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The structure of the tame kernels of quadratic number fields (II)

by

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1. Introduction. Let F be a quadratic number field and O_F the ring of its integers. Some methods of determining the structure of the 2-Sylow subgroup of the tame kernel K_2O_F have been established. The results of [13–15] give the 4-rank and the 8-rank of K_2O_F. We refer to [8] for the results on relative quadratic extensions. When the discriminant has at most three divisors, the 4-rank of K_2O_F has been given explicitly in [13] and [14].

Recently, the second author [16] introduced sign matrices via Legendre symbols to determine the 4-rank of K_2O_F. In the relative quadratic extension case, the sign matrices defined via local Hilbert symbols to compute the 4-rank of the tame kernel appeared earlier in [8]. In [16], the second author defined the type of a square-free integer d (see Section 4 below) and determined a lower bound for the 4-rank of K_2O_F (where $F = \mathbb{Q}(\sqrt{d})$) for each type of quadratic number field F. To be more precise, he found all types of real quadratic fields for which always $r_4(K_2O_F) \geq 1$, and for any other type he showed that there is a set of d of positive density for which $r_4(K_2O_F) = 0$ and a set of positive density for which $r_4(K_2O_F) \geq 1$. For imaginary quadratic fields, he also established similar results.

In this paper, we use the method developed in [16] to determine all possible values of $r_4(K_2O_F)$ for each type of real quadratic number field F. In particular, for each type of real quadratic field we determine the maximum possible value of $r_4(K_2O_F)$ and we show that each integer between the lower and upper bounds occurs as a value of the 4-rank of K_2O_F for infinitely many F.

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2. Preliminaries. We introduce the following notations:

- $F = \mathbb{Q}(\sqrt{d})$ with $d \in \mathbb{N}$ square-free.
- O_F is the ring of integers of F.
- K_2O_F is the tame kernel of F.
- $\nabla^2 = \{\alpha \in K_2O_F \mid \alpha = \beta^2 \text{ for some } \beta \in K_2O_F\}$.
- ${}_2K_2O_F = \{x \in K_2O_F \mid x^2 = 1\}$.
- $S(d) = \{\pm 1, \pm 2\}$.
- $r_4(K_2O_F)$ or r_4 for short denotes the 4-rank of K_2O_F.
- Given integers a and b with $b \neq 0$, $\left(\frac{a}{b}\right)$ denotes the Jacobi symbol. In particular, $\left(\frac{a}{b}\right)$ is the Legendre symbol if b is an odd prime.

It follows from J. Browkin and A. Schinzel [3] that ${}_2K_2O_F$ ($F = \mathbb{Q}(\sqrt{d})$) is generated by $\{-1, m\}$, $m \mid d$, together with $\{-1, u_i + \sqrt{d}\}$ if $\{-1, \pm 2\} \cap NF \neq \emptyset$, where $u_i \in \mathbb{Z}$ is such that $d = u_i^2 - c_i w_i^2$ for some $w_i \in \mathbb{Z}$ and $c_i \in \{-1, \pm 2\} \cap NF$. Suppose that $m \mid d$ and $m > 0$, and assume that $m \equiv 1 \pmod 4$ if $d \equiv 1 \pmod 8$. Then from [15] we know that there exists a prime $p \equiv 1 \pmod 4$ such that $\delta pmZ^2 = X^2 + dY^2$ is solvable for $\delta = 1$ or 2. If $2 \in NF$, then $d = u^2 - 2w^2$, where $u, w \in \mathbb{Z}$. Assume that $u > 0$ and $u + w \equiv 1 \pmod 4$; if $d \equiv 1 \pmod 8$, then from [15] we know that there is a prime $p \equiv 1 \pmod 4$ such that $pm(u+w)Z^2 = X^2 + dY^2$ is solvable. On the other hand, it is proved in [16] that: (i) $\{-1, m\} \in \nabla^2$ if and only if $\varepsilon pZ^2 = X^2 - dY^2$ is solvable for $\varepsilon \in S(d)$, where $p \equiv 1 \pmod 4$ is a prime such that $\eta pmZ^2 = X^2 + dY^2$ is solvable for $\eta = 1$ or 2; (ii) if $2 \in NF$, then $\{-1, m(u + \sqrt{d})\} \in \nabla^2$ if and only if $\varepsilon pZ^2 = X^2 - dY^2$ is solvable for $\varepsilon \in S(d)$, where $p \equiv 1 \pmod 4$ is a prime such that $pm(u+w)Z^2 = X^2 + dY^2$ is solvable. So, to determine the 4-rank of K_2O_F, we need only consider the solvability of the above indefinite equations $\varepsilon pZ^2 = X^2 - dY^2$.

Let $d = 2^\sigma l_1 \cdots l_n$ be the prime factorization, where $\sigma = 0$ or 1. Consider the vector $v(p, \varepsilon) = (\delta_1, \ldots, \delta_n)$, where $\delta_i = \left(\frac{\varepsilon p}{l_i}\right)$ with $\varepsilon \in S(d)$ for $1 \le i \le n$. Then by Legendre's Theorem on the Diophantine equation $aX^2 + bY^2 + cZ^2 = 0$ (see [10]) we have

Lemma 2.1 ([16]). *With the notation as above, we have $\{-1, m\} \in \nabla^2$ or $\{-1, m(u + \sqrt{d})\} \in \nabla^2$ if and only if $v(p, \varepsilon) = (1, \ldots, 1)$ for some $\varepsilon \in S(d)$.*

Let $n_i \mid d$ for $1 \le i \le t$. Suppose $\eta_i p_i n_i Z^2 = X^2 + dY^2$ (or $\eta_i p_i n_i (u+w) Z^2 = X^2 + dY^2$) are solvable for primes $p_i \equiv 1 \pmod 4$ and $\eta_i = 1$ or 2 ($1 \le i \le t$). We have

Lemma 2.2 ([16]). *$\{-1, n_1 \cdots n_t\} \in \nabla^2$ if and only if*

$$v(p, \varepsilon) = \left(\left(\frac{\varepsilon}{l_1}\right) \prod_{i=1}^{t} \delta_{i1}, \ldots, \left(\frac{\varepsilon}{l_n}\right) \prod_{i=1}^{t} \delta_{in} \right) = (1, \ldots, 1)$$

for some $\varepsilon \in S(d)$.

Recall from [16] that a set $S = \{m_1, \dots, m_k\}$ is called a *system of* $\bigtriangledown$-*representatives* of F if $\{-1, m_1\}, \dots, \{-1, m_k\}$ generate ${}_2K_2O_F \cap (K_2F)^2$ and $m_1 \pmod{(F^{*2} \cup 2F^{*2})}, \dots, m_k \pmod{(F^{*2} \cup 2F^{*2})}$ are multiplicatively independent. For the exact value of k and a system of $\bigtriangledown$-representatives of real quadratic number fields F, we have the following

Lemma 2.3. *Let $F = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{N}$ square-free, be a real quadratic field and $d = 2^\sigma l_1 \cdots l_n$ the prime factorization, where $\sigma = 0$ or 1. Then we can choose a system of $\bigtriangledown$-representatives as follows*:

(i) $\{l_1, \dots, l_{n-1}, u + \sqrt{d}\}$ *if either* (a) $l_i \equiv 1 \pmod 8$ $(1 \le i \le n)$ *and* $u + w \equiv 1 \pmod 4$ *or* (b) $d \not\equiv 1 \pmod 8$ *and* $2 \in NF$;

(ii) $\{l_1, \dots, l_{n-1}\}$ *if either* (a) $d \not\equiv 1 \pmod 8$ *and* $2 \notin NF$ *or* (b) $d \equiv 1 \pmod 8$, $2 \notin NF$ *and* $l_i \equiv 1 \pmod 4$ $(1 \le i \le n)$ *or* (c) $d \equiv 1 \pmod 8$, $2 \in NF$ *and* $u + w \equiv 3 \pmod 4$;

(iii) $\{l_1l_2, l_1l_3, \dots, l_1l_m, l_{m+1}, \dots, l_{n-1}, u+\sqrt{d}\}$ *if* $d \equiv 1 \pmod 8$, $2 \in NF$ *with* $u + w \equiv 1 \pmod 4$ *and* $l_i \equiv 3 \pmod 4$ $(1 \le i \le m)$, $l_j \equiv 1 \pmod 4$ $(m + 1 \le j \le n)$;

(iv) $\{l_1l_2, l_1l_3, \dots, l_1l_m, l_{m+1}, \dots, l_{n-1}, l_1(u + \sqrt{d})\}$ *if* $d \equiv 1 \pmod 8$, $2 \in NF$ *with* $u + w \equiv 3 \pmod 4$ *and* $l_i \equiv 3 \pmod 4$ $(1 \le i \le m)$, $l_j \equiv 1 \pmod 4$ $(m + 1 \le j \le n)$;

(v) $\{l_1l_2, l_1l_3, \dots, l_1l_m, l_{m+1}, \dots, l_{n-1}\}$ *if* $d \equiv 1 \pmod 8$, $2 \notin NF$ *and* $l_i \equiv 3 \pmod 4$ $(1 \le i \le m)$, $l_j \equiv 1 \pmod 4$ $(m + 1 \le j \le n)$.

Proof. The proof of this lemma can be found in [14] and [16]. ∎

For the convenience of the reader, we recall some notations from [16].

Suppose that $S = \{m_1, \dots, m_k\}$ is a system of $\bigtriangledown$-representatives of $F = \mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{N}$ is square-free and $d = 2^\sigma l_1 \cdots l_n$ $(\sigma \in \{0, 1\})$ is the prime factorization. Assume that the equations $\eta_i p_i m_i Z^2 = X^2 + dY^2$ (or $\eta_i p_i m_i (u + w) Z^2 = X^2 + dY^2$) are solvable for primes $p_i \equiv 1 \pmod 4$ and $\eta_i \in \{1, 2\}$ $(1 \le i \le k)$. Let $E = (\varepsilon_1, \dots, \varepsilon_k) \in S(d)^k$. Put $\delta_{i,j} = \big(\frac{\varepsilon_i \eta_i p_i}{l_j}\big)$ for $1 \le i \le k$ and $1 \le j \le n$. We call the matrix $M(d, S, E) = (\delta_{i,j})_{k \times n}$ the *sign matrix with respect to* $S = \{m_1, \dots, m_k\}$ and $E = (\varepsilon_1, \dots, \varepsilon_k) \in S(d)^k$. As a particular case, taking $E = (1, \dots, 1)$, we obtain the sign matrix

$$M(d, S) = \left[\left(\frac{\eta_i p_i}{l_j}\right)\right],$$

where

$$\left(\frac{\eta_i p_i}{l_j}\right) = \begin{cases} \left(\dfrac{m_i}{l_j}\right)\left(\text{resp., } \left(\dfrac{(u + w)m_i}{l_j}\right)\right) & \text{if } l_j \nmid m_i, \\ \left(\dfrac{d/m_i}{l_j}\right)\left(\text{resp., } \left(\dfrac{(u + w)d/m_i}{l_j}\right)\right) & \text{if } l_j \mid m_i, \end{cases}$$

which we call the sign matrix with respect to the set S of $\bigtriangledown$-representatives.

Sometimes we simply write $M(d)$ for $M(d,S,E)$ or $M(d,S)$ if we do not need to emphasize S and E.

Now we list some properties of sign matrices. It follows from [14] and [16, Lemmas 2.3 and 2.9] that

LEMMA 2.4. *Let S be a system of $\bigtriangledown$-representatives of $F=\mathbb{Q}(\sqrt{d})$, $d\in\mathbb{N}$ square-free, $m\in S$. Let P be the product of all entries in the row corresponding to m in each sign matrix with respect to S and $E=(1,\dots,1)$. If $m\mid d$, then*

$$P=\begin{cases}1 & \text{if either } d\equiv 1 \pmod 2 \text{ and } d\not\equiv 5 \pmod 8;\\ & \quad\text{or } d\equiv 1 \pmod 2 \text{ and } m\not\equiv 3 \pmod 4;\\ & \quad\text{or } d\equiv 0 \pmod 2,\ m\equiv 3 \pmod 8 \text{ and } d/2\equiv 1 \pmod 4;\\ & \quad\text{or } d\equiv 0 \pmod 2,\ m\equiv 7 \pmod 8 \text{ and } d/2\equiv 3 \pmod 4;\\ & \quad\text{or } d\equiv 0 \pmod 2,\ m\equiv 1 \pmod 8;\\ -1 & \text{otherwise.}\end{cases}$$

If $m=u+\sqrt{d}$ and $d\equiv 1 \pmod 2$, then the product of all entries in the row corresponding to $u+\sqrt{d}$ is 1.

Let A and B be sign matrices. A is said to be *equivalent* to B (denoted by $A\cong B$) if some of the following operations, which are called *elementary operations*, applied to A yield B:

(I) Multiplying row i of A by row j. (This corresponds to replacing m_i by m_im_j in the set of $\bigtriangledown$-representatives of F.)

(II) Interchanging the ith and jth rows. (This corresponds to interchanging m_i and m_j.)

(II′) Interchanging the ith and jth columns. (This corresponds to interchanging l_i and l_j.)

(III) Multiplying the ith row by a vector $(\varepsilon_1,\dots,\varepsilon_n)$, where $\varepsilon_i=\left(\frac{\varepsilon}{l_i}\right)$ with $\varepsilon\in S(d)$ and $l_1,\dots,l_n$ are all the odd prime divisors of d with l_i corresponding to the ith column. (This corresponds to changing the ith entry in the set $E=(\varepsilon_1,\dots,\varepsilon_k)$.)

Elementary operations (II′) are often used to fix the places of columns of a sign matrix. When applying elementary operations (II′), one must remember the congruences of l_i and l_j (mod 8) since it is possible that $\left(\frac{\varepsilon}{l_i}\right)\neq\left(\frac{\varepsilon}{l_j}\right)$ in an elementary operation (III) if $l_i\not\equiv l_j \pmod 8$.

Note that, for a real quadratic field $F=\mathbb{Q}(\sqrt{d})$, if d has exactly n odd prime divisors $l_1,\dots,l_n$, then we have only finitely many different sign matrices with respect to a system of $\bigtriangledown$-representatives of F, any two of which are equivalent. Suppose that $M(d)$ is a sign matrix with respect to a system of $\bigtriangledown$-representatives of F. It is easy to see that there exists an element $\{-1,m\}\in {}_2K_2O_F$ such that $\{-1,m\}\in\nabla^2$ if and only if there exists a totally 1 row (i.e. its entries are all 1) by applying some elementary row operations on $M(d)$ if necessary.

LEMMA 2.5 ([16]). *Let $F = \mathbb{Q}(\sqrt{d})$ be a real quadratic field, where d square-free has n odd prime divisors. Assume that a sign matrix is of size $k \times n$. We view any sign matrix as one over $\mathbb{Z}/2\mathbb{Z}$. Then $r_4(K_2O_F)$ coincides with the maximum of $k - r$, where r runs through the ranks of all sign matrices of F.*

LEMMA 2.6 ([16]). *Let $n \geq 2$ be an integer. Assume that for $1 \leq i < j \leq n$, $1 \leq k < n$ we are given $\varepsilon_{ij} \in \{\pm 1\}$ and odd integers t_k. Then there are infinitely many integers d such that d has exactly n odd prime divisors $l_1, \ldots, l_n$ with $\left(\frac{l_i}{l_j}\right) = \varepsilon_{ij}$ and $l_k \equiv t_k \pmod 8$ where $1 \leq i < j \leq n$, $1 \leq k \leq n$.*

3. Matrices over $\mathbb{F}_2$. Let M and N be matrices over $\mathbb{F}_2$. We write $M \sim N$ if N can be obtained from M by some elementary transformations.

For a matrix $A = (a_{ij})$ over $\mathbb{F}_2$, we use A^T for its transpose, $r(A)$ for the rank of A, and $1 + A$ for the matrix $(1 + a_{ij})$. In case A is of size $n \times n$, we call A *skew symmetric* if $a_{ij} + a_{ji} = 1$ for $1 \leq i \neq j \leq n$.

LEMMA 3.1. *Let $M = (\delta_{i,j})$ be an $n \times n$ skew symmetric matrix over $\mathbb{F}_2$. If n is even, then $r(M) \geq n/2$. If n is odd, then $r(M) \geq (n-1)/2$. Moreover, if n is odd and there exists a totally 1 row which can be expressed as a linear combination of some rows of M, then $r(M) \geq (n+1)/2$.*

Proof. Let $P = M + M^T = (p_{ij})$. Then $p_{ij} = 0$ if $i = j$; and $p_{ij} = 1$ if $i \neq j$.

(1) *n is even.* We have $r(P) = n$, hence $r(M) \geq n/2$.

(2) *n is odd.* It is easy to see that $r(P) = n - 1$. Hence $r(M) \geq (n-1)/2$. It is enough to prove that $r(M) \geq (n+1)/2$ when a totally 1 row can be expressed as a linear combination of some rows of M. We may assume that

$$\delta_1 + \cdots + \delta_t = (1, \ldots, 1),$$

where $\delta_i = (\delta_{i,1}, \ldots, \delta_{i,n}), i = 1, \ldots, t$.

When t is odd, adding rows 2 to t to the first row and doing the same for columns, and then adding the first (new) row to rows 2 to t, we can partition the equivalent form of M into

$$M \sim \left(\begin{array}{cccc|cccc} 1 & \cdots & \cdots & \cdots\ 1 & 1 & \cdots & \cdots & \cdots\ 1 \\ 0 & & & & & & & \\ \vdots & & A & & & & B & \\ 0 & & & & & & & \\ \hline 0 & & & & & & & \\ \vdots & & B^T & & & & C & \\ 0 & & & & & & & \end{array}\right)$$

$$\begin{cases} \delta_{1,j} = 1, \ j = 1, \ldots, n; \\ \delta_{i,1} = 0, \ \ i = 2, \ldots, n, \end{cases}$$

where the blocks A, B and C are of sizes $(t-1) \times (t-1), (t-1) \times (n-t)$ and $(n-t) \times (n-t)$, respectively. It is easy to see that both A and C are skew symmetric. The above discussion implies that $r\left(\begin{smallmatrix} A & B \\ B^T & C \end{smallmatrix}\right) \geq (n-1)/2$ since $t-1$ and $n-t$ are even. Thus $r(M) \geq (n+1)/2$.

Suppose that t is even. First we add rows 2 to t to the first row, and then do the same with columns. Next we add the last column to columns $t+1$ to $n-1$, and then do the same row transformations. Now we have

$$M \sim \left(\begin{array}{cccccc|cccccc} 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \vdots & & & A & & & & & B & & & * \\ 0 & & & & & & & & & & & \vdots \\ \hline 0 & & & & & & & & & & & \\ \vdots & & & B^T & & & & & C & & & \vdots \\ 0 & & & & & & & & & & & \vdots \\ 1 & * & \cdots & \cdots & \cdots & * & * & \cdots & \cdots & \cdots & * & * \end{array}\right)$$

$$\delta_{i,j} = \begin{cases} 1, & i = 1 \text{ and } j = 2, \ldots, t, n; \\ & \quad \text{or } i = n \text{ and } j = 1; \\ 0, & i = 1 \text{ and } j = t+2, \ldots, n-1; \\ & \quad \text{or } i = 1, \ldots, n-1 \text{ and } j = n, \end{cases}$$

where the blocks A, B and C are skew symmetric of sizes $(t-1) \times (t-1)$, $(t-1) \times (n-t-1)$ and $(n-t-1) \times (n-t-1)$, respectively. By using the same argument as in the case of t odd, we obtain

$$r \left(\begin{array}{cccc|ccc} 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & & A & & & B & \\ 0 & & & & & & \\ \hline 0 & & & & & & \\ \vdots & & B^T & & & C & \\ 0 & & & & & & \end{array}\right) \geq (n-1)/2,$$

hence $r(M) \geq (n+1)/2$. ∎

Lemma 3.2. *Suppose that* $M = (\delta_{i,j})$ *is an* $n \times n$ *skew symmetric matrix over* $\mathbb{F}_2$*. Let* t *be the number of rows of* M *with the sum of all entries* 1*. Assume that* $t \geq 1$*.*

(i) *If* n *is odd, then* $r(M) \geq (n+1)/2$*.*

(ii) *If* n *is even and the totally* 1 *row can be expressed as a linear combination of some rows of* M*, then* $r(M) \geq n/2 + 1$*.*

Proof. Applying some elementary transformations if necessary, we may assume that the sum of all entries is 1 in each of the first t rows, and 0 in others.

(i) First suppose that t is odd. Add rows 1 to $n-1$ to the last row, and next do the same with columns. Then the matrix M can be partitioned into the following equivalent form:

$$M \sim \begin{pmatrix} & & & & & & 1 \\ & M_1 & & & M_2 & & \vdots \\ & & & & & & 1 \\ & & & & & & 0 \\ & 1+M_2^T & & & M_3 & & \vdots \\ & & & & & & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where the submatrices M_1 and M_3 are skew symmetric of sizes $t \times t$ and $(n-t-1) \times (n-t-1)$ respectively. Note that

$$M \sim \begin{pmatrix} & & & & & & 1 \\ & 1+M_1 & & & M_2 & & \vdots \\ & & & & & & 1 \\ & & & & & & 0 \\ & 1+M_2^T & & & M_3 & & \vdots \\ & & & & & & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

By Lemma 3.1 we have $r(M) \geq (n-1)/2 + 1 = (n+1)/2$.

Now assume that t is even. If $\delta_{11} = 0$, with the same procedure as above and applying some elementary transformations if necessary, we see that there exist integers $1 \leq k \leq t$ and $1 \leq l \leq n-t$ such that

$$M \sim \left(\begin{array}{c|c|c} \begin{matrix} 0 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & M_1 & & & M_2 & \\ & & & & & \\ & 1+M_2^T & & & M_5 & \end{matrix} & \begin{matrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 1 \\ & M_3 & & & M_4 & & \vdots \\ & & & & & & \\ & M_6 & & & M_7 & & \vdots \\ & & & & & & 1 \end{matrix} \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \end{matrix} & \begin{matrix} & 1+M_3^T & & & 1+M_6^T & \\ & & & & & \\ & 1+M_4^T & & & 1+M_7^T & \\ 1 & \cdots & 1 & 1 & \cdots & 1 \end{matrix} & \begin{matrix} & & & & & & 0 \\ & M_8 & & & M_9 & & \vdots \\ & & & & & & \\ & 1+M_9^T & & & M_{10} & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{matrix} \end{array}\right),$$

where the submatrices M_1, M_5, M_8 and M_{10} are skew symmetric of sizes

$(k-1)\times(k-1)$, $(t-k)\times(t-k)$, $l\times l$ and $(n-t-l-1)\times(n-t-l-1)$, respectively. Add the last column to the ith $(i=k+1,\dots,k+l)$, and do the same with rows. Then

$$M \sim \left(\begin{array}{cccc|ccc}
0 & 0\ \cdots\ 0 & 0\ \cdots\ 0 & & 0\ \cdots\ 0 & 0\ \cdots\ 0 & 1 \\
1 & & & & & & \\
\vdots & M_1 & 1+M_2 & & 1+M_3 & M_4 & \vdots \\
1 & & & & & & \\
1 & & & & & & \\
\vdots & M_2^T & M_5 & & 1+M_6 & M_7 & \vdots \\
1 & & & & & & 1 \\
\hline
1 & & & & & & 0 \\
\vdots & M_3^T & M_6^T & & M_8 & M_9 & \vdots \\
1 & & & & & & \\
1 & & & & & & \\
\vdots & 1+M_4^T & 1+M_7^T & & 1+M_9^T & M_{10} & \vdots \\
1 & 1\ \cdots\ 1 & 1\ \cdots\ 1 & & 0\ \cdots\ 0 & 0\ \cdots\ 0 & 0
\end{array}\right).$$

It follows from Lemma 3.1 that $r(M)\ge(n-1)/2+1=(n+1)/2$. When $\delta_{11}=1$, the proof is similar.

(ii) Attaching to M an additional totally 1 row and an additional totally 0 (i.e. its entries are all 0) column, we obtain an $(n+1)\times(n+1)$ matrix

$$\widehat{M} = \begin{pmatrix} & & 0 \\ & M & \vdots \\ & & \vdots \\ 1 & \cdots\ 1 & 0 \end{pmatrix}.$$

Clearly, we have $r(\widehat{M})=r(M)$ and (i) implies that $r(M)\ge n/2+1$. ∎

LEMMA 3.3. *Let $M=(\delta_{i,j})$ be a $(t+n)\times(t+n+1)$ matrix over $\mathbb{F}_2$ with $\delta_{i,j}+\delta_{j,i}=1$ for $1\le i\ne j\le t+n$ and assume $t\ge 1$. Suppose that the sum of all entries in any of the first t rows is* 1, *and in any of the last n rows is* 0.

(i) *If $t+n$ is even and the two rows $(\overbrace{1,\dots,1}^{t+n},0)=:(1^{\cdots t+n},0)$ and $(1^{\cdots t},0^{\cdots n+1})$ can be expressed as linear combinations of some rows in M, then $r(M)\ge(t+n)/2+1$.*

(ii) *If $t+n$ is odd and the two rows $(1^{\cdots t+n+1})$ and $(1^{\cdots t},0^{\cdots n+1})$ can be expressed as linear combinations of some rows in M, then $r(M)\ge(t+n+1)/2+1$.*

Proof. We partition M into $\left(\begin{smallmatrix} M_1 & M_2 & \alpha_1 \\ 1+M_2^T & M_3 & \alpha_2\end{smallmatrix}\right)$, where the submatrices M_1 and M_3 are skew symmetric of sizes $t\times t$ and $n\times n$ respectively, and $\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)$ is a column vector.

(i) If $(\alpha_1) \neq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ or $(\alpha_2) \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, then by Lemma 3.2(i) we have $r(M) \geq (t+n)/2+1$. Now we assume that $(\alpha_1) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and $(\alpha_2) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. Consider the following $(t+n+1) \times (t+n+1)$ matrix:

$$\widehat{M} = \begin{pmatrix} M_1 & M_2 & \alpha_1 \\ 1+M_2^T & M_3 & \alpha_2 \\ \alpha_1^T & 1+\alpha_2^T & * \end{pmatrix},$$

where $*$ is the sum of all entries in the last column of M. It follows from the hypothesis that

$$\widehat{M} \sim \begin{pmatrix} M_1 & M_2 & \alpha_1 \\ 1+M_2^T & M_3 & \alpha_2 \\ 1+\alpha_1^T & 1+\alpha_2^T & * \end{pmatrix}.$$

By Lemma 3.2(ii) we have $r(M) = r(\widehat{M}) \geq (t+n+1+1)/2 = (t+n)/2+1$.

(ii) As above, we can obtain a $(t+n+1) \times (t+n+1)$ matrix

$$\widehat{M} = \begin{pmatrix} M_1 & M_2 & \alpha_1 \\ 1+M_2^T & M_3 & \alpha_2 \\ 1+\alpha_1^T & \alpha_2^T & * \end{pmatrix}.$$

By hypothesis, we have

$$\widehat{M} \sim \begin{pmatrix} M_1 & M_2 & \alpha_1 \\ 1+M_2^T & M_3 & \alpha_2 \\ 1+\alpha_1^T & 1+\alpha_2^T & 1+* \end{pmatrix}.$$

By Lemma 3.2(i), the result follows. ■

LEMMA 3.4. *Let $M = (\delta_{i,j})$ be a $(t+n+s+m-1) \times (t+n+s+m)$ matrix over $\mathbb{F}_2$ with $\delta_{i,j} + \delta_{j,i} = 1$ for $1 \leq i \neq j \leq t+n$. Assume that t, n and s are positive integers. Suppose that M satisfies the following conditions*:

(a) $\delta_{i,j} = \delta_{j,i}$ *when* $t+n+1 \leq i \neq j \leq t+n+s+m-1$;
(b) $t+n \equiv 1 \pmod 2$;
(c) $s+m \geq 2$;
(d) *either* $(1^{\cdots t}, 0^{\cdots n}, 1^{\cdots s}, 0^{\cdots m})$ *or* $(0^{\cdots t}, 1^{\cdots n+s}, 0^{\cdots m})$ *can be expressed as a linear combination of some rows in* M.

Then $r(M) \geq (t+n+1)/2$.

Proof. By Lemma 3.1, we may assume that each of the last $s+m-1$ rows of M can be expressed as a linear combination of some of the first $t+n$ rows. Thus it suffices to consider the submatrix A formed by the first $t+n+1$ rows and the first $t+n+1$ columns of M. As in the proof of Lemma 3.2(ii), we can show that there exist integers $1 \leq k \leq t$ and $1 \leq l \leq n-t$ such that A can be partitioned as:

$$A \sim \left(\begin{array}{cc|cc|c} A_1 & A_2 & A_3 & A_4 & \begin{matrix}1\\ \vdots \\ 1\end{matrix} \\ 1+A_2^T & A_5 & A_6 & A_7 & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 1+A_3^T & 1+A_6^T & A_8 & A_9 & \begin{matrix}1\\ \vdots \\ 1\end{matrix} \\ 1+A_4^T & 1+A_7^T & 1+A_9^T & A_{10} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ 1 \;\cdots\; 1 & 0 \;\cdots\; 0 & 1 \;\cdots\; 1 & 0 \;\cdots\; 0 & * \end{array}\right),$$

where A_1, A_5, A_8 and A_{10} are skew symmetric of sizes $k \times k$, $(t-k)\times(t-k)$, $l \times l$ and $(n-t-l)\times(n-t-l)$, respectively, and $* = \delta_{t+n+1,t+n+1}$.

Suppose that $(1^{\cdots t}, 0^{\cdots n}, 1^{\cdots s}, 0^{\cdots m})$ is a linear combination of some rows of M. Then we can attach an extra row to the above equivalent form of A to get a $(t+n+2)\times(t+n+1)$ matrix:

$$\widehat{A} = \left(\begin{array}{cc|cc|c} A_1 & A_2 & A_3 & A_4 & \begin{matrix}1\\ \vdots \\ 1\end{matrix} \\ 1+A_2^T & A_5 & A_6 & A_7 & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 1+A_3^T & 1+A_6^T & A_8 & A_9 & \begin{matrix}1\\ \vdots \\ 1\end{matrix} \\ 1+A_4^T & 1+A_7^T & 1+A_9^T & A_{10} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ 1 \;\cdots\; 1 & 0 \;\cdots\; 0 & 1 \;\cdots\; 1 & 0 \;\cdots\; 0 & * \\ 1 \;\cdots\; \cdots & \cdots\; 1 & 0 \;\cdots\; \cdots & \cdots\; 0 & 1 \end{array}\right).$$

Note that

$$\widehat{A}\sim\left(\begin{array}{cc|ccc}
1+A_1 & A_2 & 1+A_3 & A_4 & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
1+A_2^T & A_5 & A_6 & A_7 & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
\cline{1-4}
A_3^T & 1+A_6^T & 1+A_8 & A_9 & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
1+A_4^T & 1+A_7^T & 1+A_9^T & A_{10} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
1\ \cdots\ 1 & 0\ \cdots\ 0 & 1\ \cdots\ 1 & 0\ \cdots\ 0 & 1\\
1\ \cdots\ \cdots & \cdots\ 1 & 0\ \cdots\ \cdots & \cdots\ 0 & 1
\end{array}\right)\quad \text{if } *=1;$$

and

$$\widehat{A}\sim\left(\begin{array}{cc|ccc}
A_1 & 1+A_2 & 1+A_3 & A_4 & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
A_2^T & A_5 & 1+A_6 & A_7 & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
\cline{1-4}
A_3^T & A_6^T & A_8 & A_9 & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
1+A_4^T & 1+A_7^T & 1+A_9^T & A_{10} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
1\ \cdots\ 1 & 0\ \cdots\ 0 & 1\ \cdots\ 1 & 0\ \cdots\ 0 & 0\\
0\ \cdots\ 0 & 1\ \cdots\ 1 & 1\ \cdots\ 1 & 0\ \cdots\ 0 & 1
\end{array}\right)\quad \text{if } *=0.$$

By Lemma 3.1 we have $r(A)=r(\widehat{A})\geq (t+n-1)/2+1$ and the result follows.

When $(0^{\cdots t},1^{\cdots n+s},0^{\cdots m})$ is a linear combination of some rows of M, the proof is the same as above. ■

LEMMA 3.5. *Let $M=(\delta_{i,j})$ be a $(t+n+s+m-1)\times(t+n+s+m)$ matrix over $\mathbb{F}_2$ with $\delta_{i,j}+\delta_{j,i}=1$ for $1\leq i\neq j\leq t+n$, where t, n and s are positive integers. Suppose that M satisfies the following conditions*:

(a) $\delta_{i,j}=\delta_{j,i}$ *when* $t+n+1\leq i\neq j\leq t+n+s+m-1$;

(b) *the sum of all entries is* 1 *in any of the first* $t+n$ *rows*, *and is* 0 *in any of the last* $s+m-1$ *rows*;

(c) $t\equiv n\equiv s \pmod 2$;

(d) *both* $(1^{\cdots t}, 0^{\cdots n}, 1^{\cdots s}, 0^{\cdots m})$ *and* $(0^{\cdots t}, 1^{\cdots n+s}, 0^{\cdots m})$ *can be expressed as linear combinations of some rows in* M.

Then $r(M) \geq (t+n)/2 + 2$.

Proof. We divide the proof into two cases.

Case 1: $t \equiv n \equiv s \equiv 1 \pmod 2$. We construct a $(t+n+s+m-1) \times (t+n+1)$ matrix $\widehat{M} = (M_1, \alpha)$, where the submatrix M_1 is formed by the first $t+n$ columns of M, and the column vector α is the sum of the last $s+m$ columns of M. As done in Lemma 3.4, we partition the submatrix A of $\widehat{M}$ which is formed by the first $t+n$ rows of $\widehat{M}$ into

$$A \sim \left(\begin{array}{cc|cc|c} A_1 & A_2 & A_3 & A_4 & \begin{matrix}1\\ \vdots\\ 1\end{matrix} \\ 1+A_2^T & A_5 & A_6 & A_7 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 1+A_3^T & 1+A_6^T & A_8 & A_9 & \begin{matrix}1\\ \vdots\\ 1\end{matrix} \\ 1+A_4^T & 1+A_7^T & 1+A_9^T & A_{10} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \end{array}\right),$$

where the submatrices A_1, A_5, A_8 and A_{10} are skew symmetric of sizes $k \times k$, $(t-k) \times (t-k)$, $l \times l$ and $(n-t-l) \times (n-t-l)$, respectively, and $1 \leq k \leq t$, $1 \leq l \leq n$. Consider the following $(t+n+3) \times (t+n+1)$ matrix $\widehat{A}$ which is formed by attaching three additional rows to the above equivalent form of A:

$$\widehat{A} = \left(\begin{array}{cc|cc|c} A_1 & A_2 & A_3 & A_4 & \begin{matrix}1\\ \vdots\\ 1\end{matrix} \\ 1+A_2^T & A_5 & A_6 & A_7 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 1+A_3^T & 1+A_6^T & A_8 & A_9 & \begin{matrix}1\\ \vdots\\ 1\end{matrix} \\ 1+A_4^T & 1+A_7^T & 1+A_9^T & A_{10} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ 1 \ \cdots \ 1 & 0 \ \cdots \ 0 & 1 \ \cdots \ 1 & 0 \ \cdots \ 0 & * \\ 1 \ \cdots \ \cdots & \cdots \ 1 & 0 \ \cdots \ \cdots & \cdots \ 0 & 1 \\ 0 \ \cdots \ \cdots & \cdots \ 0 & 1 \ \cdots \ \cdots & \cdots \ 1 & 1 \end{array}\right).$$

Note that the $(t+n+1)$th row is the sum of all row vectors of the above equivalent form of A. We have $r(M) \geq r(\widehat{A})$ and

$$\widehat{A} \sim \left(\begin{array}{cccccc|cccccc|c}
\multicolumn{3}{c}{A_1} & \multicolumn{3}{c|}{1+A_2} & \multicolumn{3}{c}{1+A_3} & \multicolumn{3}{c|}{A_4} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\
\multicolumn{3}{c}{A_2^T} & \multicolumn{3}{c|}{A_5} & \multicolumn{3}{c}{1+A_6} & \multicolumn{3}{c|}{A_7} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\
\hline
\multicolumn{3}{c}{A_3^T} & \multicolumn{3}{c|}{A_6^T} & \multicolumn{3}{c}{1+A_8} & \multicolumn{3}{c|}{A_9} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\
\multicolumn{3}{c}{1+A_4^T} & \multicolumn{3}{c|}{1+A_7^T} & \multicolumn{3}{c}{1+A_9^T} & \multicolumn{3}{c|}{A_{10}} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\
1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\
1 & \cdots & \cdots & & \cdots & 1 & 0 & \cdots & \cdots & & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1
\end{array}\right) \quad \text{if } * = 1;$$

and

$$\widehat{A} \sim \left(\begin{array}{cccccc|cccccc|c}
\multicolumn{3}{c}{A_1} & \multicolumn{3}{c|}{A_2} & \multicolumn{3}{c}{1+A_3} & \multicolumn{3}{c|}{A_4} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\
\multicolumn{3}{c}{1+A_2^T} & \multicolumn{3}{c|}{1+A_5} & \multicolumn{3}{c}{1+A_6} & \multicolumn{3}{c|}{A_7} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\
\hline
\multicolumn{3}{c}{A_3^T} & \multicolumn{3}{c|}{A_6^T} & \multicolumn{3}{c}{A_8} & \multicolumn{3}{c|}{A_9} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\
\multicolumn{3}{c}{1+A_4^T} & \multicolumn{3}{c|}{1+A_7^T} & \multicolumn{3}{c}{1+A_9^T} & \multicolumn{3}{c|}{A_{10}} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\
1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 \\
1 & \cdots & \cdots & & \cdots & 1 & 0 & \cdots & \cdots & & \cdots & 0 & 1
\end{array}\right) \quad \text{if } * = 0.$$

By Lemma 3.2(ii) we have $r(\widehat{A}) \geq (t+n+1+1)/2+1$ and the result follows.

Case 2: $t \equiv n \equiv s \equiv 0 \pmod 2$. We construct a $(t+n+s+m-1) \times (t+n+2)$ matrix $\widehat{M} = (M_1, \alpha, \beta)$, where the submatrix M_1 is formed by the first $t+n$ columns of M, the column vector α is just the $(t+n+1)$th column of M and the column vector β is the sum of the last $s+m-1$ columns of M. We partition the submatrix A of size $(t+n+1) \times (t+n+2)$ and formed by the first $t+n+1$ rows of $\widehat{M}$ into

$$A = \begin{pmatrix} A_1 & A_2 & \alpha_1 & \beta_1 \\ 1+A_2^T & A_3 & \alpha_2 & \beta_2 \\ \alpha_1^T & \alpha_2^T & * & *' \end{pmatrix},$$

where the blocks A_1 and A_3 are skew symmetric of sizes $t \times t$ and $n \times n$ respectively, $* = \delta_{t+n+1,t+n+1}$ and $*' = \sum_{j=t+n+2}^{t+n+s+m} \delta_{i,j}$. Attaching to A two additional rows, we obtain a $(t+n+3) \times (t+n+2)$ matrix

$$\widehat{A} = \begin{pmatrix} & A_1 & & & A_2 & & \alpha_1 & \beta_1 \\ & 1 + A_2^T & & & A_3 & & \alpha_2 & \beta_2 \\ & \alpha_1^T & & & \alpha_2^T & & * & *' \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & 1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix}.$$

With the same procedure as in the first case, we have

$$\widehat{A} \sim \begin{pmatrix} & A_1' & & & A_2' & & 0 & \beta_1' \\ & 1 + (A_2')^T & & & A_3' & & 0 & \beta_2' \\ & \alpha_1^T & & & \alpha_2^T & & * & *' \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix}.$$

If $\binom{\beta_1'}{\beta_2'}$ is a totally 1 column, then by Lemmas 3.1 and 3.2(i),

$$r\begin{pmatrix} A_1' & 1 + (A_2')^T \\ A_2' & A_3' \\ (\beta_1')^T & (\beta_2')^T \end{pmatrix} \geq (t+n)/2 + 1.$$

If $\binom{\beta_1'}{\beta_2'}$ is not a totally 1 column, then by Lemma 3.3(ii),

$$r\begin{pmatrix} & A_1' & & & A_2' & & 0 \\ & 1 + (A_2')^T & & & A_3' & & 0 \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \geq (t+n)/2 + 1.$$

So $r(M) \geq r(\widehat{A}) \geq (t+n)/2 + 2$ and the lemma is proved. ∎

Similarly, we can prove

LEMMA 3.6. *Let $M = (\delta_{i,j})$ be a $(t+n+s+m-1) \times (t+n+s+m)$ matrix over $\mathbb{F}_2$ with $\delta_{i,j} + \delta_{j,i} = 1$ for $1 \leq i \neq j \leq t+n$, where t, n and s are positive integers. Suppose that M satisfies the following conditions*:

(a) $\delta_{i,j} = \delta_{j,i}$ *when* $t+n+1 \leq i \neq j \leq t+n+s+m-1$;

(b) $$\sum_{j=1}^{t+n+s+m} \delta_{i,j} = \begin{cases} 1, & i = 1, \ldots, t;\ t+n+1, \ldots, t+n+s; \\ 0, & i = t+1, \ldots, t+n;\ t+n+s+1, \ldots, \\ & \qquad t+n+s+m-1; \end{cases}$$

(c) *both* $(1^{\cdots t}, 0^{\cdots n}, 1^{\cdots s}, 0^{\cdots m})$ *and* $(0^{\cdots t}, 1^{\cdots n+s}, 0^{\cdots m})$ *can be expressed as linear combinations of some rows in* M.

Then:

(i) $r(M) \geq (t+n)/2+2$ *if* $t \equiv n \equiv s \pmod 2$;
(ii) $r(M) \geq (t+n)/2+1$ *if* $t \equiv n \not\equiv s \pmod 2$;
(iii) $r(M) \geq (t+n+1)/2+1$ *if* $t \not\equiv n \pmod 2$.

4. All values of $r_4(K_2O_F)$. Let $F = \mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{N}$ is squarefree and has at least three odd prime divisors. In this section, we recall the notion of *type* of a quadratic number field and for each type of real quadratic number field F we give all possible values of $r_4 := r_4(K_2O_F)$.

Notation. Let d have prime factorization

$$d = 2^\sigma l_1 \cdots l_m p_1 \cdots p_n q_1 \cdots q_s r_1 \cdots r_t,$$

where $\sigma \in \{0,1\}$, $l_k \equiv 1 \pmod 8$, $p_h \equiv 3 \pmod 8$, $q_j \equiv 5 \pmod 8$ and $r_i \equiv 7 \pmod 8$ are different odd primes ($0 \leq k \leq m, 0 \leq h \leq n, 0 \leq j \leq s$ and $0 \leq i \leq t$). We say that d has type $2^\sigma(m,n,s,t)$. We also say that the quadratic field $F = \mathbb{Q}(\sqrt{d})$ *has type* $2^\sigma(m,n,s,t)$.

For any given type T, let $d(T)$ denote the set of all positive integers of type T, i.e., $d(T) = \{d \mid d \in \mathbb{N} \text{ of type } T\}$. We keep the above notations throughout this section. Clearly we have

LEMMA 4.1. *Let d have type $2^\sigma(m,n,s,t)$, where $\sigma \in \{0,1\}$ and n,s,t are positive integers. In a sign matrix $M(d)$, arrange the first $t+n$ columns to correspond to $r_1,\dots,r_t,p_1,\dots,p_n$ and the last m columns to $l_1,\dots,l_m$. Then it is impossible to make any of the following rows of $M(d)$ to be a totally 1 row by applying elementary operations* (III) *only* :

(a) $(-1^{\dots t}, 1^{\dots n+s+m})$; (b) $(1^{\dots t}, -1^{\dots n}, 1^{\dots s+m})$;
(c) $(1^{\dots t}, -1^{\dots n}, 1^{\dots s+m})$; (d) $(-1^{\dots t+n+s}, 1^{\dots m})$;
(e) $(\varepsilon_1,\dots,\varepsilon_{t+n+s}, -1^{\dots m})$, $\varepsilon_i \in \{\pm 1\}$, $i = 1,\dots,t+n+s$.

We need the following

ASSUMPTION. *Notations as above. Let $n \geq 2$ be an integer. For $1 \leq k \leq n$, $1 \leq i < j \leq n$, we are given $\varepsilon_k, \varepsilon_{ij} \in \{\pm 1\}$. For any integer $0 \leq t \leq n$, there exist infinitely many $d \in \mathbb{N}$ with prime factorization $d = 2^\sigma p_1 \cdots p_n$, where $\sigma = 0$ or 1, and primes $p_i \equiv 1 \pmod 8$ $(1 \leq i \leq t)$ and $p_j \equiv -1 \pmod 8$ $(t+1 \leq j \leq n)$ such that $\left(\frac{u+w}{p_k}\right) = \varepsilon_k$ for $1 \leq k \leq n$ and $\left(\frac{p_i}{p_j}\right) = \varepsilon_{ij}$ for $1 \leq i < j \leq n$.*

We conjecture that the above Assumption always holds.

THEOREM 4.2. *Under the above Assumption for types $2^\sigma(m,0,0,0)$ and $2^\sigma(m,0,0,t)$, where $\sigma = 0$ or 1, for real quadratic fields F, we have the following tables of possible values of r_4* (*with all congruences mod* 2):

Table I

Type			min r_4	max r_4
$(m,0,s,0)$	$s\equiv 0$		1	$s+m-1$
	$s\equiv 1$		0	$s+m-1$
$(m,n,0,0)$	$n\equiv 0$		0	$n/2+m-1$
	$n\equiv 1$		0	$(n-1)/2+m$
$(m,0,0,t)$	$t\equiv 0$		1	$t/2+m$
	$t\equiv 1$		1	$(t+1)/2+m$
$(m,n,s,0)$	$n\equiv 0$	$s\equiv 0$	1	$n/2+s+m-1$
		$s\equiv 1$	1	$n/2+s+m$
	$n\equiv 1$		1	$(n-1)/2+s+m$
$(m,0,s,t)$	$t\equiv 0$	$s\equiv 0$	1	$t/2+s+m-1$
		$s\equiv 1$	1	$t/2+s+m$
	$t\equiv 1$		1	$(t-1)/2+s+m$
$(m,n,0,t)$	$t+n\equiv 0$		1	$(t+n)/2+m$
	$t+n\equiv 1$		1	$(t+n+1)/2+m$
(m,n,s,t)	$t+n\equiv 0$	$t\equiv s$	1	$(t+n)/2+s+m-1$
		$t\not\equiv s$	1	$(t+n)/2+s+m$
	$t+n\equiv 1$		1	$(t+n-1)/2+s+m$
$(m,0,0,0)$			0	m

Table II

Type			min r_4	max r_4
$2(m,0,s,0)$	$s\equiv 0$		0	$s+m-2$
	$s\equiv 1$		0	$s+m-1$
$2(m,n,0,0)$	$n\equiv 0$		1	$n/2+m$
	$n\equiv 1$		0	$(n-1)/2+m$
$2(m,0,0,t)$	$t\equiv 0$		1	$t/2+m$
	$t\equiv 1$		1	$(t+1)/2+m$
$2(m,n,s,0)$	$n\equiv 0$	$s\equiv 0$	1	$n/2+s+m-1$
		$s\equiv 1$	1	$n/2+s+m$
	$n\equiv 1$		1	$(n-1)/2+s+m$
$2(m,0,s,t)$	$t\equiv 0$	$s\equiv 0$	1	$t/2+s+m-1$
		$s\equiv 1$	1	$t/2+s+m$
	$t\equiv 1$		1	$(t-1)/2+s+m$
$2(m,n,0,t)$	$t+n\equiv 0$		1	$(t+n)/2+m$
	$t+n\equiv 1$		1	$(t+n+1)/2+m$
$2(m,n,s,t)$	$t+n\equiv 0$	$t\equiv s$	1	$(t+n)/2+s+m-1$
		$t\not\equiv s$	1	$(t+n)/2+s+m$
	$t+n\equiv 1$		1	$(t+n-1)/2+s+m$
$2(m,0,0,0)$			0	m

For each type T and each integer k between the minimum and maximum values of r_4 for this type, there exist infinitely many real quadratic number fields of type T with $r_4 = k$.

Proof. Since all the minimums of r_4 have been determined in [16], it suffices to consider the maximums and the value set of r_4. For each type T, the maximum of r_4 is ensured by the results of Section 3. For every integer k between the minimum and maximum values of r_4, we will construct a sign matrix $M_k(d)$ such that the maximal number of totally 1 rows of all equivalent forms of $M_k(d)$ is exactly k.

As remarked in [16], a totally 1 row (if any) of a sign matrix can be obtained by applying elementary operations (I) and (II), and by at most one elementary operation (III). So, for each type T we construct a sign matrix $M_k(d)$ with respect to a system of $\bigtriangledown$-representatives and $E = \{1, \ldots, 1\}$. Every equivalent form of $M_k(d)$ will be obtained by the application of elementary operations (I) and (II), and by applying an elementary operation (III) at the last step.

We may assume that n, s and t are positive integers, and only m can be 0. In a sign matrix, the first t columns will correspond to the primes $r_1, \ldots, r_t$, columns $t+1$ to $t+n$ will correspond to the primes $p_1, \ldots, p_n$, columns $t+n+1$ to $t+n+s$ will correspond to the primes $q_1, \ldots, q_s$, and columns $t+n+s+1$ to $t+n+s+m$ will correspond to the primes $l_1, \ldots, l_m$.

Case (A): $T = (m, 0, s, 0)$ *with* $m + s \geq 3$:

(A_1): *s is even.* First assume that $m > 0$. Suppose that $S = \{q_1, \ldots, q_s, l_1, \ldots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives. It follows from Lemma 2.6 and the law of quadratic reciprocity that we may choose $d \in d(T)$ such that

$$M_k(d) = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ & \ddots & & & & & \vdots \\ & & 1 & \cdots & \cdots & \cdots & 1 \\ & & & -1 & \cdots & \cdots & -1 \\ & & & & \ddots & & \vdots \\ & & & & & -1 & -1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = \delta_{i,s+m} = -1, & k+1 \leq i \leq s+m-1; \\ \delta_{i,j} = 1, & \text{otherwise,} \end{cases}$$

where $1 \leq k \leq s + m - 1$. According to Lemma 2.4 and the definition of sign matrix, the above matrices are sign matrices with respect to S and $E = \{1, \ldots, 1\}$. It is easy to see that the number of totally 1 rows in $M_k(d)$ is no less than k. This implies that $r_4 \geq k$ for the real quadratic number fields as above. If we view $M_k(d)$ as a matrix over $\mathbb{Z}/2\mathbb{Z}$, then the rank of $M_k(d)$ is $s+m-1-k$. Thus if we apply elementary operations (I) and (II) to $M_k(d)$ only, then we see easily that the rank of every equivalent form of $M_k(d)$ is also $s+m-1-k$. Therefore we must apply an elementary operation (III) to $M_k(d)$ if we want to obtain an extra totally 1 row. Since $k > 0$, when applying elementary operations (I) and (II) only, one cannot obtain the following row in any equivalent form of $M_k(d)$:

$$(-1^{\cdots s}, 1^{\cdots m}).$$

Now we see that the rank of $M_k(d)$ is unchanged under elementary operations (III). It follows from Lemma 2.5 that $r_4 = k$ for $M_k(d)$. Note that if $k = 0$ in the above sign matrix, multiplying the first row by rows from 2 to s and applying an elementary operation (III) with $\varepsilon = 2$, one can obtain an equivalent form of $M_0(d)$ whose first row has entries 1 everywhere. Therefore r_4 can be any integer k with $1 \le k \le s + m - 1$.

Now we suppose that $m = 0$ and $S = \{q_1, \dots, q_{s-1}\}$ is a system of $\bigtriangledown$-representatives. Choose d such that

$$M_k(d) = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ & \ddots & & & & & \vdots \\ & & 1 & \cdots & \cdots & \cdots & 1 \\ & & & -1 & \cdots & \cdots & -1 \\ & & & & \ddots & & \vdots \\ & & & & & -1 & -1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = \delta_{i,s} = -1, & k+1 \le i \le s-1; \\ \delta_{i,j} = 1, & \text{otherwise}, \end{cases}$$

where $1 \le k \le s - 1$. If $k = 0$, applying the same elementary operations as in the case $m > 0$, we can see that $M_0(d)$ is equivalent to a matrix whose first row is totally 1. Hence $1 \le r_4 \le s - 1$ for this type.

Observe that, for $m = 0$ and $m > 0$, the sign matrices $M_k(d)$ we described are essentially the same. So, in the following cases, we only consider $m > 0$.

(A_2): *s is odd.* We choose $S = \{q_1, \dots, q_s, l_1, \dots, l_{m-1}\}$ as a system of $\bigtriangledown$-representatives. Put

$$M_k(d) = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ & \ddots & & & & & \vdots \\ & & 1 & \cdots & \cdots & \cdots & 1 \\ & & & -1 & \cdots & \cdots & -1 \\ & & & & \ddots & & \vdots \\ & & & & & -1 & -1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = \delta_{i,s+m} = -1, & k+1 \le i \le s+m-1; \\ \delta_{i,j} = 1, & \text{otherwise}, \end{cases}$$

where $0 \le k \le s+m-1$. Thus r_4 can be any integer k for $0 \le k \le s+m-1$.

Case (B): $T = (m, n, 0, 0)$ *with* $m + n \ge 3$:

(B_1): *n is odd.* Suppose that $S = \{p_1, \dots, p_n, l_1, \dots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives. Choose $d \in d(T)$ such that

$$M_{k+m-1}(d) = \begin{pmatrix}
-1 & -1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
-1 & -1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
-1 & -1 & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & & & & & & & & & \vdots \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
-1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
-1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & (-1)^{2k+1} \\
-1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & \cdots & \cdots & \cdots & \cdots & (-1)^{2k+2} \\
\vdots & \vdots & \vdots & \vdots & & & & & \ddots & & & & & \vdots \\
-1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & \cdots & \cdots & \cdots & (-1)^n \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\
 & & & & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & & & & 1 & 1
\end{pmatrix}$$

$$\begin{cases}
\delta_{i,j} = -1, & 1 \le j \le 2(k-1) \text{ and } j+2 \le i \le n; \\
\delta_{i,j} = -1, & 2k < i \le j \le n; \\
\delta_{i,i} = (-1)^i, & 1 \le i \le 2k; \\
\delta_{i,i+1} = (-1)^i, & 1 \le i \le 2k-1; \\
\delta_{i,i-1} = (-1)^i, & 1 \le i \le 2k; \\
\delta_{i,n+m} = (-1)^i, & 2k \le i \le n; \\
\delta_{i,j} = 1, & \text{otherwise,}
\end{cases}$$

where $0 \le k \le (n-1)/2$. Applying elementary operations (I), we have the following equivalent form of $M_{k+m-1}(d)$:

$$M_{k+m-1}(d) \cong \begin{pmatrix}
-1 & -1 & & & & & & & & & & & 1 \\
-1 & -1 & & & & & & & & & & & 1 \\
 & & -1 & -1 & & & & & & & & & 1 \\
 & & -1 & -1 & & & & & & & & & 1 \\
 & & & & \ddots & & & & & & & & \vdots \\
 & & & & & -1 & -1 & & & & & & 1 \\
 & & & & & -1 & -1 & & & & & & 1 \\
 & & & & & & & -1 & & & & & -1 \\
 & & & & & & & & \ddots & & & & \vdots \\
 & & & & & & & & & -1 & & & -1 \\
 & & & & & & & & & & 1 & & 1 \\
 & & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & 1 & 1
\end{pmatrix}$$

$$\begin{cases}
\delta_{i,i} = -1, & 0 \le i \le 2k; \\
\delta_{i,i+1} = (-1)^i, & 0 \le i \le 2k-1; \\
\delta_{i,i-1} = (-1)^{i-1}, & 0 \le i \le 2k; \\
\delta_{i,i} = \delta_{i,n+m} = -1, & 2k+1 \le i \le n; \\
\delta_{i,j} = 1, & \text{otherwise,}
\end{cases}$$

where $0 \leq k \leq (n-1)/2$. It is easy to see that the number of totally 1 rows in the above matrices is $k+m-1$. If we let $\delta_{n,n} = \delta_{n,n+m} = 1$ in $M_{(n-1)/2}(d)$, then the number of totally 1 rows in the above matrix is $(n-1)/2+m$. Choose $d \in d(T)$ such that

$$M_h(d) \cong \begin{pmatrix} -1 & & & & & & & & -1 \\ & \ddots & & & & & & & \vdots \\ & & -1 & & & & & & -1 \\ & & & 1 & & & & & 1 \\ & & & & \ddots & & & & \vdots \\ & & & & & 1 & & & 1 \\ & & & & & & -1 & & -1 \\ & & & & & & & \ddots & \vdots \\ & & & & & & & -1 & -1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = \delta_{i,n+m} = -1, & 1 \leq i \leq n \text{ or } n+h+1 \leq i \leq n+m+1; \\ \delta_{i,j} = 1, & \text{otherwise}, \end{cases}$$

where $0 \leq h \leq m-1$. Then $r_4 = h, 0 \leq h \leq m-1$.

Finally, we prove that $r_4 \leq (n-1)/2+m$ for any choice of $M(d)$: First suppose that $m = 0$. Note that the product of all entries in each row of any $M \cong M(d)$, where M is obtained from $M(d)$ by applying elementary operations (I) and (II), is 1. We see that there exist no totally -1 row since n is odd. By Lemma 3.1 one can check that $r_4 \leq (n-1)/2$. Now assume that $m > 0$. We view any sign matrix $M(d)$ as a matrix over $\mathbb{F}_2$. If some row in the last $m-1$ rows cannot be expressed as a linear combination of some of the first n rows, then $r(M(d)) > (n-1)/2$. Returning to the sign matrix, we have $r_4 \leq (n-1)/2+m$. So we may assume that each of the last $m-1$ rows can be expressed as a linear combination of some of the first n rows of $M(d)$. We need only consider the submatrix of $M(d)$ formed by the first n rows and first n columns. By Lemma 3.1, we have $r_4 < (n-1)/2+m+1$.

So, the value set of r_4 is $\{0, \ldots, (n-1)/2+m\}$ for this type.

(B_2): *n is even.* Then $d \equiv 1 \pmod 8$. By Lemma 2.3 we choose $S = \{p_1p_2, \ldots, p_1p_n, l_1, \ldots, l_{m-1}\}$ as a system of $\bigtriangledown$-representatives. Let $\left(\frac{p_j}{p_1}\right) = \left(\frac{l_i}{p_1}\right) = 1$ for all $2 \leq j \leq n$ and $1 \leq i \leq m-1$. Write the sign matrix

$$M(d) = \begin{pmatrix} 1 & \\ \vdots & N \\ 1 & \end{pmatrix},$$

where the submatrix N is of size $(n+m-2) \times (n+m-1)$, as in case (B_1). Dealing with the submatrix N as in case (B_1), we obtain $0 \leq r_4 \leq (n-2)/2+m = n/2+m-1$.

Now we prove that $r_4 < n/2 + m$: Attaching to the matrix $M(d)$ an extra row $\alpha = (\alpha_1, \ldots, \alpha_{n+m})$, where $\alpha_1 = \left(\frac{d/p_1}{p_1}\right)$, $\alpha_i = \left(\frac{p_1}{p_i}\right)$ for $2 \le i \le n$, and $\alpha_{n+k} = \left(\frac{p_1}{l_k}\right)$ for $1 \le k \le m$, we obtain an $(n+m-1) \times (n+m)$ matrix $\widehat{M} = \left(\begin{smallmatrix}\alpha \\ M(d)\end{smallmatrix}\right)$. Notice that $\prod_{i=1}^{n+m} \alpha_i = -1$. By Lemma 2.4 the number of totally 1 rows in every equivalent form of $M(d)$ is the same as in $\widehat{M}$. Multiplying the ith row ($i = 2, \ldots, n$) of $\widehat{M}$ by the first row, by Lemma 3.3 we can easily show that $r_4 < n/2 + m$.

Case (C): $T = (m, 0, 0, t)$ *with* $m + t \ge 3$: We have $2 \in NF$ and $d = u^2 - 2w^2, u, w \in \mathbb{N}$. Choose a system of $\bigtriangledown$-representatives $S = \{r_1, \ldots, r_t, l_1, \ldots, l_{m-1}, u+\sqrt{d}\}$ if $d \not\equiv 1 \pmod 8$; or $\{r_1 r_2, \ldots, r_1 r_t, l_1, \ldots, l_{m-1}, u+\sqrt{d}\}$ if $d \equiv 1 \pmod 8$. We set

$$M(d) = \begin{pmatrix} & N & \\ * & * \cdots & * \end{pmatrix},$$

where the submatrix N is of size $(t+m-1) \times (t+m)$ and the last row is arranged to correspond to $u + \sqrt{d}$. It follows from Case (B) that the number of totally 1 rows in N can vary from 0 to $t/2 + m - 1$ if t is even, and to $(t-1)/2 + m$ if t is odd. We note that $\{-1, u + \sqrt{d}\} \in \nabla^2$ if and only if there exists $\varepsilon \in \{\pm 1, \pm 2\}$ such that $\left(\frac{\varepsilon(u+w)}{p}\right) = 1$ for every odd prime $p \mid d$ (see [13]). By assumption, we may choose the last row such that $1 \le r_4 \le (t+1)/2 + m$ if t is odd, or $1 \le r_4 \le t/2 + m$ if t is even.

Case (D): $T = (m, n, s, 0)$ *with* $m + n + s \ge 3$:

(D_1): $n \equiv 1 \pmod 2$. Suppose that $S = \{p_1, \ldots, p_n, q_1, \ldots, q_n, l_1, \ldots \ldots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives. By Lemma 3.1 we have $r_4 \le (n-1)/2 + s + m$. As in case (B_1), we can choose $d \in d(T)$ such that

$$M_{k+h}(d) \cong \begin{pmatrix}
-1 & -1 & & & & & & & & & & & & & 1 \\
-1 & -1 & & & & & & & & & & & & & 1 \\
 & & \ddots & & & & & & & & & & & & \vdots \\
 & & & -1 & -1 & & & & & & & & & & 1 \\
 & & & -1 & -1 & & & & & & & & & & 1 \\
 & & & & & -1 & & & & & & & & & -1 \\
 & & & & & & \ddots & & & & & & & & \vdots \\
 & & & & & & & -1 & & & & & & & -1 \\
 & & & & & & & & 1 & & & & & & 1 \\
 & & & & & & & & & \ddots & & & & & \vdots \\
 & & & & & & & & & & 1 & & & & 1 \\
 & & & & & & & & & & & -1 & & & -1 \\
 & & & & & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & & & & & -1 & -1
\end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 0 \le i \le 2k; \\ \delta_{i,i+1} = (-1)^i, & 0 \le i \le 2k-1; \\ \delta_{i,i-1} = (-1)^{i-1}, & 0 \le i \le 2k; \\ \delta_{i,i} = \delta_{i,n+s+m} = -1, & 2k+1 \le i \le n \\ & \quad \text{or } n+h+1 \le i \le n+s+m-1; \\ \delta_{i,j} = 1, & \text{otherwise,} \end{cases}$$

where $1 \le k \le (n-1)/2$ and $0 \le h \le s+m-1$. Then $r_4 = k+h$. One can easily check that r_4 is also 1 if $k = h = 0$. Hence $1 \le r_4 \le (n-1)/2+s+m-1$. When $k = (n-1)/2$ and $h = s+m-1$, if we let $\delta_{n,n} = \delta_{n,n+s+m} = 1$ in the above matrix, then $r_4 = (n-1)/2+s+m$. So $1 \le r_4 \le (n-1)/2+s+m$ for this type.

(D_2): $n \equiv 0 \pmod 2$ *and* $s \equiv 1 \pmod 2$. Then $d \equiv 5 \pmod 8$. Suppose that $S = \{p_1, \ldots, p_n, q_1, \ldots, q_n, l_1, \ldots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives. Choose $d \in d(T)$ such that $M_{k+h}(d)$ is

$$\left(\begin{array}{cccccccccc|cccccc}
-1 & -1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & & -1 & \cdots\cdots & -1 & 1 & \cdots & 1 \\
1 & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & & -1 & \cdots\cdots & -1 & 1 & \cdots & 1 \\
-1 & -1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & & -1 & \cdots\cdots & -1 & 1 & \cdots & 1 \\
-1 & -1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & & -1 & \cdots\cdots & -1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & & & & & & \vdots & & \vdots & \vdots & & \vdots \\
-1 & -1 & -1 & -1 & \cdots & \cdots & -1 & -1 & \cdots & & -1 & \cdots\cdots & -1 & 1 & \cdots & 1 \\
-1 & -1 & -1 & -1 & \cdots & \cdots & 1 & 1 & \cdots & & -1 & \cdots\cdots & -1 & 1 & \cdots & 1 \\
-1 & -1 & -1 & -1 & \cdots & \cdots & & -1 & & & -1 & \cdots\cdots & -1 & 1 & \cdots & (-1)^{2k+1} \\
 & & & & & & & & \ddots & & \vdots & & \vdots & \vdots & & \vdots \\
-1 & -1 & -1 & -1 & \cdots & \cdots & & & & -1 & -1 & \cdots\cdots & -1 & 1 & \cdots & (-1)^n \\
\hline
-1 & & & & \cdots & \cdots & \cdots & & & -1 & & & & & & \\
\vdots & & & & & & & & & \vdots & & & & & & \\
-1 & & & & \cdots & \cdots & & & & -1 & & & N_h(d) & & & \\
1 & & & & \cdots & \cdots & & & & 1 & & & & & & \\
\vdots & & & & & & & & & \vdots & & & & & & \\
1 & & & & \cdots & \cdots & & & & 1 & & & & & &
\end{array}\right)$$

$$\begin{cases} \delta_{i,j} = -1, & 1 \le j \le 2(k-1) \text{ and } j+2 \le i \le n \\ & \quad \text{or } 1 \le i \le n \text{ and } n+1 \le j \le n+s \\ & \quad \text{or } n+1 \le i \le n+s \text{ and } 1 \le j \le n; \\ \delta_{i,j} = -1, & 2k < i \le j \le n; \\ \delta_{i,i} = (-1)^i, & 1 \le i \le 2k; \\ \delta_{i,i+1} = (-1)^i, & 1 \le i \le 2k-1; \\ \delta_{i,i-1} = (-1)^i, & 1 \le i \le 2k; \\ \delta_{i,n+s+m} = (-1)^i, & 2k \le i \le n; \\ \delta_{i,j} = 1, & \text{otherwise,} \end{cases}$$

where $0 \leq k \leq n/2$ and the submatrices $N_h(d)$ $(0 \leq h \leq s+m-1)$ are the same as in case (A_2). Applying elementary operations (I), we obtain the following equivalent form:

$$M_{k+h}(d) \cong \begin{pmatrix} -1 & -1 & & & & & & & -1 & \cdots & -1 & 1 & \cdots & 1 \\ -1 & -1 & & & & & & & 1 & \cdots & 1 & 1 & \cdots & 1 \\ & & \ddots & & & & & & \vdots & & \vdots & & & \vdots \\ & & & -1 & -1 & & & & 1 & & 1 & 1 & \cdots & 1 \\ & & & -1 & -1 & & & & 1 & & 1 & 1 & \cdots & 1 \\ & & & & & -1 & & & 1 & & 1 & 1 & \cdots & -1 \\ & & & & & & \ddots & & \vdots & & \vdots & & & \vdots \\ & & & & & & & -1 & 1 & \cdots & 1 & 1 & \cdots & -1 \\ -1 & & \cdots & \cdots & \cdots & & & -1 & & & & & & \\ \vdots & & & & & & & \vdots & & & & & & \\ -1 & & \cdots & \cdots & \cdots & & & -1 & & & N_h(d) & & & \\ 1 & & \cdots & \cdots & \cdots & & & 1 & & & & & & \\ \vdots & & & & & & & \vdots & & & & & & \\ 1 & & \cdots & \cdots & \cdots & & & 1 & & & & & & \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \leq i \leq 2k; \\ \delta_{i,i+1} = (-1)^i, & 1 \leq i \leq 2k; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \leq i \leq 2k; \\ \delta_{i,i} = \delta_{i,n+s+m} = -1, & 2k+1 \leq i \leq n; \\ \delta_{1,j} = -1, & n+1 \leq j \leq n+s; \\ \delta_{i,j} = -1, & n+1 \leq i \leq n+s \text{ and } 1 \leq j \leq n; \\ \delta_{i,j} = 1, & \text{otherwise.} \end{cases}$$

Multiplying the first two rows by all rows $2i$ $(i = 2, \ldots, k)$ and rows $2k+1$ to n, we see that they will be $(-1^{\cdots n+s}, 1^{\cdots m})$ and $(-1^{\cdots n}, 1^{\cdots s+m})$, respectively. We can convert them to be totally 1 rows by applying an elementary operation (III) with $\varepsilon = 2$ and $\varepsilon = -2$, respectively. So we have $1 \leq r_4 \leq n/2 + s + m$. The fact that $r_4 < n/2 + s + m + 1$ follows from Lemmas 3.1 and 3.2(ii).

(D_3): $n \equiv s \equiv 0 \pmod 2$. So $d \equiv 1 \pmod 8$. Choose $S = \{p_1p_2, \ldots, p_1p_n, q_1, \ldots, q_s, l_1, \ldots, l_{m-1}\}$ as a system of $\bigtriangledown$-representatives. As in case (B_2), we write

$$M(d) = \begin{pmatrix} 1 & \\ \vdots & N \\ 1 & \end{pmatrix},$$

where the submatrix N is of size $(n+m-2) \times (n+m-1)$, as in case (D_1). One can easily check that $1 \leq r_4 \leq n/2 + s + m - 1$.

As done in case (B_2), we consider the $(n+s+m-1)\times(n+s+m)$ matrix

$$\widehat{M}=\begin{pmatrix}\alpha\\ M(d)\end{pmatrix},$$

where

$$\alpha=\left(\left(\frac{d/p_1}{p_1}\right),\left(\frac{p_1}{p_2}\right),\dots,\left(\frac{p_1}{p_n}\right),\left(\frac{p_1}{q_1}\right),\dots,\left(\frac{p_1}{q_s}\right),\dots,\left(\frac{p_1}{l_1}\right),\dots,\left(\frac{p_1}{l_m}\right)\right).$$

Multiplying the ith rows $(i=2,\dots,n)$ of $\widehat{M}$ by the first row, we obtain a new equivalent form of $\widehat{M}$ and partition it into

$$\begin{pmatrix} M_1 & M_2 & \alpha_1\\ M_2^T & M_3 & \alpha_2\end{pmatrix},$$

where the blocks M_1 and M_3 are of sizes $n\times n$ and $(s+m-1)\times(s+m-1)$ respectively, and $\binom{\alpha_1}{\alpha_2}$ is the last column. We view the above equivalent form of $\widehat{M}$ as a matrix over $\mathbb{F}_2$. Consider the $n\times(n+1)$ matrix (M_1,β), where the column vector β is the sum of all columns of M_2 and the column vector α_1. It follows from Lemma 3.3 that $r_4<n/2+s+m$.

So, for this type, we have $1\le r_4\le n/2+s+m-1$.

Case (E): *$T=(m,0,s,t)$ with $m+s+t\ge 3$*: Everything here is the same as in Case (D). With similar constructions of sign matrices $M(d)$, we have

$$1\le r_4\le\begin{cases} t/2+s+m-1 & \text{if } t\equiv 0 \pmod 2 \text{ and } s\equiv 0 \pmod 2;\\ t/2+s+m & \text{if } t\equiv 0 \pmod 2 \text{ and } s\equiv 1 \pmod 2;\\ (t-1)/2+s+m & \text{if } t\equiv 1 \pmod 2.\end{cases}$$

Case (F): *$T=(m,n,0,t)$ with $m+n+t\ge 3$*:

(F_1): $t+n\equiv 1 \pmod 2$. Let $S=\{r_1,\dots,r_t,p_1,\dots,p_n,l_1,\dots,l_{m-1}\}$ be a system of $\bigtriangledown$-representatives. First we assume that t is even. As in case (B_1), we have $1\le r_4\le(t+n-1)/2+m$. Choose $d\in d(T)$ such that

$$M(d)\cong\begin{pmatrix} -1 & -1 & & & & & & & 1\\ -1 & -1 & & & & & & & 1\\ & & \ddots & & & & & & \vdots\\ & & & -1 & -1 & & & & 1\\ & & & -1 & -1 & & & & 1\\ & & & & & 1 & & & 1\\ & & & & & & 1 & & 1\\ & & & & & & & \ddots & \vdots\\ & & & & & & & 1 & 1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \le i \le t+n-1; \\ \delta_{i,i+1} = (-1)^i, & 1 \le i \le t+n-1; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le t+n-1; \\ \delta_{i,j} = 1, & \text{otherwise.} \end{cases}$$

Multiplying both rows $t-1$ and t by all rows $2i+1$ ($i = 0, \ldots, t/2-2$), we obtain a new equivalent form of $M(d)$, in which both row $t-1$ and row t are

$$(-1^{\ldots t}, 1^{\ldots n+m}).$$

They will be totally 1 rows if one applies an elementary operation (III) with $\varepsilon = -2$. Then $r_4 = (t+n+1)/2 + m$. It follows from Lemmas 2.4 and 3.1 that $r_4 < (t+n+1)/2 + m + 1$. The proof for the case $t \equiv 1 \pmod 2$ is similar.

So, for this type, we have $1 \le r_4 \le (t+n+1)/2 + m$.

(F_2): $t \equiv n \equiv 1 \pmod 2$. Then $d \equiv 5 \pmod 8$. Suppose that $S = \{r_1, \ldots, r_t, p_1, \ldots, p_n, l_1, \ldots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives. By a discussion similar to case ($\mathbb{C}_1$) (see below), one can easily see that r_4 can be any k for $1 \le k \le (t+n)/2 + m - 1$. That r_4 be $(t+n)/2 + m$ follows by putting

$$M(d) \cong \begin{pmatrix} -1 & -1 & & & & & & & & & & & 1 \\ -1 & -1 & & & & & & & & & & & 1 \\ & & \ddots & & & & & & & & & & \vdots \\ & & & -1 & -1 & & & & & & & & 1 \\ & & & -1 & -1 & & & & & & & & 1 \\ & & & & & -1 & & & & & & & 1 \\ & & & & & & -1 & -1 & & & & & 1 \\ & & & & & & -1 & -1 & & & & & 1 \\ & & & & & & & & \ddots & & & & \vdots \\ & & & & & & & & & -1 & -1 & & 1 \\ & & & & & & & & & -1 & -1 & & 1 \\ & & & & & & & & & & & 1 & 1 \\ & & & & & & & & & & & 1 & 1 \\ & & & & & & & & & & & \ddots & \vdots \\ & & & & & & & & & & & 1 & 1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \le i \le t-1 \\ & \quad \text{or } t+1 \le i \le t+n-1; \\ \delta_{i,i+1} = (-1)^i, & 1 \le i \le t-1; \\ \delta_{i,i-1} = (-1)^{i-1}, & 2 \le i \le t; \\ \delta_{i,i+1} = (-1)^{i+1}, & t+1 \le i \le t+n-1; \\ \delta_{i,i-1} = (-1)^i, & t+1 \le i \le t+n; \\ \delta_{i,j} = 1, & \text{otherwise.} \end{cases}$$

On the other hand, Lemma 3.2(ii) implies that $r_4 < (t+n)/2 + m + 1$. So we have $1 \le r_4 \le (t+n)/2 + m$.

(F_3): $t \equiv n \equiv 0 \pmod 2$. Then $d \equiv 1 \pmod 8$. Choose $S = \{r_1r_2, r_1r_3, \dots, r_1r_t, r_1p_1, \dots, r_1p_n, l_1, \dots, l_{m-1}\}$ as a system of $\bigtriangledown$-representatives. As in case (B_2), by Lemma 3.2(i) one can easily show that $r_4 \le (t+n)/2 + m$. As in cases (B_2) and (F_1), r_4 can be any integer k with $1 \le k \le (t+n)/2 + m$.

Case (G): $T = (m, n, s, t)$ *with* $m + n + s + t \ge 3$:

(G_1): $t \equiv n \not\equiv s \pmod 2$. Then $d \equiv 5 \pmod 8$. Let $S = \{r_1, r_2, \dots, r_t, p_1, \dots, p_n, q_1, \dots, q_s, l_1, \dots, l_{m-1}\}$ be a system of $\bigtriangledown$-representatives. With a similar choice of the entries of $M(d)$ as in case ($\mathbb{C}_1$), we see that r_4 can be any integer k with $1 \le k \le (t+n)/2 + s + m - 1$. When s is even, we choose $d \in d(T)$ such that

$$M(d) \cong \left(\begin{array}{cccccc|cccccc|cccccc}
-1 & -1 & & & & 1 & 1 & & \cdots & & & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
-1 & -1 & & & & 1 & 1 & & \cdots & & & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
 & & \ddots & & & \vdots & \vdots & & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 & & & -1 & -1 & 1 & \vdots & & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 & & & -1 & -1 & 1 & \vdots & & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 & & & & & -1 & 1 & & \cdots & & & 1 & -1 & \cdots & -1 & 1 & \cdots & 1 \\
\hline
-1 & \cdots & \cdots & \cdots & \cdots & -1 & -1 & -1 & & & & & -1 & \cdots & -1 & 1 & \cdots & 1 \\
1 & \cdots & \cdots & \cdots & \cdots & 1 & -1 & -1 & & & & & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & & & & & \vdots & & & \ddots & & & & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & & & & & \vdots & & & & -1 & -1 & & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & & & & & \vdots & & & & -1 & -1 & & \vdots & & \vdots & \vdots & & \vdots \\
1 & \cdots & \cdots & \cdots & \cdots & 1 & & & & & & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\hline
-1 & \cdots & \cdots & \cdots & \cdots & -1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & & & & & \vdots & \vdots & & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
-1 & \cdots & \cdots & \cdots & \cdots & -1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
 & & & & & & & & & & & & & & & 1 & \cdots & 1 \\
 & & & & & & & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & & & & & & 1 & 1
\end{array}\right)$$

$$\begin{cases}
\delta_{i,i} = -1, & 1 \le i \le t+n-1; \\
\delta_{i,i+1} = (-1)^i, & 1 \le i \le t-1; \\
\delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le t-1; \\
\delta_{i,i+1} = (-1)^{i-1}, & t+1 \le i \le t+n-1; \\
\delta_{i,i-1} = (-1)^i, & t+1 \le i \le t+n-1; \\
\delta_{i,j} = -1, & t+n+1 \le i \le t+n+s \text{ and } 1 \le j \le t+n; \\
\delta_{t,j} = -1, & t+n+1 \le j \le t+n+s; \\
\delta_{t+1,j} = -1, & 1 \le j \le t \text{ or } t+n+1 \le j \le t+n+s; \\
\delta_{i,j} = 1, & \text{otherwise.}
\end{cases}$$

It is easy to check that $r_4 = (t+n)/2 + s + m$ for the above choice. When s is odd, choose $d \in d(T)$ such that

$$M(d) \cong \left(\begin{array}{cccc|cccc|ccc|ccc}
-1 & -1 & & & & & & & -1 & \cdots & -1 & 1 & \cdots & 1 \\
-1 & -1 & & & & & & & 1 & \cdots & 1 & 1 & \cdots & 1 \\
 & & \ddots & & & & & & \vdots & & \vdots & \vdots & & \vdots \\
 & & -1 & -1 & & & & & \vdots & & \vdots & \vdots & & \vdots \\
 & & -1 & -1 & & & & & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
 & & & & -1 & -1 & & & 1 & \cdots & 1 & 1 & \cdots & 1 \\
 & & & & -1 & -1 & & & 1 & \cdots & 1 & 1 & \cdots & 1 \\
 & & & & & & \ddots & & \vdots & & \vdots & \vdots & & \vdots \\
 & & & & & & -1 & -1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
 & & & & & & -1 & -1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
-1 & & \cdots & -1 & -1 & \cdots\cdots & \cdots & -1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
-1 & & \cdots & -1 & -1 & \cdots\cdots & \cdots & -1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
 & & & & & & & & & & & 1 & \cdots & 1 \\
 & & & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & & 1 & 1
\end{array}\right)$$

$$\begin{cases}
\delta_{i,i} = -1, & 1 \le i \le t+n; \\
\delta_{i,i+1} = (-1)^i, & 1 \le i \le t+n; \\
\delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le t+n; \\
\delta_{1,j} = -1, & t+n+1 \le j \le t+n+s; \\
\delta_{i,j} = -1, & t+n+1 \le i \le t+n+s \text{ and } 1 \le j \le t+n; \\
\delta_{i,j} = 1, & \text{otherwise.}
\end{cases}$$

Multiplying both rows $t+n-1$ and $t+n$ by all rows $2i+1$ ($i = 0, \ldots, (t+n)/2-2$), we obtain a new equivalent form of $M(d)$, in which both row $t+n-1$ and row $t+n$ are

$$(-1^{\ldots t+n}, 1^{\ldots s+m}).$$

Applying an elementary operation (III) with $\varepsilon = -1$, we see that all -1 entries in this row are transformed into 1. Next, multiplying the first row by all rows $2i$ ($i = 2, \ldots, t/2$), we see that it will become

$$(-1^{\ldots t}, 1^{\ldots n}, -1^{\ldots s}, 1^{\ldots m}).$$

Applying an elementary operation (III) with $\varepsilon = -2$, we see that it becomes a totally 1 row. Thus $r_4 = (t+n)/2 + s + m$ for the above choice.

By Lemmas 3.2(ii) and 4.1, one can easily prove that $r_4 < (t+n)/2 + s + m + 1$. So, for this type, $1 \le r_4 \le (t+n)/2 + s + m$.

(G_2): $t \equiv n \equiv s \pmod 2$. We have $d \equiv 1 \pmod 8$. Choose $S = \{r_1r_2, \ldots, r_1r_t, r_1p_1, \ldots, r_1p_n, q_1, \ldots, q_s, l_1, \ldots, l_{m-1}\}$ as a system of $\bigtriangledown$-representatives. As in case (B_2), by Lemmas 3.5 and 4.1, one can prove that $r_4 < (t+n)/2 + s + m$. With similar choices of entries of $M(d)$ to those in case (B_2), we have $1 \le r_4 \le (t+n)/2 + s + m - 1$.

(G_3): $t + n \equiv 1 \pmod 2$. Suppose that $S = \{r_1, r_2, \ldots, r_t, p_1, \ldots, p_n, q_1, \ldots, q_s, l_1, \ldots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives. By similar constructions of $M(d)$ to that in case (D_1), we have $1 \le r_4 \le (t+n-1)/2 + s + m$. The maximums of r_4 are ensured by Lemmas 3.3, 3.4 and 4.1.

Case (H): $T = (m, 0, 0, 0)$ *with* $m \ge 3$: We have $2 \in NF$ and $d = u^2 - 2w^2$, $u, w \in \mathbb{Z}$. Choose a system of $\bigtriangledown$-representatives $S = \{l_1, \ldots, l_{m-1}, u + \sqrt{d}\}$ if $u + w \equiv 1 \pmod 4$, or $S = \{l_1, \ldots, l_{m-1}\}$ if $u + w \equiv 3 \pmod 4$. Choose $d \in d(T)$ such that

$$M_k(d) = \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & & \vdots \\ & & 1 & & & & 1 \\ & & & -1 & & & -1 \\ & & & & \ddots & & \vdots \\ & & & & & -1 & -1 \\ * & \cdots & \cdots & \cdots & \cdots & * & * \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = \delta_{i,n+m} = -1, & k+1 \le i \le m-1,\ 0 \le k \le m-1; \\ \delta_{i,j} = 1, & \text{otherwise}, \end{cases}$$

where the last row is arranged to correspond to $u + \sqrt{d}$ if $u + w \equiv 1 \pmod 4$, and it will disappear if $u + w \equiv 3 \pmod 4$. It is easy to check that $0 \le r_4 \le m$.

Case ($\mathbb{A}$): $T = 2(m, 0, s, 0)$ *with* $m + s \ge 3$: Suppose that $S = \{q_1, \ldots, q_s, l_1, \ldots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives.

($\mathbb{A}_1$): $s \equiv 0 \pmod 2$. We choose $d \in d(T)$ such that

$$M_{k+m-1}(d) = \begin{pmatrix} 1 & & & & & & & & -1 \\ & \ddots & & & & & & & \vdots \\ & & 1 & & & & & & -1 \\ & & & -1 & & & & & 1 \\ & & & & \ddots & & & & \vdots \\ & & & & & -1 & & & 1 \\ & & & & & & 1 & & 1 \\ & & & & & & & \ddots & \vdots \\ & & & & & & & 1 & 1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,s+m} = -1, & 0 \le i \le k; \\ \delta_{i,i} = -1, & k+1 \le i \le s; \\ \delta_{i,j} = 1, & \text{otherwise}, \end{cases}$$

where $1 \le k \le s$. Then $r_4 = k - 1 + m - 1,\ 1 \le k \le s$. Choose $d \in d(T)$ such that

$$M_h(d) = \begin{pmatrix} 1 & & & & & & & & & -1 \\ & -1 & & & & & & & & 1 \\ & & \ddots & & & & & & & \vdots \\ & & & -1 & & & & & & 1 \\ & & & & 1 & & & & & 1 \\ & & & & & \ddots & & & & \vdots \\ & & & & & & 1 & & & 1 \\ & & & & & & & -1 & & -1 \\ & & & & & & & & \ddots & \vdots \\ & & & & & & & & -1 & -1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 2 \le i \le s-1; \\ \delta_{i,i} = \delta_{i,s+m} = -1, & s+h+1 \le i \le s+m-1; \\ \delta_{1,s+m} = -1; & \\ \delta_{i,j} = 1, & \text{otherwise}, \end{cases}$$

where $0 \le h \le m-1$. Thus r_4 can be any integer h for $0 \le h \le m-1$. Since the product of all entries in any of the first s rows is -1 and s is even, $r_4 \le s + m - 2$. Hence $0 \le r_4 \le s + m - 2$.

($\mathbb{A}_2$): $s \equiv 1 \pmod 2$. With the above constructions, we have $0 \le r_4 \le s + m - 2$. It is sufficient to show that $r_4 = s + m - 1$ for some $M(d)$. In fact, if we choose $d \in d(T)$ such that

$$M(d) = \begin{pmatrix} -1 & \cdots & -1 & & & & 1 \\ \vdots & \ddots & \vdots & & & & \vdots \\ -1 & \cdots & -1 & & & & 1 \\ & & & 1 & & & 1 \\ & & & & \ddots & & \vdots \\ & & & & & 1 & 1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,j} = -1, & 1 \le i \le s \text{ and } 1 \le j \le s; \\ \delta_{i,j} = 1, & \text{otherwise} \end{cases}$$

and apply an elementary operation (III) with $\varepsilon = 2$, then $M(d)$ is equivalent to a matrix with all entries 1. So $r_4 = s + m - 1$.

Case ($\mathbb{B}$): $T = 2(m, n, 0, 0)$ with $m + n \ge 3$: We have $d \not\equiv 1 \pmod 8$ and $2 \notin NF$. Suppose that $S = \{p_1, \dots, p_n, l_1, \dots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives.

($\mathbb{B}_1$): $n \equiv 0 \pmod 2$. Then $d/2 \equiv 1 \pmod 4$. Choose $d \in d(T)$ such that

$$M_{k+e}(d) \cong \begin{pmatrix} -1 & -1 & & & & & & & & & & & & 1 \\ -1 & -1 & & & & & & & & & & & & 1 \\ & & \ddots & & & & & & & & & & & \vdots \\ & & & -1 & -1 & & & & & & & & & 1 \\ & & & -1 & -1 & & & & & & & & & 1 \\ & & & & & -1 & & & & & & & & -1 \\ & & & & & & \ddots & & & & & & & \vdots \\ & & & & & & & -1 & & & & & & -1 \\ & & & & & & & & 1 & & & & & 1 \\ & & & & & & & & & \ddots & & & & \vdots \\ & & & & & & & & & & 1 & & & 1 \\ & & & & & & & & & & & -1 & & -1 \\ & & & & & & & & & & & & \ddots & \vdots \\ & & & & & & & & & & & & -1 & -1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 0 \le i \le 2k; \\ \delta_{i,i+1} = (-1)^i, & 0 \le i \le 2k; \\ \delta_{i,i-1} = (-1)^{i-1}, & 0 \le i \le 2k; \\ \delta_{i,i} = \delta_{i,n+m} = -1, & 2k+1 \le i \le n \\ & \quad \text{or } n+e+1 \le i \le n+m-1; \\ \delta_{i,j} = 1, & \text{otherwise}, \end{cases}$$

where $0 \le k \le n/2$ and $0 \le e \le m-1$. When $k \le n/2 - 1$, multiplying the nth row by all rows $2j$ $(j = 1, \dots, k)$ and rows $2k+1$ to $n-1$, we see that it will be

$$(-1^{\cdots n}, 1^{\cdots m}).$$

When $k = n/2$, multiplying both rows $n-1$ and n by all rows $2j$ $(j = 1, \dots, k-1)$, one can see that they will be

$$(-1^{\cdots n}, 1^{\cdots m}).$$

Applying an elementary operation (III) with $\varepsilon = 2$, we see that the above rows will become totally 1. It follows that $r_4 = k + e + 1$, i.e. $1 \le r_4 \le n/2 + m$.

Finally, by Lemma 3.1 one can easily prove that $r_4 < n/2 + m + 1$ for any choice of $M(d)$.

($\mathbb{B}_2$): $n \equiv 1 \pmod 2$. Then $d/2 \equiv 3 \pmod 4$. Choose $d \in d(T)$ such that $M_{k+e}(d)$ is

$$\begin{pmatrix}
-1 & -1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & 1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & 1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & 1 & 1 & 1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
 & & & & \ddots & & & & & & \vdots & \vdots & & & & & & \vdots \\
 & & & & & -1 & -1 & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
 & & & & & 1 & 1 & -1 & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
 & & & & & & & -1 & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
 & & & & & & & & -1 & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & (-1)^{2k+1} \\
 & & & & & & & & & \ddots & \vdots & \vdots & & & & & & \vdots \\
 & & & & & & & & & & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & (-1)^{n+1} \\
 & & & & & & & & & & & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
 & & & & & & & & & & & & \ddots & & & & & \vdots \\
 & & & & & & & & & & & & & 1 & & & & 1 \\
 & & & & & & & & & & & & & & -1 & & & -1 \\
 & & & & & & & & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & & & & & & & & -1 & -1
\end{pmatrix}$$

$$\begin{cases}
\delta_{i,i} = (-1)^i, & 1 \le i \le 2k; \\
\delta_{i,j} = -1, & 1 \le i < j \le n; \\
\delta_{i,i} = -1, & 2k+1 \le i \le n; \\
\delta_{i,n+m} = (-1)^{i-1}, & 2k+1 \le i \le n; \\
\delta_{i,i} = \delta_{i,n+m} = -1, & n+e+1 \le i \le n+m-1; \\
\delta_{i,j} = 1, & \text{otherwise},
\end{cases}$$

where $0 \le k \le (n-1)/2$ and $0 \le e \le m-1$. Applying elementary operations (I) and (II), we have the following equivalent form:

$$M_{k+e}(d) \cong \begin{pmatrix}
-1 & -1 & & & & & & & & & & & & & & 1 \\
-1 & -1 & & & & & & & & & & & & & & 1 \\
 & & \ddots & & & & & & & & & & & & & \vdots \\
 & & & -1 & -1 & & & & & & & & & & & 1 \\
 & & & -1 & -1 & & & & & & & & & & & 1 \\
 & & & & & -1 & & & & & & & & & & -1 \\
 & & & & & & \ddots & & & & & & & & & \vdots \\
 & & & & & & & -1 & & & & & & & & -1 \\
 & & & & & & & & -1 & & & & & & & 1 \\
 & & & & & & & & & 1 & & & & & & 1 \\
 & & & & & & & & & & \ddots & & & & & \vdots \\
 & & & & & & & & & & & 1 & & & & 1 \\
 & & & & & & & & & & & & -1 & & & -1 \\
 & & & & & & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & & & & & & -1 & -1
\end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \le i \le 2k; \\ \delta_{i,i+1} = (-1)^i, & 1 \le i \le 2k; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le 2k; \\ \delta_{i,i} = \delta_{i,n+m} = -1, & 2k+1 \le i \le n-1 \\ & \quad \text{or } n+e+1 \le i \le n+m-1; \\ \delta_{n,n} = -1; & \\ \delta_{i,j} = 1, & \text{otherwise,} \end{cases}$$

where $0 \le k \le (n-1)/2$, $0 \le e \le m-1$. Multiplying row n by all rows $2l-1$ $(l = 1, \ldots, k)$ and rows $2k+1$ to $n-1$, we see that it will be $(-1^{\cdots n}, 1^{\cdots m})$. It will become a totally 1 row after an elementary operation (III) with $\varepsilon = -2$. So $r_4 = k+e+1$, i.e., $1 \le r_4 \le (n-1)/2+m$. Note that if we put $\delta_{n,n} = 1$, $\delta_{n,n+m} = -1$ in the above matrix when $k = e = 0$, then $r_4 = 0$.

When $m \ge 1$, the fact that $r_4 < (n-1)/2+m+1$ follows from Lemma 3.1. When $m = 0$, consider the $n \times n$ matrix $\widehat{M} = \binom{M(d)}{\alpha}$, where the row vector α is the sum of all rows of $M(d)$. By Lemma 3.2 we have $r_4 < (n-1)/2+1$.

Hence, for this type, $0 \le r_4 \le (n-1)/2+m$.

Case ($\mathbb{C}$): $T = 2(m, 0, 0, t)$ *with* $m + t \ge 3$: We have $d \equiv 1 \pmod 8$ and $d = u^2 - 2w^2, u, w \in \mathbb{Z}$. Let $S = \{r_1, \ldots, r_t, l_1, \ldots, l_{m-1}, u + \sqrt{d}\}$ be a system of $\bigtriangledown$-representatives. We write

$$M(d) = \begin{pmatrix} & N(d) & \\ * & \cdots & * \end{pmatrix},$$

where the submatrix $N(d)$ is of size $(t+m-1) \times (t+m)$, and the last row is arranged to correspond to $u + \sqrt{d}$.

($\mathbb{C}_1$): $t \equiv 0 \pmod 2$. Then $d/2 \equiv 1 \pmod 4$. By Lemma 3.3, we have $r_4 \le t/2 + m$. Choose $d \in d(T)$ such that $N_{k+h}(d)$ is

$$\begin{pmatrix}
-1 & -1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\
1 & 1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\
1 & 1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\
1 & 1 & 1 & 1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\
 & & & & \ddots & & & & & & \vdots & \vdots & & & & & & \vdots \\
 & & & & & -1 & -1 & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\
 & & & & & 1 & 1 & -1 & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\
 & & & & & & & -1 & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\
 & & & & & & & & -1 & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & (-1)^{2k} \\
 & & & & & & & & & \ddots & \vdots & \vdots & & & & & & \vdots \\
 & & & & & & & & & & -1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & (-1)^t \\
 & & & & & & & & & & & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
 & & & & & & & & & & & & \ddots & & & & & \vdots \\
 & & & & & & & & & & & & & 1 & & & \cdots & 1 \\
 & & & & & & & & & & & & & & -1 & & \cdots & -1 \\
 & & & & & & & & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & & & & & & & & -1 & -1
\end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = (-1)^i, & 1 \le i \le 2k; \\ \delta_{i,j} = -1, & 1 \le i < j \le t; \\ \delta_{i,i} = -1, & 2k+1 \le i \le t; \\ \delta_{i,t+m} = -1, & 1 \le i \le 2k; \\ \delta_{i,t+m} = (-1)^i, & 2k+1 \le i \le t; \\ \delta_{i,i} = \delta_{i,t+m} = -1, & t+h+1 \le i \le t+m-1; \\ \delta_{i,j} = 1, & \text{otherwise,} \end{cases}$$

where $0 \le k \le t/2$ and $0 \le h \le m-1$. By elementary operations (I) and (II), we have the following equivalent form:

$$N_{k+h}(d) \cong \left(\begin{array}{cccccccccccccccc}
-1 & -1 &&&&&&&&&&&&&& 1 \\
-1 & -1 &&&&&&&&&&&&&& 1 \\
&& \ddots &&&&&&&&&&&&& \vdots \\
&&& -1 & -1 &&&&&&&&&&& 1 \\
&&& -1 & -1 &&&&&&&&&&& 1 \\
&&&&& -1 &&&&&&&&&& -1 \\
&&&&&& \ddots &&&&&&&&& \vdots \\
&&&&&&& -1 &&&&&&&& -1 \\
&&&&&&&& -1 &&&&&&& 1 \\
&&&&&&&&& 1 &&&&&& 1 \\
&&&&&&&&&& \ddots &&&&& \vdots \\
&&&&&&&&&&& 1 &&&& 1 \\
&&&&&&&&&&&& -1 &&& -1 \\
&&&&&&&&&&&&& \ddots && \vdots \\
&&&&&&&&&&&&&& -1 & -1
\end{array}\right)$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \le i \le 2k; \\ \delta_{i,i+1} = (-1)^i, & 1 \le i \le 2k; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le 2k; \\ \delta_{i,i} = \delta_{i,t+m} = -1, & 2k+1 \le i \le t-1 \\ & \quad \text{or } t+h+1 \le i \le t+m-1; \\ \delta_{t,t} = -1; & \\ \delta_{i,j} = 1, & \text{otherwise,} \end{cases}$$

where $1 \le k \le t/2$ and $0 \le h \le m-1$. One can easily check that $r_4 = k+h$ for the above choice. So, by assumption, we may choose the last row of $M(d)$ such that $1 \le r_4 \le t/2+m$.

($\mathbb{C}_2$): $t \equiv 1 \pmod 2$. Then $d/2 \equiv 3 \pmod 4$. This is the same as the case $t \equiv 1 \pmod 2$ of case (C).

Case ($\mathbb{D}$): $T = 2(m,n,s,0)$ *with* $m+n+s \ge 3$: Then $2 \notin NF$. Suppose that $S = \{p_1, \ldots, p_n, q_1, \ldots, q_s, l_1, \ldots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives. Deal with $M(d)$ as in case ($\mathbb{B}$).

($\mathbb{D}_1$): $n \equiv 0 \pmod 2$. Then $d/2 \equiv 1 \pmod 4$. With suitable choices of $M(d)$ as in cases ($\mathbb{A}$) and ($\mathbb{B}_1$), we can easily check that r_4 can be any integer k with

$$1 \le k \le \begin{cases} n/2 + s + m & \text{if } s \equiv 1 \pmod 2; \\ n/2 + s + m - 1 & \text{if } s \equiv 0 \pmod 2). \end{cases}$$

It follows from Lemma 3.1 that $r_4 < n/2 + s + m + 1$ if $s \equiv 1 \pmod 2$. On the other hand, when $s \equiv 0 \pmod 2$, the product of all entries in any of the first n rows of $M(d)$ is 1 and the product of all entries in row i $(i = n+1, \ldots, n+s)$ is -1. By Lemmas 3.1 and 3.2, one can easily prove that $r_4 < n/2 + s + m$.

($\mathbb{D}_2$): $n \equiv 1 \pmod 2$. Then $d/2 \equiv 3 \pmod 4$. By a similar discussion to that in case ($\mathbb{B}_2$), the number of totally 1 rows in the first n rows of $M(d)$ can range from 1 to $(n-1)/2+1$. If $s \equiv 1 \pmod 2$, by a similar construction of $M(d)$ to that in case ($\mathbb{A}_2$), the number of totally 1 rows in the first $n+s$ rows is $(n-1)/2 + s + 1$. So $1 \le r_4 \le (n-1)/2 + s + m$. Now, we assume $s \equiv 0 \pmod 2$. We want to verify that r_4 can be $(n-1)/2 + s + m$. In fact, choose $d \in d(T)$ such that

$$M(d) \cong \begin{pmatrix}
-1 & -1 & & & & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
-1 & -1 & & & & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
& & \ddots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
& & & -1 & -1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
& & & -1 & -1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
& & & & & -1 & -1 & \cdots & -1 & 1 & \cdots & 1 \\
-1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & & & & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\
-1 & \cdots & \cdots & \cdots & \cdots & -1 & & & 1 & 1 & \cdots & 1 \\
& & & & & & & & & 1 & \cdots & 1 \\
& & & & & & & & & & \ddots & \vdots \\
& & & & & & & & & & 1 & 1
\end{pmatrix}$$

$$\begin{cases}
\delta_{i,i} = -1, & 1 \le i \le n; \\
\delta_{i,i+1} = (-1)^i, & 1 \le i \le n-1; \\
\delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le n-1; \\
\delta_{n,j} = -1, & n+1 \le j \le n+s; \\
\delta_{i,j} = -1, & n+1 \le i \le n+s \text{ and } 1 \le j \le n; \\
\delta_{i,j} = 1, & \text{otherwise.}
\end{cases}$$

Hence $r_4 = (n-1)/2 + s + m$. So, for this type, r_4 can be any integer k with $1 \le k \le (n-1)/2 + s + m$. Note that $r_4 \le (n-1)/2 + s + m$ is ensured by Lemma 3.1.

Case ($\mathbb{E}$): *$T = 2(m,0,s,t)$ with $m+s+t \geq 3$*: Suppose that $S = \{r_1, \dots, r_t, q_1, \dots, q_s, l_1, \dots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives. Deal with $M(d)$ as in case ($\mathbb{D}$).

($\mathbb{E}_1$): *$t \equiv 0 \pmod 2$ and $s \equiv 0 \pmod 2$*. Then $d/2 \equiv 1 \pmod 4$. With similar choices of entries of $M(d)$ to those in cases ($\mathbb{A}_1$) and (B_2), we find that r_4 can be any integer k with $1 \leq k \leq t/2 + s + m - 2$. Taking $n = 0$ in Lemma 3.3(i), we can prove that $r_4 \leq t/2 + s + m - 1$. Now we are going to show that r_4 can be $t/2 + s + m - 1$. Consider the matrix

$$M(d) \cong \begin{pmatrix} -1 & -1 & & & & & & & & & & 1 \\ -1 & -1 & & & & & & & & & & 1 \\ & & \ddots & & & & & & & & & \vdots \\ & & & -1 & -1 & & & & & & & 1 \\ & & & -1 & -1 & & & & & & & -1 \\ & & & & & 1 & & & & & & -1 \\ & & & & & & \ddots & & & & & \vdots \\ & & & & & & & 1 & & & & -1 \\ & & & & & & & & 1 & & & 1 \\ & & & & & & & & & \ddots & & \vdots \\ & & & & & & & & & & 1 & 1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \leq i \leq t; \\ \delta_{i,i+1} = (-1)^i, & 1 \leq i \leq t; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \leq i \leq t; \\ \delta_{i,t+s+m} = -1, & t \leq i \leq t+s; \\ \delta_{i,j} = 1, & \text{otherwise.} \end{cases}$$

Multiplying row t by row $t+1$, we obtain an equivalent form

$$\begin{pmatrix} -1 & -1 & & & & & & & & & & 1 \\ -1 & -1 & & & & & & & & & & 1 \\ & & \ddots & & & & & & & & & \vdots \\ & & & -1 & -1 & & & & & & & 1 \\ & & & -1 & -1 & & & & & & & 1 \\ & & & & & 1 & & & & & & -1 \\ & & & & & & \ddots & & & & & \vdots \\ & & & & & & & 1 & & & & -1 \\ & & & & & & & & 1 & & & 1 \\ & & & & & & & & & \ddots & & \vdots \\ & & & & & & & & & & 1 & 1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \leq i \leq t; \\ \delta_{i,i+1} = (-1)^i, & 1 \leq i \leq t; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \leq i \leq t; \\ \delta_{i,t+s+m} = -1, & t+1 \leq i \leq t+s; \\ \delta_{i,j} = 1, & \text{otherwise.} \end{cases}$$

Hence $r_4 = t/2 + s + m - 1$. So, for this type, we have $1 \le r_4 \le t/2 + s + m - 1$.

$(\mathbb{E}_2)$: $t \equiv 0 \pmod 2$ and $s \equiv 1 \pmod 2$. Then $d/2 \equiv 1 \pmod 4$. It follows from Lemma 3.1 that $r_4 \le t/2 + s + m$. Choose $d \in d(T)$ such that

$$M_{k+h}(d) \cong \begin{pmatrix} -1 & -1 & & & & & & -1 & \cdots & -1 & 1 & \cdots & 1 \\ -1 & -1 & & & & & & 1 & \cdots & 1 & 1 & \cdots & 1 \\ & & \ddots & & & & & \vdots & & \vdots & \vdots & & \vdots \\ & & & -1 & -1 & & & 1 & & 1 & 1 & \cdots & 1 \\ & & & -1 & -1 & & & 1 & & 1 & 1 & \cdots & 1 \\ & & & & & -1 & & 1 & & 1 & 1 & \cdots & -1 \\ & & & & & & \ddots & \vdots & & \vdots & \vdots & & \vdots \\ & & & & & & -1 & 1 & \cdots & 1 & 1 & \cdots & -1 \\ -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & & & & & & \\ \vdots & & & & & & \vdots & & & & & & \\ -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & & & N_h(d) & & & \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & & & & & & \\ \vdots & & & & & & \vdots & & & & & & \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & & & & & & \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \le i \le 2k; \\ \delta_{i,i+1} = (-1)^i, & 1 \le i \le 2k; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le 2k; \\ \delta_{i,i} = \delta_{i,t+s+m} = -1, & 2k+1 \le i \le t; \\ \delta_{1,j} = -1, & t+1 \le j \le t+s; \\ \delta_{i,j} = -1, & t+1 \le i \le t+s \text{ and } 1 \le j \le t; \\ \delta_{i,j} = 1, & \text{otherwise}, \end{cases}$$

where $0 \le k \le t/2$ and the submatrices N_h $(0 \le h \le s + m - 1)$ are the same as in case $(\mathbb{A}_2)$. Then $r_4 = k + h + 1$, i.e., $1 \le r_4 \le t/2 + s + m$.

$(\mathbb{E}_3)$: $t \equiv 1 \pmod 2$. Then $d/2 \equiv 3 \pmod 4$. Choose $d \in d(T)$ such that

$$M_{k+h}(d) \cong \begin{pmatrix} -1 & -1 & & & & \cdots & \cdots & \cdots & & \cdots & & 1 \\ -1 & -1 & & & & \cdots & \cdots & \cdots & & \cdots & & 1 \\ & & \ddots & & & & & & & & & \vdots \\ & & & -1 & -1 & & & & \cdots & \cdots & & 1 \\ & & & -1 & -1 & & & & \cdots & \cdots & & 1 \\ & & & & & -1 & & & \cdots & \cdots & & -1 \\ & & & & & & \ddots & & & & & \vdots \\ & & & & & & & -1 & & \cdots & & -1 \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & & & & \\ \vdots & & & & & & & \vdots & & N_h & & \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & & & & \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \le i \le 2k; \\ \delta_{i,i+1} = (-1)^i, & 1 \le i \le 2k; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le 2k; \\ \delta_{i,i} = \delta_{i,t+s+m} = -1, & 2k+1 \le i \le t; \\ \delta_{i,j} = 1, & \text{otherwise}, \end{cases}$$

where $0 \le k \le (t-1)/2$ and the submatrices N_h ($0 \le h \le s+m-2$ if s is even, or $0 \le h \le s+m-1$ if s is odd) are the same as in case ($\mathbb{A}$). When s is odd, we have $1 \le r_4 \le (t-1)/2 + s + m - 1$. Furthermore, taking $k = (t-1)/2$ and putting $\delta_{t,t} = \delta_{t,t+s+m} = 1$ in the above sign matrix, we have $r_4 = (t-1)/2 + s + m$. Now we assume that s is even. It follows from the above choices of $M(d)$ that $1 \le r_4 \le (t-1)/2 + s + m - 2$, and r_4 is $(t-1)/2 + s + m - 1$ if we take $k = (t-1)/2$ and put $\delta_{t,t} = \delta_{t,t+s+m} = 1$ in the above sign matrix. Choose $d \in d(T)$ such that

$$M(d) \cong \begin{pmatrix} -1 & -1 & & \cdots & & 1 & -1 & \cdots & -1 & 1 & \cdots & 1 \\ -1 & -1 & & \cdots & & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ & & \ddots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ & & & -1 & -1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ & & & -1 & -1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ & & & & & 1 & 1 & \cdots & 1 & 1 & & 1 \\ -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & & & & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\ -1 & \cdots & \cdots & \cdots & \cdots & -1 & & & 1 & 1 & \cdots & 1 \\ & & & & & & & & & 1 & \cdots & 1 \\ & & & & & & & & & & \ddots & \vdots \\ & & & & & & & & & & 1 & 1 \end{pmatrix}$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \le i \le t-1; \\ \delta_{i,i+1} = (-1)^i, & 1 \le i \le t-1; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le t-1; \\ \delta_{1,j} = -1, & t+1 \le j \le t+s; \\ \delta_{i,j} = -1, & t+1 \le i \le t+s \text{ and } 1 \le j \le t; \\ \delta_{i,j} = 1, & \text{otherwise}. \end{cases}$$

Then we have $r_4 = (t-1)/2 + s + m$.

So, for this type, Lemma 3.1 implies that $1 \le r_4 \le (t-1)/2 + s + m$.

Case ($\mathbb{F}$): *$T = 2(m, n, 0, t)$ with $t + n + m \ge 3$*: Let $S = \{r_1, \ldots, r_t, p_1, \ldots, p_n, l_1, \ldots, l_{m-1}\}$ be a system of $\bigtriangledown$-representatives.

($\mathbb{F}_1$): $t + n \equiv 1 \pmod 2$. Then $d/2 \equiv 3 \pmod 4$. Applying elementary operations (I) and (II) to $M(d)$, we see from Lemma 3.1 that the number of totally 1 (or totally -1) rows in any equivalent form of $M(d)$ is no more than $(t+n+1)/2 + m$. Suppose that $M'(d)$ is one of equivalent forms of

$M(d)$ whose number of totally 1 (or totally -1) rows is $(t+n+1)/2+m$. If $(1^{\cdots t}, -1^{\cdots n}, 1^{\cdots m})$ or $(-1^{\cdots t}, 1^{\cdots n+m})$ appears in $M'(d)$, then by an elementary operation (III) we can obtain an extra totally 1 row. So $r_4 \leq (t+n+1)/2+m$. Now we show that every value of r_4 between 1 and $(t+n+1)/2+m$ occurs. First assume that $t \equiv 1 \pmod 2$. With a similar argument to that in case $(\mathbb{B})$, we choose $d \in d(T)$ such that $M_{k+e+h}(d)$ is equivalent to

$$\left(\begin{array}{ccccccc|ccccccc|ccccccc}
-1 & -1 & & & & & & & & & \cdots & & & & & & & & & & 1 \\
-1 & -1 & & & & & & & & & \cdots & & & & & & & & & & 1 \\
 & & \ddots & & & & & & & & \cdots & & & & & & & & & & \vdots \\
 & & & -1 & -1 & & & & & & \cdots & & & & & & & & & & 1 \\
 & & & -1 & -1 & & & & & & \cdots & & & & & & & & & & 1 \\
 & & & & & -1 & & & & & \cdots & & & & & & & & & & -1 \\
 & & & & & & \ddots & & & & & & & & & & & & & & \vdots \\
 & & & & & & -1 & & & & \cdots & & & & & & & & & & -1 \\
\hline
-1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & -1 & & & & & & & & & & & \cdots & 1 \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & -1 & -1 & & & & & & & & & & & \cdots & 1 \\
 & & & & & & & & & \ddots & & & & & & & & & & & \vdots \\
\vdots & & & & & & \vdots & & & & -1 & -1 & & & & & & & & \cdots & 1 \\
\vdots & & & & & & \vdots & & & & -1 & -1 & & & & & & & & \cdots & 1 \\
\vdots & & & & & & \vdots & & & & & & -1 & & & & & & & \cdots & -1 \\
 & & & & & & & & & & & & & \ddots & & & & & & & \vdots \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & & & & & & & -1 & & & & & & \cdots & -1 \\
\hline
 & & & & & & & & & & & & & & 1 & & & & & \cdots & 1 \\
 & & & & & & & & & & & & & & & \ddots & & & & & \vdots \\
 & & & & & & & & & & & & & & & & 1 & & & \cdots & 1 \\
 & & & & & & & & & & & & & & & & & -1 & & \cdots & -1 \\
 & & & & & & & & & & & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & & & & & & & & & & & -1 & -1
\end{array}\right)$$

$$\begin{cases}
\delta_{i,i} = -1, & 1 \leq i \leq t+n \\
 & \quad \text{or } t+n+h+1 \leq i \leq t+n+m-1; \\
\delta_{i,i+1} = (-1)^i, & 1 \leq i \leq 2k; \\
\delta_{i,i-1} = (-1)^{i-1}, & 1 \leq i \leq 2k; \\
\delta_{i,i+1} = (-1)^{i-1}, & t+1 \leq i \leq t+2e; \\
\delta_{i,i-1} = (-1)^i, & t+1 \leq i \leq t+2e; \\
\delta_{t+1,j} = -1, & 1 \leq j \leq t; \\
\delta_{i,t+n+m} = -1, & 2k+1 \leq i \leq t \\
 & \quad \text{or } t+2e+1 \leq i \leq t+n \\
 & \quad \text{or } t+n+h+1 \leq i \leq t+n+m-1; \\
\delta_{i,j} = 1, & \text{otherwise},
\end{cases}$$

where $0 \leq k \leq (t-1)/2$, $0 \leq e \leq n/2$ and $0 \leq h \leq m-1$. When $e = n/2$, multiplying both rows $t+n-1$ and $t+n$ by all rows $t+2i$ $(i = 1, \ldots, n/2-1)$, we obtain a new equivalent form of $M_{k+e+h}(d)$ whose $(t+n-1)$th and

$(t+n)$th rows both are

$$(1^{\cdots t}, -1^{\cdots n}, 1^{\cdots m}).$$

By an elementary operation (III) with $\varepsilon = 2$, they will be totally 1. Therefore one can easily check that $r_4 = k + e + h + 1$ for the above choices. So $1 \le r_4 \le (t+n+1)/2+m-1$. If we take $k = (t-1)/2$, $e = n/2$, $h = m-1$ and $\delta_{t,t} = \delta_{t,t+n+m} = 1$ in the above sign matrix, we obtain $r_4 = (t+n+1)/2+m$.

Similarly, we can prove that $1 \le r_4 \le (t+n+1)/2+m$ when $t \equiv 1 \pmod 2$.

($\mathbb{F}_2$): $t+n \equiv 0 \pmod 2$. Then $d/2 \equiv 1 \pmod 4$. A similar argument to those in cases ($\mathbb{B}$) and ($\mathbb{E}$) implies that the number of totally 1 rows in the equivalent forms of $M(d)$ can range from 1 to $(t+n)/2+m$. So, for this type, by Lemma 3.3 we have $1 \le r_4 \le (t+n)/2+m$.

Case ($\mathbb{G}$): *$T = 2(m,n,s,t)$ with $m+n+s+t \ge 3$*: Suppose that $S = \{r_1, \dots, r_t, p_1, \dots, p_n, q_1, \dots, q_s, l_1, \dots, l_{m-1}\}$ is a system of $\bigtriangledown$-representatives. Since the upper bound can be determined by Lemmas 3.6 and 4.1, it suffices to show that every value of r_4 between the lower and upper bounds can occur.

($\mathbb{G}_1$): $s \equiv 0 \pmod 2$. We consider the following subcases:

($\mathbb{G}_{11}$): $t+n \equiv 1 \pmod 2$. Then $d/2 \equiv 3 \pmod 4$. First assume that $t \equiv 0 \pmod 2$. We choose $d \in d(T)$ such that $M_{k+e+f+h}(d)$ is equivalent to

$$\left(\begin{array}{ccccccc|ccccc|ccccc}
-1 & -1 & & \cdots & & & & & & \cdots\cdots\cdots & & & & & \cdots & & 1 \\
-1 & -1 & & & & & & & & & & & & & \cdots & & 1 \\
& & \ddots & & & & & & & & & & & & & & \vdots \\
& & & -1 & -1 & & & & & & & & & & \cdots & & 1 \\
& & & -1 & -1 & & & & & & & & & & \cdots & & 1 \\
& & & & & -1 & & & & & & & & & \cdots & & -1 \\
& & & & & & \ddots & & & & & & & & & & \vdots \\
& & & & & & -1 & & & \cdots\cdots\cdots & & & & & \cdots & & -1 \\
\hline
-1 & \cdots & & \cdots & & \cdots & -1 & -1 & -1 & & & & -1 & \cdots & -1 & 1 \cdots & 1 \\
1 & \cdots & & \cdots & & \cdots & 1 & -1 & -1 & & & & 1 & \cdots & 1 & 1 \cdots & 1 \\
\vdots & & & & & & \vdots & & & \ddots & & & \vdots & & \vdots & \vdots & \vdots \\
& & & & & & & & & -1 \; -1 & & & \vdots & & \vdots & \vdots & \vdots \\
\vdots & & & & & & \vdots & & & -1 \; -1 & & & 1 & \cdots & 1 & 1 \cdots & 1 \\
& & & & & & & & & & -1 & & 1 & \cdots & 1 & 1 \cdots & -1 \\
\vdots & & & & & & \vdots & & & & & \ddots & \vdots & & \vdots & \vdots & \vdots \\
& & & & & & & & & & & -1 & 1 & \cdots & 1 & 1 \cdots & -1 \\
1 & \cdots & & \cdots & & \cdots & 1 & & & & & -1 & 1 & \cdots & 1 & 1 \cdots & 1 \\
\hline
& & & & & & & -1 & \cdots & \cdots & \cdots & -1 & & & & & \\
& & & & & & & \vdots & & & & \vdots & & & & & \\
& & & & & & & -1 & \cdots & \cdots & \cdots & -1 & & & N_{f+h} & & \\
& & & & & & & 1 & \cdots & \cdots & \cdots & 1 & & & & & \\
& & & & & & & \vdots & & & & \vdots & & & & & \\
& & & & & & & 1 & \cdots & \cdots & \cdots & 1 & & & & &
\end{array}\right)$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \le i \le t+n; \\ \delta_{i,i+1} = (-1)^i, & 1 \le i \le 2k \\ & \quad \text{or } t+1 \le i \le t+2e; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le 2k \\ & \quad \text{or } t+1 \le i \le t+2e; \\ \delta_{t+1,j} = -1, & 1 \le j \le t \\ & \quad \text{or } t+n+1 \le j \le t+n+s; \\ \delta_{i,t+n+s+m} = -1, & 2k+1 \le i \le t \\ & \quad \text{or } t+2e+1 \le i \le t+n-1; \\ \delta_{i,j} = -1, & t+n+1 \le i \le t+n+s \text{ and } t+1 \le j \le t+n; \\ \delta_{i,j} = 1, & \text{otherwise,} \end{cases}$$

where $0 \le k \le t/2$ and $0 \le e \le (n-1)/2$, and the submatrix

$$N_{f+h} = \left(\begin{array}{ccc|cccc} -1 & \cdots & -1 & 1 & \cdots & & 1 \\ \vdots & & \vdots & & & & \vdots \\ -1 & \cdots & -1 & 1 & \cdots & & 1 \\ & -1 & & & \cdots & & 1 \\ & & \ddots & & & & \vdots \\ & & -1 & 1 & \cdots & & 1 \\ \hline & & & 1 & \cdots & & 1 \\ & & & & \ddots & & \vdots \\ & & & & 1 & \cdots & 1 \\ & & & & -1 & \cdots & -1 \\ & & & & & \ddots & \vdots \\ & & & & & -1 & -1 \end{array}\right)$$

$$\begin{cases} \delta_{i,j} = -1, & 1 \le i \le f \text{ and } 1 \le j \le s; \\ \delta_{i,i} = \delta_{i,t+n+s+m} = -1, & f+1 \le i \le s \\ & \quad \text{or } s+h+1 \le i \le s+m-1; \\ \delta_{i,j} = 1, & \text{otherwise,} \end{cases}$$

where $0 \le f \le s$ and $0 \le h \le m-1$. Note that, multiplying row $t+n$ by all rows $2i$ ($i = 1, \ldots, k, t+1, \ldots, t+e$) and rows $k+1$ to t, and then multiplying it by rows $e+1$ to $t+n-1$, we easily see that row $t+n$ will be

$$(-1^{\ldots t+n}, 1^{\ldots s+m}).$$

Applying an elementary operation (III) with $\varepsilon = -1$, one can see that it will become totally 1. By Lemma 4.1, we have $r_4 = k+e+f+h+1$, i.e., $1 \le r_4 \le (t+n-1)/2 + m + s$.

When t is odd, with a similar choice of entries of $M(d)$ as above, we can show that $1 \le r_4 \le (t+n-1)/2 + m + s$.

($\mathbb{G}_{12}$): $t \equiv n \equiv 0 \pmod 2$. As in cases ($\mathbb{E}_1$) and ($\mathbb{G}_{11}$), we can check that the number of totally 1 rows in the equivalent forms of $M(d)$ can range from 1 to $(t+n)/2+s+m-1$. So $1 \leq r_4 \leq (t+n)/2+s+m-1$.

($\mathbb{G}_{13}$): $t \equiv n \equiv 1 \pmod 2$. With a similar argument to those in cases ($\mathbb{B}_2$) and ($\mathbb{G}_{11}$), one can easily verify that the number of totally 1 rows in the equivalent forms of $M(d)$ can range from 1 to $(t+n)/2+s+m$. The assertion that r_4 can be $(t+n)/2+s+m$ follows from putting

$$M(d) \cong \left(\begin{array}{cccccc|cccccc|cccccc}
-1 & -1 & & & & 1 & -1 & & \cdots & & & -1 & -1 & \cdots & -1 & 1 & \cdots & 1 \\
-1 & -1 & & & & 1 & 1 & & \cdots & & & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
 & & \ddots & & & \vdots & \vdots & & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 & & & -1 & -1 & 1 & \vdots & & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 & & & -1 & -1 & 1 & \vdots & & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 & & & & & 1 & 1 & & \cdots & & & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\hline
 & & & & & & -1 & -1 & & & & & & & & & & \vdots \\
 & & & & & & -1 & -1 & & & & & & & & & & \vdots \\
 & & & & & & & & \ddots & & & & & & & & & \vdots \\
 & & & & & & & & & -1 & -1 & & & & & & & \vdots \\
 & & & & & & & & & -1 & -1 & & & & & & & \vdots \\
 & & & & & & & & & & & 1 & & & & & & \vdots \\
\hline
-1 & \cdots & \cdots & \cdots & \cdots & -1 & & & & & & & -1 & \cdots & -1 & 1 & \cdots & 1 \\
\vdots & & & & & \vdots & & & & & & & \vdots & & \vdots & \vdots & & \vdots \\
-1 & \cdots & \cdots & \cdots & \cdots & -1 & & & & & & & -1 & \cdots & -1 & 1 & \cdots & 1 \\
 & & & & & & & & & & & & & & & 1 & \cdots & 1 \\
 & & & & & & & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & & & & & & 1 & 1
\end{array}\right)$$

$$\begin{cases}
\delta_{i,i} = -1, & 1 \leq i \leq t-1 \\
 & \quad \text{or } t+1 \leq i \leq t+n-1 \\
 & \quad \text{or } t+n+1 \leq i \leq t+n+s; \\
\delta_{i,i+1} = (-1)^i, & 1 \leq i \leq t-1; \\
\delta_{i,i-1} = (-1)^{i-1}, & 1 \leq i \leq t-1; \\
\delta_{i,i+1} = (-1)^{i-1}, & t+1 \leq i \leq t+n-1; \\
\delta_{i,i-1} = (-1)^i, & t+1 \leq i \leq t+n-1; \\
\delta_{i,j} = -1, & t+n+1 \leq i \leq t+n+s \text{ and } 1 \leq j \leq t \\
 & \quad \text{or } t+n+1 \leq i \leq t+n+s \\
 & \quad \text{and } t+n+1 \leq j \leq t+n+s; \\
\delta_{1,j} = -1 & t+1 \leq j \leq t+n+s; \\
\delta_{i,j} = 1, & \text{otherwise.}
\end{cases}$$

($\mathbb{G}_2$): $s \equiv 1 \pmod 2$. We consider the following subcases:

($\mathbb{G}_{21}$): $t+n \equiv 1 \pmod 2$. Then $d/2 \equiv 3 \pmod 4$. First assume that $t \equiv 1 \pmod 2$. We choose $d \in d(T)$ such that $M_{k+e+f+h}(d)$ is equivalent to

$$
\left(\begin{array}{cccccccc|cccccccc|ccccccccccccc}
-1&-1&&&&&& & &&&&&&& & &\cdots\cdots\cdots&&&&&&&&&&\cdots&1 \\
-1&-1&&&&&& & &&&&&&& & &&&&&&&&&&&\cdots&1 \\
&&\ddots&&&&& & &&&&&&& & &&&&&&&&&&&&\vdots \\
&&&-1&-1&&& & &&&&&&& & &&&&&&&&&&&\cdots&1 \\
&&&-1&-1&&& & &&&&&&& & &&&&&&&&&&&\cdots&1 \\
&&&&&-1&& & &&&&&&& & &&&&&&&&&&&\cdots&-1 \\
&&&&&&\ddots& & &&&&&&& & &&&&&&&&&&&&\vdots \\
&&&&&&&-1 & &&&&&&& & &\cdots\cdots\cdots&&&&&&&&&&\cdots&-1 \\
\hline
-1&&\cdots&&&&\cdots&-1 & -1&-1&&&&&& & &\cdots\cdots\cdots&&&&&&&&&&\cdots&1 \\
1&&\cdots&&&&\cdots&1 & -1&-1&&&&&& & &\cdots\cdots\cdots&&&&&&&&&&\cdots&1 \\
\vdots&&&&&&&\vdots & &&\ddots&&&&& & &&&&&&&&&&&&\vdots \\
&&&&&&& & &&&-1&-1&&& & &&&&&&&&&&&\cdots&1 \\
\vdots&&&&&&&\vdots & &&&-1&-1&&& & &&&&&&&&&&&\cdots&1 \\
&&&&&&& & &&&&&-1&& & &&&&&&&&&&&\cdots&-1 \\
\vdots&&&&&&&\vdots & &&&&&&\ddots& & &&&&&&&&&&&&\vdots \\
1&&\cdots&&&&\cdots&1 & &&\cdots&&&\cdots&&-1 & &\cdots\cdots\cdots&&&&&&&&&&\cdots&-1 \\
\hline
&&&&&&& & &&&&&&& & -1&\cdots\cdots\cdots&-1&&&&&&&&&\cdots&1 \\
&&&&&&& & &&&&&&& & \vdots&&\vdots&&&&&&&&&&\vdots \\
&&&&&&& & &&&&&&& & -1&\cdots\cdots\cdots&-1&&&&&&&&&\cdots&1 \\
&&&&&&& & &&&&&&& & &&&-1&&&&&&&&\cdots&1 \\
&&&&&&& & &&&&&&& & &&&&\ddots&&&&&&&&\vdots \\
&&&&&&& & &&&&&&& & &&&&&-1&&&&&&\cdots&1 \\
&&&&&&& & &&&&&&& & &&&&&&1&&&&&\cdots&1 \\
&&&&&&& & &&&&&&& & &&&&&&&\ddots&&&&&\vdots \\
&&&&&&& & &&&&&&& & &&&&&&&&1&&&\cdots&1 \\
&&&&&&& & &&&&&&& & &&&&&&&&&-1&&\cdots&-1 \\
&&&&&&& & &&&&&&& & &&&&&&&&&&\ddots&&\vdots \\
&&&&&&& & &&&&&&& & &&&&&&&&&&&-1&-1
\end{array}\right)
$$

$$
\begin{cases}
\delta_{i,i} = -1, & 1 \le i \le t+n+s \\
 & \quad \text{or } t+n+s+h+1 \le i \le t+n+s+m-1; \\
\delta_{i,i+1} = (-1)^i, & 1 \le i \le 2k; \\
\delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le 2k; \\
\delta_{i,i+1} = (-1)^{i-1}, & t+1 \le i \le t+2e; \\
\delta_{i,i-1} = (-1)^i, & t+1 \le i \le t+2e; \\
\delta_{t+1,j} = -1, & 1 \le j \le t; \\
\delta_{i,j} = -1, & t+n+1 \le i \le t+n+f \text{ and } t+n+1 \le j \le t+n+s; \\
\delta_{i,t+n+s+m} = -1, & 2k+1 \le i \le t \\
 & \quad \text{or } t+2e+1 \le i \le t+n \\
 & \quad \text{or } t+n+s+h \le t+n+s+m-1; \\
\delta_{i,j} = 1, & \text{otherwise,}
\end{cases}
$$

where $0 \le k \le (t-1)/2$, $0 \le e \le n/2$, $0 \le f \le s$ and $0 \le h \le m-1$. Multiplying row $t+1$ by all rows $t+2i$ $(i = 2, \dots, e)$ and rows $t+2e+1$ to $t+n$, we see that row $t+1$ will be

$$(-1^{\dots t}, -1^{\dots n}, 1^{\dots s}, 1^{\dots m}).$$

Applying an elementary operation (III) with $\varepsilon = -1$, we see that it will become totally 1. Note that, when $e \ge 1$, multiplying row $t+2$ by all rows $t+2i$ $(i = 2, \dots, e)$ and all rows j $(j = t+2e+1, \dots, t+n+1$ if $f \ge 1$, or $j = t+2e+1, \dots, t+n+s$ if $f = 0)$, we see that row $t+2$ will be

$$(1^{\dots t}, -1^{\dots n+s}, 1^{\dots m}).$$

By an elementary operation (III) with $\varepsilon = 2$, it will be totally 1. So, by Lemma 4.1, we can easily check that $r_4 = k+e+f+h+1$ and $1 \le r_4 \le (t+n-1)/2 + s + m$ for this type.

Now assume that $t \equiv 0 \pmod 2$. Choose $d \in d(T)$ such that $M_{k+e+h}(d)$ is equivalent to

$$\left(\begin{array}{cccccccc|cccccccc|cccccc}
-1 & -1 & & & & & & & & & & \cdots & \cdots & & & & & & & & & 1 \\
-1 & -1 & & & & & & & & & & & & & & & & & & & & 1 \\
 & & \ddots & & & & & & & & & & & & & & & & & & & \vdots \\
 & & & -1 & -1 & & & & & & & & & & & & & & & & & 1 \\
 & & & -1 & -1 & & & & & & & & & & & & & & & & & 1 \\
 & & & & & -1 & & & & & & & & & & & & & & & & -1 \\
 & & & & & & \ddots & & & & & & & & & & & & & & & \vdots \\
 & & & & & & & -1 & & & & \cdots & \cdots & & & & & & & & & -1 \\
\hline
-1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & -1 & -1 & & \cdots & \cdots & & & & -1 & \cdots & -1 & 1 & \cdots & 1 \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & -1 & -1 & & & & & & & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & & & & & & & \vdots & & & \ddots & & & & & & \vdots & & \vdots & \vdots & & \vdots \\
 & & & & & & & & & & & -1 & -1 & & & & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & & & & & & & \vdots & & & & -1 & -1 & & & & & \cdots & & & \cdots & 1 \\
 & & & & & & & & & & & & & -1 & & & & \cdots & & & \cdots & -1 \\
\vdots & & & & & & & \vdots & & & & & & & \ddots & & & & & & & \vdots \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & & & \cdots & \cdots & \cdots & & & -1 & 1 & \cdots & 1 & & \cdots & -1 \\
\hline
 & & & & & & & & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & & & & & & \\
 & & & & & & & & \vdots & & & & & & & \vdots & & & & & & \\
 & & & & & & & & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & & & N_h & & & \\
 & & & & & & & & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & & & & & & \\
 & & & & & & & & \vdots & & & & & & & \vdots & & & & & & \\
 & & & & & & & & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & & & & & &
\end{array}\right)$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \le i \le t+n; \\ \delta_{i,i+1} = (-1)^{i}, & 1 \le i \le 2k \\ & \quad \text{or } t+1 \le i \le t+2e; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le 2k \\ & \quad \text{or } t+1 \le i \le t+2e; \\ \delta_{t+1,j} = -1, & 1 \le j \le t \\ & \quad \text{or } t+n+1 \le j \le t+n+s; \\ \delta_{i,t+n+s+m} = -1, & 2k+1 \le i \le t \\ & \quad \text{or } t+2e+1 \le i \le t+n; \\ \delta_{i,j} = -1, & t+n+1 \le i \le t+n+s \text{ and } t+1 \le j \le t+n; \\ \delta_{i,j} = 1, & \text{otherwise,} \end{cases}$$

where $0 \le k \le t/2$, $0 \le e \le (n-1)/2$ and the submatrices N_h $(0 \le h \le s+m-1)$ are the same as in case $(\mathbb{A}_2)$. The previous discussion implies that $r_4 = k+e+h+1$. So $1 \le r_4 \le (t+n-1)/2+s+m$ for this type.

$(\mathbb{G}_{22})$: $t+n \equiv 0 \pmod 2$. Then $d/2 \equiv 1 \pmod 4$. It is easy to check that $1 \le r_4 \le (t+n)/2+s+m-1$ by a similar choice of entries of $M(d)$ to those in cases $(\mathbb{E}_1)$ and $(\mathbb{G}_{11})$.

When $t \equiv n \equiv 0 \pmod 2$, if we choose $d \in d(T)$ such that

$$M(d) \cong \left(\begin{array}{cccc|cccc|ccc|ccc}
-1 & -1 & & & & \cdots & & & -1 & \cdots\cdots\cdots & -1 & 1 & & \cdots\ 1 \\
-1 & -1 & & & & \cdots & & & 1 & \cdots\cdots & 1 & 1 & & \cdots\ 1 \\
 & & \ddots & & & & & & \vdots & & \vdots & \vdots & & \vdots \\
 & & & -1\ -1 & & & & & \vdots & & \vdots & \vdots & & \vdots \\
 & & & -1\ -1 & & & & & 1 & \cdots\cdots & 1 & 1 & & \cdots\ 1 \\
\hline
 & & & & -1 & -1 & & & 1 & \cdots\cdots & 1 & 1 & & \cdots\ 1 \\
 & & & & -1 & -1 & & & 1 & \cdots\cdots & 1 & 1 & & \cdots\ 1 \\
 & & & & & & \ddots & & \vdots & & \vdots & \vdots & & \vdots \\
 & & & & & & & -1\ -1 & \vdots & & \vdots & \vdots & & \vdots \\
 & & & & & & & -1\ -1 & 1 & \cdots\cdots & 1 & 1 & & \cdots\ 1 \\
\hline
-1 & & \cdots & -1 & 1 & \cdots\cdots & & 1 & -1 & \cdots\cdots\cdots & -1 & 1 & & \cdots\ 1 \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
-1 & & \cdots & -1 & 1 & \cdots\cdots & & 1 & -1 & \cdots\cdots\cdots & -1 & 1 & & \cdots\ 1 \\
 & & & & & & & & & & & 1 & & \cdots\ 1 \\
 & & & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & & & 1\ 1
\end{array}\right)$$

$$\begin{cases} \delta_{i,i} = -1, & 1 \le i \le t+n; \\ \delta_{i,i+1} = (-1)^i, & 1 \le i \le t+n; \\ \delta_{i,i-1} = (-1)^{i-1}, & 1 \le i \le t+n; \\ \delta_{1,j} = -1, & t+n+1 \le j \le t+n+s; \\ \delta_{i,j} = -1, & t+n+1 \le i \le t+n+s,\ 1 \le j \le t \\ & \quad \text{or } t+n+1 \le i \le t+n+s,\ t+n+1 \le j \le t+n+s; \\ \delta_{i,j} = 1, & \text{otherwise,} \end{cases}$$

then $r_4 = (t+n)/2 + s + m$.

When $t \equiv n \equiv 1 \pmod 2$, the procedure is analogous.

CASE ($\mathbb{H}$): $T = 2(m,0,0,0)$ with $m \ge 3$. This situation is the same as in case (H).

This completes the proof of the theorem. ∎

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