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Chen's double sieve, Goldbach's conjecture and the twin prime problem, 2

by

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1. Introduction. Let $\Omega(n)$ be the number of all prime factors of the integer n with the convention $\Omega(1) = 0$. For each even integer $N \geq 4$, we define

$$D(N) := |\{p \leq N : \Omega(N-p) = 1\}|;$$

here and in what follows, the letter p, with or without subscript, denotes a prime number. The well known Goldbach conjecture can be stated as $D(N) \geq 1$ for all even integers $N \geq 4$. A more precise version of this conjecture was proposed by Hardy & Littlewood [10]:

$$D(N) \sim 2\Theta(N) \quad (N \to \infty), \tag{1.1}$$

where

$$\Theta(N) := \frac{C_N N}{(\log N)^2}, \quad C_N := \prod_{p \mid N,\, p>2} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right). \tag{1.2}$$

Certainly, the asymptotic formula (1.1) is extremely difficult. One way of approaching the lower bound problem in (1.1) is to give a non-trivial lower bound for the quantity

$$D_{1,2}(N) := |\{p \leq N : \Omega(N-p) \leq 2\}|.$$

In this direction, Chen [5] proved, by his system of weights and the switching principle, the following famous theorem: *Every sufficiently large even integer can be written as sum of a prime and an integer having at most two prime factors.* More precisely, he established

$$D_{1,2}(N) \geq 0.67\, \Theta(N) \tag{1.3}$$

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for $N \geq N_0$. As Halberstam & Richert indicated in [9], it would be interesting to know whether a more elaborate weighting procedure could be adapted to the purpose of (1.3). This might lead to numerical improvements and could be important. Chen's constant 0.67 has been improved by many authors. The historical record is as follows:

0.689	by Halberstam & Richert [9],
0.754	by Chen [6],
0.81	by Chen [7],
0.828	by Cai & Lu [4],
0.836	by Wu [13],
0.867	by Cai [2].

The aim of this paper is to propose a better constant.

THEOREM. *For sufficiently large* N, *we have*

$$D_{1,2}(N) \geq 0.899\,\Theta(N).$$

Our improvement comes from a delicate application of Chen's double sieve ([8], [12], [13]), which can be described as follows: With standard notation in the theory of sieve methods, the linear sieve formulas (see [9], or Lemma 2.2 of [13]) can be stated as

$$XV(z)f\left(\frac{\log Q}{\log z}\right) + \text{error} \leq S(\mathcal{A};\mathcal{P},z) \leq XV(z)F\left(\frac{\log Q}{\log z}\right) + \text{error}. \tag{1.4}$$

These inequalities are the best possible in the sense that for

$$\mathcal{A} = \mathcal{B}_\nu := \{n \leq x : \Omega(n) \equiv \nu \pmod 2\} \qquad (\nu = 1, 2),$$

the upper and lower bounds in (1.4) are respectively attained by $\nu = 1$ and $\nu = 2$ (see [9, p. 239]). Aiming at improving Bombieri–Davenport's upper bound [1]

$$D(N) \leq \{8 + o(1)\}\Theta(N),$$

Chen [8] found an improvement for (1.4) for some special sequences $\mathcal{A}$. Roughly speaking, for the sequence

$$\mathcal{A} = \{N - p : p \leq N\}$$

he narrowed down the gap in (1.4) by introducing two functions $h(s)$ and $H(s)$ such that the functions $sf(s)/(2e^\gamma)$ and $sF(s)/(2e^\gamma)$ are replaced by $sf(s)/(2e^\gamma) + h(s)$ and $sF(s)/(2e^\gamma) - H(s)$ respectively, where γ is the Euler constant. The key point is thus to prove $h(s) > 0$ and $H(s) > 0$. Chen's proof is very long and difficult to follow, but his innovative idea is clear (see [11] for example). In [13], we gave a more comprehensive treatment of this method and named it *Chen's double sieve*. Indeed, our treatment is not only simpler but even more powerful than Chen's. Our approach improved Chen's upper estimate $D(N) \le 7.8342\,\Theta(N)$ to $D(N) \le 7.8209\,\Theta(N)$. It is worth indicating that Chen's record stood for 26 years before our work [13].

To prove our Theorem, we first simplify and improve Chen's weight system (cf. (12) of [7] and Lemma 2.2 below), and then apply Chen's double sieve, as the classical linear sieve, to handle terms such as Υ_2, Υ_3, Υ_4, Υ_5 and Υ_6 in Propositions 4.1–4.4 below. The idea of using Chen's double sieve to treat sums of the type

$$\sum_{\substack{N^{\phi_1} \le p < N^{\phi_2} \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa}) \tag{1.5}$$

first appeared in [12]. However, due to the first condition in (3.1) below, our Chen's double sieve can only handle the initial part of the sum over small p in (1.5) (i.e. $p \le N^{1/4}$). On the other hand, very recently Cai [2] used a similar idea to control the sum over large p in (1.5). Actually his method can be viewed as a simplified version of Chen's double sieve (see Proposition 4.4 below and the comments before it). Here we shall combine both versions and refine them to obtain our result. As is apparent from the proof, the first version gives a saving of 0.0211 while the second saves 0.0078. Without Chen's double sieve technique, we still obtain 0.870 in place of 0.899, which is slightly better than Cai's 0.867.

Clearly our method can be used to refine the corresponding constants in the conjugate problems ([2] and [3]). The proofs are very similar and even easier and simpler. Hence we omit the relevant discussion. Maybe this is a good exercise for senior graduate students in analytic number theory.

2. Chen's system of weights. This section is devoted to discussing the weighted sieve of Chen type. Let

$$\mathcal{A} := \{N - p : p \le N\} \quad \text{and} \quad \mathcal{P}(N) := \{p : (p, N) = 1\}.$$

The sieve function is defined as

$$S(\mathcal{A}; \mathcal{P}(N), z) := |\{a \in \mathcal{A} : (a, P(z)) = 1\}|,$$

where $P(z) := \prod_{p \le z,\, p \in \mathcal{P}(N)} p$.

LEMMA 2.1. *Let* $0 < \kappa < \sigma \le 1/3$. *Then*

$$\begin{aligned} 2D_{1,2}(N) \ge{}& 2S(\mathcal{A}; \mathcal{P}(N), N^{\kappa}) - S_1(\kappa, \sigma) - 2S_2(\kappa, \sigma) \\ &- S_3(\kappa, \sigma) + S_4(\kappa, \sigma) + O(N^{1-\kappa}), \end{aligned} \tag{2.1}$$

where

$$S_1(\kappa,\sigma) := \sum_{\substack{N^\kappa \le p < N^\sigma \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^\kappa),$$

$$S_2(\kappa,\sigma) := \sum\sum_{\substack{N^\sigma \le p_1 < p_2 < (N/p_1)^{1/2} \\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), p_2),$$

$$S_3(\kappa,\sigma) := \sum\sum_{\substack{N^\kappa \le p_1 < N^\sigma \le p_2 < (N/p_1)^{1/2} \\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), p_2),$$

$$S_4(\kappa,\sigma) := \sum\sum\sum_{\substack{N^\kappa \le p_1 < p_2 < p_3 < N^\sigma \\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(Np_1), p_2).$$

The inequality (2.1) first appeared in [7, p. 479, (11)] with $(\kappa,\sigma) = \left(\frac{1}{12}, \frac{1}{3.047}\right), \left(\frac{1}{9.2}, \frac{1}{3.41}\right)$ without proof. Cai & [Lu] [4] gave a proof with an extra assumption $3\sigma + \kappa > 1$. In [13], we proved (2.1) under the hypothesis $0 < \kappa < \sigma < 1/3$. Clearly the proof there is also valid for $\sigma = 1/3$. Very recently Cai [2] gave another proof for Lemma 2.1.

As in [7], we shall apply (2.1) with two different pairs of parameters (κ,σ) to take advantage of $S_4(\kappa,\sigma)$. Our weighted sieve is simpler and more powerful than those of Chen ([7, (12)]) and Cai ([2, Lemma 6]).

LEMMA 2.2. *Let* $\kappa_2 > \kappa_1 \ge 1/18$ *be such that*

$$3\kappa_1 + \kappa_2 < 1/2 \quad \text{and} \quad 3\kappa_1 - \kappa_2 < 1/6.$$

Then

$$4D_{1,2}(N) \ge 3\Upsilon_1 + \Upsilon_2 - \Upsilon_3 - \Upsilon_4 + \Upsilon_5 + \Upsilon_6 - 2\Upsilon_7 - \Upsilon_8 - \Upsilon_9 - \Upsilon_{10} - \Upsilon_{11} + O(N^{1-\kappa_1}), \tag{2.2}$$

where

$$\Upsilon_i := S(\mathcal{A}; \mathcal{P}(N), N^{\kappa_i}) \qquad (i = 1, 2),$$

$$\Upsilon_3 := \sum_{\substack{N^{\kappa_1} \le p < N^{1/3} \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa_1}),$$

$$\Upsilon_4 := \sum_{\substack{N^{\kappa_1} \le p < N^{1/2-3\kappa_1} \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa_1}),$$

$$\Upsilon_5 := \sum\sum_{\substack{N^{\kappa_1} \le p_1 < p_2 < N^{\kappa_2} \\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\kappa_1}),$$

$$\Upsilon_6 := \sum_{\substack{N^{\kappa_1}\le p_1<N^{\kappa_2}\le p_2<N^{1/2-3\kappa_1}\\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\kappa_1}),$$

$$\Upsilon_7 := \sum_{\substack{N^{1/2-3\kappa_1}\le p_1<p_2<(N/p_1)^{1/2}\\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),p_2),$$

$$\Upsilon_8 := \sum_{\substack{N^{\kappa_1}\le p_1<N^{1/3}\le p_2<(N/p_1)^{1/2}\\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),p_2),$$

$$\Upsilon_9 := \sum_{\substack{N^{\kappa_2}\le p_1<N^{1/2-3\kappa_1}\le p_2<(N/p_1)^{1/2}\\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),(N/p_1p_2)^{1/2}),$$

$$\Upsilon_{10} := \sum\sum\sum\sum_{\substack{N^{\kappa_1}\le p_1<p_2<p_3<p_4<N^{\kappa_2}\\ (p_1p_2p_3p_4,N)=1}} S(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(N),p_2),$$

$$\Upsilon_{11} := \sum\sum\sum\sum_{\substack{N^{\kappa_1}\le p_1<p_2<p_3<N^{\kappa_2}\le p_4<N^{1/2-2\kappa_1}/p_3\\ (p_1p_2p_3p_4,N)=1}} S(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(N),p_2).$$

Proof. By noticing that our hypothesis implies $\kappa_2 < 1/2 - 3\kappa_1 \le 1/3$, we can apply (2.1) with $(\kappa,\sigma) = (\kappa_2, 1/2 - 3\kappa_1)$ to obtain

$$2D_{1,2}(N) \ge 2\Upsilon_2 - S_1(\kappa_2, 1/2-3\kappa_1) - 2\Upsilon_7 - S_3(\kappa_2,1/2-3\kappa_1) + O(N^{1-\kappa_2}), \tag{2.3}$$

where the term $S_4(\kappa_2, 1/2-3\kappa_1)$ is dropped by non-negativity.

Buchstab's identity, applied three times, gives the equality

$$\Upsilon_2 = \Upsilon_1 - \sum_{\substack{N^{\kappa_1}\le p<N^{\kappa_2}\\ (p,N)=1}} S(\mathcal{A}_p;\mathcal{P}(N),N^{\kappa_1}) + \Upsilon_5 - \sum\sum\sum_{\substack{N^{\kappa_1}\le p_1<p_2<p_3<N^{\kappa_2}\\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_1).$$

Similarly, a double application of Buchstab's identity yields

$$S_1(\kappa_2,1/2-3\kappa_1) = \sum_{\substack{N^{\kappa_2}\le p<N^{1/2-3\kappa_1}\\ (p,N)=1}} S(\mathcal{A}_p;\mathcal{P}(N),N^{\kappa_1}) - \Upsilon_6 + \sum\sum\sum_{\substack{N^{\kappa_1}\le p_1<p_2<N^{\kappa_2}\le p_3<N^{1/2-3\kappa_1}\\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_1).$$

By Buchstab's identity, we can prove

$$S_3(\kappa_2, 1/2-3\kappa_1) \le \Upsilon_9 + \sum\sum\sum_{\substack{N^{\kappa_2}\le p_1<N^{1/2-3\kappa_1}\le p_2<p_3<(N/p_1p_2)^{1/2}\\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(Np_1),p_3).$$

Inserting them into (2.3), we find that

$$2D_{1,2}(N) \ge \Upsilon_1+\Upsilon_2-\Upsilon_4+\Upsilon_5+\Upsilon_6-2\Upsilon_7-\Upsilon_9-\Delta_1+O(N^{1-\kappa_2}), \tag{2.4}$$

where

$$\begin{aligned}\Delta_1 := &\sum\sum\sum_{\substack{N^{\kappa_1}\le p_1<p_2<p_3<N^{\kappa_2}\\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_1)\\ &+\sum\sum\sum_{\substack{N^{\kappa_1}\le p_1<p_2<N^{\kappa_2}\le p_3<N^{1/2-3\kappa_1}\\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_1)\\ &+\sum\sum\sum_{\substack{N^{\kappa_2}\le p_1<N^{1/2-3\kappa_1}\le p_2<p_3<(N/p_1p_2)^{1/2}\\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(Np_1),p_3).\end{aligned}$$

The inequality (2.1) with $(\kappa,\sigma)=(\kappa_1,1/3)$ gives

$$2D_{1,2}(N) \ge 2\Upsilon_1-\Upsilon_3-\Upsilon_8+S_4(\kappa_1,1/3)+O(N^{1-\kappa_1}), \tag{2.5}$$

where we have used the fact that $S_2(\kappa_1,1/3)=0$.

Adding (2.4) to (2.5) yields

$$\begin{aligned}4D_{1,2}(N) \ge{}& 3\Upsilon_1+\Upsilon_2-\Upsilon_3-\Upsilon_4+\Upsilon_5+\Upsilon_6\\ &-2\Upsilon_7-\Upsilon_8-\Upsilon_9+\Delta_2+O(N^{1-\kappa_1}),\end{aligned} \tag{2.6}$$

where

$$\Delta_2 := \sum\sum\sum_{\substack{N^{\kappa_1}\le p_1<p_2<p_3<N^{1/3}\\ (p_1p_2p_3,N)=1}} S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_2)-\Delta_1.$$

Clearly all the summation ranges in the three triple sums of Δ_1 are distinct and the first two are covered by the range of the triple sum in Δ_2 (since our hypothesis on κ_1 and κ_2 implies $\max\{\kappa_2, 1/2-3\kappa_1\}\le 1/3$). On the other hand, we easily see that the range of summation in the third triple sum of Δ_1 is equivalent to $N^{\kappa_2}\le p_1<N^{1/2-3\kappa_1}\le p_2\le (N/p_1)^{1/3}$ and $p_2<p_3<(N/p_1p_2)^{1/2}$. From this we deduce that $(N/p_1p_2)^{1/2}\le N^{(1/2+3\kappa_1-\kappa_2)/2}\le N^{1/3}$, since $3\kappa_1-\kappa_2<1/6$. Thus this range is also contained in the triple sum of Δ_2. Therefore we have

$$\begin{aligned}\Delta_2 \geq & - \sum_{\substack{N^{\kappa_1}\le p_1<p_2<p_3<N^{\kappa_2}\\(p_1p_2p_3,N)=1}}\sum\sum \Delta_2' \\ & - \sum_{\substack{N^{\kappa_1}\le p_1<p_2<N^{\kappa_2}\le p_3<N^{1/2-2\kappa_1}/p_2\\(p_1p_2p_3,N)=1}}\sum\sum \Delta_2' \\ & + \sum_{\substack{N^{\kappa_2}\le p_1<N^{1/2-3\kappa_1}\le p_2<p_3<(N/p_1p_2)^{1/2}\\(p_1p_2p_3,N)=1}}\sum\sum \Delta_2'' \\ \geq & -\Upsilon_{10}-\Upsilon_{11}+O(N^{1-\kappa_1}),\end{aligned}$$

where

$$\Delta_2' := S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_1) - S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_2),$$
$$\Delta_2'' := S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_2) - S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_3).$$

Combining this with (2.6), we obtain the required result. ∎

REMARK 1. In the proof, we have chosen $(\kappa,\sigma) = (\kappa_1, 1/2-3\kappa_1)$, $(\kappa_2, 1/3)$ when applying Lemma 2.1. It is possible to optimize the choice of σ. But this increases the number of terms of (2.2), and the numerical improvement for the Theorem is quite small.

3. Chen's double sieve. In this section, we recall Chen's double sieve described in [13] and give numerical lower bounds for $H(s)$ and $h(s)$ for later use.

For any large even integer N, we write

$$\mathcal{A} := \{N-p : p\le N\}, \qquad \mathcal{P}(N) := \{p : (p,N)=1\}.$$

Let $\delta>0$ be a sufficiently small number ([1]) and $k\in\mathbb{Z}^+$. Put

$$Q := N^{1/2-\delta}, \qquad \underline{d} := Q/d, \qquad \mathcal{L} := \log N, \qquad W_k := N^{\delta^{1+k}}.$$

Denote by $\pi_{[Y,Z)}$ the characteristic function of the set $\mathcal{P}(N)\cap[Y,Z)$. For $k\in\mathbb{Z}^+$ and $N\ge 2$, let $\mathfrak{U}_k(N)$ be the set of all arithmetical functions σ which can be written in the form

$$\sigma = \pi_{[V_1/\Delta,V_1)} * \cdots * \pi_{[V_i/\Delta,V_i)},$$

where Δ is a real number with $1+\mathcal{L}^{-4}\le\Delta<1+2\mathcal{L}^{-4}$, i is an integer with $0\le i\le k$, and $V_1,\ldots,V_i$ are real numbers satisfying

([1]) In numerical computation, we can formally take $\delta=0$.

$$\begin{cases} V_1^2 \le Q, \\ V_1 V_2^2 \le Q, \\ \cdots\cdots\cdots\cdots \\ V_1 \cdots V_{i-1} V_i^2 \le Q, \\ V_1 \ge \cdots \ge V_i \ge W_k. \end{cases} \tag{3.1}$$

We adopt the convention that σ is the characteristic function of the set $\{1\}$ if $i = 0$.

Let F and f be defined by

$$\begin{aligned} F(s) &= 2e^\gamma/s, \quad & f(s) &= 0 \qquad & (0 < s \le 2), \\ (sF(s))' &= f(s-1), \quad & (sf(s))' &= F(s-1) \qquad & (s > 2), \end{aligned} \tag{3.2}$$

where γ is Euler's constant. Moreover, we set

$$A(s) := sF(s)/(2e^\gamma), \qquad a(s) := sf(s)/(2e^\gamma), \tag{3.3}$$

and introduce the notation

$$\Phi(N, \sigma, s) := \sum_d \sigma(d) S(\mathcal{A}_d; \mathcal{P}(dN), \underline{d}^{1/s}), \tag{3.4}$$

$$\Theta(N, \sigma) := 4\,\mathrm{li}(N) \sum_d \frac{\sigma(d) C_{dN}}{\varphi(d) \log \underline{d}}, \tag{3.5}$$

where $\varphi(d)$ is the Euler function.

For $k \in \mathbb{Z}^+$, $N_0 \ge 2$ and $s \in [1, 10]$, we define $H_{k,N_0}(s)$ and $h_{k,N_0}(s)$ to be the supremum of $h \ge -\infty$ such that for all $N \ge N_0$ and $\sigma \in \mathfrak{U}_k(N)$, the inequalities

$$\Phi(N, \sigma, s) \le \{A(s) - h\} \Theta(N, \sigma),$$
$$\Phi(N, \sigma, s) \ge \{a(s) + h\} \Theta(N, \sigma)$$

hold true respectively. Obviously $H_{k,N_0}(s)$ and $h_{k,N_0}(s)$ are decreasing in N_0, as well as decreasing in k by Lemma 3.1. Hence their limits at infinity exist (in the extended real line), and we write

$$H_k(s) := \lim_{N_0 \to \infty} H_{k,N_0}(s), \qquad H(s) := \lim_{k \to \infty} H_k(s),$$
$$h_k(s) := \lim_{N_0 \to \infty} h_{k,N_0}(s), \qquad h(s) := \lim_{k \to \infty} h_k(s).$$

The next lemma collects the relevant properties of these functions (see [13, Lemma 3.2, Propositions 1 & 2 and Corollary 1]).

LEMMA 3.1.

(i) *For $k \in \mathbb{Z}^+, N \ge N_0, s \in [1, 10]$ and $\sigma \in \mathfrak{U}_k(N)$, we have*

$$\Phi(N, \sigma, s) \le \{A(s) - H_{k,N_0}(s)\} \Theta(N, \sigma), \tag{3.6}$$

$$\Phi(N, \sigma, s) \ge \{a(s) + h_{k,N_0}(s)\} \Theta(N, \sigma). \tag{3.7}$$

(ii) *For* $k \in \mathbb{Z}^+$ *and* $s \in [1, 10]$, *we have* $H_k(s) \ge 0$ *and* $h_k(s) \ge 0$.

(iii) *For* $2 \le s \le s' \le 10$, *we have*

$$h(s) \ge h(s') + \int_{s-1}^{s'-1} \frac{H(t)}{t}\,dt, \tag{3.8}$$

$$H(s) \ge H(s') + \int_{s-1}^{s'-1} \frac{h(t)}{t}\,dt. \tag{3.9}$$

(iv) *The function* $H(s)$ *is decreasing on* $[1, 10]$. *The function* $h(s)$ *is increasing on* $[1, 2]$ *and decreasing on* $[2, 10]$.

We cannot give explicit expressions for $H(s)$ and $h(s)$. But it is possible to obtain numerical lower bounds for these two functions. Let

$$s_i := 2 + 0.1\,i \qquad (i \ge 0). \tag{3.10}$$

By [13, §7], we have numerical lower bounds of $H(s_i)$ for $2 \le i \le 10$. Next we shall consider the case of $11 \le i \le 29$ and the lower bounds of $h(s_i)$ for $0 \le i \le 29$. These will be used in the proof of the Theorem.

Let $\mathbf{1}_{[a,b]}(t)$ be the characteristic function of the interval $[a, b]$. Define

$$\sigma(a, b, c) := \int_a^b \log\left(\frac{c}{t-1}\right)\frac{dt}{t}, \qquad \sigma_0(t) := \frac{\sigma(3, t+2, t+1)}{1 - \sigma(3, 5, 4)}.$$

From (6.2) of [13] and the decreasing of $H(s)$, we deduce that

$$H(s_j) \ge \sum_{2 \le i \le 10} c_{i,j} H(s_i) \tag{3.11}$$

for $11 \le j \le 29$, where

$$c_{2,j} := \int_1^{s_2} \left\{ \frac{\sigma_0(t)}{t} \log\left(\frac{4}{s_j - 1}\right) + \frac{\mathbf{1}_{[s_j-2,3]}(t)}{t} \log\left(\frac{t+1}{s_j - 1}\right) \right\} dt$$

and

$$c_{i,j} := \int_{s_{i-1}}^{s_i} \left\{ \frac{\sigma_0(t)}{t} \log\left(\frac{4}{s_j - 1}\right) + \frac{\mathbf{1}_{[s_j-2,3]}(t)}{t} \log\left(\frac{t+1}{s_j - 1}\right) \right\} dt$$

for $3 \le i \le 10$. From (3.8) and the fact that $h(s) \ge 0$, we also derive

$$h(s_j) \ge \int_{s_j-1}^{5} \frac{H(t)}{t}\,dt \ge H(s_2) \log\left(\frac{s_{\max\{2,j-10\}}}{s_j - 1}\right) + \sum_{\max\{3,j-9\} \le i \le 29} H(s_i) \log\left(\frac{s_i}{s_{i-1}}\right) \tag{3.12}$$

for $0 \le j \le 29$.

Using the numerical lower bounds of $H(s_i)$ for $2 \le i \le 10$ given in [13, §7], (3.11) and (3.12), we get via a numerical computation the following results.

Table 1. Numerical lower bounds for $H(s_i)$

i	s_i	$H(s_i) \ge$	i	s_i	$H(s_i) \ge$	i	s_i	$H(s_i) \ge$
			10	3.0	0.0072943	20	4.0	0.0010835
			11	3.1	0.0061642	21	4.1	0.0008451
2	2.2	0.0223939	12	3.2	0.0052233	22	4.2	0.0006482
3	2.3	0.0217196	13	3.3	0.0044073	23	4.3	0.0004882
4	2.4	0.0202876	14	3.4	0.0036995	24	4.4	0.0003602
5	2.5	0.0181433	15	3.5	0.0030860	25	4.5	0.0002592
6	2.6	0.0158644	16	3.6	0.0025551	26	4.6	0.0001803
7	2.7	0.0129923	17	3.7	0.0020972	27	4.7	0.0001187
8	2.8	0.0100686	18	3.8	0.0017038	28	4.8	0.0000702
9	2.9	0.0078162	19	3.9	0.0013680	29	4.9	0.0000313

Table 2. Numerical lower bounds for $h(s_i)$

i	s_i	$h(s_i) \ge$	i	s_i	$h(s_i) \ge$	i	s_i	$h(s_i) \ge$
0	2.0	0.0232385	10	3.0	0.0077162	20	4.0	0.0010120
1	2.1	0.0211041	11	3.1	0.0066236	21	4.1	0.0008099
2	2.2	0.0191556	12	3.2	0.0055818	22	4.2	0.0006440
3	2.3	0.0173631	13	3.3	0.0046164	23	4.3	0.0005084
4	2.4	0.0157035	14	3.4	0.0037529	24	4.4	0.0003980
5	2.5	0.0141585	15	3.5	0.0030123	25	4.5	0.0003085
6	2.6	0.0127132	16	3.6	0.0023901	26	4.6	0.0002365
7	2.7	0.0113556	17	3.7	0.0018997	27	4.7	0.0001791
8	2.8	0.0100756	18	3.8	0.0015336	28	4.8	0.0001336
9	2.9	0.0088648	19	3.9	0.0012593	29	4.9	0.0000981

REMARK 2. It is possible to get better numerical lower bounds for $H(s_i)$ and $h(s_i)$ by applying (3.8) and (3.9) repeatedly. But the improvement will be small.

4. Application of Chen's double sieve. In this section, we apply Chen's double sieve to estimate the terms Υ_3, Υ_4, Υ_5 and Υ_6 in (2.2). Propositions 4.1–4.4 below concern a general context. These estimates are better than those obtained by the classical linear sieve, since $H(s), h(s) > 0$.

PROPOSITION 4.1. *Let* $0 < \phi_1 < \phi_2 < 1/4$ *and* $\kappa > 0$ *be such that*

$$\phi_2 + \kappa \le 1/2.$$

Then for $N \to \infty$, we have

$$\sum_{\substack{N^{\phi_1} \le p < N^{\phi_2} \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^\kappa) \le \left\{ 8 \int_{(1/2-\phi_2)/\kappa}^{(1/2-\phi_1)/\kappa} \frac{A(t) - H(t)}{t(1-2\kappa t)}\, dt + o(1) \right\} \Theta(N).$$

Proof. We keep the previous notation. Denote by S the sum in the proposition. Let $\alpha_j := N^{\phi_1}\Delta^j$ and J be the integer such that $\alpha_J \le N^{\phi_2} < \alpha_{J+1}$. We write

$$S = \sum_{1 \le j \le J} \sum_p \pi_{[\alpha_{j-1},\alpha_j)}(p) S(\mathcal{A}_p; \mathcal{P}(pN), \underline{p}^{1/\tau_p}) + R_1, \tag{4.1}$$

where $\tau_p := (\log \underline{p})/(\kappa \log N)$ and

$$R_1 := \sum_{\alpha_J \le p < N^{\phi_2}} S(\mathcal{A}_p; \mathcal{P}(N), N^\kappa) \ll \sum_{\alpha_J \le p < N^{\phi_2}} N/p \ll \Theta(N)\mathcal{L}^{-3}. \tag{4.2}$$

Introducing

$$\tau_j := (\log \underline{\alpha_j})/(\kappa \log N),$$

we easily see that $\pi_{[\alpha_{j-1},\alpha_j)}(p) \neq 0 \Rightarrow \tau_j \le \tau_p \le \tau_{j-1}$. Thus we can deduce from (4.1) and (4.2) that

$$S \le \sum_{1 \le j \le J} \sum_p \pi_{[\alpha_{j-1},\alpha_j)}(p) S(\mathcal{A}_p; \mathcal{P}(pN), \underline{p}^{1/\tau_j}) + O(\Theta(N)\mathcal{L}^{-3}), \tag{4.3}$$

where we have used the following estimates:

$$\begin{aligned}
&\sum_{1 \le j \le J} \sum_p \pi_{[\alpha_{j-1},\alpha_j)}(p) \{ S(\mathcal{A}_p; \mathcal{P}(pN), \underline{p}^{1/\tau_p}) - S(\mathcal{A}_p; \mathcal{P}(pN), \underline{p}^{1/\tau_j}) \} \\
&\qquad \le \sum_{1 \le j \le J} \sum_{\alpha_{j-1} \le p < \alpha_j} \sum_{\underline{p}^{1/\tau_p} \le p' < \underline{p}^{1/\tau_j}} N/(pp') \\
&\qquad \ll N\mathcal{L}^{-5} \sum_{1 \le j \le J} \sum_{\alpha_{j-1} \le p < \alpha_j} 1/p \ll \Theta(N)\mathcal{L}^{-3}.
\end{aligned}$$

Next we treat the inner sum (over p) in (4.3). Clearly for each $j \in \{1, \ldots, J\}$, our hypothesis on ϕ_1, ϕ_2 and κ ensures that $\pi_{[\alpha_{j-1},\alpha_j)} \in \mathfrak{U}_k(N)$ for all $k \ge 0$, $N_0 \ge 2$ and $N \ge N_0$, and $\tau_j \ge 1$. Thus we can apply (3.6) of Lemma 3.1 to estimate the sum over p (which is $\Phi(N, \pi_{[\alpha_{j-1},\alpha_j)}, \tau_j)$):

$$\begin{aligned}
S &\le \sum_{1 \le j \le J} \{A(\tau_j) - H_{k,N_0}(\tau_j)\} \Theta(N, \pi_{[\alpha_{j-1},\alpha_j)}) + O(\Theta(N)\mathcal{L}^{-3}) \\
&\le 4\,\mathrm{li}(N)\, \frac{C_N}{\log \underline{1}} \sum_{\alpha_0 \le p < \alpha_J} \frac{A(\tau_p) - H_{k,N_0}(\tau_p)}{(p-2)(1 - \log p/\log \underline{1})} + O(\Theta(N)\mathcal{L}^{-3}) \\
&\le 4\,\mathrm{li}(N)\, \frac{C_N}{\log \underline{1}} \sum_{N^{\phi_1} \le p < N^{\phi_2}} \frac{A(\tau_p) - H_{k,N_0}(\tau_p)}{(p-2)(1 - \log p/\log \underline{1})} + O(\Theta(N)\mathcal{L}^{-3}),
\end{aligned}$$

where we have used the fact that $A(s) - H_{k,N_0}(s)$ is increasing in s. An integration by parts with the prime number theorem shows that

$$\sum_{N^{\phi_1} \le p < N^{\phi_2}} \frac{A(\tau_p) - H_{k,N_0}(\tau_p)}{(p-2)(1-\log p/\log \underline{1})} = \int_{(1/2-\phi_2)/\kappa}^{(1/2-\phi_1)/\kappa} \frac{A(t) - H_{k,N_0}(t)}{t(1-2\kappa t)}\, dt + O_{\delta,k}(\varepsilon).$$

Hence

$$S \le 8 \left\{ \int_{(1/2-\phi_2)/\kappa}^{(1/2-\phi_1)/\kappa} \frac{A(t) - H_{k,N_0}(t)}{t(1-2\kappa t)}\, dt + O_{\delta,k}(\varepsilon) \right\} \Theta(N)$$

for $N \ge N_0$. From this, we infer that

$$\limsup_{N\to\infty} \frac{S}{\Theta(N)} \le 8 \int_{(1/2-\phi_2)/\kappa}^{(1/2-\phi_1)/\kappa} \frac{A(t) - H_{k,N_0}(t)}{t(1-2\kappa t)}\, dt + O_{\delta,k}(\varepsilon),$$

which implies, by taking $N \to \infty$, $k \to \infty$ and $\varepsilon \to 0$,

$$\limsup_{N\to\infty} \frac{S}{\Theta(N)} \le 8 \int_{(1/2-\phi_2)/\kappa}^{(1/2-\phi_1)/\kappa} \frac{A(t) - H(t)}{t(1-2\kappa t)}\, dt.$$

Clearly this is equivalent to the required inequality. ∎

In a similar fashion we can prove the following results.

Proposition 4.2. *Let $0 < \phi_1 < \phi_2 < 1/6$ and $\kappa > 0$ be such that*

$$2\phi_2 + \kappa \le 1/2.$$

Then for $N \to \infty$, we have

$$\sum_{\substack{N^{\phi_1} \le p_1 < p_2 < N^{\phi_2} \\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\kappa}) \ge \left\{ 8 \int_{\phi_1}^{\phi_2} \int_{(1/2-\phi_2-t)/\kappa}^{(1/2-2t)/\kappa} \frac{a(u) + h(u)}{t(1-2t-2\kappa u)}\, dt\, du + o(1) \right\} \Theta(N).$$

Proposition 4.3. *Let $0 < \phi_1 < \phi_2 \le \phi_3 < \phi_4 < 1/4$ and $\kappa > 0$ be such that*

$$2\phi_2 + \phi_4 < 1/2 \quad \textit{and} \quad \phi_2 + \phi_4 + \kappa \le 1/2.$$

Then for $N \to \infty$, we have

$$\sum_{N^{\phi_1} \le p_1 < N^{\phi_2}} \sum_{\substack{N^{\phi_3} \le p_1 < N^{\phi_4} \\ (p_1p_2,N)=1}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\kappa}) \ge \left\{ 8 \int_{\phi_1}^{\phi_2} \int_{(1/2-\phi_4-t)/\kappa}^{(1/2-\phi_3-t)/\kappa} \frac{a(u) + h(u)}{tu(1-2t-2\kappa u)}\, dt\, du + o(1) \right\} \Theta(N).$$

Finally, we estimate the sum (1.5) with $\phi_1 \geq 1/4$. In this case, we cannot directly apply our delicate Chen's double sieve because of the first condition of (3.1). As remarked by Cai [2], it is possible to use a simplified version of Chen's double sieve. This approach will give a result better than using the classical linear sieve but weaker than Proposition 4.1, since, without iteration, $\Psi_1(s)$ or $\Psi_2(s)$ are principal contributions of $H(s)$. (See Lemmas 5.1 and 5.2 of [13] and compare Proposition 4.4 below and Proposition 4.1.)

PROPOSITION 4.4. *Let $\kappa > 0$, $\phi > 0$ and $2 \leq s \leq 3 \leq s' \leq 5$ be such that*

$$1/4 \leq 1/2 - s\kappa < \phi.$$

Then for $N \to \infty$, we have

$$\sum_{\substack{N^{1/2-s\kappa} \leq p < N^{\phi} \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), N^{\kappa}) \leq \bigg\{ 8 \int_{(1/2-\phi)/\kappa}^{s} \frac{A(t) - \Psi_1(s)}{t(1-2\kappa t)}\, dt + o(1) \bigg\} \Theta(N),$$

where

$$\Psi_1(s) := -\int_2^{s'-1} \frac{\log(t-1)}{t}\, dt + \frac{1}{2} \int_{1-1/s}^{1-1/s'} \frac{\log(s't-1)}{t(1-t)}\, dt - \max_{\phi \geq 2} \iiint_{1/s' \leq t \leq u \leq v \leq 1/s} \omega\bigg(\frac{\phi - t - u - v}{u}\bigg) \frac{dt\, du\, dv}{tu^2 v}$$

and $\omega(u)$ is Buchstab's function. The same result holds if we replace $\Psi_1(s)$ by $\Psi_2(s)$, where $\Psi_2(s)$ is defined as in Lemma 5.2 *of* [13].

Proof. For simplicity, we denote the sum by S. Since $N^{\kappa} \geq \underline{p}^{1/s}$ for $p \geq N^{1/2-s\kappa}$, we can write

$$S \leq \sum_{\substack{N^{1/2-s\kappa} \leq p < N^{\phi} \\ (p,N)=1}} S(\mathcal{A}_p; \mathcal{P}(N), \underline{p}^{1/s}) \leq \sum_{1 \leq j \leq J} \sum_{p} \pi_{[\alpha_{j-1}, \alpha_j)}(p) S(\mathcal{A}_p; \mathcal{P}(N), \underline{p}^{1/s}),$$

where $\alpha_j := N^{1/2-s\kappa} \Delta^j$ and J is the integer such that $\alpha_{J-1} \leq N^{\phi} < \alpha_J$.

Similar to Lemma 4.1 of [13], we can prove that there is a constant $\eta > 0$ such that

$$S \leq \sum_{1 \leq j \leq J} \sum_{p} \pi_{[\alpha_{j-1}, \alpha_j)}(p) \bigg(\Omega_1(p) - \frac{1}{2}\, \Omega_2(p) + \frac{1}{2}\, \Omega_3(p) \bigg) + O(N^{1-\eta}), \tag{4.4}$$

where

$$\Omega_1(p) := S(\mathcal{A}_p; \mathcal{P}(pN), \underline{p}^{1/s'}),$$

$$\Omega_2(p) := \sum_{\substack{\underline{p}^{1/s'} \le p_1 < \underline{p}^{1/s} \\ (p_1, N)=1}} S(\mathcal{A}_{pp_1}; \mathcal{P}(pN), \underline{p}^{1/s'}),$$

$$\Omega_3(p) := \sum\sum\sum_{\substack{\underline{p}^{1/s'} \le p_1 < p_2 < p_3 < \underline{p}^{1/s} \\ (p_1p_2p_3, N)=1}} S(\mathcal{A}_{pp_1p_2p_3}; \mathcal{P}(pp_1N), p_2).$$

Similar to (5.1), (5.2) and (5.9) of [13], we can prove, uniformly for $N \ge 10$ and for $1 \le i \le 3$, $1 \le j \le J$,

$$\sum_p \pi_{[\alpha_{j-1},\alpha_j)}(p)\Omega_i(p) \le \{\widetilde{\Omega}_i(s,s') + o(1)\}\Theta(N, \pi_{[\alpha_{j-1},\alpha_j)}),$$

where

$$\widetilde{\Omega}_1(s,s') := A(s'),$$

$$\widetilde{\Omega}_2(s,s') := \int_{1-1/s}^{1-1/s'} \frac{a(s't)}{t(1-t)}\,dt,$$

$$\widetilde{\Omega}_3(s,s') := 2\max_{\phi\ge 2} \iiint_{1/s' \le t \le u \le v \le 1/s} \omega\Big(\frac{\phi - t - u - v}{u}\Big)\frac{dt\,du\,dv}{tu^2v}.$$

Inserting these into (4.4) and noticing that

$$A(s') = 1 + \int_2^{s'-1} \frac{\log(t-1)}{t}\,dt, \qquad a(s't) = \log(s't - 1),$$

we find that

$$\begin{aligned} S &\le \{1 - \Psi_1(s) + o(1)\}\sum_{1\le j\le J}\Theta(N, \pi_{[\alpha_{j-1},\alpha_j)}) + O(N^{1-\eta}) \\ &\le \bigg\{8(1-\Psi_1(s))\int_{1/2-s\kappa}^{\phi}\frac{dt}{t(1-2t)} + o(1)\bigg\}\Theta(N), \end{aligned}$$

which is equivalent to the required result for the case of $\Psi_1(s)$, since

$$\int_{(1/2-\phi)/\kappa}^{s} \frac{A(t)}{t(1-2\kappa t)}\,dt = \int_{(1/2-\phi)/\kappa}^{s} \frac{dt}{t(1-2\kappa t)} = \int_{1/2-s\kappa}^{\phi} \frac{dt}{t(1-2t)}.$$

The case of $\Psi_2(s)$ can be treated in the same way. The main difference is the use of Lemma 4.2 of [13] in place of Lemma 4.1 there. We omit the details. ∎

5. Proof of the Theorem. Set

$$\kappa_1 = 1/13.27 \quad \text{and} \quad \kappa_2 = 1/8.24, \tag{5.1}$$

which satisfy the hypothesis of Lemma 2.2. Next we estimate all the terms Υ_i in (2.2).

1° *Lower bounds of* Υ_1 *and* Υ_2. Write $N^\kappa = \underline{1}^{\kappa'}$ with $\kappa' := \kappa/(1/2-\delta)$. By using (4.2) with $\sigma := \mathbf{1}_{\{1\}}$ (the characteristic function of $\{1\}$), it follows that

$$\begin{aligned}\Upsilon_i = \Phi(N, \mathbf{1}_{\{1\}}, 1/\kappa_i') &\ge \{a(1/\kappa_i') + h_{k,N_0}(1/\kappa_i')\}\Theta(N, \mathbf{1}_{\{1\}}) \\ &\ge \{F_i + o(1)\}\Theta(N)\end{aligned} \tag{5.2}$$

with

$$F_i := 8a(1/(2\kappa_i)) + G_i \quad (i = 1, 2)$$

and

$$G_i := 8h(1/(2\kappa_i)) \quad (i = 1, 2).$$

2° *Upper bounds of* Υ_3 *and* Υ_4. We divide the sum Υ_3 (resp. Υ_4) into subsums according to

(a) $N^{\kappa_1} \le p < N^{1/4}$,
(b) $N^{1/4} \le p < N^{1/2 - s_9\kappa_1}$,
(c) $N^{1/2 - s_j\kappa_1} \le p < N^{1/2 - s_{j-1}\kappa_1}$ $(9 \ge j \ge 4)$,
(d) $N^{1/2 - s_3\kappa_1} \le p < N^{1/3}$

(resp. $N^{\kappa_1} \le p < N^{1/4}$ or $N^{1/4} \le p < N^{1/2-3\kappa_1}$), where s_i is defined by (3.10). The contribution of (a) is estimated by Proposition 4.1 and we evaluate (b) (resp. $N^{1/4} \le p < N^{1/2-3\kappa_1}$) by the classical linear sieve. The remaining subsums are treated by Proposition 4.4. It is worth pointing out that case (b) requires another treatment because $\Psi_1(s_{10}) = 0$ (see Table 3 below). Thus we obtain

$$\Upsilon_i \le \{F_i + o(1)\}\Theta(N) \quad (i = 3, 4), \tag{5.3}$$

where

$$F_3 := 8 \int_{1/(6\kappa_1)}^{1/(2\kappa_1)-1} \frac{A(t)}{t(1-2\kappa_1 t)}\, dt - G_3,$$

$$F_4 := 8 \int_{3}^{1/(2\kappa_1)-1} \frac{A(t)}{t(1-2\kappa_1 t)}\, dt - G_4,$$

and

$$G_4 := 8 \int_{1/(4\kappa_1)}^{1/(2\kappa_1)-1} \frac{H(t)}{t(1-2\kappa_1 t)}\, dt,$$

$$G_3 := 8 \int_{1/(4\kappa_1)}^{1/(2\kappa_1)-1} \frac{H(t)}{t(1-2\kappa_1 t)}\, dt + 8 \int_{1/(6\kappa_1)}^{s_3} \frac{\Psi_2(s_3)}{t(1-2\kappa t)}\, dt$$
$$+ 8 \sum_{4\le i\le 5} \int_{s_{i-1}}^{s_i} \frac{\Psi_2(s_i)}{t(1-2\kappa t)}\, dt + 8 \sum_{6\le i\le 9} \int_{s_{i-1}}^{s_i} \frac{\Psi_1(s_i)}{t(1-2\kappa t)}\, dt.$$

3° *Lower bounds of* Υ_5 *and* Υ_6. Since $\kappa_1 + 2\kappa_2 = 0.318\ldots < 1/2$, Proposition 4.2 yields

$$\Upsilon_5 \ge \{F_5 + o(1)\}\Theta(N), \tag{5.4}$$

where

$$F_5 := 8 \int_{\kappa_1}^{\kappa_2} \int_{(1/2-\kappa_2-t)/\kappa_1}^{(1/2-2t)/\kappa_1} \frac{a(u)\, dt\, du}{tu(1-2t-2\kappa_1 u)} + G_5,$$

$$G_5 := 8 \int_{\kappa_1}^{\kappa_2} \int_{(1/2-\kappa_2-t)/\kappa_1}^{(1/2-2t)/\kappa_1} \frac{h(u)\, dt\, du}{tu(1-2t-2\kappa_1 u)}.$$

We divide the double sum Υ_6 into three subsums according to

(a) $N^{\kappa_1} \le p_1 < N^{\kappa_2} \le p_2 < N^{1/2-2\kappa_2}$,

(b) $N^{\kappa_1} \le p_1 < N^{3\kappa_1/2}$ and $N^{1/2-2\kappa_2} \le p_2 < N^{1/2-3\kappa_1}$,

(c) $N^{3\kappa_1/2} \le p_1 < N^{\kappa_2}$ and $N^{1/2-2\kappa_2} \le p_2 < N^{1/2-3\kappa_1}$.

The first two subsums can be estimated by Proposition 4.3 and the last one by the classical linear sieve. Thus we obtain

$$\Upsilon_6 \ge \{F_6 + o(1)\}\Theta(N), \tag{5.5}$$

where

$$F_6 := 8 \int_{\kappa_1}^{\kappa_2} \int_{(3\kappa_1-t)/\kappa_1}^{(1/2-\kappa_2-t)/\kappa_1} \frac{a(u)\, dt\, du}{tu(1-2t-2\kappa_1 u)} + G_6,$$

$$G_6 := 8 \int_{\kappa_1}^{\kappa_2} \int_{(2\kappa_2-t)/\kappa_1}^{(1/2-\kappa_2-t)/\kappa_1} \frac{h(u)\, dt\, du}{tu(1-2t-2\kappa_1 u)}$$
$$+ 8 \int_{\kappa_1}^{3\kappa_1/2} \int_{(3\kappa_1-t)/\kappa_1}^{(2\kappa_2-t)/\kappa_1} \frac{h(u)\, dt\, du}{tu(1-2t-2\kappa_1 u)}.$$

4° *Upper bounds of* Υ_i *for* $i = 7, 8, 9, 10, 11$. Clearly the terms Υ_7, Υ_8, Υ_9, Υ_{10} and Υ_{11} here are the terms Υ_7 (with $\sigma_1 = 1/2 - 3\kappa_1$), Υ_9 (with $\sigma_1 = 1/3$), Υ_{10} (with $\sigma_2 = 1/2 - 3\kappa_1$), Υ_{13} and Υ_{14} of (9.4) in [13]. Thus (10.10), (10.11), (10.12) of [13] give us the estimates

$$\Upsilon_i \le \{F_i + o(1)\}\Theta(N) \qquad (i = 7, 8, 9, 10, 11), \tag{5.6}$$

where

$$F_7 := 8 \int_2^{2/(1-6\kappa_1)-1} \frac{\log(t-1)}{t}\, dt,$$

$$F_8 := \frac{36}{5} \int_{\kappa_1}^{1/10} \frac{\log(2-3t)}{t(1-t)^2}\, dt + 8 \int_{1/10}^{1/3} \frac{\log(2-3t)}{t(1-t)}\, dt,$$

$$F_9 := 8 \int_{\kappa_2}^{1/2-3\kappa_1} \frac{\log\{(1+6\kappa_1-2t)/(1-6\kappa_1)\}}{t(1-t)}\, dt,$$

$$\begin{aligned} F_{10} := \frac{36}{5} \int_{\kappa_1}^{1/10} \frac{dt_1}{t_1(1-t_1)} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{t_3}^{\kappa_2} \omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big) \frac{dt_4}{t_4} \\ + 8 \int_{1/10}^{\kappa_2} \frac{dt_1}{t_1} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{t_3}^{\kappa_2} \omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big) \frac{dt_4}{t_4}, \end{aligned}$$

$$\begin{aligned} F_{11} := \frac{36}{5} \int_{\kappa_1}^{1/10} \frac{dt_1}{t_1(1-t_1)} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \\ \times \int_{\kappa_2}^{1/2-2\kappa_1-t_3} \omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big) \frac{dt_4}{t_4} \\ + 8 \int_{1/10}^{\kappa_2} \frac{dt_1}{t_1} \int_{t_1}^{\kappa_2} \frac{dt_2}{t_2^2} \int_{t_2}^{\kappa_2} \frac{dt_3}{t_3} \int_{\kappa_2}^{1/2-2\kappa_1-t_3} \omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big) \frac{dt_4}{t_4}, \end{aligned}$$

and $\omega(t)$ is the Buchstab function (see Lemma 2.10 of [13]).

Inserting (5.2)–(5.6) into (2.2), we get

$$D_{1,2}(N) \ge \{F(\kappa_1, \kappa_2) + o(1)\}\Theta(N),$$

where

$$\begin{aligned} &F(\kappa_1, \kappa_2) \\ &\quad := \tfrac{1}{4}(3F_1 + F_2 - F_3 - F_4 + F_5 + F_6 - 2F_7 - F_8 - F_9 - F_{10} - F_{11}). \end{aligned}$$

5° *Numerical computation.* From (3.2) and (3.3), we deduce easily that

$$a(s)=\begin{cases}0 & (0<s\le 2),\\ \log(s-1) & (2<s\le 4),\\ \log(s-1)+\int_3^{s-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(u-1)}{u}\,du & (4<s\le 6),\\ \log(s-1)+\int_3^{s-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(u-1)}{u}\,du & \\ \quad+\int_5^{s-1}\frac{dt}{t}\int_4^{t-1}\frac{du}{u}\int_3^{u-1}\frac{dv}{v}\int_2^{v-1}\frac{\log(w-1)}{w}\,dw & (6<s\le 8),\end{cases}$$

and

$$A(s)=\begin{cases}1 & (0<s\le 3),\\ 1+\int_2^{s-1}\frac{\log(t-1)}{t}\,dt & (3<s\le 5),\\ 1+\int_2^{s-1}\frac{\log(t-1)}{t}\,dt & \\ \quad+\int_4^{s-1}\frac{dt}{t}\int_3^{t-1}\frac{du}{u}\int_2^{u-1}\frac{\log(v-1)}{v}\,dv & (5<s\le 7).\end{cases}$$

By using (3.8), we have

$$G_2\ge 8\biggl(h(s_{22})+\int_{s_{22}-1}^{1/(2\kappa_2)-1}\frac{H(t)}{t}\,dt\biggr)\ge 0.005283.$$

In order to estimate G_4, we use Table 1 and the decreasing of $H(s)$ to obtain

$$G_4=8\int_{1/(4\kappa_1)}^{1/(2\kappa_1)-1}\frac{H(t)}{t(1-2\kappa_1 t)}\,dt\ge 8\sum_{14\le i\le 29}g_4^i H(s_i)\ge 0.008860$$

with

$$g_4^{14}:=\log\biggl(\frac{2\kappa_1 s_{14}}{1-2\kappa_1 s_{14}}\biggr),$$
$$g_4^{i}:=\log\biggl(\frac{s_i(1-2\kappa_1 s_{i-1})}{s_{i-1}(1-2\kappa_1 s_i)}\biggr)\qquad(15\le i\le 29).$$

With a simpler calculation, we get

$$G_3=G_4+8\sum_{3\le i\le 5}g_3^i\Psi_2(s_i)+8\sum_{6\le i\le 9}g_3^i\Psi_1(s_i)\ge 0.039890$$

with

$$g_3^3 := \log\left(\frac{4\kappa_1 s_3}{1-2\kappa_1 s_3}\right),$$

$$g_3^i := \log\left(\frac{s_i(1-2\kappa_1 s_{i-1})}{s_{i-1}(1-2\kappa_1 s_i)}\right) \qquad (4 \le i \le 9).$$

Here we have used Table 1 of [13] on the lower bounds for $\Psi_2(s_i)$ $(3 \le i \le 5)$ and $\Psi_1(s_i)$ $(6 \le i \le 9)$:

Table 3. Lower bounds for $\Psi_1(s_i)$ and $\Psi_2(s_i)$

i	s_i	s_i'	$\kappa_{1,i}$	$\kappa_{2,i}$	$\kappa_{3,i}$	$\Psi_1(s_i)$	$\Psi_2(s_i)$
3	2.3	4.50	3.54	2.88	2.43		0.015247971
4	2.4	4.46	3.57	2.87	2.40		0.013898757
5	2.5	4.12	3.56	2.91	2.50		0.011776059
6	2.6	3.58				0.009405211	
7	2.7	3.47				0.006558950	
8	2.8	3.34				0.003536751	
9	2.9	3.19				0.001056651	
10	3.0	3.00				0	

Similarly

$$\begin{aligned} G_5 &= 8\int_{\kappa_1}^{\kappa_2}\int_{(1/2-\kappa_2-t)/\kappa_1}^{(1/2-2t)/\kappa_1} \frac{h(u)\,dt\,du}{tu(1-2t-2\kappa_1 u)} \\ &= 8\int_{(1/2-2\kappa_2)/\kappa_1}^{(1/2-\kappa_1-\kappa_2)/\kappa_1} h(u)\log\left(\frac{2\kappa_2}{1-2\kappa_2-2\kappa_1 u}\right)\frac{du}{u(1-2\kappa_1 u)} \\ &\quad + 8\int_{(1/2-\kappa_1-\kappa_2)/\kappa_1}^{(1/2-2\kappa_1)/\kappa_1} h(u)\log\left(\frac{1-2\kappa_1-2\kappa_1 u}{2\kappa_1}\right)\frac{du}{u(1-2\kappa_1 u)} \\ &\ge 8\sum_{15\le i\le 27} g_5^i h(s_i) \ge 0.001359 \end{aligned}$$

with

$$g_5^{15} := \int_{(1/2-2\kappa_2)/\kappa_1}^{s_{15}} \log\left(\frac{2\kappa_2}{1-2\kappa_2-2\kappa_1 u}\right)\frac{du}{u(1-2\kappa_1 u)},$$

$$g_5^i := \int_{s_{i-1}}^{s_i} \log\left(\frac{2\kappa_2}{1-2\kappa_2-2\kappa_1 u}\right)\frac{du}{u(1-2\kappa_1 u)} \qquad (16 \le i \le 20),$$

$$g_5^{21} := \int_{s_{20}}^{(1/2-\kappa_1-\kappa_2)/\kappa_1} \log\left(\frac{2\kappa_2}{1-2\kappa_2-2\kappa_1 u}\right)\frac{du}{u(1-2\kappa_1 u)} + \int_{(1/2-\kappa_1-\kappa_2)/\kappa_1}^{s_{21}} \frac{\log(1/(2\kappa_1)-1-u)}{u(1-2\kappa_1 u)}\,du,$$

$$g_5^{i} := \int_{s_{i-1}}^{s_i} \frac{\log(1/(2\kappa_1)-1-u)}{u(1-2\kappa_1 u)}\,du \qquad (22 \le i \le 26),$$

$$g_5^{27} := \int_{s_{26}}^{(1/2-2\kappa_1)/\kappa_1} \frac{\log(1/(2\kappa_1)-1-u)}{u(1-2\kappa_1 u)}\,du;$$

and

$$\begin{aligned}
G_6 &= 8\int_{\kappa_1}^{\kappa_2}\int_{(2\kappa_2-t)/\kappa_1}^{(1/2-\kappa_2-t)/\kappa_1} \frac{h(u)\,dt\,du}{tu(1-2t-2\kappa_1 u)} \\
&\quad + 8\int_{\kappa_1}^{3\kappa_1/2}\int_{(3\kappa_1-t)/\kappa_1}^{(2\kappa_2-t)/\kappa_1} \frac{h(u)\,dt\,du}{tu(1-2t-2\kappa_1 u)} \\
&\ge 8\int_{2}^{(1/2-2\kappa_2)/\kappa_1} \log\left(\frac{\kappa_2(1-2\kappa_1-2\kappa_1 u)}{\kappa_1(1-2\kappa_2-2\kappa_1 u)}\right)\frac{h(u)\,du}{u(1-2\kappa_1 u)} \\
&\quad + 8\int_{(1/2-2\kappa_2)/\kappa_1}^{(1/2-\kappa_1-\kappa_2)/\kappa_1} \log\left(\frac{(1-2\kappa_1-2\kappa_1 u)(1-2\kappa_2-2\kappa_1 u)}{4\kappa_1\kappa_2}\right)\frac{h(u)\,du}{u(1-2\kappa_1 u)} \\
&\ge 8\sum_{1\le i\le 21} g_6^i h(s_i) \ge 0.060469
\end{aligned}$$

with

$$g_6^{i} := \int_{s_{i-1}}^{s_i} \log\left(\frac{\kappa_2(1-2\kappa_1-2\kappa_1 u)}{\kappa_1(1-2\kappa_2-2\kappa_1 u)}\right)\frac{du}{u(1-2\kappa_1 u)} \qquad (1 \le i \le 14),$$

$$g_6^{15} := \int_{s_{14}}^{(1/2-2\kappa_2)/\kappa_1} \log\left(\frac{\kappa_2(1-2\kappa_1-2\kappa_1 u)}{\kappa_1(1-2\kappa_2-2\kappa_1 u)}\right)\frac{du}{u(1-2\kappa_1 u)} + \int_{(1/2-2\kappa_2)/\kappa_1}^{s_{15}} \log\left(\frac{(1-2\kappa_1-2\kappa_1 u)(1-2\kappa_2-2\kappa_1 u)}{4\kappa_1\kappa_2}\right)\frac{du}{u(1-2\kappa_1 u)},$$

$$g_6^{i} := \int_{s_{i-1}}^{s_i} \log\left(\frac{(1-2\kappa_1-2\kappa_1 u)(1-2\kappa_2-2\kappa_1 u)}{4\kappa_1\kappa_2}\right)\frac{du}{u(1-2\kappa_1 u)} \qquad (16 \le i \le 20),$$

$$g_6^{21} := \int_{s_{20}}^{(1/2-\kappa_1-\kappa_2)/\kappa_1} \log\left(\frac{(1-2\kappa_1-2\kappa_1 u)(1-2\kappa_2-2\kappa_1 u)}{4\kappa_1\kappa_2}\right)\frac{du}{u(1-2\kappa_1 u)}.$$

To simplify the computation of F_{10} and F_{11}, we make use of the fact that $\omega(t) \le 0.561522$ for $t \ge 3.4$.

Finally, a numerical computation yields

$$\begin{aligned} F(\kappa_1,\kappa_2) \ge \tfrac{1}{4}\{&3\cdot 14.900897 + (9.103015+0.005283) \\ &- (23.652925-0.039890) - (19.643510-0.008860) \\ &+ (1.654808+0.001359) + (3.819092+0.060469) \\ &- 2\cdot 0.585179 - 5.279581 - 5.372410 - 0.104305 - 0.543858\} \\ > 0.899.& \end{aligned}$$

This completes the proof of the Theorem. ■

References

[1] E. Bombieri and H. Davenport, *Small differences between prime numbers*, Proc. Roy. Soc. Ser. A 293 (1966), 1–18.

[2] Y. C. Cai, *On Chen's theorem (II)*, J. Number Theory, to appear.

[3] —, *Chen's theorem with small primes*, Acta Math. Sinica (Chin. Ser.) 48 (2005), 593–598 (in Chinese).

[4] Y. C. Cai and M. G. Lu, *On Chen's theorem*, in: Analytic Number Theory (Beijing/Kyoto, 1999), Dev. Math. 6, Kluwer, Dordrecht, 2002, 99–119.

[5] J. R. Chen, *On the representation of a large even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica 16 (1973), 157–176.

[6] —, *On the representation of a large even integer as the sum of a prime and the product of at most two primes (II)*, ibid. 21 (1978), 421–430.

[7] —, *Further improvement on the constant in the proposition '1+2': On the representation of a large even integer as the sum of a prime and the product of at most two primes (II)*, ibid. 21 (1978), 477–494 (in Chinese).

[8] —, *On the Goldbach's problem and the sieve methods*, ibid. 21 (1978), 701–739.

[9] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[10] G. H. Hardy and J. E. Littlewood, *Some problems of 'partitio numerorum' III: On the expression of a number as a sum of primes*, Acta Math. 44 (1923), 1–70.

[11] C. D. Pan and C. B. Pan, *Goldbach Conjecture*, Science Press, Beijing, 1992.

[12] J. Wu, *Sur la suite des nombres premiers jumeaux*, Acta Arith. 55 (1990), 365–394.

[13] —, *Chen's double sieve, Goldbach's conjecture and the twin prime problem*, ibid. 114 (2004), 215–273.

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