ACTA ARITHMETICA
107.3 (2003)

Additive inhomogeneous Diophantine inequalities

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1. Introduction. In the study of Diophantine equations and inequalities, most results concern homogeneous polynomials. For example, suppose that $G(\mathbf{x})$ is a homogeneous polynomial, of odd degree k in s variables, with real coefficients. Schmidt [16] has given the impressive result that there exists a positive integer $s_0(k)$, which depends only on the degree k, so that if $s \geq s_0(k)$, then there is a vector $\mathbf{x} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ satisfying the inequality

$$|G(\mathbf{x})| < 1. \tag{1}$$

In other words, if there are enough variables, in terms of the degree only, then there is a non-trivial solution of the Diophantine inequality (1). Earlier, Birch [4] had proved a similar theorem for Diophantine equations. But there are still very few known results about inhomogeneous polynomials.

For Diophantine equations, this is perhaps to be expected, for reasons more convincing than simply because homogeneous polynomials have a nicer form. For if one considers equations such as

$$(G(\mathbf{x}))^3 - 2 = 0, \tag{2}$$

where $G(\mathbf{x})$ is an integral polynomial, we can see that there is good reason to restrict to the homogeneous case: the number of variables s here can be chosen as large as we like, and $G(\mathbf{x})$ can even be chosen so that the equation (2) has real solutions, yet there are clearly no integral solutions of (2).

Diophantine equations are specific cases of Diophantine inequalities, so there is still cause to be careful in the inequality case. Moreover, if $F(\mathbf{x})$ is a real multiple of an integral form, then the inequality $|F(\mathbf{x})| < \varepsilon$ reduces to the equation $F(\mathbf{x}) = 0$, for sufficiently small ε. Now consider the Diophantine inequality $|F(\mathbf{x})| < \varepsilon$ in the alternative case, when F is not a real multiple of an integral form, or equivalently, when the coefficients of F are not all in

2000 *Mathematics Subject Classification*: Primary 11D75; Secondary 11D41, 11D72, 11P55.

Key words and phrases: Diophantine inequalities, inhomogeneous polynomials, forms in many variables, applications of the Hardy–Littlewood method.

Supported by a National Science Foundation Postdoctoral Fellowship.

rational ratio. Perhaps one can more often solve Diophantine inequalities in such cases.

As a natural first step towards considering such inhomogeneous inequalities, we look at inhomogeneous polynomials which are themselves sums of polynomials in one variable. Thus suppose that κ is a real number and that $h_1(y),\ldots,h_s(y)$ are real polynomials, each in one variable. We look for integral solutions $(y_1,\ldots,y_s)$ to Diophantine inequalities of the type

$$|h_1(y_1)+\ldots+h_s(y_s)-\kappa|<\varepsilon. \tag{3}$$

We give two definitions so that we may state our result more easily.

Definition. Suppose that k and s are positive integers. Also, for $1\le i\le s$, suppose that $h_i(y)$ is a polynomial with real coefficients, given by

$$h_i(y)=\beta_{ik}y^k+\beta_{i(k-1)}y^{k-1}+\ldots+\beta_{i1}y+\beta_{i0}.$$

Then we say that the polynomials $h_1,\ldots,h_s$ satisfy the *irrationality condition* if there exist integers i_1 and i_2 with $1\le i_1,i_2\le s$ and integers j_1 and j_2 with $1\le j_1,j_2\le k$ for which one has

$$\frac{\beta_{i_1j_1}}{\beta_{i_2j_2}}\notin\mathbb{Q}.$$

Of course here we assume that $\beta_{i_2j_2}$ is non-zero.

This condition will serve to guarantee that one of the ratios of the coefficients of the sum polynomial $h_1(y_1)+\ldots+h_s(y_s)$ is irrational, and to ensure that we are not essentially considering a Diophantine equation. We make a few observations. Note that we could have $i_1=i_2$, so that the coefficients whose ratio is irrational could come from only one polynomial. Another observation, which is very important, is that we require $j_1\ge 1$ and $j_2\ge 1$. In other words, neither of the coefficients whose ratio is irrational is one of the constant terms. To see why we make such a requirement, consider the simple inequality

$$|y_1^3+\ldots+y_{s-1}^3+(y_s^3+\pi)|<\varepsilon.$$

This is an inequality of the form (3) with $\kappa=0$ and with $h_s(y_s)=y_s^3+\pi$. For $\varepsilon\le 1/10$, say, this inequality has no solutions in integers $y_1,\ldots,y_s$, even though there is clearly an irrational ratio among the coefficients. The problem here is that we are essentially still dealing with an integral polynomial in this case. We give one more definition.

Definition. Suppose that $H(\mathbf{x})$ is a sum of non-constant polynomials $h_i(x_i)$, each of degree at most k, where k is a positive integer. Then we say that $H(\mathbf{x})$ is an *indefinite polynomial* if not all of the leading coefficients of the polynomials h_i are of the same sign, or if any of the polynomials h_i has odd degree. We say that $H(\mathbf{x})$ is a *positive-definite polynomial* (respectively, *negative-definite polynomial*) if all of the leading coefficients of the polynomials h_i are positive (respectively, negative) and all of the polynomials h_i are of even degree.

We note that a definition of a somewhat similar nature has been given by Cook and Raghavan [7].

Having given the above definitions, we may now state our main result.

THEOREM 1. *Suppose that k is a positive integer. Then there is a positive integer $s_0(k)$ such that any integer s with $s \geq s_0(k)$ has the following property*:

Suppose that, *for $1 \leq i \leq s$, the polynomial $h_i(y)$ has real coefficients*, *is non-constant and is of degree at most k*, *and that the polynomials $h_1, \ldots, h_s$ satisfy the irrationality condition. Fix a positive number ε and any real number κ. Set*

$$H(\mathbf{y}) = H(y_1, \ldots, y_s) = \sum_{i=1}^{s} h_i(y_i).$$

Finally, *suppose that $H(\mathbf{y})$ is an indefinite polynomial. Then there exist infinitely many s-tuples of integers $\mathbf{z} = (z_1, \ldots, z_s)$ for which*

$$|H(\mathbf{z}) - \kappa| < \varepsilon. \tag{4}$$

Moreover, *we in fact may take $s_0(k)$ to satisfy*

$$s_0(k) \sim 4k \log k.$$

Note that, for example, Theorem 1 states that the values taken at integer points by a sum of $s_0(k)$ polynomials, which are of odd degree k and satisfy the irrationality condition, are dense on the real line.

For sums of general polynomials, Theorem 1 is one of the first results of its kind. However, the theorem is essentially already known in the special case in which every polynomial h_i takes the simple form $h_i(y_i) = \beta_{ik} y_i^k$. (See the paper of Brüdern and Cook [6]. They only prove the theorem in this special case for $\kappa = 0$, but their argument extends to prove this special case for all κ.) In fact, in this case, one can take $s_0(k)$ asymptotic to $k \log k$ by combining the smooth number methods of Wooley with the work in [6]. Currently, we cannot use these methods for general polynomials however, so we need the constant 4 here. We also note that we could follow our proof carefully and give an error term of order k in the asymptotic formula for $s_0(k)$, and we could also likely fine-tune our method to give better bounds of this type, but we choose not to, for ease of exposition. Such techniques would not allow us to reduce the constant factor 4 at any rate, without some new ideas.

Before stating another result, we must give some notation and definitions. For real vectors $\mathbf{x} \in \mathbb{R}^s$, we define

$$|\mathbf{x}| = \max_{1 \le i \le s} |x_i|.$$

Also, as is usual, by a non-trivial solution $\mathbf{x}$ of an inequality we mean that $\mathbf{x}$ is a solution which is not the zero vector.

We give a more technical version of the above result, which in fact implies Theorem 1. In Theorem 1, we only considered polynomials which were indefinite. Now we consider both definite and indefinite polynomials.

THEOREM 2. *Suppose that k is a positive integer. Then there is a positive integer $s_0(k)$ such that any integer s with $s \ge s_0(k)$ has the following property*:

Suppose that, *for $1 \le i \le s$*, *the polynomial*

$$h_i(y) = \beta_{ik} y^k + \beta_{i(k-1)} y^{k-1} + \ldots + \beta_{i1} y + \beta_{i0}$$

has real coefficients, *is non-constant*, *and is of degree at most k*, *and that $h_1, \ldots, h_s$ satisfy the irrationality condition. Set*

$$H(\mathbf{y}) = H(y_1, \ldots, y_s) = \sum_{i=1}^{s} h_i(y_i).$$

Now fix a positive number ε. Then there are positive constants C_1, C_2 and C_3 with $C_2 < C_3$, which depend only on k and the coefficients β_{ij}, such that, *given any positive number P which is sufficiently large in terms of k and ε and the coefficients β_{ij}, the following two statements hold*:

(i) *Suppose that $H(\mathbf{y})$ is an indefinite polynomial. If M is a real number with $|M| \le C_1 P$, then there exists a non-trivial s-tuple of integers $\mathbf{z} = (z_1, \ldots, z_s)$ with $|\mathbf{z}| \le P$ for which*

$$|H(\mathbf{z}) - M| < \varepsilon. \tag{5}$$

(ii) *If $H(\mathbf{y})$ is a positive-definite polynomial and if M is a real number with $C_2 P \le M \le C_3 P$, then there exists an s-tuple of integers $\mathbf{z} = (z_1, \ldots, z_s)$ with $|\mathbf{z}| \le P$ for which*

$$|H(\mathbf{z}) - M| < \varepsilon. \tag{6}$$

Moreover, *we in fact may take $s_0(k)$ to satisfy*

$$s_0(k) \sim 4k \log k.$$

We observe that we also of course have a result similar to case (ii) of this theorem if $H(\mathbf{x})$ is a negative-definite polynomial where we instead assume that $-C_3 P \le M \le -C_2 P$ holds. Such a result can of course be obtained by applying case (ii) of this theorem to $-H(\mathbf{x})$. We also note that if we assumed that each of the polynomials h_i were of degree k, then we could take the solutions $\mathbf{z}$ in cases (i) and (ii) to satisfy $|\mathbf{z}| \le P^{1/k}$ instead of $|\mathbf{z}| \le P$. We note as well that we could give some sort of bounds for the constants C_1, C_2 and C_3. However, in our proof, we focus on keeping the number $s_0(k)$ of variables necessary from being too large, and this comes at the price of any chance of determining the best possible constants here, so we do not concern ourselves with such bounds.

We also note, as Professor Schmidt observed, that in case (ii) one can show by a straightforward argument that we can assume only that M satisfies $M \ge C_4$ for some positive constant C_4, if one allows C_4 to depend on k, the coefficients β_{ij}, and ε.

We turn now to considering some related results. We do not mention any more results on Diophantine inequalities involving homogeneous polynomials, having already mentioned the work of Schmidt [16] above. We now consider instead results about Diophantine inequalities which involve inhomogeneous polynomials. First, there is the recent work of Bentkus and Götze. Let k be an even integer and suppose that s satisfies $s \ge s_1(k)$, where $s_1(k)$ is a function they give which satisfies $s_1(k) \ll k^4 4^k$. Then let

$$F(\mathbf{x}) = \lambda_1 x_1^k + \ldots + \lambda_s x_s^k + R(\mathbf{x}),$$

where $R(\mathbf{x})$ is a polynomial in $\mathbf{x} = (x_1, \ldots, x_s)$ of degree strictly less than k. Suppose that $\lambda_i > 0$ for $1 \le i \le s$. Also, suppose that for some $1 \le i < j \le s$, we have $\lambda_i/\lambda_j \notin \mathbb{Q}$. Next, fix a positive number ε. Then for any positive number M which is sufficiently large in terms of k and s and ε and the coefficients of $F(\mathbf{x})$, they prove that there is a solution of

$$|F(\mathbf{x}) - M| < \varepsilon.$$

(See [3]. In fact, they give a much stronger result concerning the distribution of the values of $F(\mathbf{x})$.) This is one of the first forays into the study of inhomogeneous Diophantine inequalities. And indeed, as they remark in their paper, their methods can most likely be used to non-trivially solve inequalities of the type $|F(\mathbf{x})| < \varepsilon$, where $F(\mathbf{x})$ is as above but is indefinite in nature.

There are fundamental differences between their result and ours, and also in the methods used. The polynomial $F(\mathbf{x})$ is of course of a more general type than $H(\mathbf{x})$, the sum of polynomials which we consider. However, if one restricts $F(\mathbf{x})$ to be a sum of polynomials, the work of Bentkus and Götze requires one to assume that there is an irrational ratio among the coefficients of the highest degree terms of these polynomials, rather than our more relaxed assumption that there is an irrational ratio among the coefficients which are not constant terms. Of course, they require many more variables for large k, but this is hardly surprising concerning the more general form of the polynomial $F(\mathbf{x})$.

Most other known results about Diophantine inequalities involving inhomogeneous polynomials concern sums of constant multiples of mixed powers. We consider, for example, the following result due to Brüdern [5], which we state in a simplified form. Suppose that $\lambda_1, \ldots, \lambda_6$ are non-zero real numbers such that at least one of the ratios λ_i/λ_j is irrational. Then for any real number $\varepsilon > 0$ and any real number μ, there are integers $x_1, \ldots, x_6$, not all zero, such that one has

$$|\lambda_1 x_1^2 + \lambda_2 x_2^3 + \lambda_3 x_3^3 + \lambda_4 x_4^3 + \lambda_5 x_5^3 + \lambda_6 x_6^3 - \mu| < \varepsilon.$$

There are other results of a similar nature. (See for example [11], or see also [5] for references to other such results.) Finally, we mention that there are a few other sporadic results which concern inhomogeneous Diophantine inequalities. We direct the reader to the work of Cook and Raghavan [7] and the work of Watson [19], both concerning inequalities involving quadratic polynomials.

There has also been some work on inhomogeneous Diophantine equations. Let $f(x)$ be a polynomial of degree k with integer coefficients which satisfies the property that if d is a positive integer which divides $f(x)$ for all integers x, then $d = 1$. Kamke [14] showed in 1921 that there is an integer s, depending on the polynomial f, such that for sufficiently large n there is an integer solution of the equation

$$f(x_1) + \ldots + f(x_s) = n.$$

See the work of Wooley ([23], Theorem 9), Ford [12] and Nathanson ([15], Sections 11.4 and 12.4) for recent work on this type of problem and references to earlier results.

Now suppose that $F(\mathbf{x})$ is a general polynomial, not necessarily homogeneous, of degree k in s variables. Then we call the sum of the terms of degree 3 the cubic part of the polynomial, and we refer to the sum of the terms of degree 2 as the quadratic part. Watson [20] has given a result about quadratic integral polynomials whose quadratic part is a positive-definite quadratic form, in particular concerning the values the polynomials take at integer points, under certain congruence conditions. Davenport and Lewis [9] have given conditions under which one can solve the Diophantine equation $C(\mathbf{x}) = 0$, where $C(\mathbf{x})$ is a cubic integral polynomial. They require $C(\mathbf{x})$ to satisfy certain congruence conditions, and require certain algebraic restrictions on the cubic part of $C(\mathbf{x})$.

We briefly discuss the methods we use to prove Theorems 1 and 2. We use the Davenport–Heilbronn method, with variations based on the ideas of Bentkus and Götze [2]. We note that we need not use the ideas of Bentkus and Götze to attack the case in which $H(\mathbf{x})$ is an indefinite polynomial. However, we use the method since it also gives us a result in the case in which $H(\mathbf{x})$ is positive-definite. We note that it also allows us to obtain an asymptotic lower bound of a certain kind for the number of solutions in a box of size P for every large positive number P, which the usual Davenport–Heilbronn method does not yield.

As in most applications of the Davenport–Heilbronn method, there are two key ingredients: an analogue of Hua's inequality and an analogue of Weyl's inequality. To obtain our analogue of Hua's inequality, we use a diminishing ranges argument and bounds essentially due to Vinogradov, and some ideas of Davenport and Roth [10]. We must take more care, however, because of the nature of the techniques of Bentkus and Götze. Often, when using the Hardy–Littlewood circle method or the Davenport–Heilbronn method, one considers an exponential sum $T(\alpha)$ of length P, say. Then an analogue of Hua's inequality is usually of the form

$$\int_0^1 |T(\alpha)|^s \, d\alpha \ll P^{s-k+\eta},$$

where η can be taken to be any positive number. When using the techniques of Bentkus and Götze, one instead must be able to replace the right hand side above by simply P^{s-k}. We call such a bound an "exact Hua inequality", as often this is the best possible bound (up to a constant factor) that one can expect. We obtain our exact Hua inequality by first proving a more typical analogue of Hua's inequality, and then applying the Hardy–Littlewood method with some mild twists. Here we also use some techniques of Baker, Hua and Vaughan.

To obtain our analogue of Weyl's inequality, we must also do a bit more because we have chosen to use the techniques of Bentkus and Götze. In fact, as remarked above, it would be possible (in the indefinite case) to not use these techniques; to do so, we would employ a result of Baker (Theorem 5.1 of [1]) as our analogue of Weyl's inequality. However, as with most analogues of (technically, the contrapositive of) Weyl's inequality, Baker's result states that if an exponential sum $T(\alpha)$ is large, then one has good rational approximations to α, but where the quality of these rational approximations is described by bounds which have a factor of P^{ε} in them. We cannot use such results because of these factors. When we have similar results without such factors, we call them "exact Weyl inequalities". In a previous version of the paper, we developed work from [13], using techniques of Schmidt [17], in order to obtain such an exact Weyl inequality. However, Professor Wooley has pointed out a much more straightforward proof, which we give below. This simplifies the proof to a very large extent.

Finally, we note that we could also take $s_0(k) = k2^{k-1} + 1$ in Theorems 1 and 2, and in this case we could even find an asymptotic lower bound of the expected order of magnitude for the number of solutions of our inequality. We briefly note that to do so, one would use a largely similar method but without the diminishing ranges, and one would give some alterations to obtain a different version of Lemma 5. To obtain the required version of Lemma 5, one would only need to treat differently the version of the minor arcs given in the proof of Lemma 5. This may be done by using a slight modification of the result of Baker mentioned above. (See Theorem 5.1 of [1].)

I would like to take this opportunity to thank Professor Schmidt for a conversation during which this question arose. I would also like to thank Professor Wooley for making an important observation which led to a considerable simplification of the proof of Lemma 6. I am grateful to him as well for pointing out that my original results, in which all of the polynomials h_i were of degree k, could be readily extended to yield the current result.

2. A proposition. In this section, we give a technical proposition which implies Theorems 1 and 2. In fact, it will be enough to show that the proposition implies Theorem 2, as we now demonstrate by proving that Theorem 2 implies Theorem 1. This is clear, except for the infinitude of solutions of the inequality (4). Even this part is fairly straightforward, but we prove it for completeness.

To see this, fix a real number κ and a positive number ε. We construct a sequence of distinct integral solutions $\mathbf{z}_n$ of (4) as follows. For positive integers n, let

$$M_n = \kappa + \varepsilon(1 - 2/4^n) \quad \text{and} \quad \varepsilon_n = \varepsilon/4^n. \tag{7}$$

For each $n \in \mathbb{Z}^+$, we apply Theorem 2 with P large enough in terms of ε, k and the coefficients β_{ij} and with $|M_n| \le C_1 P$. In this manner, we have $\mathbf{z}_n \in \mathbb{Z}^s$ with

$$|H(\mathbf{z}_n) - M_n| < \varepsilon_n. \tag{8}$$

Observe from (7) and (8) that

$$H(\mathbf{z}_{n-1}) < \kappa + \varepsilon(1 - 1/4^{n-1}) < \kappa + \varepsilon(1 - 3/4^n) < H(\mathbf{z}_n) \tag{9}$$

and

$$\kappa - \varepsilon \le \kappa + \varepsilon(1 - 3/4^n) < H(\mathbf{z}_n) < \kappa + \varepsilon(1 - 1/4^n) \le \kappa + \varepsilon. \tag{10}$$

By (9), the vectors $\mathbf{z}_n$ are distinct, and by (10), the vectors $\mathbf{z}_n$ are solutions of (4). Thus Theorem 2 implies Theorem 1.

We now state the proposition which implies Theorems 1 and 2. In this section, we demonstrate how the proposition implies Theorem 2. The remainder of the paper is dedicated to the proof of the proposition.

Proposition 1. *Fix a positive number ε. Suppose that k is an integer satisfying $k \geq 2$. Define*

$$\phi = \phi(k) = \bigl(8k^2\bigl(\log k + \tfrac{1}{2}\log\log k + 2\bigr)\bigr)^{-1}. \tag{11}$$

Also, for convenience, set $m = m(k) = 2k+4$. Define t to be the smallest positive integer which satisfies

$$(1-1/k)^t < (2k+1)\phi/k. \tag{12}$$

Now, for $1 \leq i \leq 2t$ and $1 \leq n \leq m$, suppose that v_i and w_n are integers satisfying $2 \leq v_i, w_n \leq k$. As well, for $1 \leq i \leq 2t$ and $1 \leq n \leq m$, suppose that the polynomials

$$f_i(x) = \lambda_{iv_i}x^{v_i} + \lambda_{i(v_i-1)}x^{v_i-1} + \ldots + \lambda_{i1}x,$$
$$g_n(y) = \mu_{nw_n}y^{w_n} + \mu_{n(w_n-1)}y^{w_n-1} + \ldots + \mu_{n1}y$$

have real coefficients, and are of degree v_i and w_n, respectively. Assume that for every j with $2 \leq j \leq k$, there are an even number of the degrees v_i which equal j. Also assume that for some j_1 and j_2 with $1 \leq j_1 \leq w_1$ and $1 \leq j_2 \leq w_2$, we have

$$\mu_{1j_1}/\mu_{2j_2} \notin \mathbb{Q},$$

where of course here we assume that μ_{2j_2} is non-zero. Suppose as well that P is a positive number which is sufficiently large in terms of k and ε and the coefficients λ_{ij} and μ_{nj}. Consider the inequality

$$|f_1(x_1) + \ldots + f_{2t}(x_{2t}) + g_1(y_1) + \ldots + g_m(y_m) - M| < \varepsilon. \tag{13}$$

The following two statements hold.

(i) *Suppose that $\mu_{1w_1} > 0$ and $\mu_{2w_2} < 0$. Suppose in addition that $1 \leq |\mu_{1w_1}/\mu_{2w_2}| \leq 2$. Also suppose that*

$$\sum_{i=1}^{2t}\sum_{j=1}^{v_i}|\lambda_{ij}| + \sum_{n=3}^{m}\sum_{j=1}^{w_n}|\mu_{nj}| \leq \frac{\mu_{1w_1}}{2^{k+3}}. \tag{14}$$

Finally, suppose that M is a real number with

$$|M| \leq \mu_{1w_1}P/8. \tag{15}$$

Then there exists a non-trivial $(2t+m)$-tuple of integers

$$(\mathbf{x},\mathbf{y}) = (x_1,\ldots,x_{2t},y_1,\ldots,y_m) \quad \textit{with } |\mathbf{x}|,|\mathbf{y}| \leq 2P$$

such that (13) *holds.*

(ii) *Suppose that $\mu_{1w_1} > 0$ for $1 \leq i \leq s$ and that M is a real number with*

$$\mu_{1w_1}P/4 \leq M \leq 3\mu_{1w_1}P/4. \tag{16}$$

Additionally, suppose that

$$\sum_{i=1}^{2t}\sum_{j=1}^{v_i}|\lambda_{ij}|+\sum_{n=2}^{m}\sum_{j=1}^{w_n}|\mu_{nj}|\le\frac{\mu_{1w_1}}{2^{k+3}}. \tag{17}$$

Then there exists a non-trivial $(2t+m)$*-tuple of integers*

$$(\mathbf{x},\mathbf{y})=(x_1,\dots,x_{2t},y_1,\dots,y_m)\quad \textit{with } |\mathbf{x}|,|\mathbf{y}|\le 2P$$

such that (13) *holds.*

We note that the numerical constants appearing in conditions (14)–(17) could all doubtlessly be improved, but we do not concern ourselves with the optimal choices for these constants.

We show now that Proposition 1 implies Theorem 2. Note first of all that it is enough to prove Theorem 2 in the case in which the coefficients β_{i0} are all zero. For suppose that we have established the theorem in this special case. We may choose a positive constant C_1', for P large in terms of the coefficients β_{i0}, such that $|M|\le C_1'P$ implies that the required condition $|M-\sum_{i=1}^{s}\beta_{i0}|\le C_1P$ holds. Thus case (i) of Theorem 2, in the special case in which the coefficients are all zero, implies case (i) of Theorem 2 in the general case, albeit with a different constant C_1. In a similar manner, we may deduce case (ii) of Theorem 2 in general, if we have established it in the special case in which all of the coefficients β_{i0} are zero. Thus, in what remains, we assume that all of these constant terms are zero.

Now observe that Theorem 2 holds in the case in which at least one of the polynomials h_i has degree one. To see this, it is enough to show that the theorem holds for a sum of polynomials $\beta_{11}y_1+h_2(y_2)$, where $\beta_{11}y_1$ and h_2 satisfy the irrationality condition, as one may set the variables corresponding to all other polynomials equal to zero, while ensuring that the irrationality condition still holds; after all, the ratios of β_{11} to the other (non-constant term) coefficients cannot all be rational. It follows from the irrationality condition that $g_2(y_2)=h_2(y_2)/\beta_{11}$ has at least one irrational coefficient. By a famous result of Weyl [21], the values of $g_2(y_2)$ are uniformly distributed modulo one for integers y_2. Thus for sufficiently large P, and M with $M\le(\max(C_1,C_3))P$, we may find an integer y_2 with $|y_2|\le c_0P$, for some positive constant c_0, such that

$$\|g_2(y_2)-M/\beta_{11}\|<\varepsilon/|\beta_{11}|;$$

here, as usual, we denote the nearest integer to a real number x by $\|x\|$. For sufficiently small choices of c_0, C_1 and C_3, we may choose $|y_1|\le P$ so that

$$|y_1+g_2(y_2)-M/\beta_{11}|<\varepsilon/|\beta_{11}|.$$

Clearly we then have $|\beta_{11}y_1+h_2(y_2)-M|<\varepsilon$. Observe that in this special case, which includes the case $k=1$ of the theorem, we could take $s_0(k)=2$.

We now turn to the deduction of Theorem 2 in the special case in which all of the polynomials h_i have degree at least 2, and, as remarked above, in which all of their constant terms are zero.

We first observe that, for $k \geq 2$, there exists a function $s_0(k)$ for which

$$2t + m \leq s_0(k) \quad \text{and} \quad s_0(k) \sim 4k \log k.$$

Although this may be seen fairly readily, we give a simple elementary proof for completeness. Observe from calculus that $1 - x \leq e^{-x}$ for all real numbers x, and thus for all real numbers y with $y \geq 1$, we have

$$(1 - 1/y)^y \leq 1/e.$$

For any real number C' with $C' \geq 1$, it follows that if we set

$$w = k(2 \log k + \log C' + \log \log k),$$

we have $(1 - 1/k)^w \leq (C' k^2 \log k)^{-1}$. Thus for a sufficiently large number C', we can see that w satisfies

$$(1 - 1/k)^w < (2k + 1)\phi/k.$$

Now let $t' = \lceil w \rceil$. Note that $w \sim 2k \log k$, whence we certainly have $t' \sim 2k \log k$. But recall that t is the least integer for which (12) holds, whence $t \leq t'$. Recalling as well that $m = 2k + 4$, we see that we may take $2t + m \leq s_0(k)$, where $s_0(k) \sim 4k \log k$. We now assume that s is an integer which satisfies $s \geq s_0(k)$, and thus

$$s \geq 2t + m. \tag{18}$$

Now, for $1 \leq i \leq s$, suppose that the polynomial h_i is of degree d_i, where $2 \leq d_i \leq k$. Note that therefore β_{id_i} is non-zero for $1 \leq i \leq s$. The polynomials $h_1, \ldots, h_s$ satisfy the irrationality condition, so there exist i_1, i_2, j_1 and j_2 with $1 \leq i_1, i_2 \leq s$ and $1 \leq j_1 \leq d_{i_1}$ and $1 \leq j_2 \leq d_{i_2}$ such that $\beta_{i_1 j_1}/\beta_{i_2 j_2}$ is irrational. Suppose that $i_1 = i_2$. Fix some i_0 with $i_0 \neq i_1$ and $1 \leq i_0 \leq s$. Then at least one of $\beta_{i_1 j_1}/\beta_{i_0 d_{i_0}}$ and $\beta_{i_1 j_2}/\beta_{i_0 d_{i_0}}$ must be irrational. In this manner, we can assume that $i_1 \neq i_2$.

Now suppose that we are in the setting of case (i) of Theorem 2, so that $H(\mathbf{y})$ is an indefinite polynomial. We claim that there exists i_0 with $1 \leq i_0 \leq s$ for which $\beta_{i_1 d_{i_1}}/\beta_{i_0 d_{i_0}}$ is negative. In the case in which d_i is even for $1 \leq i \leq s$, this is clear, as not all of the coefficients β_{id_i} can have the same sign. If there exists some i_0 for which d_{i_0} is odd, then by replacing z_{i_0} by $-z_{i_0}$ if necessary, we may assume that $\beta_{i_1 d_{i_1}}/\beta_{i_0 d_{i_0}}$ is negative. Fix this i_0. Observe that we must have $i_0 \neq i_1$.

Now assume as well that $\beta_{i_1 d_{i_1}}/\beta_{i_2 d_{i_2}}$ is positive; note in particular that i_0 then must be distinct from both i_1 and i_2. In this case, both $\beta_{i_1 d_{i_1}}/\beta_{i_0 d_{i_0}}$ and $\beta_{i_2 d_{i_2}}/\beta_{i_0 d_{i_0}}$ are negative, while at least one of $\beta_{i_1 j_1}/\beta_{i_0 d_{i_0}}$ and $\beta_{i_2 j_2}/\beta_{i_0 d_{i_0}}$ is irrational. So without loss of generality, in case (i) of Theorem 2, we may assume that $\beta_{i_1j_1}/\beta_{i_2j_2}$ is irrational and that $\beta_{i_1d_{i_1}}/\beta_{i_2d_{i_2}}$ is negative. By relabeling variables, we may assume that both $\beta_{(2t+1)j_1}/\beta_{(2t+2)j_2} \notin \mathbb{Q}$ for some j_1 and j_2 with $1 \le j_1 \le d_{j_1}$ and $1 \le j_2 \le d_{j_2}$ and also, in case (i) of Theorem 2, that $\beta_{(2t+1)d_{2t+1}} > 0$ and $\beta_{(2t+2)d_{2t+2}} < 0$.

Recall now that by (18) and the definition of m, we have $s \ge 2t+2k+4$. We will now relabel the polynomials $h_1,\dots,h_{2t}$ so that we may assume that, for each j with $2 \le j \le k$, there are an even number of polynomials h_i among $h_1,\dots,h_{2t}$ of degree j. To do so, we consider each j with $2 \le j \le k$. We start with $j=2$, and work our way up to $j=k$. If there is an even number of polynomials of degree j among $h_1,\dots,h_{2t}$, we proceed and consider the polynomials of degree $j+1$. Now suppose that there is an odd number of polynomials of degree j among $h_1,\dots,h_{2t}$. As $2t$ is of course even, there is another degree j' such that there is an odd number of polynomials of degree j' among $h_1,\dots,h_{2t}$. Let one of these polynomials of degree j' be $h_{i_{j'}}$, say. If there is some polynomial of degree j among $h_{2t+3},h_{2t+4},\dots,h_s$, say h_{i_j}, then we switch the indices of $h_{i_{j'}}$ and h_{i_j}. In so doing, we are left with an even number of polynomials of both degrees j' and j among the polynomials $h_1,\dots,h_{2t}$. Now, alternatively, suppose that there is an odd number of polynomials of degree j among $h_1,\dots,h_{2t}$ but that none of the polynomials $h_{2t+3},\dots,h_s$ have degree j. Then, as $s \ge 2t+2k+4$, there is certainly at least one degree j'' with $2 \le j'' \le k$ for which there are at least two polynomials h_i of degree j'' among $h_{2t+3},\dots,h_s$. We exchange these two polynomials with a polynomial of degree j and a polynomial of degree j', each among $h_1,\dots,h_{2t}$, where $j' \ne j$ is such that there is an odd number of polynomials of degree j' among $h_1,\dots,h_{2t}$. Then among $h_1,\dots,h_{2t}$, we are left with an even number of polynomials of degrees j and j', and the parity of the number of polynomials of degree j'' is unchanged. Now we may proceed to consider the polynomials of degree $j+1$, repeating this step until we reach the degree $j=k$. In this manner, we are left with an even number of polynomials of every degree j with $2 \le j \le k$, among the polynomials $h_1,\dots,h_{2t}$.

We now define sets of polynomials $f_1,\dots,f_{2t}$ and $g_1,\dots,g_m$ such that if

$$|f_1(x_1)+\ldots+f_{2t}(x_{2t})+g_1(y_1)+\ldots+g_m(y_m)-M| < \varepsilon \tag{19}$$

has an integral solution $(\mathbf{x},\mathbf{y})$, then there is a corresponding integral solution of

$$|H(\mathbf{z})-M| < \varepsilon. \tag{20}$$

We will then apply Proposition 1 to the inequality (19), and thus show that (20) has an integral solution. Our polynomials f_i and g_n are similar in both cases (i) and (ii) of Theorem 2, although with slight differences.

Now recall that the polynomials h_i are of degree d_i for $1 \le i \le s$. We define

$$v_i = d_i \quad \text{for } 1 \le i \le 2t \quad \text{and} \quad w_n = d_{2t+n} \quad \text{for } 1 \le n \le m.$$

In both cases (i) and (ii) of Theorem 2, let c_1 and c_2 be positive integers, to be chosen later, and set

$$\lambda_{ij} = \beta_{ij} \quad \text{for } 1 \le i \le 2t \text{ and } 1 \le j \le v_i, \tag{21}$$

while, for $1 \le n \le m$ and $1 \le j \le w_n$, set

$$\mu_{nj} = \begin{cases} c_1^j \beta_{(2t+1)j} & \text{for } n = 1, \\ c_2^j \beta_{(2t+2)j} & \text{for } n = 2, \\ \beta_{(2t+n)j} & \text{for } 3 \le n \le m. \end{cases} \tag{22}$$

We then define, for $1 \le i \le 2t$ and $1 \le n \le m$, the polynomials $f_i(x)$ and $g_n(y)$ by

$$f_i(x) = \lambda_{iv_i} x^{v_i} + \lambda_{i(v_i-1)} x^{v_i-1} + \ldots + \lambda_{i1} x,$$
$$g_n(y) = \mu_{nw_n} y^{w_n} + \mu_{n(w_n-1)} y^{w_n-1} + \ldots + \mu_{n1} y.$$

Note that the polynomials f_i are of degree v_i for $1 \le i \le 2t$ and the polynomials g_n are of degree w_n for $1 \le n \le m$, that is, their leading coefficients are non-zero.

Now, in case (i) of Theorem 2, we choose large positive integers c_1' and c_2' so that

$$1 \le \frac{(c_1')^{w_1}}{(c_2')^{w_2}} \left| \frac{\beta_{(2t+1)w_1}}{\beta_{(2t+2)w_2}} \right| \le 2; \tag{23}$$

to do so, first find a positive number L that is large enough so that $(L+1)^{w_1} \le 2L^{w_1}$. Then choose c_2' to be a positive integer which is large enough so that

$$(c_2')^{w_2} \left| \frac{\beta_{(2t+2)w_2}}{\beta_{(2t+1)w_1}} \right| \ge L^{w_1}.$$

By our choice of L, we may choose a positive integer c_1' with $c_1' \ge L$ for which (23) holds. Now set $c_1 = uc_1'$ and $c_2 = uc_2'$, where u is a positive integer. Recalling the definitions (21) and (22), we may ensure that (14) holds by choosing u sufficiently large. Note that by (22) and (23), we have $1 \le |\mu_{1w_1}/\mu_{2w_2}| \le 2$.

In case (ii) of Theorem 2, our choices are easier. We simply set $c_2 = 1$, and, recalling the definitions (21) and (22), we see that by choosing c_1 sufficiently large, we may ensure that (17) holds.

Note that in either case (i) or (ii), by (22), we have

$$\frac{\mu_{1j_1}}{\mu_{2j_2}} = \frac{c_1^{j_1} \beta_{(2t+1)j_1}}{c_2^{j_2} \beta_{(2t+2)j_2}},$$

which is irrational as c_1 and c_2 are positive integers. Also, recall that we have assumed that $\beta_{(2t+1)d_{2t+1}} > 0$ and $\beta_{(2t+2)d_{2t+2}} < 0$ in case (i) of Theorem 2.

In case (ii) of Theorem 2, we must have $\beta_{(2t+1)d_{2t+1}} > 0$, as the polynomial $H(\mathbf{y})$ is positive-definite. It follows from the fact that c_1 and c_2 are positive integers, and the definition (22), that $\mu_{1w_1} > 0$ and $\mu_{2w_2} < 0$ in case (i) of Theorem 2, and that $\mu_{1w_1} > 0$ in case (ii) of Theorem 2.

Now we apply cases (i) and (ii) of Proposition 1 to deduce, respectively, cases (i) and (ii) of Theorem 2. Note that our work above shows that the set of polynomials $f_1, \ldots, f_{2t}, g_1, \ldots, g_m$ satisfy the respective assumptions of each case of the proposition. Thus, setting $P' = P/(2c_1c_2)$, and applying Proposition 1 for large P', we may see that for $|M| \le \mu_{1w_1}P'/8$ in case (i), and for $\mu_{1w_1}P'/4 \le M \le 3\mu_{1w_1}P'/4$ in case (ii), we may find an integral solution $(\mathbf{x}, \mathbf{y})$ of the inequality

$$\Big|\sum_{i=1}^{2t} f_i(x_i) + \sum_{n=1}^{m} g_n(y_n) - M\Big| < \varepsilon.$$

Then set

$$z_i = \begin{cases} x_i & \text{for } 1 \le i \le 2t, \\ c_1y_1 & \text{for } i = 2t+1, \\ c_2y_2 & \text{for } i = 2t+2, \\ y_{i-2t} & \text{for } 2t+3 \le i \le 2t+m, \\ 0 & \text{for } i \ge 2t+m+1. \end{cases} \tag{24}$$

(Observe that here we have again implicitly used the condition $s \ge s_0(k) \ge 2t+m$.) By (21), (22) and (24), noting that c_1 and c_2 are integers, we find that $\mathbf{z}$ is an integral solution of $|H(\mathbf{z}) - M| < \varepsilon$. Note also that if $(\mathbf{x}, \mathbf{y})$ is a non-trivial solution, then by (24), $\mathbf{z}$ is a non-trivial solution as well. Finally observe that because $|\mathbf{x}|, |\mathbf{y}| \le 2P'$, the condition $|\mathbf{z}| \le P$ follows from our choice of P'.

This completes the proof that Proposition 1 implies Theorem 2. As we have remarked, Proposition 1 thus also implies Theorem 1.

3. The Davenport–Heilbronn method. We now start the proof of Proposition 1, which comprises the rest of the paper. We use the Davenport–Heilbronn method, and in addition some of the recent ideas of Bentkus and Götze [2]. We also rely heavily on the methods of Davenport and Roth [10].

Before we begin, we need some standard notation. For real numbers x, we set

$$e(x) = e^{2\pi i x}.$$

At this point, we also take a moment to note that throughout the proof, all sums run only over integers. Also, implicit constants in the notations $\ll$, $\gg$, $o()$ and $O()$ may depend throughout on k, t, ε and the coefficients λ_{ij} and μ_{nj}.

As is usual with the method, we need a special kernel function which allows us to give a lower bound for the number of solutions of the inequality (13). We have the following lemma, essentially due to Davenport and Heilbronn. (See Lemma 4 of [8].)

LEMMA 1 (Davenport and Heilbronn). *Fix a positive number η. Then for any real number α, define*

$$K(\alpha)=\frac{(\sin(\pi\eta\alpha))^2}{\pi^2\alpha^2\eta}.$$

Observe that $K(\alpha)$ is a real-valued function which is positive and even. $K(\alpha)$ satisfies

$$\int_{\mathbb{R}} e(\alpha t)K(\alpha)\,d\alpha=\max\left(0,1-\left|\frac{t}{\eta}\right|\right) \tag{25}$$

for all real numbers t. Also, $K(\alpha)$ satisfies the bound

$$K(\alpha)\ll_\eta \min(1,|\alpha|^{-2}). \tag{26}$$

Proof. The lemma can be deduced from the original result of Davenport and Heilbronn by a trivial change of variable.

For the remainder of the paper, we fix K as in the lemma with the choice $\eta=\varepsilon$. Also, let P be a large positive number; we shall require it to be sufficiently large at various points throughout the proof.

We now relabel the polynomials $f_1,\ldots,f_{2t}$ to make our notation more convenient. Suppose that for each integer j satisfying $2\le j\le k$, there are $2a_j$ polynomials of degree j, where a_j is a non-negative integer. We assume, by relabeling, that the first a_k polynomials $f_1,\ldots,f_{a_k}$ are of degree k, the next a_{k-1} are of degree $k-1$ and so on, so that finally the polynomials $f_{t-a_2+1},\ldots,f_t$ are of degree 2. We relabel the polynomials $f_{t+1},\ldots,f_{2t}$ similarly, whence the degree of f_{t+i} is the same as the degree of f_i for $1\le i\le t$. Observe that $a_k+a_{k-1}+\ldots+a_2=t$.

Now, for $2\le j\le k$, define

$$L_j=a_k+a_{k-1}+\ldots+a_{j+1}.$$

Observe that the polynomials f_i with $L_j+1\le i\le L_j+a_j$ are the polynomials of degree j among $f_1,\ldots,f_t$, if there are any at all, i.e., if a_j is positive. Now we define exponents κ_i for $1\le i\le t$, as follows. For i satisfying $L_j+1\le i\le L_j+a_j$, we define

$$\kappa_i=\kappa_{t+i}=\frac{1}{j-1}\left(\frac{j-1}{j}\right)^{i-L_j}\prod_{l=j+1}^{k}\left(\frac{l-1}{l}\right)^{a_l}. \tag{27}$$

Observe that we of course only define such exponents if a_j is positive, and the product above is of course understood to be one in the case when $j=k$. Essentially, these are the natural exponents to be used in a diminishing ranges argument involving polynomials of different degrees. Also, for

$1 \le i \le t$, define

$$P_i = P^{\kappa_i}. \tag{28}$$

Now we collect some observations about the exponents κ_i and the numbers P_i which will be important for future reference. Note from (27) that for $L_j + 1 \le i \le L_j + a_j$, recalling that v_i is the degree of the polynomial f_i, we have

$$\kappa_i v_i \le \frac{j}{j-1}\left(\frac{j-1}{j}\right)^{i-L_j} \le 1.$$

It follows that

$$P_i^{v_i} \le P \quad \text{for } 1 \le i \le t. \tag{29}$$

Observe that for $L_j + 1 \le i \le L_j + a_j - 1$, we have $\kappa_{i+1} = \kappa_i((j-1)/j)$. Note that in this case, $v_{i+1} = v_i = j$, so $\kappa_{i+1}v_{i+1} = \kappa_i(v_i - 1)$. For $i = L_j + a_j$, assuming that $1 \le i \le t-1$ and that r is the largest integer less than j for which $a_r > 0$, we have $\kappa_{i+1} = \kappa_i((j-1)/r)$. In this case, $v_i = j$, whereas $v_{i+1} = r$, so $\kappa_{i+1}v_{i+1} = \kappa_i(v_i - 1)$ again. Thus

$$\kappa_{i+1}v_{i+1} = \kappa_i(v_i - 1) \quad \text{for } 1 \le i \le t-1.$$

By the definition (28) of P_i, we therefore have

$$P_{i+1}^{v_{i+1}} = P_i^{v_i - 1} \quad \text{for } 1 \le i \le t-1. \tag{30}$$

Finally, we consider the sum $\sum_{i=1}^{t} \kappa_i$. One obtains

$$\begin{aligned}
\sum_{i=1}^{t} \kappa_i &= \sum_{j=2}^{k} \frac{1}{j-1}\left(\prod_{l=j+1}^{k}\left(\frac{l-1}{l}\right)^{a_l}\right)\sum_{i=1}^{a_j}\left(\frac{j-1}{j}\right)^{i} \\
&= \sum_{j=2}^{k} \frac{1}{j-1}\left(\prod_{l=j+1}^{k}\left(\frac{l-1}{l}\right)^{a_l}\right)(j-1)\left(1-\left(\frac{j-1}{j}\right)^{a_j}\right) \\
&= \sum_{j=2}^{k}\left(\prod_{l=j+1}^{k}\left(\frac{l-1}{l}\right)^{a_l} - \prod_{l=j}^{k}\left(\frac{l-1}{l}\right)^{a_l}\right) \\
&= \left(1-\prod_{l=2}^{k}\left(\frac{l-1}{l}\right)^{a_l}\right).
\end{aligned}$$

From this calculation, we clearly have

$$\sum_{i=1}^{t} \kappa_i = \left(1-\prod_{l=2}^{k}\left(\frac{l-1}{l}\right)^{a_l}\right) \ge \left(1-\left(\frac{k-1}{k}\right)^{t}\right).$$

Therefore from (12) one obtains

$$\sum_{i=1}^{t} \kappa_i > 1 - \frac{(2k+1)\phi}{k}. \tag{31}$$

Now we define the generating functions we shall use. For $1 \le i \le t$ and any real number α, we define

$$\begin{aligned} S_i(\alpha) = S_i(\alpha, P) &= \sum_{P_i \le x \le 2P_i} e(\alpha f_i(x)), \\ S_{t+i}(\alpha) = S_{t+i}(\alpha, P) &= \sum_{P_i \le x \le 2P_i} e(\alpha f_{t+i}(x)). \end{aligned} \tag{32}$$

Also, for any real numbers α and N, and any polynomial $g(x)$, we set

$$U(\alpha, g) = U(\alpha, g, N) = \sum_{1 \le x \le N} e(\alpha g(x)). \tag{33}$$

Recall that w_n is the degree of the polynomial g_n. For $1 \le n \le m$ and any real number α, we define

$$U_n(\alpha) = U_n(\alpha, P) = U(\alpha, g_n, P^{1/w_n}). \tag{34}$$

Also define

$$W = \sum_{n=1}^{m} \frac{1}{w_n}. \tag{35}$$

Note that $1/w_n \ge 1/k$ and $m = 2k + 4$, whence clearly

$$W \ge (2k + 4)/k. \tag{36}$$

Considering all of the above definitions and the identity (25), one may see in a standard manner that the number of integral solutions $(\mathbf{x}, \mathbf{y})$ of the inequality (13) with $|\mathbf{x}| \le 2P$ and $|\mathbf{y}| \le 2P$ is at least

$$\int_{\mathbb{R}} \Big(\prod_{i=1}^{2t} S_i(\alpha) \Big) \Big(\prod_{n=1}^{m} U_n(\alpha) \Big) e(-M\alpha) K(\alpha) \, d\alpha. \tag{37}$$

We will show that in fact this integral is $\gg (P_1 \dots P_t)^2 P^{W-1}$ for large P, which in turn certainly shows that there is a non-trivial solution of the inequality (13). So for all large P, we will in fact have an asymptotic lower bound of this type for the number of solutions of the inequality (13); this is one of the benefits of using the techniques of Bentkus and Götze. With the standard Davenport–Heilbronn method, we would only obtain such an asymptotic lower bound for a sequence of large P tending to infinity.

As is usual, our strategy will be to give a dissection of the real line into three regions, and consider the contributions to the integral (37) from each of the three regions. For the remainder of the paper, we fix a positive number δ satisfying

$$0 < \delta < 1/k. \tag{38}$$

We could fix a value of δ for definiteness if we so desired, but we choose not to. We then define the region $\mathcal{M}$, which we call the *major arc*, by setting

$$\mathcal{M} = \{\alpha \in \mathbb{R} : |\alpha| \leq P^{\delta-1}\}. \tag{39}$$

In Section 7, we will show that the contribution to the integral (37) from the major arc is $\gg (P_1 \dots P_t)^2 P^{W-1}$ for large P. During the course of the proof, we shall define a function $T(P)$ which depends only on the coefficients μ_{nj}, and tends to infinity as P tends to infinity. Using this function, we define the so-called *minor arcs* $\mathfrak{m}$ by

$$\mathfrak{m} = \{\alpha \in \mathbb{R} : P^{\delta-1} < |\alpha| \leq T(P)\}. \tag{40}$$

This will be the most difficult region to treat. Finally, we define the so-called *trivial arcs* $\mathfrak{t}$ by

$$\mathfrak{t} = \{\alpha \in \mathbb{R} : |\alpha| > T(P)\}. \tag{41}$$

We will show in Section 6 that the contribution to the integral (37) from each of the last two regions is $o((P_1 \dots P_t)^2 P^{W-1})$. Combined with our treatment of the major arc, this will show that the integral (37) is $\gg (P_1 \dots P_t)^2 P^{W-1}$ for large P. We now proceed to the treatment of the minor arcs and trivial arcs, by establishing an analogue of Hua's inequality.

4. An analogue of Hua's inequality. We first give a version of a lemma essentially proved originally by Vinogradov. We state the lemma in a form very close to that given by Baker. (See Theorem 4.4 of [1].)

LEMMA 2. *Fix a positive number η. Suppose that k is an integer satisfying $k \geq 2$. Let $J(k) = 8k^2(\log k + (1/2)\log\log k + 2)$. Suppose that N is a positive number which is sufficiently large in terms of k. For real numbers $\alpha_1, \dots, \alpha_k$, let $f(x) = \alpha_k x^k + \alpha_{k-1}x^{k-1} + \dots + \alpha_1 x$. Suppose also that*

$$\Big|\sum_{1 \leq x \leq N} e(f(x))\Big| \geq \gamma N,$$

where γ is a positive number satisfying

$$\gamma \geq N^{-1/J(k)}. \tag{42}$$

Then there are integers $y, u_1, \dots, u_k$ with

$$1 \leq y \leq \gamma^{-k} N^{\eta} \quad \text{and} \quad |y\alpha_j - u_j| \leq \gamma^{-k} N^{\eta - j} \quad \text{for } 1 \leq j \leq k,$$

and

$$(y, u_k, u_{k-1}, \dots, u_2) \leq 2k^2, \qquad (y, u_k, u_{k-1}, \dots, u_1) = 1.$$

The lemma can be obtained by adjustments of the proof of the case $M = 1$ of Theorem 4.4 of [1]; we give a proof for the sake of completeness, although we will be brief. We also note that the bounds of the lemma could be improved by combining the proof with the work of Wooley [22], but a superior version of this type does not yield significant improvement of our final results.

Proof of Lemma 2. We first consider the case $k \geq 4$. As in the proof of Theorem 4.4 of [1], we choose $l = [k \log(4k^2 \log k)] + 1$. One has $J(k) > 4kl(1-2\theta)^{-1}$ exactly as in that proof, where

$$\theta = \theta(k) = \frac{1}{2}(k-1)^2\left(\frac{k-2}{k-1}\right)^l.$$

Set $A = \gamma N$. Then it follows from (42) that

$$(NA^{-1})^{4(k-1)l} \ll N^{4(k-1)l/J(k)} \ll N^{1-2\theta-4\eta} \tag{43}$$

for sufficiently small η.

Now consider the cases $k = 2$ and $k = 3$. We set $l = 3$ in these cases, whence $\theta(2) = 0$ and $\theta(3) = 1/4$, where $\theta(k)$ is defined as above. We set $A = \gamma N$. A calculation reveals that $J(2) \geq 80$ and $J(3) \geq 226$. Thus, from (42), one can check that the condition (43) also holds in these cases.

Now we consider all of the cases $k \geq 2$ simultaneously. As (43) holds for $k \geq 2$, we can apply Theorem 4.3 of [1] with $A = \gamma N$ for large N. It follows that there are coprime pairs of integers q_j and a_j for $2 \leq j \leq k$ such that

$$q_j \geq 1 \quad \text{and} \quad |q_j\alpha_j - a_j| \leq N^{\eta-j+\theta}(NA^{-1})^{2(k-1)l} \qquad \text{for } 2 \leq j \leq k,$$

and such that the least common multiple q_0 of $q_2, \ldots, q_k$ satisfies

$$q_0 \leq (NA^{-1})^{2(k-1)l} N^{\theta+\eta}.$$

As the condition (43) holds for $k \geq 2$, it follows that for sufficiently small η we have

$$1 \leq q_0 \leq N^{1-2\eta},$$

and that there are integers b_j for $2 \leq j \leq k$ such that

$$|q_0\alpha_j - b_j| \ll N^{1-j-2\eta} \qquad \text{for } 2 \leq j \leq k;$$

in fact, one simply chooses $b_j = q_0 a_j/q_j$ for $2 \leq j \leq k$. We note that by a simple argument one also has

$$(q_0, b_k, \ldots, b_2) = 1.$$

Now, following the proof of Theorem 4.4 of [1], for large N we may apply Lemma 4.6 of [1] with $r = q_0$ and $H = A$, noting that $d = 1$. We then set $y = tq_0$, and $u_j = tb_j$ for $2 \leq j \leq k$, and let u_1 be the integer closest to $tq_0\alpha_1$. If necessary, we can remove a factor so that we also have $(y, u_k, u_{k-1}, \ldots, u_1) = 1$. This completes the proof of Lemma 2.

We now need a generalization of a lemma due to Davenport and Roth. (See Lemma 3 of [10].) We give a proof for completeness, although our proof is similar to theirs.

LEMMA 3. *Suppose that we are in the setting of Proposition* 1, *and that* $S_i(\alpha)$ *is defined as in* (32). *Then*

$$\int_{\mathbb{R}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big) K(\alpha)\, d\alpha \ll P_1 \ldots P_t. \tag{44}$$

Proof. We first use the Cauchy–Schwarz inequality, bearing in mind that K is positive, whence we see that the left hand side of (44) is

$$\ll \Big(\int_{\mathbb{R}} \Big(\prod_{i=1}^{t} |S_i(\alpha)|^2\Big) K(\alpha)\, d\alpha\Big)^{1/2} \Big(\int_{\mathbb{R}} \Big(\prod_{i=t+1}^{2t} |S_i(\alpha)|^2\Big) K(\alpha)\, d\alpha\Big)^{1/2}. \tag{45}$$

We shall show that

$$\int_{\mathbb{R}} \Big(\prod_{i=1}^{t} |S_i(\alpha)|^2\Big) K(\alpha)\, d\alpha \ll P_1 \ldots P_t. \tag{46}$$

A similar bound holds for the other integral in (45). We omit its proof as the method is the same.

Using the identity (25) and the definition (32), one can see that the integral in (46) is less than or equal to the number of integral solutions $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_t, y_1, \ldots, y_t)$ of the inequality

$$|f_1(x_1) - f_1(y_1) + f_2(x_2) - f_2(y_2) + \ldots + f_t(x_t) - f_t(y_t)| < \varepsilon, \tag{47}$$

where one has $P_i \le x_i, y_i \le 2P_i$ for $1 \le i \le t$. We bound the number of such solutions by induction.

First, we make a general observation. If x and y are positive numbers with $x > y$ and l is a positive integer, then

$$x^l - y^l \ge l(x-y)y^{l-1}. \tag{48}$$

Also, recall from (30) that $P_{i+1}^{v_{i+1}} = P_i^{v_i - 1}$ for $1 \le i \le t-1$. It follows by induction that given an integer i_0 with $1 \le i_0 \le t-1$, we have

$$P_i^{v_i} \le P_{i_0}^{v_{i_0} - 1} \quad \text{for } i_0 < i \le t. \tag{49}$$

Now fix any choice of y_1 which satisfies $P_1 \le y_1 \le 2P_1$. Suppose that there are integers $x_1, x_2, \ldots, x_t, y_2, \ldots, y_t$, with $P_i \le x_i, y_i \le 2P_i$ for $2 \le i \le t$ and $P_1 \le x_1 \le 2P_1$, such that $(\mathbf{x}, \mathbf{y})$ is a solution of (47). By (49), one has

$$\Big|\sum_{i=2}^{t} (f_i(x_i) - f_i(y_i))\Big| \ll P_1^{v_1 - 1}; \tag{50}$$

here the implied constant in Vinogradov's notation may depend on the coefficients λ_{ij}, but not on P. Now we also have $|f_1(x_1) - \lambda_{1v_1} x_1^{v_1}| \ll P_1^{v_1-1}$ and $|f_1(y_1) - \lambda_{1v_1} y_1^{v_1}| \ll P_1^{v_1-1}$, whence

$$|\lambda_{1v_1}(x_1^{v_1} - y_1^{v_1})| \ll P_1^{v_1-1}.$$

As $y_1 \ge P_1$ and $x_1 \ge P_1$ and λ_{1v_1} is non-zero, by the observation (48) there are only finitely many possible choices x_1, given this fixed choice of y_1, for which the vector $(x_1,\dots,x_t,y_1,\dots,y_t)$ could possibly be a solution of the inequality (47). So there are $\ll P_1$ choices of the integral vector (x_1,y_1), with $P_1\le x_1,y_1\le 2P_1$, which could extend to an integral solution $(x_1,\dots,x_t,y_1,\dots,y_t)$ of (47), with $P_i\le x_i,y_i\le 2P_i$ for $1\le i\le t$.

Now assume that for some j with $2\le j\le t$, there are $\ll P_1\dots P_{j-1}$ choices of the integral vector $(x_1,\dots,x_{j-1},y_1,\dots,y_{j-1})$, with $P_i\le x_i,y_i\le 2P_i$ for $1\le i\le j-1$, which could extend to an integral solution $(x_1,\dots,x_{j-1},\dots,x_t,y_1,\dots,y_{j-1},\dots,y_t)$ of (47), with $P_i\le x_i,y_i\le 2P_i$ for $1\le i\le t$. Note this does indeed hold in the case $j=2$, as we have demonstrated. We aim of course to establish this result for the case $j+1$.

To this end, fix a choice $(x_1,\dots,x_{j-1},y_1,\dots,y_{j-1})$ as above. Also fix a choice of y_j with $P_j\le y_j\le 2P_j$. Then suppose that the vector given by $(x_1,\dots,x_j,\dots,x_t,y_1,\dots,y_j,\dots,y_t)$ is a solution of the inequality (47) with $P_i\le x_i,y_i\le 2P_i$ for $j+1\le i\le t$ and with $P_j\le x_j\le 2P_j$. Then, as above,

$$\Big|\lambda_{jv_j}(x_j^{v_j}-y_j^{v_j})+\sum_{i=1}^{j-1}(f_i(x_i)-f_i(y_i))\Big|\ll P_j^{v_j-1}.$$

By (48), the integers z^{v_j} with $P_j\le z\le 2P_j$ are spaced apart by at least some constant multiple of $P_j^{v_j-1}$, whence, because λ_{jv_j} is non-zero, for a fixed choice of y_j, there are finitely many choices of x_j for which the last bound could possibly hold. Thus there are $\ll P_1\dots P_j$ possible choices of the $(2j)$-tuple $(x_1,\dots,x_j,y_1,\dots,y_j)$ which could possibly extend to a solution $(\mathbf{x},\mathbf{y})\in\mathbb{Z}^{2t}$ of the inequality (47) with the constraint $P_i\le x_i,y_i\le 2P_i$ on the variables.

Therefore by induction we obtain the desired bound (46) and hence complete the proof of Lemma 3.

We now quote a lemma. (See Lemma 4.4 of [1].)

LEMMA 4 (Baker). *Fix a positive number η. Suppose that k is an integer with $k\ge 2$ and that N is a real number with $N\ge 1$. For real numbers $\alpha_1,\dots,\alpha_k$, let $f(x)=\alpha_kx^k+\dots+\alpha_1x$ and suppose that $q,a_1,\dots,a_k$ are integers with*

$$|q\alpha_j-a_j|\le\frac{N^{1-j}}{2k^2}\quad\textit{for } 1\le j\le k.$$

Set $d=(q,a_k,\dots,a_2)$ and write

$$\beta_j=\alpha_j-\frac{a_j}{q}\quad\textit{for } 1\le j\le k,\qquad g(x)=\sum_{j=1}^k\beta_jx^j,$$

$$G(x)=\sum_{j=1}^k a_jx^j,\qquad S(q)=\sum_{v=1}^q e\Big(\frac{G(v)}{q}\Big).$$

Then

$$\sum_{x=1}^{T} e(f(x)) = \frac{S(q)}{q} \int_0^T e(g(y))\,dy + O(q^{1-1/k+\eta} d^{1/k})$$

for all real numbers T with $1 \le T \le N$.

Note that Baker only states the above lemma for integers T, but the result clearly extends to real numbers T by absorbing a number of size at most 1 on each side into the error term.

We can now give the central lemma of this section. It is our version of Hua's inequality.

LEMMA 5. *Suppose that we are in the setting of Proposition* 1*. Define the functions $S_i(\alpha)$ as in* (32) *and the functions $U(\alpha, g)$ as in* (33)*. Also let $K(\alpha)$ be as in Lemma* 1 *with $\eta = \varepsilon$. Suppose that r is an integer satisfying $r \ge 2k+1$. Then for $1 \le n \le r$, let w_n be an integer satisfying $2 \le w_n \le k$ and let*

$$h_n(x) = \gamma_{n w_n} x^{w_n} + \gamma_{n(w_n-1)} x^{w_n-1} + \ldots + \gamma_{n1} x$$

be a real polynomial with zero constant term, where $\gamma_{n w_n} \ne 0$ for $1 \le n \le r$. Let

$$V = \sum_{n=1}^{r} \frac{1}{w_n}.$$

Then

$$\int_{\mathbb{R}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big)\Big(\prod_{n=1}^{r} |U(\alpha, h_n, P^{1/w_n})|\Big) K(\alpha)\,d\alpha \ll (P_1 \ldots P_t)^2 P^{V-1}. \tag{51}$$

We note that for many applications of such an inequality, a weaker bound of the type $(P_1 \ldots P_t)^2 P^{V-1+\eta}$, where this holds for any small positive number η, would be sufficient. Such a bound does not suffice for our purposes however, since we are using some of the techniques of Bentkus and Götze.

Proof of Lemma 5. We essentially use the Hardy–Littlewood method, in a way which is not exactly standard. Thus we shall give the whole proof.

Observe that

$$\sum_{n=1}^{r} \frac{1}{V w_n} = 1.$$

Thus by Hölder's inequality, and the fact that $K(\alpha)$ is positive, we may see that the integral in (51) satisfies

$$\begin{aligned}&\int_{\mathbb{R}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big)\Big(\prod_{n=1}^{r} |U(\alpha, h_n, P^{1/w_n})|\Big) K(\alpha)\,d\alpha \\ &\qquad \ll \prod_{n=1}^{r} \Big(\int_{\mathbb{R}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big) |U(\alpha, h_n, P^{1/w_n})|^{V w_n} K(\alpha)\,d\alpha\Big)^{1/(V w_n)}.\end{aligned}$$

Therefore, in order to prove the lemma, it suffices to show that

$$(52)\qquad \int_{\mathbb{R}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big) |U(\alpha, h_n, P^{1/w_n})|^{Vw_n} K(\alpha)\, d\alpha \ll (P_1 \dots P_t)^2 P^{V-1}$$

for $1 \le n \le r$.

We thus fix n with $1 \le n \le r$ and prove (52). Fix as well a positive number η, which we shall later choose to be sufficiently small. For convenience of notation throughout the proof, we set

$$u = w_n.$$

Recall the definition (11) of ϕ. We now give some other definitions. For integers q and $a_1, \dots, a_u$ with $q \ge 1$, we write $\mathbf{a} = (a_1, \dots, a_u)$ and define

$$\mathcal{N}(q, \mathbf{a}) = \{\alpha \in \mathbb{R} : |\gamma_{nj} \alpha q - a_j| < P^{\phi - j/u} \text{ for } 1 \le j \le u\}.$$

For any integers q and a_u with $q \ge 1$, we define

$$\mathcal{N}_0(q, a_u) = \{\alpha \in \mathbb{R} : |\gamma_{nu} \alpha q - a_u| < P^{\phi - 1}\}.$$

Now, set

$$\mathcal{N} = \bigcup_{1 \le q \le P^{\phi}} \bigcup_{\substack{\mathbf{a} : (q, a_u, \dots, a_1) = 1 \\ (q, a_u, \dots, a_2) \le 2u^2}} \mathcal{N}(q, \mathbf{a}).$$

For the purposes of this lemma, we think of $\mathcal{N}$ as a version of the major arcs. Now we define a corresponding version of the minor arcs, namely the set

$$\mathfrak{n} = \mathbb{R} \setminus \mathcal{N}.$$

We show first that

$$(53)\qquad \int_{\mathfrak{n}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big) |U(\alpha, h_n, P^{1/u})|^{uV} K(\alpha)\, d\alpha \ll (P_1 \dots P_t)^2 P^{V-1}.$$

In fact, we show that the left side is strictly smaller in order than the right side, although this is not needed.

Suppose now that $\alpha \in \mathfrak{n}$, and that $|U(\alpha, h_n, P^{1/u})| \ge P^{1/u - \phi/u + \eta}$. We can then apply Lemma 2 with $N = P^{1/u}$, with $\gamma = P^{-\phi/u + \eta}$ and $k = u$, and with $\alpha_j = \alpha \gamma_{nj}$. Note that we have assumed $u \ge 2$. We obtain integers $q, a_1, \dots, a_u$ with

$$1 \le q \le P^{\phi}, \qquad |\gamma_{nj} \alpha q - a_j| < P^{\phi - j/u} \quad \text{for } 1 \le j \le u,$$

and

$$(q, a_u, \dots, a_2) \le 2u^2, \qquad (q, a_u, \dots, a_1) = 1.$$

But then $\alpha \in \mathcal{N}$, which contradicts the assumption $\alpha \in \mathfrak{n}$. Therefore

$$|U(\alpha, h_n, P^{1/u})| \le P^{1/u - \phi/u + \eta} \qquad \text{for } \alpha \in \mathfrak{n}.$$

It follows from Lemma 3 that

$$\int_{\mathfrak{n}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big) |U(\alpha, h_n, P^{1/u})|^{uV} K(\alpha)\, d\alpha \tag{54}$$
$$\ll \sup_{\alpha\in\mathfrak{n}} |U(\alpha, h_n, P^{1/u})|^{uV} \int_{\mathfrak{n}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big) K(\alpha)\, d\alpha$$
$$\ll P^{V-V\phi+uV\eta} P_1 \ldots P_t.$$

Now, by (28), we have

$$P^{V-V\phi+uV\eta} P_1 \ldots P_t = P^{\Gamma} (P_1 \ldots P_t)^2, \tag{55}$$

where

$$\Gamma = V - V\phi + uV\eta - \sum_{i=1}^{t} \kappa_i.$$

By (31), for sufficiently small η we have

$$\Gamma < V - V\phi - 1 + (2k+1)\phi/k.$$

Observe that

$$V = \sum_{n=1}^{r} \frac{1}{w_n} \ge \sum_{n=1}^{r} \frac{1}{k} \ge \frac{2k+1}{k}. \tag{56}$$

Thus

$$\Gamma < V - 1.$$

By (54) and (55), the bound (53) follows.

To obtain the bound (52) and thus to finish the proof of the lemma, it remains only to prove that

$$\int_{\mathcal{N}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big) |U(\alpha, h_n, P^{1/u})|^{uV} K(\alpha)\, d\alpha \ll (P_1 \ldots P_t)^2 P^{V-1}. \tag{57}$$

We make a few observations which enable us to prove this bound.

First, suppose that $\alpha \in \mathcal{N}(q, \mathbf{a}) \subseteq \mathcal{N}$. We may see from (11) that $u\phi \le k\phi < 1$. Moreover, P is large, so we may apply Lemma 4 with $k = u$ and $N = P^{1/u}$, and thus we obtain

$$U(\alpha, h_n, P^{1/u}) = \frac{S(q)}{q} \int_0^{P^{1/u}} e(g(y))\, dy + O(q^{1-1/u+\eta}),$$

where

$$\beta_j = \gamma_{nj}\alpha - \frac{a_j}{q} \quad \text{for } 1 \le j \le u, \qquad g(y) = \sum_{j=1}^{u} \beta_j y^j,$$

$$G(x) = \sum_{j=1}^{u} a_j x^j, \qquad S(q) = \sum_{v=1}^{q} e\Big(\frac{G(v)}{q}\Big).$$

By Theorems 7.1 and 7.3 of [18], for $\alpha \in \mathcal{N}(q, \mathbf{a}) \subseteq \mathcal{N}$, we therefore have

$$U(\alpha, h_n, P^{1/u}) \ll q^{-1/u+\eta} P^{1/u} (1 + |\beta_1| P^{1/u} + |\beta_2| P^{2/u} + \ldots + |\beta_u| P)^{-1/u} + q^{1-1/u+\eta}.$$

It certainly follows that

$$U(\alpha, h_n, P^{1/u}) \ll q^{-1/u+\eta} (\min(P^{1/u}, |\beta_u|^{-1/u}) + q),$$

whence also

$$U(\alpha, h_n, P^{1/u})^{uV} \ll q^{-V+uV\eta} ((\min(P^{1/u}, |\beta_u|^{-1/u}))^{uV} + q^{uV}). \tag{58}$$

For any integers q and a_u with $q \geq 1$, define

$$\mathcal{R}(q, a_u) = \bigcup \mathcal{N}(q, \mathbf{a}),$$

where the union is taken over all integer $(u-1)$-tuples $(a_1, a_2, \ldots, a_{u-1})$ such that

$$(q, a_u, \ldots, a_1) = 1 \quad \text{and} \quad (q, a_u, \ldots, a_2) \leq 2u^2.$$

Then we certainly have

$$\mathcal{R}(q, a_u) \subset \mathcal{N}_0(q, a_u). \tag{59}$$

We need to make one more observation. Suppose that $\alpha \in \mathcal{N}(q, \mathbf{a}) \subseteq \mathcal{N}$. Then $|\gamma_{nu} \alpha q - a_u| < P^{\phi-1}$. But γ_{nu} is non-zero, as the polynomial h_n has degree u. One certainly has $\phi < 1$, whence for non-zero a_u and sufficiently large P we have

$$|\alpha| \geq \frac{|a_u|}{2q|\gamma_{nu}|} \gg \frac{|a_u|}{q}. \tag{60}$$

Note on the other hand that this statement trivially follows if $a_u = 0$. So it holds regardless of the value of a_u.

We now consider the integral in (57). By the definitions of $\mathcal{N}$ and $\mathcal{R}(q, a_u)$, by trivial bounds, and by noting that K is positive, we have

$$\int_{\mathcal{N}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)| \Big) |U(\alpha, h_n, P^{1/u})|^{uV} K(\alpha)\, d\alpha \ll (P_1 \ldots P_t)^2 \sum_{1 \leq q \leq P^\phi} \sum_{a_u \in \mathbb{Z}} \int_{\mathcal{R}(q, a_u)} |U(\alpha, h_n, P^{1/u})|^{uV} K(\alpha)\, d\alpha.$$

Now for $\alpha \in \mathcal{R}(q, a_u)$, one has $\alpha \in \mathcal{N}(q, \mathbf{a})$ for some integers $a_1, \ldots, a_{u-1}$ which satisfy the conditions $(q, a_u, \ldots, a_1) = 1$ and $(q, a_u, \ldots, a_2) \leq 2u^2$. Thus (60) holds, whence by the bound (26) we have

$$\int_{\mathcal{N}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)| \Big) |U(\alpha, h_n, P^{1/u})|^{uV} K(\alpha)\, d\alpha \ll (P_1 \ldots P_t)^2 \sum_{1 \leq q \leq P^\phi} \sum_{a_u \in \mathbb{Z}} \min(1, |a_u|^{-2} q^2) \int_{\mathcal{R}(q, a_u)} |U(\alpha, h_n, P^{1/u})|^{uV}\, d\alpha. \tag{61}$$

Now for $\alpha \in \mathcal{R}(q, a_u)$ and $q \le P^{\phi}$, we have $\alpha \in \mathcal{N}(q, \mathbf{a})$ for some integers $a_1, \ldots, a_{u-1}$ satisfying the conditions $(q, a_u, \ldots, a_1) = 1$ and $(q, a_u, \ldots, a_2) \le 2u^2$, so the bound (58) holds. Using also (59) and the fact that $q \le P^{\phi}$, we obtain

$$\begin{aligned}
&\int\limits_{\mathcal{R}(q,a_u)} |U(\alpha, h_n, P^{1/u})|^{uV}\, d\alpha \\
&\qquad \ll \int\limits_{\mathcal{N}_0(q,a_u)} q^{-V+uV\eta}((\min(P^{1/u}, |\beta_u|^{-1/u}))^{uV} + q^{uV})\, d\alpha \\
&\qquad \ll \int\limits_{0}^{q^{-1}P^{\phi-1}} q^{-V+uV\eta}((\min(P^{1/u}, \beta_u^{-1/u}))^{uV} + q^{uV})\, d\beta_u \\
&\qquad \ll q^{-V+uV\eta}\Big[\int\limits_{0}^{P^{-1}} P^V\, d\beta_u + \int\limits_{P^{-1}}^{\infty} \beta_u^{-V}\, d\beta_u + q^{uV-1}P^{\phi-1}\Big] \\
&\qquad \ll q^{-V+uV\eta}[P^{V-1} + P^{uV\phi-1}],
\end{aligned}$$

as we have $V > 1$ certainly, from (56). But $u\phi \le k\phi < 1$, whence

$$\int\limits_{\mathcal{R}(q,a_u)} |U(\alpha, h_n, P^{1/u})|^{uV}\, d\alpha \ll q^{-V+uV\eta}P^{V-1}.$$

By combining with (61), we have

$$\begin{aligned}
&\int\limits_{\mathcal{N}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big) |U(\alpha, h_n, P^{1/u})|^{uV} K(\alpha)\, d\alpha \\
&\qquad \ll (P_1 \ldots P_t)^2 \sum_{1\le q\le P^{\phi}} \sum_{a_u \in \mathbb{Z}} \min(1, |a_u|^{-2}q^2) q^{-V+uV\eta} P^{V-1} \\
&\qquad \ll (P_1 \ldots P_t)^2 P^{V-1} \sum_{1\le q\le P^{\phi}} q^{-V+uV\eta}\Big[\sum_{a_u=0}^{q} 1 + \sum_{l=1}^{\infty} \sum_{a_u=lq+1}^{(l+1)q} \frac{1}{l^2}\Big] \\
&\qquad \ll (P_1 \ldots P_t)^2 P^{V-1} \sum_{1\le q\le P^{\phi}} q^{1-V+uV\eta} \\
&\qquad \ll (P_1 \ldots P_t)^2 P^{V-1}
\end{aligned}$$

for sufficiently small η, since $V > 2$ from (56). If we recall that we have already established the bound (53), the proof of Lemma 5 is complete.

5. An analogue of Weyl's inequality. We now proceed to establish an analogue of Weyl's inequality. (For the traditional Weyl inequality, see, for example, Lemma 2.4 of [18].) Our aim is to show, following the method of [2], that there is a function $T(P)$, which tends to infinity as P tends to infinity, such that we have

$$\sup_{P^{\delta-1}\le|\alpha|\le T(P)} |U_1(\alpha,P)U_2(\alpha,P)| = o(P^{1/w_1+1/w_2}). \tag{62}$$

We now give a lemma which allows us to find good rational approximations to the coefficients of a polynomial if the corresponding exponential sum is large in absolute value. In the following lemma, it is crucial to eliminate extra factors of N^η that might appear on the right side of the first inequality in (65). When using the ideas of Bentkus and Götze, it is crucial to eliminate this extra factor. Otherwise, we could simply quote Theorem 5.1 of [1]. We note that an observation by Professor Wooley has enabled us to give a much shorter proof. The following proof is based on his ideas, and I am very grateful to him for pointing out the improvement.

LEMMA 6. *Suppose that k is a positive integer and that $r(x)$ is a polynomial of degree k with real coefficients. We write*

$$r(x) = \lambda_k x^k + \lambda_{k-1}x^{k-1} + \ldots + \lambda_1 x + \lambda_0.$$

Fix any positive number η. Suppose as well that N is a positive number which is sufficiently large in terms of k and η. Define the exponential sum

$$S(N) = \sum_{1\le x\le N} e(r(x)).$$

Suppose that γ is a positive number for which

$$|S(N)| \ge \gamma N. \tag{63}$$

Then there are positive constants C_4 and C_5, each of which depends only on k, such that if γ satisfies

$$N^\eta \max(N^{-2^{1-k}}, N^{-1/(k+1)}) \le \gamma \le 1, \tag{64}$$

then there exists a positive integer q and integers $a_1,\ldots,a_k$, satisfying $(a_1,\ldots,a_k,q)=1$ with

$$q < C_4\gamma^{-k-2k^2\eta}, \qquad |\lambda_j - a_j/q| < C_5\gamma^{-k}N^{-j} \quad \textit{for } 1\le j\le k. \tag{65}$$

We observe that the case $k=1$ of Lemma 6 can be proved easily from the standard estimate

$$\sum_{1\le x\le N} e(\lambda_1 x) \ll \min(N, \|\lambda_1\|^{-1}).$$

In fact, in this case, one can prove a much stronger result, namely that if (63) holds, then $\|\lambda_1\| \ll \gamma^{-1}N^{-1}$. In this case, we can replace (64) with the simple condition that γ is a positive number. Thus we no longer consider the case $k=1$.

Proof of Lemma 6. We may clearly assume that $\eta \le 1/(2k)$. As $\gamma \ge N^{-2^{1-k}+\eta}$, we can apply Theorem 5.1 of [1], whence there are integers

$q, a_1, \ldots, a_k$ with

$$(a_1, \ldots, a_k, q) = 1, \quad 1 \le q < \gamma^{-k} N^{\eta}, \qquad (66)$$
$$|\lambda_j q - a_j| < \gamma^{-k} N^{\eta - j} \quad \text{for } 1 \le j \le k.$$

For real vectors $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)$, positive integers q and integral vectors $\mathbf{a} = (a_1, \ldots, a_k)$, define

$$I(\boldsymbol{\beta}) = \int_0^N e(\beta_1 \gamma + \beta_2 \gamma^2 + \ldots + \beta_k \gamma^k)\, d\gamma,$$
$$S(q, \mathbf{a}) = \sum_{x=1}^{q} e\bigg(\frac{a_1 x + a_2 x^2 + \ldots + a_k x^k}{q}\bigg).$$

Then, if we write $\beta_j = \lambda_j - a_j/q$ for $1 \le j \le k$, by Theorem 7.2 of [18], it follows that

$$S(N) = q^{-1} S(q, \mathbf{a}) I(\boldsymbol{\beta}) + O\Big(q\Big(1 + \sum_{j=1}^{k} |\beta_j| N^j\Big)\Big).$$

By (66), it follows that

$$S(N) = q^{-1} S(q, \mathbf{a}) I(\boldsymbol{\beta}) + O(\gamma^{-k} N^{\eta}).$$

As $\gamma \ge N^{\eta - 1/(k+1)}$, for large N we must have

$$\gamma N \le |S(N)| \le |q^{-1} S(q, \mathbf{a}) I(\boldsymbol{\beta})|.$$

By Theorems 7.1 and 7.3 of [18],

$$\gamma N \ll q^{\eta - 1/k} N \bigg(1 + \sum_{j=1}^{k} \bigg|\lambda_j - \frac{a_j}{q}\bigg| N^j\bigg)^{-1/k},$$

whence

$$q^{1/k - \eta} \ll \gamma^{-1}, \quad |\lambda_j - a_j/q| N^j \ll \gamma^{-k} \quad \text{for } 1 \le j \le k.$$

The proof of Lemma 6 follows.

From this point onward, we follow [13] quite closely, which we note was in turn motivated by the work of Bentkus and Götze [2]. We have the following lemma, which is similar to Lemma 3 of [13] and to Theorem 6.1 of [2]. The proof follows that of Lemma 3 of [13] quite closely, so we omit it.

LEMMA 7. *Suppose that we are in the setting of Proposition* 1. *Suppose that T_0 and T are real numbers with $0 < T_0 \le 1 \le T$. Then*

$$\sup_{T_0 \le |\alpha| \le T} |U_1(\alpha, P) U_2(\alpha, P)| = o(P^{1/w_1 + 1/w_2}).$$

Now we give a result which is an almost exact analogue of Lemma 4 of [13]. The present lemma follows in much the same manner as Lemma 4 of [13], so we omit its proof.

LEMMA 8. *Suppose that we are in the setting of Proposition* 1. *Then there are positive real-valued functions $T_0(P)$ and $T(P)$, depending only on the coefficients μ_{1j_1} and μ_{2j_2}, for which*

$$\lim_{P\to\infty} T_0(P) = 0 \quad \text{and} \quad \lim_{P\to\infty} T(P) = \infty, \tag{67}$$

and so that

$$P^{-\delta} \le T_0(P) \le 1 \quad \text{for } P \ge 3, \tag{68}$$

and

$$\sup_{T_0(P)\le|\alpha|\le T(P)} |U_1(\alpha, P)U_2(\alpha, P)| = o(P^{1/w_1+1/w_2}). \tag{69}$$

This completes our analogue of Weyl's inequality for one part of the minor arcs. We note that now we have chosen the function $T(P)$ which is used to define the minor arcs.

Now we need to handle the remaining, easier, region of the minor arcs. The following lemma is very similar to Lemma 5 of [13], and is proved in much the same manner. We omit the proof.

LEMMA 9. *Suppose that we are in the setting of Proposition* 1. *Then*

$$\sup_{P^{\delta-1}\le|\alpha|\le T_0(P)} |U_1(\alpha, P)| = o(P^{1/w_1}).$$

6. The minor arcs and trivial arcs. We first apply our analogues of Hua's inequality and Weyl's inequality to treat the minor arcs. We have the following lemma.

LEMMA 10. *Suppose that we are in the setting of Proposition* 1, *and that the functions $S_i(\alpha)$ and $U_n(\alpha)$ and $K(\alpha)$ are defined as above. Then*

$$\int_{\mathfrak{m}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big)\Big(\prod_{n=1}^{m} |U_n(\alpha)|\Big) K(\alpha)\, d\alpha = o(P^{W-1}(P_1\ldots P_t)^2).$$

Proof. Observe first that

$$\begin{aligned}\int_{\mathfrak{m}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big)\Big(\prod_{n=1}^{m} |U_n(\alpha)|\Big) K(\alpha)\, d\alpha \qquad\qquad\qquad\qquad \\ \ll \big(\sup_{\alpha\in\mathfrak{m}} |U_1(\alpha, P)U_2(\alpha, P)|\big) \int_{\mathbb{R}} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|\Big)\Big(\prod_{n=3}^{m} |U_n(\alpha)|\Big) K(\alpha)\, d\alpha.\end{aligned} \tag{70}$$

The integral on the right side of (70) is $O((P_1\ldots P_t)^2 P^{W-1/w_1-1/w_2-1})$, by Lemma 5. On the other hand, by applying Lemmas 8 and 9, and recalling the definition (40) of $\mathfrak{m}$, we have

$$\sup_{\alpha\in\mathfrak{m}} |U_1(\alpha, P)U_2(\alpha, P)| = o(P^{1/w_1+1/w_2}).$$

Inserting these two observations in the bound (70) proves Lemma 10.

At this point, we treat the contribution to the integral (37) from the trivial arcs. We first must state a slight generalization of a lemma due to Davenport and Roth. (See Lemma 2 of [10].)

LEMMA 11 (Davenport and Roth). *Fix a positive number η. Let*

$$F(\alpha)=\sum e(\alpha f(x_1,\dots,x_s)),$$

where f is any real function of s variables, and the summation is over any finite set of values of $x_1,\dots,x_s$. Define the function L by

$$L(\alpha)=\frac{(\sin(\pi\eta\alpha))^2}{\pi^2\alpha^2\eta}.$$

Then for any real number A with $A>4/\eta$, we have

$$\int_{|\alpha|>A}|F(\alpha)|^2L(\alpha)\,d\alpha\le\frac{16}{\eta A}\int_{\mathbb{R}}|F(\alpha)|^2L(\alpha)\,d\alpha.$$

We observe that in the case $\eta=1$, the lemma is exactly Lemma 2 of [10]. The proof of the general case can be deduced easily from the case $\eta=1$ by a change of variable, so we omit the details.

Now we can complete the treatment of the trivial arcs. We have the following result.

LEMMA 12. *Suppose that we are in the setting of Proposition 1, and that the functions $S_i(\alpha)$ and $U_n(\alpha)$ and $K(\alpha)$ are defined as above. Then*

$$\int_{\mathfrak{t}}\Big(\prod_{i=1}^{2t}|S_i(\alpha)|\Big)\Big(\prod_{n=1}^{m}|U_n(\alpha)|\Big)K(\alpha)\,d\alpha=o(P^{W-1}(P_1\dots P_t)^2).\tag{71}$$

Proof. Recall from the statement of Proposition 1 that $m=2k+4$. Thus we may use Hölder's inequality and the fact that $K(\alpha)$ is positive to see that the left hand side of (71) is

$$\begin{aligned}&\ll\Big(\int_{|\alpha|>T(P)}\Big(\prod_{i=1}^{t}|S_i(\alpha)|^2\Big)\Big(\prod_{n=1}^{m/2}|U_n(\alpha)|^2\Big)K(\alpha)\,d\alpha\Big)^{1/2}\\&\quad\times\Big(\int_{|\alpha|>T(P)}\Big(\prod_{i=t+1}^{2t}|S_i(\alpha)|^2\Big)\Big(\prod_{n=m/2+1}^{m}|U_n(\alpha)|^2\Big)K(\alpha)\,d\alpha\Big)^{1/2}.\end{aligned}\tag{72}$$

We will show only that

$$\int_{|\alpha|>T(P)}\Big(\prod_{i=1}^{t}|S_i(\alpha)|^2\Big)\Big(\prod_{n=1}^{m/2}|U_n(\alpha)|^2\Big)K(\alpha)\,d\alpha=o(P^{(\sum_{n=1}^{m/2}2/w_n)-1}(P_1\dots P_t)^2).\tag{73}$$

One can show the corresponding bound

$$\int_{|\alpha|>T(P)} \Big(\prod_{i=t+1}^{2t} |S_i(\alpha)|^2\Big)\Big(\prod_{n=m/2+1}^{m} |U_n(\alpha)|^2\Big) K(\alpha)\,d\alpha = o(P^{(\sum_{n=m/2+1}^{m} 2/w_n)-1}(P_1\dots P_t)^2)$$

for the other integral in (72) in a very similar fashion, so we omit that part of the proof.

Note that our choice of $K(\alpha)$, made using Lemma 1, is of course exactly the function $L(\alpha)$ of Lemma 11 with $\eta=\varepsilon$. Thus, for sufficiently large P, if we recall that $T(P)$ tends to infinity as P tends to infinity, Lemma 11 yields

$$\int_{|\alpha|>T(P)} \Big(\prod_{i=1}^{t} |S_i(\alpha)|^2\Big)\Big(\prod_{n=1}^{m/2} |U_n(\alpha)|^2\Big) K(\alpha)\,d\alpha \ll_\varepsilon \frac{1}{T(P)} \int_{\mathbb{R}} \Big(\prod_{i=1}^{t} |S_i(\alpha)|^2\Big)\Big(\prod_{n=1}^{m/2} |U_n(\alpha)|^2\Big) K(\alpha)\,d\alpha.$$

Combining this bound with Lemma 5 yields

$$\int_{|\alpha|>T(P)} \Big(\prod_{i=1}^{2t} |S_i(\alpha)|^2\Big)\Big(\prod_{n=1}^{m/2} |U_n(\alpha)|^2\Big) K(\alpha)\,d\alpha \ll \frac{1}{T(P)}(P_1\dots P_t)^2 P^{(\sum_{n=1}^{m/2} 2/w_n)-1};$$

observe that here we have used the special case of Lemma 5 in which $\lambda_{(t+i)j}=\lambda_{ij}$ for $1\le i\le t$ and $1\le j\le v_i$. But $T(P)$ was chosen in Lemma 8 to satisfy $\lim_{P\to\infty} T(P)=\infty$, whence the bound (73) follows. Therefore the proof of Lemma 12 is complete.

7. The major arc. We now come to the contribution of the major arc to the integral (37). We show that

$$\int_{\mathcal{M}} \Big(\prod_{i=1}^{2t} S_i(\alpha)\Big)\Big(\prod_{n=1}^{m} U_n(\alpha)\Big) e(-M\alpha)K(\alpha)\,d\alpha \gg (P_1\dots P_t)^2 P^{W-1} \tag{74}$$

for sufficiently large P. This will complete the proof of Proposition 1, in view of Lemmas 10 and 12. The following treatment is largely very standard, but we give the proof for the sake of completeness.

We start off by applying Lemma 4 to the generating functions $U_n(\alpha)$. For $\alpha\in\mathcal{M}$, we have $|\alpha|\le P^{\delta-1}$. From (38), we have $\delta<1/k\le 1/w_n$. Thus for $\alpha \in \mathcal{M}$, for $1 \le n \le m$, and for large P, we certainly have

$$|\alpha\mu_{nj}| \le \frac{P^{(1-j)/w_n}}{2w_n^2} \quad \text{for } 1 \le j \le w_n.$$

Therefore, for $\alpha \in \mathcal{M}$ and $1 \le n \le m$, Lemma 4 yields

$$U_n(\alpha) = I_n(\alpha) + O(1),$$

where

$$I_n(\alpha) = \int_0^{P^{1/w_n}} e\Big(\sum_{j=1}^{w_n} \alpha\mu_{nj}x^j\Big)\,dx.$$

It follows by a telescoping series argument that for $\alpha \in \mathcal{M}$,

$$\prod_{n=1}^{m} U_n(\alpha) = \prod_{n=1}^{m} I_n(\alpha) + O(P^{W-1/k}).$$

Using this approximation, trivial estimates and the bound (26), we obtain

$$\begin{aligned}(75)\quad &\int_{\mathcal{M}} \Big(\prod_{i=1}^{2t} S_i(\alpha)\Big)\Big(\prod_{n=1}^{m} U_n(\alpha)\Big) e(-M\alpha)K(\alpha)\,d\alpha \\ &= \int_{\mathcal{M}} \Big(\prod_{i=1}^{2t} S_i(\alpha)\Big)\Big(\prod_{n=1}^{m} I_n(\alpha)\Big) e(-M\alpha)K(\alpha)\,d\alpha \\ &\quad + O((P_1\ldots P_t)^2 P^{W-1+\delta-1/k}).\end{aligned}$$

But now, using (26), trivial estimates and Theorem 7.3 of [18], we also have

$$\begin{aligned}(76)\quad &\int_{\mathbb{R}\setminus\mathcal{M}} \Big(\prod_{i=1}^{2t} S_i(\alpha)\Big)\Big(\prod_{n=1}^{m} I_n(\alpha)\Big) e(-M\alpha)K(\alpha)\,d\alpha \\ &\ll (P_1\ldots P_t)^2 \int_{\mathbb{R}\setminus\mathcal{M}} \prod_{n=1}^{m} |\mu_{nw_n}\alpha|^{-1/w_n}\,d\alpha \\ &\ll (P_1\ldots P_t)^2 (P^{\delta-1})^{1-W} \\ &\ll (P_1\ldots P_t)^2 P^{W-1} P^{\delta(1-W)};\end{aligned}$$

here we have used the fact that $W > 1$, which clearly follows from (36).

Combining (75) and (76) and using (38), we have

$$\begin{aligned}&\int_{\mathcal{M}} \Big(\prod_{i=1}^{2t} S_i(\alpha)\Big)\Big(\prod_{n=1}^{m} U_n(\alpha)\Big) e(-M\alpha)K(\alpha)\,d\alpha \\ &= \int_{\mathbb{R}} \Big(\prod_{i=1}^{2t} S_i(\alpha)\Big)\Big(\prod_{n=1}^{m} I_n(\alpha)\Big) e(-M\alpha)K(\alpha)\,d\alpha + o((P_1\ldots P_t)^2 P^{W-1}).\end{aligned}$$

Thus to prove that (74) holds for large P, it suffices to show that for large P we have

$$\int_{\mathbb{R}} \Big(\prod_{i=1}^{2t} S_i(\alpha)\Big)\Big(\prod_{n=1}^{m} I_n(\alpha)\Big) e(-M\alpha)K(\alpha)\,d\alpha \gg (P_1\ldots P_t)^2 P^{W-1}. \tag{77}$$

Now, by (26) and trivial estimates, the integral on the left side of (77) is absolutely convergent. Thus we may rewrite this integral as

$$\sum_{P_1\le x_1,x_{t+1}\le 2P_1} \cdots \sum_{P_t\le x_t,x_{2t}\le 2P_t} \int_{\prod_{n=1}^m [0,P^{1/w_n}]} \int_{\mathbb{R}} e(\alpha(f_1(x_1)+\ldots+f_{2t}(x_{2t})+g_1(y_1)+\ldots+g_m(y_m)-M))K(\alpha)\,d\alpha\,d\mathbf{y}.$$

Therefore, by the identity (25), we may see that the integral on the left side of (77) is bounded below by

$$\frac{1}{2}\sum_{P_1\le x_1,x_{t+1}\le 2P_1} \cdots \sum_{P_t\le x_t,x_{2t}\le 2P_t} \int_{\mathcal{R}} d\mathbf{y},$$

where one defines $\mathcal{R}=\mathcal{R}(x_1,\ldots,x_{2t})$ to be the set

$$\Big\{\mathbf{y}\in\prod_{n=1}^{m}[0,P^{1/w_n}] : \Big|\sum_{i=1}^{2t} f_i(x_i)+\sum_{n=1}^{m} g_n(y_n)-M\Big| < \frac{\varepsilon}{2}\Big\}.$$

So to prove (77), it is enough to show that for all choices of $(2t)$-tuples $(x_1,\ldots,x_{2t})$ with $P_i\le x_i,x_{t+i}\le 2P_i$ for $1\le i\le t$,

$$\mu(\mathcal{R}(x_1,\ldots,x_{2t})) \gg P^{W-1}, \tag{78}$$

where μ denotes m-dimensional measure. We turn now to proving the bound (78) in each case of Proposition 1.

Consider case (i) of Proposition 1. Recall that we have assumed that condition (14) holds, namely that

$$\sum_{i=1}^{2t}\sum_{j=1}^{v_i}|\lambda_{ij}| + \sum_{n=3}^{m}\sum_{j=1}^{w_n}|\mu_{nj}| \le \frac{\mu_{1w_1}}{2^{k+3}}.$$

Recall as well from (29) that $P_i^{v_i}\le P$ for $1\le i\le t$. It follows that given any real vectors $\mathbf{x}\in\mathbb{R}^{2t}$ and $\mathbf{y}=(y_2,\ldots,y_m)\in\mathbb{R}^{m-1}$ with $P_i\le x_i,x_{t+i}\le 2P_i$ for $1\le i\le t$ and with $0\le y_n\le P^{1/w_n}$ for $2\le n\le m$, for sufficiently large P we have

$$\Big|\sum_{i=1}^{2t} f_i(x_i)+\sum_{n=3}^{m} g_n(y_n)+\sum_{j=1}^{w_2-1}\mu_{2j}y_2^j\Big| \le \frac{\mu_{1w_1}P}{8}. \tag{79}$$

Now fix any integer vector $\mathbf{x}\in\mathbb{Z}^{2t}$ and real vector $(y_3,\ldots,y_m)$ with

$$P_i\le x_i,x_{t+i}\le 2P_i \quad \text{for } 1\le i\le t, \qquad 0\le y_n\le P^{1/w_n} \quad \text{for } 3\le n\le m.$$

Then consider any real number y_2 satisfying

$$(7/12)^{1/w_2}P^{1/w_2} \le y_2 \le (2/3)^{1/w_2}P^{1/w_2}.$$

Recall that we have assumed in case (i) of Proposition 1 that μ_{2w_2} is negative. Thus we have

$$-\frac{7}{12}\mu_{2w_2}P \le -\mu_{2w_2}y_2^{w_2} \le -\frac{2}{3}\mu_{2w_2}P.$$

Recall also the condition $1 \le |\mu_{1w_1}/\mu_{2w_2}| \le 2$ from case (i) of the proposition. It follows that

$$\frac{7}{24}\mu_{1w_1}P \le -\mu_{2w_2}y_2^{w_2} \le \frac{2}{3}\mu_{1w_1}P.$$

Also, recall from (15) that we have assumed $|M| \le (\mu_{1w_1}P)/8$, whence from (79) we have

$$\frac{1}{24}\mu_{1w_1}P \le \Big(M - g_2(y_2) - \sum_{i=1}^{2t} f_i(x_i) - \sum_{n=3}^{m} g_n(y_n)\Big) \le \frac{11}{12}\mu_{1w_1}P.$$

It follows for large P that there is a real number y_1 with

$$(1/25)^{1/w_1}P^{1/w_1} \le y_1 \le (12/13)^{1/w_1}P^{1/w_1}$$

which satisfies

$$g_1(y_1) = M - g_2(y_2) - \sum_{i=1}^{2t} f_i(x_i) - \sum_{n=3}^{m} g_n(y_n).$$

Now, for this choice of y_1 and any real number L with $|L| \le 1$, and for sufficiently large P, we have

$$|g_1(y_1 + L) - g_1(y_1)| \ll LP^{1-1/w_1}.$$

Thus it follows that for each choice as above of $(x_1, \ldots, x_{2t}, y_2, \ldots, y_n)$, there is an interval $\mathcal{I} \subseteq [0, P^{1/w_1}]$ of choices y_1 of length $\gg_\varepsilon P^{1/w_1-1}$ for which

$$\Big|\sum_{i=1}^{2t} f_i(x_i) + \sum_{n=1}^{m} g_n(y_n) - M\Big| < \frac{\varepsilon}{2}.$$

Thus for each fixed choice of $(x_1, \ldots, x_{2t})$ as above, there is a set of m-dimensional measure

$$\gg_\varepsilon P^{1/w_1-1}P^{1/w_2}\prod_{n=3}^{m} P^{1/w_n} \gg_\varepsilon P^{W-1}$$

which is contained in $\mathcal{R}$. This completes our proof in case (i).

The proof of case (ii) of Proposition 1 is similar, and in fact slightly more simple, so we omit it. In both cases of Proposition 1, it follows that the contribution to the integral (37) from the major arc is $\gg (P_1 \dots P_t)^2 P^{W-1}$. As noted above, this is enough to complete the proof of Proposition 1. As also observed above, this completes the proof of Theorems 1 and 2.

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Received on 20.11.2001 (4154)