s)&\le\log(r+1)+s\log p_s\\&\le\varepsilon\log n+(1-2\varepsilon)\log n<\log n.\end{aligned}$$

Hence, $Q\in\Delta_n^k$ and Q is invisible from any point of $Q_1,\ldots,Q_r$. Therefore

$$f_k(n)>r\ge\zeta(k)\,\frac{\log n}{\log\log n}\,(1-6\varepsilon).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let $0<\varepsilon<1/4$. Let t be an integer with

$$2^{k-1}\sum_{p>p_t}\frac{1}{p^{k-1}}<\varepsilon\zeta(k-1)^{-1}.$$

For $n\ge n_4(\varepsilon,k)$ there exists an integer r with

$$\zeta(k-1)\,\frac{\log n}{\log\log n}\,\frac{1+2\varepsilon}{1-\varepsilon}\le r^{k-1}\le\zeta(k-1)\,\frac{\log n}{\log\log n}\,\frac{1+3\varepsilon}{1-\varepsilon},\qquad p_1p_2\ldots p_t\mid r.$$

Let

$$G_n = \{(a_1,\ldots,a_{k-1},1) : a_i \text{ integers}, 1 \le a_i \le r \ (1 \le i \le k-1)\} \\ \cup \{(2,2,\ldots,2)\}.$$

Given any point $(x_1,\ldots,x_k) \in \Delta_n^k$. If $x_k = 1$, then $(x_1,\ldots,x_k)$ is visible from $(2,2,\ldots,2)$. Now we assume that $x_k > 1$. We will show that there exists at least one point $(a_1,\ldots,a_{k-1},1) \in G_n$ such that

$$(x_1 - a_1,\ldots,x_{k-1} - a_{k-1}, x_k - 1) = 1.$$

In order to prove this, we use a simple sieving argument. Let $q_1,\ldots,q_m$ be the prime divisors of $x_k - 1$. We know that

$$m = \omega(x_k - 1) \le (1 + o(1)) \log n/\log\log n.$$

We want to find $(a_1,\ldots,a_{k-1},1) \in G_n$ so that no q_i divides each $x_j - a_j, 1 \le j \le k-1$.

For the primes q_j that are among $p_1,\ldots,p_t$ we use the combinatorial sieve. We find that the number of remaining vectors is

$$r^{k-1}\prod(1 - p_i^{-(k-1)}) > \zeta(k-1)^{-1}r^{k-1}.$$

Each prime q_j with $p_t < q_j \le r$ excludes at most $(1+[r/q_j])^{k-1} < (2r/q_j)^{k-1}$ vectors. The total number of these is

$$< (2r)^{k-1}\sum_{q_j > p_t} q_j^{-(k-1)} < \varepsilon\zeta(k-1)^{-1}r^{k-1}.$$

Finally, a $q_j > r$ excludes at most one, altogether $(1 + o(1)) \log n/\log\log n$ at most. Since

$$\zeta(k-1)^{-1}r^{k-1} > (1+\varepsilon)\,\frac{\log n}{\log\log n} + \varepsilon\zeta(k-1)^{-1}r^{k-1},$$

we are done.

Therefore

$$F_k(n) \le |G_n| + 1 \le r^{k-1} + 1 \le \zeta(k-1)\,\frac{1+3\varepsilon}{1-\varepsilon}\,\frac{\log n}{\log\log n} + 1.$$

This completes the proof.

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