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Abelian surfaces of GL_2-type as Jacobians of curves

by

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1. Introduction. Let C/k be a smooth and irreducible projective algebraic curve over a field k and let $(J(C),\Theta)$ be its principally polarized Jacobian variety. By Torelli's theorem, C is completely determined by $(J(C),\Theta)$ up to isomorphism over a fixed algebraic closure $\overline{k}$ of k. The question then arises of whether the unpolarized abelian variety $J(C)$ already determines C. It turns out that the answer has its roots in the arithmetic of the ring of endomorphisms $\mathrm{End}(J(C))$ of $J(C)$. Indeed, in the generic case where $\mathrm{End}(J(C))=\mathbb{Z}$, the curve C can be recovered from $J(C)$, but as soon as $\mathrm{End}(J(C))\supsetneq\mathbb{Z}$, it may very well be the case that there exist finitely many pairwise nonisomorphic curves $C_1,\ldots,C_\tau$ such that

$$J(C_1)\simeq\cdots\simeq J(C_\tau)$$

as unpolarized abelian varieties. This phenomenon was first observed by Humbert in [12], where he proved the existence of two nonisomorphic Riemann surfaces C_1 and C_2 of genus two sharing the same simple Jacobian variety. Humbert's method was generalized by Lange in [13]. Finally, it has recently been shown in [23] that there exist arbitrarily large sets $\{C_1,\ldots,C_\tau\}$, $\tau\gg 0$, of pairwise nonisomorphic curves of genus two such that $J(C_i)$ are isomorphic simple abelian surfaces.

The aim of this paper is to present an arithmetical and effective approach to this question by considering abelian surfaces A_f defined over $\mathbb{Q}$ attached by Shimura to a newform f in $S_2(\Gamma_1(N))$. These surfaces are viewed as optimal quotients of the Jacobian $J_1(N)$ of the modular curve $X_1(N)$ and are canonically polarized with the polarization $\mathcal{L}$ over $\mathbb{Q}$ pushed out from the principal polarization $\Theta_{X_1(N)}$ on $J_1(N)$. We provide an effective crite-

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rion for determining all their principal polarizations defined over $\mathbb{Q}$ and for determining which of them correspond to the canonical polarization of the Jacobian of a curve.

In Section 2, we consider abelian varieties A of GL_2-type over k, i.e. such that their algebra of endomorphisms $F = \mathbb{Q} \otimes \mathrm{End}_k A$ is a number field of degree $[F : \mathbb{Q}] = \dim A$. In Theorem 2.10, we determine the set of isomorphism classes of principal polarizations on A over k in an explicit and computable way, while in Corollary 2.12 we provide necessary and sufficient conditions for A to be principally polarizable over the base field of definition k.

Section 3 is devoted to the study of abelian surfaces of GL_2-type over a number field k and their principal polarizations from an arithmetical point of view. As we discuss in detail, these questions are closely related to the problem of finding nonisomorphic curves whose Jacobian varieties are pairwise isomorphic as unpolarized abelian varieties. One of our main results can be rephrased as follows.

THEOREM 1.1. *Let A be a principally polarizable abelian surface over $\mathbb{Q}$ such that $\mathrm{End}_{\mathbb{Q}} A = R$ is an order in a quadratic field. Let R_0 be the subring of R fixed by complex conjugation. Then*:

(1) *If either $R_0 = \mathbb{Z}$ or R^* contains a unit of negative norm, there is a single isomorphism class of principal polarizations on A over $\mathbb{Q}$.*
(2) *Otherwise, there are exactly two isomorphism classes of principal polarizations on A over $\mathbb{Q}$. More precisely, either*
   - *there exists a curve $C/\mathbb{Q}$ of genus two and an elliptic curve C' over a quadratic extension K of $\mathbb{Q}$ such that A is both isomorphic to $J(C)$ and to Weil's restriction $\mathrm{Res}_{K/\mathbb{Q}}(C')$ of C' as unpolarized abelian surfaces over $\mathbb{Q}$, or*
   - *there exist two curves C, $C'/\mathbb{Q}$ of genus two nonisomorphic over $\mathbb{Q}$ such that A is isomorphic over $\mathbb{Q}$ to their Jacobian varieties.*

As a consequence, we prove the existence of infinitely many abelian surfaces $A/\mathbb{Q}$ of GL_2-type which are simultaneously the Jacobian variety of a curve $C/\mathbb{Q}$ of genus two and Weil's restriction of an elliptic curve over a quadratic field.

In the last section explicit examples are presented. First, we summarize well known results about modular abelian varieties arising from modular newforms; in the rest of the section we show how our results can be effectively applied to abelian surfaces arising from newforms of trivial Nebentypus by using the procedure described in [3], which allows us to obtain an equation of a genus two curve from a period matrix in a symplectic basis. In this way, we construct an explicit example of two nonisomorphic (over $\overline{\mathbb{Q}}$) curves C, $C'/\mathbb{Q}$ of genus two such that their Jacobian varieties are isomorphic over

$\mathbb{Q}$ to an absolutely simple quotient of the Jacobian of the modular curve $X_0(65)$. We also provide explicit examples of curves C of genus two over $\mathbb{Q}$ whose Jacobian variety $J(C)$ is of GL_2-type and isomorphic to Weil's restriction over $\mathbb{Q}$ of an elliptic curve E over a quadratic field. Finally, we exhibit several pairwise nonisomorphic curves $C_i/\mathbb{Q}$ of genus two such that their respective Jacobian varieties $J(C_i)$ are mutually isogenous over $\mathbb{Q}$.

Alternative methods to ours were proposed by Howe ([9], [8]) and Villegas ([22]) and explicit equations of similar phenomena have been given in [10] and [11]. However, Howe's examples correspond to abelian varieties which decompose as the product of elliptic curves with complex multiplication (CM). As he remarked himself in [10], no explicit examples were known of absolutely simple abelian varieties which could be realized as the simultaneous Jacobian variety of several nonisomorphic curves.

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2. Abelian varieties of GL_2-type

2.1. *The Néron–Severi group of an abelian variety.* We begin by recalling some basic facts on abelian varieties (cf. [17], [18]). Let k be a field, k^{s} a separable closure of k and let $G_k = \mathrm{Gal}(k^{\mathrm{s}}/k)$. For any field extension K/k, let $A_K = A \times_k K$ be the abelian variety obtained by base change of A to $\operatorname{Spec} K$. Let $\mathrm{Pic}(A)$ be the group of invertible sheaves on A, $\mathrm{Pic}^0(A_{k^{\mathrm{s}}})$ be the subgroup of $\mathrm{Pic}(A_{k^{\mathrm{s}}})$ of invertible sheaves algebraically equivalent to 0 and $\mathrm{Pic}^0(A) = \mathrm{Pic}(A) \cap \mathrm{Pic}^0(A_{k^{\mathrm{s}}})$. The *Néron–Severi* group of A is $\mathrm{NS}(A) = \mathrm{Pic}(A)/\mathrm{Pic}^0(A)$. We shall denote by $\mathrm{NS}(A_{k^{\mathrm{s}}})^{G_k} = H^0(G_k, \mathrm{NS}(A_{k^{\mathrm{s}}}))$ the group of k-rational algebraic equivalence classes of invertible sheaves. Note that in general not all elements in $\mathrm{NS}(A_{k^{\mathrm{s}}})^{G_k}$ are represented by an element in $\mathrm{Pic}(A)$.

For any $P \in A(k^{\mathrm{s}})$, let τ_P denote the translation by P map on A. Every invertible sheaf $\mathcal{L} \in \mathrm{NS}(A_{k^{\mathrm{s}}})^{G_k}$ defines a morphism $\varphi_{\mathcal{L}} : A \to \widehat{A}$ over k given by $\varphi_{\mathcal{L}}(P) = \tau_P^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$, which is an isogeny if $\mathcal{L}$ is nondegenerate. A polarization on A defined over k is the class of algebraic equivalence of an ample invertible sheaf $\mathcal{L} \in \mathrm{NS}(A_{k^{\mathrm{s}}})^{G_k}$. Equivalently, a polarization on A over k is an isogeny $\lambda : A \to \widehat{A}$ defined over k such that $\lambda \otimes k^{\mathrm{s}} = \varphi_{\mathcal{L}}$ for some ample line bundle $\mathcal{L}$ on $A_{k^{\mathrm{s}}}$.

A nondegenerate invertible sheaf $\mathcal{L}$ on A induces an anti-involution on the algebra of endomorphisms

$$* : \mathbb{Q} \otimes \mathrm{End}_k A \xrightarrow{\sim} \mathbb{Q} \otimes \mathrm{End}_k A, \quad t \mapsto \varphi_{\mathcal{L}}^{-1} \cdot \widehat{t} \cdot \varphi_{\mathcal{L}}.$$

Let $\mathrm{End}_k^{\mathrm{s}} A = \{\beta \in \mathrm{End}_k A : \beta^* = \beta\}$ denote the subgroup of symmetric endomorphisms and $\operatorname{End}^{\mathrm{s}}_{k+} A$ be the set of positive symmetric endomorphisms of A.

From now on, we assume that k is a subfield of a fixed algebraic closure $\overline{\mathbb{Q}} = \overline{k}$ of $\mathbb{Q}$.

Proposition 2.1. *Let A/k be an abelian variety and let $\mathcal{L} \in \mathrm{NS}(A_{\overline{k}})^{G_k}$ be nondegenerate. Then, for any endomorphism $t \in \operatorname{End}^{\mathrm{s}}_k A$, there exists a unique $\mathcal{L}^{(t)} \in \mathrm{NS}(A_{\overline{k}})^{G_k}$ such that $\varphi_{\mathcal{L}^{(t)}} = \varphi_{\mathcal{L}} \cdot t$. The following properties are satisfied*:

(i) *Let E and E_t denote the alternating Riemann forms attached to $\mathcal{L}$ and $\mathcal{L}^{(t)}$ respectively. Then*

$$E_t(x, y) = E(x, ty) = E(tx, y).$$

(ii) *If t is a totally positive element, then $\mathcal{L}$ is a polarization if and only if $\mathcal{L}^{(t)}$ is.*

(iii) *For any $t \in \operatorname{End}^{\mathrm{s}}_k A$ and $d \in \mathbb{Z}$, $t^*(\mathcal{L}) = \mathcal{L}^{(t^2)}$ and $\mathcal{L}^{(d)} = \mathcal{L}^{\otimes d}$.*

Proof. Let $A(\mathbb{C}) = V/\Lambda$ for a complex vector space V and a lattice Λ. Since the involution induced by $\mathcal{L}$ on $\operatorname{End}^{\mathrm{s}}_k A$ is the identity map, it may be checked that, for any endomorphism $t \in \operatorname{End}^{\mathrm{s}}_k A$, $E_t : V \times V \to \mathbb{R}$, $(x, y) \mapsto E(tx, y) = E(x, ty)$ is again an alternating Riemann form on A. Then, by the Appell–Humbert theorem, there exists a unique invertible sheaf $\mathcal{L}^{(t)}$ up to algebraic equivalence such that $E_{\mathcal{L}^{(t)}} = E_t$. It follows from the analytical representation of the morphism $\varphi_{\mathcal{L}}$ in terms of E that $\varphi_{\mathcal{L}^{(t)}} = \varphi_{\mathcal{L}} \circ t$. Since both t and $\mathcal{L}$ are defined over k, the same holds for $\mathcal{L}^{(t)}$.

As for (ii), let us denote by H the hermitian form on V attached to $\mathcal{L}$. Then the matrix of the hermitian form H_t attached to $\mathcal{L}^{(t)}$, with respect to any basis of V, is the product of the matrices of H and t. Since t is a self-adjoint endomorphism on the hermitian space (V, H), there is an orthogonal basis of V over which t has diagonal form. It is clear that if t is totally positive, then H_t is positive definite if and only if H is.

Finally, we know that for any endomorphism $t \in \operatorname{End}^{\mathrm{s}}_k A$ and any invertible sheaf $\mathcal{L}$ on A, $E_{t^*(\mathcal{L})}(x, y) = E(tx, ty) = E(t^2 x, y) = E_{\mathcal{L}^{(t^2)}}(x, y)$ and thus $t^*(\mathcal{L}) = \mathcal{L}^{(t^2)}$. Since $E_{\mathcal{L}^{\otimes d}} = dE$, we also have $\mathcal{L}^{(d)} = \mathcal{L}^{\otimes d}$. ∎

Remark 2.2. Note that our construction of $\mathcal{L}^{(t)}$ extends to arbitrary $t \in \mathbb{Q} \otimes \operatorname{End}_k A$ to produce elements $\mathcal{L}^{(t)} \in \mathbb{Q} \otimes \mathrm{NS}(A_{\overline{k}})^{G_k}$.

Theorem 2.3. *Assume there exists a principal polarization $\mathcal{L}_0$ on A defined over k and let $\operatorname{End}^{\mathrm{s}}_k A$ denote the subgroup of symmetric endomorphisms with respect to $\mathcal{L}_0$. Then there is an isomorphism of groups*

$$\varepsilon : \mathrm{NS}(A_{\overline{k}})^{G_k} \xrightarrow{\sim} \operatorname{End}^{\mathrm{s}}_k A, \qquad \mathcal{L} \mapsto \varphi_{\mathcal{L}_0}^{-1} \cdot \varphi_{\mathcal{L}},$$

such that $\mathcal{L} \in \mathrm{NS}(A_{\overline{k}})^{G_k}$ *is a polarization if and only if* $\varepsilon(\mathcal{L}) \in \mathrm{End}^{\mathrm{s}}_{k+} A$ *and it is principal if and only if* $\varepsilon(\mathcal{L}) \in \mathrm{Aut}^{\mathrm{s}}_{k+} A$. *Moreover*, $\varepsilon^{-1}(t) = \mathcal{L}_0^{(t)}$.

Proof. The first part of the theorem is well known if we replace k by $\overline{k}$. It is clear that if $\mathcal{L}_0 \in \mathrm{NS}(A_{\overline{k}})^{G_k}$, then $\varepsilon(\mathcal{L}) \in \mathrm{End}^{\mathrm{s}}_k A$ for any $\mathcal{L} \in \mathrm{NS}(A_{\overline{k}})^{G_k}$. From the relation

$$\varepsilon(\mathcal{L}_0^{(t)}) = \varphi_{\mathcal{L}_0}^{-1} \cdot \varphi_{\mathcal{L}_0^{(t)}} = \varphi_{\mathcal{L}_0}^{-1} \cdot \varphi_{\mathcal{L}_0} \cdot t = t,$$

we conclude that the morphism ε is surjective and the statement is immediate. ∎

Definition 2.4. We shall say that two polarizations $\mathcal{L}_1, \mathcal{L}_2$ defined over a field extension K/k are K-*isomorphic*, $\mathcal{L}_1 \overset{K}{\simeq} \mathcal{L}_2$, if there exists $u \in \mathrm{Aut}_K A$ such that $u^*(\mathcal{L}_2) = \mathcal{L}_1$. We denote by $\Pi(A_k)$ the set of k-isomorphism classes of principal polarizations on A_k defined over k, i.e.,

$$\Pi(A_k) = \{\text{principal polarizations } \mathcal{L} \text{ defined over } k\}/\overset{k}{\simeq}\ .$$

Proposition 2.5 ([17]). *The set* $\Pi(A_k)$ *is finite.*

We will denote by $\pi(A_k)$ its cardinality. Two positive symmetric endomorphisms $\beta_1, \beta_2 \in \mathrm{End}^{\mathrm{s}}_{k+} A$ are *equivalent*, $\beta_1 \sim \beta_2$, if $\beta_1 = \beta^* \beta_2 \beta$ for some $\beta \in \mathrm{Aut}_k A$. The next result follows from Theorem 2.3.

Corollary 2.6. *The morphism* ε *induces a bijection of finite sets*

$$\Pi(A_k) \leftrightarrow \mathrm{Aut}^{\mathrm{s}}_{k+} A/\sim\ .$$

2.2. *Polarizations on abelian varieties of* GL_2*-type.* We now describe explicitly the isomorphism of Theorem 2.3 for a certain class of abelian varieties. This will provide a procedure for determining all isomorphism classes of principal polarizations on them in a computable way. We will illustrate it with several examples in Section 4.

Definition 2.7. An abelian variety A is of GL_2-*type* over a field k if A is defined over k and $\mathrm{End}_k A$ is an order in a number field F of degree $[F:\mathbb{Q}] = \dim A$. If F is totally real, we then say that A is of *real* GL_2*-type* over k.

Remark 2.8. By Albert's classification of involuting division algebras (see [18]), the algebra of endomorphisms $F = \mathbb{Q} \otimes \mathrm{End}_k A$ of an abelian variety A of GL_2-type over k is isomorphic either to a totally real field or a CM-field. The Rosati involution with respect to any polarization $\mathcal{L}$ on A over k acts as complex conjugation on the number field $F = \mathbb{Q} \otimes \mathrm{End}_k A$ (see [18]) and it can be checked that the same holds for the involution with respect to a not necessarily ample invertible sheaf $\mathcal{L}$. In particular, $\mathbb{Q} \otimes \mathrm{End}^{\mathrm{s}}_k A$ is isomorphic either to F or to the maximal totally real subfield F_0 of F, depending on whether A is of real GL_2-type or not, and independently of the choice of $\mathcal{L}$. This allows us in this section to refer to $\mathrm{End}_k^{\mathrm{s}}(A)$ without further mention of the chosen polarization.

Remark 2.9. An abelian variety A/k is said to have real multiplication by a field F if F is a totally real field contained in $\mathbb{Q}\otimes\mathrm{End}_k A$ with $[F:\mathbb{Q}] = \dim A$. It is clear that the condition of being of real GL_2-type is stronger than that of being of real multiplication.

The main sources of abelian varieties of GL_2-type over $\mathbb{Q}$ are the abelian varieties A_f attached by Shimura to an eigenform f of weight 2 for the modular congruence subgroup $\Gamma_1(N)$ and the real case occurs when $f \in S_2(\Gamma_0(N))$. Actually, Ribet has proved in [21] that, assuming Serre's conjecture (3.2.4) of [24], these are all the abelian varieties of GL_2-type over $\mathbb{Q}$.

For the rest of this section, we fix the following notation. For any totally real number field F_0 and any subset S of F_0, we shall denote by S_+ the set of totally positive elements of S.

The next theorem provides an explicit description of all polarizations on an abelian variety of GL_2-type.

Theorem 2.10. *Let A be an abelian variety of GL_2-type over k. Let $R = \mathrm{End}_k A$, R_0 be the subring of R fixed by complex conjugation, $F = \mathbb{Q}\otimes R$ and $F_0 = \mathbb{Q}\otimes R_0$. Suppose that A admits a principal polarization $\mathcal{L}_0$ on A defined over k and let ε be as in Theorem 2.3. Then:*

(i) *For any $\mathcal{L} \in \mathrm{NS}(A_{\bar{k}})^{G_k}$ and any endomorphism $s \in R$,*

$$\deg\mathcal{L} = |\mathrm{Norm}_{F/\mathbb{Q}}(\varepsilon(\mathcal{L}))|,$$

$$\varepsilon(s^*(\mathcal{L})) = \begin{cases} s^2\cdot\varepsilon(\mathcal{L}) & \text{if } F = F_0,\\ \mathrm{Norm}_{F/F_0}(s)\cdot\varepsilon(\mathcal{L}) & \text{if } [F:F_0] = 2.\end{cases}$$

(ii) *The class of an invertible sheaf $\mathcal{L}\in\mathrm{NS}(A_{\bar{k}})^{G_k}$ is a polarization if and only if $\varepsilon(\mathcal{L})\in(R_0)_+$. In particular, $\mathcal{L}$ is a principal polarization if and only if $\varepsilon(\mathcal{L})\in(R_0^*)_+$.*

(iii) *Let $P(R)$ be the multiplicative group*

$$P(R) = \begin{cases} (R_0)_+^*/R_0^{*2} & \text{if } F = F_0,\\ (R_0)_+^*/\,\mathrm{Norm}_{F/F_0}(R^*) & \text{if } [F:F_0] = 2.\end{cases}$$

Then there is a bijective correspondence between the sets $\Pi(A_k)$ and $P(R)$ and hence

$$\pi(A_k) = 2^N \quad \text{with} \quad 0\le N\le[F_0:\mathbb{Q}]-1.$$

The 2^N isomorphism classes of principal polarizations on A over k are represented by the invertible sheaves $\mathcal{L}_0^{(u_j)}$, where $\{u_j\}_{j=1}^{2^N}$ is a system of representatives of $P(R)$.

Proof. By Theorem 2.3, for every $\mathcal{L} \in \mathrm{NS}(A_{\overline{k}})^{G_k}$, there exists $t \in R_0$ such that $\mathcal{L} = \mathcal{L}_0^{(t)}$. Thus, $\varepsilon(s^*(\mathcal{L})) = \varepsilon(s^*(\mathcal{L}_0^{(t)})) = \varepsilon(\mathcal{L}_0^{(s\overline{s}t)}) = s\overline{s} \cdot \varepsilon(\mathcal{L})$, where $^-$ denotes complex conjugation. Moreover, $\deg \mathcal{L} = \deg(\varepsilon(\mathcal{L}))^{1/2} = |\mathrm{Norm}_{F/\mathbb{Q}}(\varepsilon(\mathcal{L}))|$. This yields (i).

Part (ii) follows immediately from Theorem 2.3 and part (i).

Finally, the bijection $\Pi(A_k) \simeq P(R)$ is now a consequence of Corollary 2.6. Dirichlet's unit theorem then implies that $\pi(A_k) = 2^N$ for $0 \leq N \leq [F_0 : \mathbb{Q}] - 1$. From (i) we deduce that any two polarizations $\mathcal{L}$, $\mathcal{L}'$ on A defined over k are isomorphic if and only if $\varepsilon(\mathcal{L}) = u\overline{u} \cdot \varepsilon(\mathcal{L}')$ for some $u \in R^*$. Hence, representatives of the isomorphism classes of principal polarizations on A over k are provided by $\mathcal{L}_0^{(u_j)}$ for representatives u_j of $P(R)$. ∎

Remark 2.11. If $\mathcal{L}_0 \in \mathrm{NS}(A_{\overline{k}})^{G_k}$ is a not necessarily ample invertible sheaf on A over k of degree 1, then the map $\varepsilon : \mathrm{NS}(A_{\overline{k}})^{G_k} \to \mathrm{End}_k A$, defined by $\varepsilon(\mathcal{L}) = \varphi_{\mathcal{L}_0}^{-1} \cdot \varphi(\mathcal{L})$, is also an isomorphism that still satisfies $\varepsilon^{-1}(t) = \mathcal{L}_0^{(t)}$ and part (i) of the theorem above.

Corollary 2.12. *Let A be an abelian variety of* GL_2*-type over k with a polarization $\mathcal{L}$ over k of degree $d \geq 1$. Then A is principally polarizable over k if and only if there exists $t \in (R_0)_+$ satisfying* $\mathrm{Norm}_{F/\mathbb{Q}}(t) = d$ *and* $\mathcal{L}^{(t^{-1})} \in \mathrm{NS}(A_{\overline{k}})^{G_k}$.

Proof. Assume $\mathcal{L}^{(t^{-1})} \in \mathrm{NS}(A_{\overline{k}})^{G_k}$ for some $t \in (R_0)_+^*$ such that $\mathrm{Norm}_{F/\mathbb{Q}}(t) = d$. Then $\mathcal{L}^{(t^{-1})}$ must be ample, because t is totally positive and principal, as $\deg(\mathcal{L}^{(t^{-1})}) = \deg(\mathcal{L}) \,\mathrm{Norm}_{F/\mathbb{Q}}(t^{-1}) = 1$. Conversely, if $\mathcal{L}_0$ is a principal polarization on A over k, it must be of the form $\mathcal{L}_0 = \mathcal{L}^{(t^{-1})}$ for some $t \in (R_0)_+$ whose norm from F over $\mathbb{Q}$ is d. ∎

The above corollary provides an effective criterion to decide whether a polarized abelian variety $(A, \mathcal{L})$ of real GL_2-type over k is principally polarizable over k. Indeed, denote by M and T the matrices of the alternating Riemann form E attached to $\mathcal{L}$ and $t \in \mathrm{End}_k^+ A$ with respect to a fixed basis of $H_1(A, \mathbb{Z})$ respectively. It then suffices to check whether $M \cdot T^{-1} \in \mathrm{GL}_{2g}(\mathbb{Z})$ for any $t \in (R_0)_+$ (up to multiplicative elements in R_0^{*2} if $F = F_0$ or $\mathrm{Norm}_{F/F_0}(R^*)$ if $[F : F_0] = 2$) such that $\mathrm{Norm}_{F/\mathbb{Q}}(t) = d$. Note that if $(A, \mathcal{L})$ is an abelian variety of GL_2-type together with a polarization $\mathcal{L}$ of primitive type $(1, d_2, \ldots, d_g)$ and degree $d = d_2 \cdots d_g$, we only need to consider those $t \in (R_0)_+$ of norm d such that $t/m \notin R_0$ for any $m > 1$.

3. Abelian surfaces of GL_2-type

3.1. *Principal polarizations on abelian surfaces.* Let C/k be a smooth projective curve defined over a number field k and let $A = J(C) = \mathrm{Pic}^0(C)/k$ denote the Jacobian variety of C. The sheaf of sections of the Theta divisor

Θ_C associated to the curve is a principal polarization $\mathcal{L}(\Theta_C)$ defined over k. For any other curve C'/k such that $A \overset{k}{\simeq} J(C) \overset{k}{\simeq} J(C')$, Torelli's theorem asserts that C and C' are isomorphic over k if and only if there exists $u \in \operatorname{Aut}_k A$ such that $u^*(\Theta_{C'}) = \Theta_C$ in $\mathrm{NS}(A_{\overline{k}})^{G_k}$.

For an abelian variety A/k, this leads us to introduce the set $T(A_k)$ of k-isomorphism classes of smooth algebraic curves C/k such that $J(C) \overset{k}{\simeq} A$. Let $\tau(A_k) = |T(A_k)|$. The Torelli map $C \mapsto \mathcal{L}(\Theta_C)$ induces an inclusion of finite sets

$$T(A_k) \subseteq \Pi(A_k).$$

While a principally polarized abelian surface over an algebraic closed field can only be the Jacobian of a curve or the product of two elliptic curves, the panorama is a little wider from the arithmetical point of view. We refer the reader to [16] for an account of Weil's restriction of scalars of abelian varieties over number fields.

THEOREM 3.1. *Let A/k be an abelian surface with a principal polarization $\mathcal{L}$ defined over k. The polarized abelian variety $(A, \mathcal{L})$ is of one of the following three types*:

(1) $(A, \mathcal{L}) \overset{k}{\simeq} (J(C), \mathcal{L}(\Theta_C))$, *where C/k is a smooth curve of genus two.*
(2) $(A, \mathcal{L}) \overset{k}{\simeq} (C_1 \times C_2, \mathcal{L}_{\mathrm{can}})$, *where C_1 and C_2 are elliptic curves over k and $\mathcal{L}_{\mathrm{can}}$ is the natural product polarization on $C_1 \times C_2$.*
(3) $(A, \mathcal{L}) \overset{k}{\simeq} (\operatorname{Res}_{K/k} C, \mathcal{L}_{\mathrm{can}})$, *where $\operatorname{Res}_{K/k} C$ is the Weil restriction of an elliptic curve C over a quadratic extension K/k, and $\mathcal{L}_{\mathrm{can}}$ is the polarization over k isomorphic over K to the canonical polarization of $C \times {}^{\sigma}C$.*

Proof. It is well known that $(A, \mathcal{L})$ is isomorphic over $\overline{k}$ to either the canonically polarized Jacobian variety $(J(C), \mathcal{L}(\Theta_C))$ of a smooth curve of genus two or to the canonically polarized product of two elliptic curves (cf. [28]).

Let us first assume that $(A, \mathcal{L})$ is irreducible. We then know that there exists a curve $C/\overline{\mathbb{Q}}$ of genus two such that $(A, \mathcal{L}) \overset{\overline{\mathbb{Q}}}{\simeq} (J(C), \mathcal{L}(\Theta_C))$. Since $\mathcal{L} = \mathcal{L}(\Theta_C) \in \mathrm{NS}(A_{\overline{k}})^{G_k}$, we claim that C admits a k-isomorphism onto all its Galois conjugates ${}^{\sigma}C$ for $\sigma \in G_k$. More precisely, if we regard C as an embedded curve in $\operatorname{Pic}^1(C)$, then ${}^{\sigma}C = C + a_\sigma$ for some $a_\sigma \in \operatorname{Pic}^0(C)(\overline{\mathbb{Q}})$. Indeed, this follows from the fact that the sheaves ${}^{\sigma}\mathcal{L} = \mathcal{L}(\Theta_{{}^{\sigma}C})$ are all algebraically equivalent and $h^0(\mathcal{L}) = 1$. In particular, we infer that the field of moduli k_C of C is contained in k. Let us now show that C does admit a projective model over k. We distinguish two cases depending on whether the group $\operatorname{Aut} C$ is trivial or not.

If $\operatorname{Aut} C \not\simeq \mathbb{Z}/2\mathbb{Z}$, Cardona has recently proved that C always admits a model over its field of moduli (cf. [1]).

We now consider the case that the hyperelliptic involution v on C generates the group of the automorphisms of C. Then, as shown by Mestre in [15], there is a projective model C/K of C over a quadratic extension K/k. Let $\sigma \in G_k$ be such that σ does not act trivially on K. There is an isomorphism $\varphi_\sigma : C \xrightarrow{\simeq} {}^\sigma C$ of C/K onto C^σ given by the translation by a_σ map on $\operatorname{Pic}^1(C)$. The map $\operatorname{Pic}^1(C) \to \operatorname{Pic}^1(C)$, $D \mapsto D + a_\sigma + {}^\sigma a_\sigma$ descends to an automorphism of C, ${}^\sigma\varphi_\sigma \circ \varphi_\sigma$, which cannot be the hyperelliptic involution v, since $v = -1_{J(C)}$ on $J(C)$. As we are assuming that $\operatorname{Aut} C \simeq \mathbb{Z}/2\mathbb{Z}$, we obtain ${}^\sigma\varphi_\sigma = \varphi_\sigma^{-1}$. Now, Weil's criterion on the field of definition of an algebraic variety applies to ensure that C admits a projective model over k (cf. [27]).

Now, assume that $(A, \mathcal{L})$ is reducible over $\overline{k}$, i.e., $A \overset{\overline{k}}{\simeq} C_1 \times C_2$ for some elliptic curves C_1 and C_2. If both C_1, C_2 and the isomorphism are defined over k, then $(A, \mathcal{L}) \overset{k}{\simeq} (C_1 \times C_2, \mathcal{L}_{\mathrm{can}})$. Otherwise, C_i/K must be defined over a quadratic extension K/k and $C_1 = {}^\sigma C_2$ where $\operatorname{Gal}(K/k) = \langle \sigma \rangle$, since the product $C_1 \times C_2$ is defined over k. This is equivalent to saying that $A \overset{k}{\simeq} \operatorname{Res}_{K/k}(C_1)$. ∎

Definition 3.2. We say that a principal polarization $\mathcal{L}$ on an abelian surface A over k is *split* if $(A, \mathcal{L}) \overset{\overline{k}}{\simeq} (C_1 \times C_2, \mathcal{L}_{\mathrm{can}})$ for some elliptic curves $C_i/\overline{k}$. We shall denote by $\sigma(A_k)$ the number of k-isomorphism classes of split principal polarizations on A over k.

Corollary 3.3. *Let A/k be an abelian surface. Then*

$$\pi(A_k) = \tau(A_k) + \sigma(A_k).$$

It may very well be the case that $\sigma(A_k) \geq 2$ for some abelian surface A/k. This amounts to saying that $A \simeq C_1 \times C_2 \simeq C_3 \times C_4$ as unpolarized abelian varieties for two different nonordered pairs of elliptic curves (C_1, C_2) and (C_3, C_4) (cf. [14, p. 318]).

3.2. *Principal polarizations on abelian surfaces of* GL_2*-type.* We now focus our attention on abelian surfaces of GL_2-type. As an immediate consequence of Theorem 2.10, we obtain the following.

Corollary 3.4. *Let A be a principally polarizable abelian surface over k such that* $\operatorname{End}_k A = R$ *is an order in a quadratic field F. Then*

$$\pi(A_k) = \begin{cases} 2 & \text{if } F \text{ is real and all units in } R \text{ have positive norm,} \\ 1 & \text{otherwise.} \end{cases}$$

Since every abelian variety of GL_2-type over k is simple over k, note that the second possibility of Theorem 3.1 cannot occur when the surface is of this type. The following definition was first introduced in [2].

Definition 3.5. An elliptic curve $C/\overline{k}$ is a *k-curve* if C is isogenous to all its Galois conjugates C^σ, $\sigma \in \mathrm{Gal}(\overline{k}/k)$. A k-curve C is completely defined over an extension K/k if C is defined over K and it is isogenous over K to all its Galois conjugates.

Lemma 3.6. *Let A be an abelian surface of real GL_2-type over k, which is k-isogenous to the Weil restriction $\mathrm{Res}_{K/k} C$ of an elliptic curve C/K, where K/k is a quadratic extension. Then C is a k-curve completely defined over K, its j-invariant $j(C) \notin k$ and $\mathbb{Q} \otimes \mathrm{End}_K A$ is isomorphic to $M_2(\mathbb{Q})$.*

Proof. Let σ be the nontrivial automorphism of $\mathrm{Gal}(K/k)$. We have $\mathbb{Q} \otimes \mathrm{End}_K C = \mathbb{Q}$ and there is an isogeny $\mu : C \to {}^\sigma C$ defined over K, since otherwise $F = \mathbb{Q} \otimes \mathrm{End}_k A$ must be either $\mathbb{Q}$ or contain an imaginary quadratic field. Therefore, C is a k-curve completely defined over K and $\mu \circ {}^\sigma\mu \in \mathbb{Q}$. Then $\mathbb{Q} \otimes \mathrm{End}_K A \simeq M_2(\mathbb{Q})$ and $\mu \circ {}^\sigma\mu = \pm \deg \mu$. Since F is a totally real quadratic field, $\mu \circ {}^\sigma\mu = \deg \mu$ and $F = \mathbb{Q}(\sqrt{\deg \mu})$. Consequently, μ cannot be an isomorphism and hence $j(C) \notin k$. ∎

Proposition 3.7. *Let C_1, C_2 be two elliptic curves defined over the quadratic fields K_1, K_2 respectively. If $\mathrm{Res}_{K_1/\mathbb{Q}}(C_1)$ and $\mathrm{Res}_{K_2/\mathbb{Q}}(C_2)$ are $\mathbb{Q}$-isomorphic and they are of real GL_2-type over $\mathbb{Q}$, then $K_1 = K_2$ and C_1 is isomorphic over K_1 either to C_2 or to its Galois conjugate.*

Proof. Let $A = \mathrm{Res}_{K_i/\mathbb{Q}}(C_i)$. First, we show that $K_1 = K_2$. If C_i does not have complex multiplication, we then know by the previous lemma that $\mathbb{Q} \otimes \mathrm{End}_{K_i} A = \mathbb{Q} \otimes \mathrm{End}_{\overline{\mathbb{Q}}} A \simeq M_2(\mathbb{Q})$ and hence $\mathbb{Q} \otimes \mathrm{End}_{K_1 \cap K_2} A \simeq M_2(\mathbb{Q})$. It follows that $K_1 = K_2$.

Assume then that both C_1 and C_2 have complex multiplication by the same imaginary quadratic field L because C_1 and C_2 must be isogenous. Again by the lemma, $K_i = \mathbb{Q}(j(C_i))$. We have $j(C_i) = j(\mathfrak{a}_i)$ for an invertible ideal $\mathfrak{a}_i$ of the order $\mathrm{End}(C_i)$ of L. Since K_i is quadratic, the ideal classes of $\mathfrak{a}_i$ and $\mathfrak{a}_i^{-1}$ are equal, and therefore, $j(\mathfrak{a}_i) = j(\mathfrak{a}_i^{-1}) = j(\overline{\mathfrak{a}_i}) = \overline{j(\mathfrak{a}_i)}$, where $\bar{\ }$ denotes the complex conjugation. It follows that $K_i = \mathbb{Q}(j(C_i))$ is a real quadratic field. Assume that $K_1 \neq K_2$ and let $\mathrm{Gal}(K_i/\mathbb{Q}) = \langle \sigma_i \rangle$. Since $C_1 \times C_1^{\sigma_1} \simeq C_2 \times C_2^{\sigma_2}$ over $K_1 \cdot K_2$ but not over K_i, there exists an isogeny $\mu : C_1 \to C_2$ defined over $K_1 \cdot K_2$ but not over K_i. Let $\tau \in \mathrm{Gal}(K_1 \cdot K_2/\mathbb{Q})$ be the automorphism which does not act trivially over K_1 or over K_2. Then $\widehat{\mu} = \mu \times {}^\tau\mu \in \mathbb{Q} \otimes \mathrm{End}_{K_1 \cdot K_2} A \setminus \mathbb{Q} \otimes \mathrm{End}_{K_i} A$. Denote by $\mathcal{A}$ the $\mathbb{Q}$-algebra generated by $\mathbb{Q} \otimes \mathrm{End}_{K_i} A$ and $\widehat{\mu}$. From the inclusions

$$\mathbb{Q} \otimes \mathrm{End}_{K_i} A \simeq M_2(\mathbb{Q}) \subsetneq \mathcal{A} \subseteq \mathbb{Q} \otimes \mathrm{End}_{\overline{\mathbb{Q}}} A \simeq M_2(L),$$

we obtain $\mathcal{A} = \mathbb{Q} \otimes \mathrm{End}_{\overline{\mathbb{Q}}} A$ and, thus, all endomorphisms of A are defined over $K_1 \cdot K_2$. In particular, $\mathbb{Q} \otimes \operatorname{End}_{K_1 \cdot K_2}(C_i) \simeq L$, but this leads to a contradiction since the totally real field $K_1 \cdot K_2$ cannot contain L. Therefore, $K_1 = K_2$.

Set $K = K_i$, $C = C_1$ and $\operatorname{Gal}(K/\mathbb{Q}) = \langle \sigma \rangle$. Now, we will prove that C is isomorphic either to C_2 or to ${}^\sigma C_2$ over K. Denote by $\pi : A \to C$ the natural projection over K, so that $\ker \pi = {}^\sigma C$. Then the period lattice of C is $\Lambda = \{\int_\gamma \pi^*(\omega) \mid \gamma \in H_1(A, \mathbb{Z})\}$, where ω is the invariant differential of C. Given $\alpha \in \mathbb{Q} \otimes \operatorname{End}_K A$, we will denote by Λ_α the set $\{\int_\gamma \alpha^*(\pi^*(\omega)) \mid \gamma \in H_1(A, \mathbb{Z})\}$. When $\alpha^*(\pi^*(\omega)) \neq 0$, Λ_α is the period lattice of a certain elliptic curve over K; these lattices cover all the K-isomorphism classes of elliptic curves which are optimal quotients of A over K. We will prove that all these classes are also obtained when α only runs over F^*. Let $w \in \operatorname{End}_K A$ be the composition of the morphisms

$$A \xrightarrow{\pi} C \xhookrightarrow{i} A = C \times {}^\sigma C,$$

where i is the natural inclusion. We have $w^* H^0(A, \Omega^1_{A/K}) = K\pi^*(\omega)$ and $w^2 = w$. Moreover, $\mathbb{Q} \otimes \operatorname{End}_K A = F \oplus w \cdot F$ since $w \notin F$ and $\mathbb{Q} \otimes \operatorname{End}_K A$ is a F-algebra of dimension 2. Now, it suffices to use the equality $w \cdot \mathbb{Q} \otimes \operatorname{End}_K A = w \cdot F$. Since for every integer $m \neq 0$, the classes corresponding to Λ_α and $\Lambda_{m\alpha}$ are isomorphic, we can assume that $\alpha \in \operatorname{End}_{\mathbb{Q}} A$ and in this case Λ_α is a sublattice of Λ.

There exists a cyclic isogeny between C and ${}^\sigma C$ over K of a certain degree n, which extends to an endomorphism $\beta \in \operatorname{End}_{\mathbb{Q}} A$ with the following properties:

(1) β restricted to ${}^\sigma C$ (resp. C) provides a cyclic isogeny between ${}^\sigma C$ and C (resp. C and ${}^\sigma C$) of degree n.
(2) Λ_β is the period lattice of the elliptic curve isomorphic to ${}^\sigma C$ over K which satisfies $\Lambda/\Lambda_\beta \simeq \mathbb{Z}/n\mathbb{Z}$.
(3) $\beta^2 = n$, $F = \mathbb{Q}(\beta)$, and moreover, for all integers $m > 1$ we have $\beta/m \notin \operatorname{End}_{\mathbb{Q}} A$.
(4) Due to the previous step, β also provides a cyclic isogeny between ${}^\sigma C_2$ and C_2 of degree n.
(5) For all integers a, b we have $\Lambda_{a+b\beta} = a\Lambda + b\Lambda_\beta$.

Let Λ_α be a lattice corresponding to the K-isomorphism class of C_2 and write $\alpha = a + b\beta$ with a, b integers which we can assume to be coprime. We can also assume that $a, b \neq 0$, since otherwise the statement is obvious. Set $d = \gcd(a, n)$. Using the fact that $\Lambda/\Lambda_\beta \simeq \mathbb{Z}/n\mathbb{Z}$, it can be checked that $\Lambda_\alpha = a\Lambda + b\Lambda_\beta$ is the sublattice $d\Lambda + \Lambda_\beta$ of Λ of degree d. Then we can take $a = d$ and $b = 1$. Now, we can see that $\Lambda_{\alpha\cdot\beta} = n\Lambda + d\Lambda_\beta$ is a sublattice of Λ_α of degree $d \cdot n$. Therefore, d must be 1 and C and C_2 are isomorphic over K. ∎

Corollary 3.8. *Let A be a principally polarizable abelian surface of* GL_2*-type over* $\mathbb{Q}$. *Then* $\sigma(A_{\mathbb{Q}}) \leq 1$ *and, in particular,*

$$\tau(A_{\mathbb{Q}}) = \begin{cases} \pi(A_{\mathbb{Q}}) - 1 & \text{if } A \overset{\mathbb{Q}}{\simeq} \mathrm{Res}_{K/\mathbb{Q}} C \text{ for a } \mathbb{Q}\text{-curve } C/K, \\ \pi(A_{\mathbb{Q}}) & \text{otherwise.} \end{cases}$$

Remark 3.9. Note that if $\mathrm{End}_{\overline{\mathbb{Q}}} A = \mathrm{End}_{\mathbb{Q}} A$, then $\pi(A_{\mathbb{Q}}) = \pi(A_{\overline{\mathbb{Q}}})$ and $\tau(A_{\mathbb{Q}}) = \tau(A_{\overline{\mathbb{Q}}})$. Furthermore, if $\tau(A_{\mathbb{Q}}) = 2$ then the two curves over $\mathbb{Q}$ which share A as Jacobian are nonisomorphic over $\overline{\mathbb{Q}}$.

Now, Theorem 1.1 is a consequence of Corollaries 3.4 and 3.8.

Corollary 3.10. *There are infinitely many genus two curves over* $\mathbb{Q}$ *such that their Jacobians are of real* GL_2*-type and isomorphic to the product of two elliptic curves as unpolarized abelian varieties.*

Proof. As a consequence of Corollary 3.4 and Corollary 3.8, for every quadratic $\mathbb{Q}$-curve C such that $A = \mathrm{Res}_{K/\mathbb{Q}} C$ is of real GL_2-type and the order $\mathrm{End}_{\mathbb{Q}} A$ contains no units of negative norm, there is a genus two curve over $\mathbb{Q}$ whose Jacobian is isomorphic to A as unpolarized abelian varieties. Fix $d = 3$ or 7. It is known that for every quadratic field K there exists a $\mathbb{Q}$-curve C'/K without CM such that $K = \mathbb{Q}(j(C'))$ and an isogeny from C' onto its Galois conjugate of degree d. For such a curve there is an isomorphic curve C/K with $A = \mathrm{Res}_{K/\mathbb{Q}} C$ being of real GL_2-type if and only if d is a norm of K (see [20]). Now, the statement follows from the fact that $\mathbb{Z}[\sqrt{d}]$ contains no units of negative norm and from the existence of infinitely many quadratic fields K such that $d \in \mathrm{Norm}_{K/\mathbb{Q}}(K^*)$. ∎

In Subsection 4.4, we will show how for some quadratic $\mathbb{Q}$-curves Theorem 1.1 allows us to construct a genus two curve over $\mathbb{Q}$ whose Jacobian is isomorphic over $\mathbb{Q}$ to its Weil's restriction.

In view of the above, it would be useful to have a criterion for deciding whether an abelian surface of real GL_2-type is principally polarizable or not. In addition to Corollary 2.12, we now characterize the existence of a principal polarization on an abelian surface of real GL_2-type under some arithmetical restrictions on the ring of the endomorphisms.

Proposition 3.11. *Let A be an abelian surface of* GL_2*-type over k by a real quadratic field F of class number* $h(F) = 1$ *and assume that* $\mathrm{End}_k A = \mathcal{O}$ *is the ring of integers of F. Then*:

(1) *If $\mathcal{O}^*$ contains some unit of negative norm, then A is principally polarized over k and, in particular, the degree of any polarization on A over k is a norm of F.*
(2) *If $\mathcal{O}^*$ contains no units of negative norm and $\mathcal{L}$ is a polarization of degree d over k, then either d or $-d$ is a norm of F. In the first case, A is principally polarizable over k.*

Proof. First, we will prove that there is an invertible sheaf on A in $\mathrm{NS}(A_{\overline{k}})^{G_k}$ of degree 1. Since A is defined over k, there is a polarization $\mathcal{L}$ on A defined over k. Let us denote by E the corresponding alternating Riemann form on $H_1(A,\mathbb{Z})$. Since $H_1(A,\mathbb{Z})$ is a free $\mathcal{O}$-module of rank 2 and $h(F)=1$, we have $H_1(A,\mathbb{Z})=\mathcal{O}\gamma_1\oplus\mathcal{O}\gamma_2$ for some $\gamma_1,\gamma_2\in H_1(A,\mathbb{Z})$. Consider the morphism of groups

$$\phi:\mathcal{O}\to\mathbb{Z},\qquad \alpha\mapsto E(\alpha\gamma_1,\gamma_2).$$

By the nondegeneracy of the trace, there exists a $\delta\in F$ such that $\phi(\alpha)=\mathrm{Tr}_{F/\mathbb{Q}}(\alpha\cdot\delta)$. Due to the fact that δ lies in the codifferent of $\mathcal{O}$, there is $v\in\mathcal{O}$ such that $\delta=v/\sqrt{\Delta}$, where Δ is the discriminant of F. Set $\mathcal{L}_0=\mathcal{L}^{(v^{-1})}\in\mathbb{Q}\otimes\mathrm{NS}(A_{\overline{k}})^{G_k}$ and denote by E_0 its alternating Riemann form. From the relations

$$E_0(\alpha\gamma_1,\gamma_2)=\mathrm{Tr}_{F/\mathbb{Q}}(\alpha/\sqrt{\Delta})\in\mathbb{Z},\qquad E_0(\gamma,\alpha\gamma)=0$$

for all $\alpha\in\mathcal{O}$ and $\gamma\in H_1(A,\mathbb{Z})$, we obtain $\mathcal{L}_0\in\mathrm{NS}(A_{\overline{k}})^{G_k}$. Moreover, taking a basis $\gamma_1,\alpha\gamma_1,\gamma_2,\alpha\gamma_2$ of $H_1(A,\mathbb{Z})$ for a suitable α, an easy check shows that the matrix of E_0 with respect to this basis has determinant 1.

Now, assume that $\mathcal{O}^*$ contains some unit u of negative norm. Then either $\mathcal{L}_0$, $\mathcal{L}_0^{(-1)}$, $\mathcal{L}_0^{(u)}$ or $\mathcal{L}_0^{(-u)}$ must be a principal polarization. By Corollary 2.12, we also see that the degree of any polarization on A over k is a norm of F.

In the case $-1\notin\mathrm{Norm}_{F/\mathbb{Q}}(\mathcal{O}^*)$, by Remark 2.11, there is $t\in\mathcal{O}$ such that $\mathcal{L}_0^{(t)}=\mathcal{L}$, with $\mathrm{Norm}_{F/\mathbb{Q}}(t)=\pm\deg\mathcal{L}$. If $\deg\mathcal{L}$ is a norm of F, then $\mathrm{Norm}_{F/\mathbb{Q}}(t)=\deg\mathcal{L}$. Therefore, either t or $-t$ lies in $\mathcal{O}_+$ and, thus, either $\mathcal{L}_0$ or $\mathcal{L}_0^{(-1)}$ is a polarization. ■

4. Explicit examples. In the previous section we have shown the different possibilities for abelian surfaces of GL_2-type over $\mathbb{Q}$. We now illustrate these possibilities with explicit examples for the case of real GL_2-type. We will not consider abelian surfaces of CM GL_2-type because they are less interesting. Indeed, these abelian surfaces have a unique primitive polarization $\mathcal{L}$ and they are principally polarizable if and only if $\mathcal{L}$ is principal. Assuming Serre's conjecture to be true, we must look for these examples among the abelian varieties A_f attached by Shimura to an eigenform $f\in S_2(\Gamma_0(N))$. In order to present these abelian varieties as Jacobians of curves, we will use the procedure described in [3], which is based on Jacobian Thetanullwerte (see [7]).

All the computations were performed with `Magma v.2.7` using the package `MAV` written by E. González and J. Guàrdia ([4]). Both the package and some files required to reproduce them are available via the web pages of the authors.

4.1. *Modular case.* We summarize here some facts about modular abelian varieties, fixing also the notation used in the examples in the following subsections. Let $f = \sum_{n\ge 1} a_n q^n$ be a normalized newform of $S_2(\Gamma_0(N))$ where, as usual, $q = e^{2\pi i z}$. Attached to f, Shimura constructed an abelian variety A_f over $\mathbb{Q}$ in two different ways. In [25], this abelian variety is presented as a subvariety of $J_0(N)$ while in [26] it is constructed as an optimal quotient of $J_0(N)$, and it is proved that both are dual. We will take A_f as subvariety, and this will not be a restriction because we will only consider the principally polarized case and in this situation both abelian varieties are isomorphic (over $\mathbb{Q}$). More precisely, let $\mathbf{T}$ be the Hecke algebra of endomorphisms of the Jacobian variety $J_0(N)$ of $X_0(N)$, and let I_f be the kernel of the map $\mathbf{T} \to \mathbb{Z}[a_1, a_2, \ldots]$ which identifies every Hecke operator with the corresponding eigenvalue of f. Then $A = J_0(N)/I_f J_0(N)$ is the abelian variety attached by Shimura as an optimal quotient. We recall that $K_f = \mathbb{Q}(\{a_n\})$ is a number field of degree $n = [K_f : \mathbb{Q}] = \dim A$, the endomorphism algebra $\mathbb{Q}\otimes \operatorname{End}_{\mathbb{Q}} A$ is the $\mathbb{Q}$-algebra generated by $\mathbf{T}$ acting on A, $\mathbb{Q}\otimes \mathbf{T}/I_f$, and this is isomorphic to K_f. Denote by $\pi : J_0(N) \to A$ the natural projection over $\mathbb{Q}$. There is a $\mathbb{Z}$-submodule H of $H_1(J_0(N), \mathbb{Z})$ of rank $2n$ such that $H_1(J_0(N), \mathbb{Z}) = \ker \pi_* \oplus H$. Note that $\ker \pi_* = I_f H_1(J_0(N), \mathbb{Z})$.

It is well known that $\pi^* H^0(A, \Omega^1_{A/\mathbb{C}})$ is the $\mathbb{C}$-vector space generated by the Galois conjugates of $f(q)\, dq/q$ and $\pi^* H^0(A, \Omega^1_{A/\mathbb{Q}})$ is the subspace obtained by taking the modular forms with rational q-expansion. For a fixed rational basis $h_1, \ldots, h_n$ of $\pi^* H^0(A, \Omega^1_{A_f/\mathbb{Q}})$, the Abel–Jacobi map induces an isomorphism of complex torus:

$$A(\mathbb{C}) \to \mathbb{C}^n/\Lambda, \qquad P \mapsto \Big(\int_0^P h_1, \ldots, \int_0^P h_n\Big),$$

where

$$\Lambda = \Big\{\Big(\int_\gamma h_1, \ldots, \int_\gamma h_n\Big) \,\Big|\, \gamma \in H_1(X_0(N), \mathbb{Z})\Big\} = \Big\{\Big(\int_\gamma h_1, \ldots, \int_\gamma h_n\Big) \,\Big|\, \gamma \in H\Big\}.$$

The abelian variety A_f, viewed as subvariety of $J_0(N)$, is described by

$$A_f(\mathbb{C}) \to \mathbb{C}^n/\Lambda_f, \qquad P \mapsto \Big(\int_0^P h_1, \ldots, \int_0^P h_n\Big),$$

where $\Lambda_f = \{(\int_\gamma h_1, \ldots, \int_\gamma h_n) \mid \gamma \in H_f\}$ and $H_f = \{\gamma \in H_1(X_0(N), \mathbb{Z}) \mid I_f \gamma = \{0\}\}$. Note that Λ_f is a sublattice of Λ. Obviously, given $T \in \mathbb{Q} \otimes \mathbf{T}$, we have $T \in \operatorname{End}_{\mathbb{Q}}(A_f)$ if and only if T leaves Λ stable or equivalently T leaves H_f stable. Let Θ be the canonical polarization on $J_0(N)$ and E its corresponding Riemann form, which is obtained from the intersection numbers on $H_1(X_0(N), \mathbb{Z})$. From now on, we shall call the polarization $\mathcal{L}$ on A_f

obtained from the canonical polarization Θ on $J_0(N)$ the *canonical polarization* on A_f. Note that the corresponding Riemann form E_Λ on Λ is obtained as the restriction of E to Λ, and although Θ is principal, $\mathcal{L}$ may not be.

We know by Ogg (see [19]) that the cusps of $X_0(N)$ associated to $1/d$ for $d \mid N$ and $\gcd(d, N/d) = 1$ are rational points on $X_0(N)$. Moreover, the divisors of degree 0 generated by these cusps are torsion points on $J_0(N)(\mathbb{Q})$, and the same holds for their projections on A_f, although their orders can decrease. For a given torsion $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable subgroup G of A_f, the abelian variety A_f/G is isogenous to A_f over $\mathbb{Q}$. Moreover, both abelian varieties can be isomorphic over $\mathbb{Q}$ only if G is the kernel of some endomorphism in $\mathrm{End}_{\mathbb{Q}}(A_f)$. As complex tori, if we identify $A_f(\mathbb{C}) = \mathbb{C}^g/\Lambda_f$, then $A_f/G(\mathbb{C}) = \mathbb{C}^g/\Lambda_G$, where $\Lambda_G = \langle \Lambda_f, G\rangle$. The polarization $\mathcal{L}$ induces a polarization $\mathcal{L}_G$ on A_f/G, whose alternating Riemann form E_G is given by $E_G = \#G \cdot E_\Lambda : \Lambda_G \times \Lambda_G \to \mathbb{Z}$ and its degree is $\#G \cdot \deg \mathcal{L}$. Notice that $\mathrm{End}_{\mathbb{Q}}(A_f)$ and $\mathrm{End}_{\mathbb{Q}}(A_f/G)$ can be different.

4.2. *Nonisomorphic genus two curves with* $\mathbb{Q}$*-isomorphic Jacobian.* We begin by studying the unique two-dimensional factor $S_{65,B}$ of $J_0(65)$, given by the newform

$$f = q + aq^2 + (1-a)q^3 + q^4 - q^5 + (a-3)q^6 + \cdots$$

with $a = \sqrt{3}$. We know that $\mathrm{End}_{\mathbb{Q}}(A_f) = \mathbb{Z}[\sqrt{3}]$ and moreover A_f is simple in its isogeny class over $\overline{\mathbb{Q}}$, because f does not admit any extra-twist. Hence, each principal polarization on A_f is the sheaf of sections of the Theta divisor of a smooth curve C of genus two such that $A_f \overset{\mathbb{Q}}{\simeq} J(C)$. A basis of the $\mathbb{Z}$-module H_f is spanned by the modular symbols

$$\begin{aligned}
\gamma_1 &= \{-\tfrac{1}{15}, 0\} - \{-\tfrac{1}{30}, 0\} + \{-\tfrac{1}{40}, 0\} - \{-\tfrac{1}{60}, 0\},\\
\gamma_2 &= \{-\tfrac{1}{20}, 0\} - \{-\tfrac{1}{35}, 0\} + \{-\tfrac{1}{50}, 0\} - \{-\tfrac{1}{55}, 0\},\\
\gamma_3 &= \{-\tfrac{1}{15}, 0\} - \{-\tfrac{1}{26}, 0\} + \{-\tfrac{1}{40}, 0\} - \{-\tfrac{1}{50}, 0\} - \{-\tfrac{2}{5}, -\tfrac{5}{13}\},\\
\gamma_4 &= \{-\tfrac{1}{30}, 0\} - \{-\tfrac{1}{45}, 0\} + \{-\tfrac{1}{52}, 0\} - \{-\tfrac{1}{55}, 0\} + \{-\tfrac{2}{5}, -\tfrac{5}{13}\}.
\end{aligned}$$

An integral basis of $H^0(A_f, \Omega^1)$ is given by the forms $h_1 = ({}^\sigma f - f)/\sqrt{3}$, $h_2 = (f + {}^\sigma f)/2$. By integrating these differentials along the paths $\gamma_1, \ldots, \gamma_4$, we obtain an analytic presentation of the abelian surface A_f as a complex torus $\mathbb{C}^2/\Lambda$. The restriction of E to H_f is the Riemann form of a polarization on A_f, given by the following Riemann matrix:

$$M_E = (E(\gamma_i, \gamma_j))_{i,j} = \begin{pmatrix} 0 & 2 & -2 & 0\\ -2 & 0 & -2 & 2\\ 2 & 2 & 0 & 0\\ 0 & -2 & 0 & 0 \end{pmatrix}.$$

The type of this polarization is $(2,2)$. Thus, the primitive polarization $\mathcal{L}_0$ associated to it is principal. In [3] it has been checked that $(S_{65,B},\mathcal{L}_0)$ is the polarized Jacobian of the hyperelliptic curve

$$C_{65,B,1} : Y^2 = -X^6 - 4X^5 + 3X^4 + 28X^3 - 7X^2 - 62X + 42,$$

whose absolute Igusa invariants are

$$\{i_1, i_2, i_3\} = \left\{ -\frac{2^6 5 \cdot 313^5}{13^3}, \frac{139 \cdot 313^3 701}{5 \cdot 13^3}, \frac{7 \cdot 313^2 59104229}{2^3 5^2 13^3} \right\}.$$

Following Theorem 2.10, we now build a second polarization on $S_{65,B}$, which will exhibit $S_{65,B}$ as the Jacobian of a second curve $C_{65,B,2}$ nonisomorphic to $C_{65,B,1}$ over $\overline{\mathbb{Q}}$. Let us consider the Hecke operator $u = 2+T_2 = 2+\sqrt{3} \in \mathrm{End}_{\mathbb{Q}}(A_f)$. This is a nonsquare totally positive unit in the ring of integers $\mathcal{O}$ of K_f. The action of u on $H_1(A_f,\mathbb{Z})$ with respect to the basis $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ is given by

$$M_u = \begin{pmatrix} 3 & 2 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 4 & 0 \\ 2 & 1 & 1 & 1 \end{pmatrix}.$$

We now take the polarization $\mathcal{L}_0^{(u)}$, which is given by the Riemann form E_u whose alternating Riemann matrix is $M_{E_u} = \frac{1}{2} \cdot M_E \cdot M_u$. It is a principal polarization on A_f and nonisomorphic to $\mathcal{L}_0$. Since A_f is simple (over $\overline{\mathbb{Q}}$), the polarized abelian variety $(A_f,\mathcal{L}_0^{(u)})$ is the Jacobian of a curve $C_{65,B,2}$ defined over $\mathbb{Q}$. A symplectic basis with respect to E_u is

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}.$$

In order to identify the polarized abelian variety $(A_f,\mathcal{L}^{(u)})$, we compute the period matrix $\Omega = (\Omega_1 \,|\, \Omega_2) = ((\int_{\delta_k} h_j dq/q)_{k=1,2} \,|\, (\int_{\delta_k} h_j dq/q)_{k=3,4})$ of the forms h_1,h_2 along these paths. Let $Z = \Omega_1^{-1}\Omega_2 \in \mathcal{H}_2$, which belongs to the Siegel upper half space. We can now apply the method of [3] to recover an equation of this curve. We obtain

$$C_{65,B,2} : Y^2 = -(X^2+3X+1)(7X^4+37X^3+71X^2+44X+8).$$

The absolute Igusa invariants of this curve are

$$\{i_1, i_2, i_3\} = \left\{ -\frac{2^3 3109^5}{5^3 13^4}, \frac{3109^3 206639}{2 \cdot 5^3 13^4}, \frac{7 \cdot 3109^2 123916753}{2^3 5^3 13^4} \right\}$$

and this makes it clear that the curves $C_{65,B,1}$ and $C_{65,B,2}$ are nonisomorphic over $\overline{\mathbb{Q}}$, although their Jacobians are isomorphic as unpolarized abelian varieties over $\mathbb{Q}$.

4.3. *A genus two curve with Jacobian isomorphic to the Weil restriction of a quadratic* $\mathbb{Q}$*-curve.* Let us now consider the factor A_f of $J_0(63)$ corresponding to the newform

$$f = q + aq^2 + q^4 - 2aq^5 + q^7 - aq^8 + 6q^{10} + 2q^{11} + 2q^{13} + \cdots,$$

where $a = \sqrt{3}$. Again $\mathrm{End}_{\mathbb{Q}}(A_f) \simeq \mathbb{Z}[\sqrt{3}] =: \mathcal{O}$ is the integer ring of $K_f = \mathbb{Q}(\sqrt{3})$. Let $\mathcal{L}_0$ be the primitive canonical polarization on A_f induced from $J_0(63)$. It turns out that $\mathcal{L}_0$ is principal and that the polarized abelian variety $(A_f, \mathcal{L})$ is the Jacobian of the hyperelliptic curve (see [3])

$$C_{63,B} : Y^2 = -3X^6 + 162X^3 + 81.$$

We see that f does not have complex multiplication and the quadratic character of $L = \mathbb{Q}(\sqrt{-3})$ is an extra-twist for f. Since the discriminant of L is a norm of K_f, we know (see [6]) that, up to Galois conjugation, there is a unique quadratic $\mathbb{Q}$-curve C defined over L such that C is an optimal quotient of A_f defined over L. In fact, the invariant differentials of the optimal quotients of A_f, when pulled back to $H^0(A_f, \Omega^1_{A_f/L})$, are given by $\langle \big(\frac{1-i}{2}(a+b\sqrt{3})f + \frac{1+i}{2}(a-b\sqrt{3})^\sigma f\big)dq/q \rangle$, with $a+b\sqrt{3}$ running over K_f^*.

In order to obtain the other principal polarization on A_f, we proceed as in the previous example. Let $u = T_2 + T_{13} = \sqrt{3} + 2 \in \mathcal{O}^* \setminus \mathcal{O}^{*2}$ be the Hecke operator acting on A_f and let $\mathcal{L}_0^{(u)}$ be the principal polarization on A_f associated to it by Proposition 2.1. A symplectic basis of $H_1(A_f, \mathbb{Z})$ with respect to the Riemann form E_u associated to $\mathcal{L}_0^{(u)}$ is

$$\gamma_1 = \{-\tfrac{1}{24}, 0\} - \{-\tfrac{1}{28}, 0\} + \{-\tfrac{1}{30}, 0\} - \{-\tfrac{1}{51}, 0\} - \{-\tfrac{1}{3}, -\tfrac{2}{7}\},$$
$$\begin{aligned}\gamma_2 = {} & 2\big(\{-\tfrac{1}{24}, 0\} - \{-\tfrac{1}{28}, 0\} + \{-\tfrac{1}{39}, 0\} - \{-\tfrac{1}{6}, -\tfrac{1}{7}\}\big) - 5\{-\tfrac{1}{57}, 0\} \\ & + 3\big(-\{-\tfrac{1}{36}, 0\} + \{-\tfrac{1}{49}, 0\} - \{-\tfrac{1}{51}, 0\} + \{-\tfrac{1}{54}, 0\} + \{-\tfrac{1}{60}, 0\} \\ & + \{-\tfrac{1}{3}, -\tfrac{2}{7}\}\big),\end{aligned}$$
$$\begin{aligned}\gamma_3 = {} & -\{-\tfrac{1}{28}, 0\} + \{-\tfrac{1}{36}, 0\} + \{-\tfrac{1}{45}, 0\} - \{-\tfrac{1}{49}, 0\} + \{-\tfrac{1}{51}, 0\} - \{-\tfrac{1}{54}, 0\} \\ & - \{-\tfrac{1}{6}, -\tfrac{1}{7}\} + \{\tfrac{3}{7}, \tfrac{4}{9}\},\end{aligned}$$
$$\begin{aligned}\gamma_4 = {} & 2\big(-\{-\tfrac{1}{36}, 0\} + \{-\tfrac{1}{49}, 0\} - \{-\tfrac{1}{51}, 0\} + \{-\tfrac{1}{54}, 0\} - \{-\tfrac{1}{57}, 0\}\big) \\ & + \{-\tfrac{1}{24}, 0\} + \{-\tfrac{1}{39}, 0\} - \{-\tfrac{1}{45}, 0\} + \{-\tfrac{1}{60}, 0\} + \{-\tfrac{1}{3}, -\tfrac{2}{7}\} - \{\tfrac{3}{7}, \tfrac{4}{9}\}.\end{aligned}$$

We take the basis of $H^0(A_f, \Omega^1_{A_f/L})$ given by

$$\begin{aligned} g_1 &= \left(\frac{1-i}{2}(1+\sqrt{3})f + \frac{1+i}{2}(1-\sqrt{3})^\sigma f\right) dq/q \\ &= ((1-i\sqrt{3})q + (3-i\sqrt{3})q^2 + (1-\sqrt{3})q^4 + \cdots)dq/q \end{aligned}$$

and its conjugate $\overline{g}_1$. Let $\Omega = (\Omega_1 \mid \Omega_2)$ be the period matrix of this basis $g_1, \overline{g}_1$ with respect to $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and take $Z = \Omega_1^{-1}\Omega_2$. When we compute the even Thetanullwerte corresponding to Z, we find that, up to high accuracy, exactly one of them vanishes. This suggests that $(A_f, \mathcal{L}^{(u)})$ is not irreducible. We may confirm this by giving its explicit decomposition. Let

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}$$

so that $(\delta_1, \delta_2, \delta_3, \delta_4)$ is a new symplectic basis for E_u. The period matrix of $g_1, \overline{g}_1$ with respect to it is $\Omega' = (\Omega_1' \mid \Omega_2')$ with

$$\Omega_1' = \begin{pmatrix} 6.584352\ldots + 8.916903\ldots i & 0 \\ 0 & -15.444529\ldots + 11.404432\ldots i \end{pmatrix},$$

$$\Omega_2' = \begin{pmatrix} -2.275825\ldots + 6.429374\ldots i & 0 \\ 0 & -8.860177\ldots + 2.487529\ldots i \end{pmatrix}.$$

The lattices $\Lambda = \langle 6.584352 + 8.916903i, -2.275825 + 6.429374i\rangle$, $\Lambda_\sigma = \langle -15.444529 + 11.404432i, -8.860177 + 2.487529i\rangle$ correspond to a pair of Galois conjugate elliptic curves C and ${}^\sigma C$ and the invariants of C are

$$c_4(C) = \frac{9(5 - 12\sqrt{-3})}{2^8}, \qquad c_6(C) = \frac{27(43 + 42\sqrt{-3})}{2^{12}},$$
$$j(C) = \frac{27(121171 - 36627\sqrt{-3})}{686}.$$

Hence,

$$C \times {}^\sigma C \overset{\mathbb{Q}(\sqrt{-3})}{\simeq} (A_f, \mathcal{L}^{(u)}) \overset{\mathbb{Q}}{\simeq} \mathrm{Res}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}\, C.$$

4.4. *Constructing a genus two curve from a quadratic* $\mathbb{Q}$*-curve.* In the example above, we have modified the canonical polarization on the Jacobian of a hyperelliptic curve to present it as the Weil restriction of a $\mathbb{Q}$-curve. We now perform the reverse process, i.e., we depart from a quadratic $\mathbb{Q}$-curve and construct a polarization on its Weil restriction which transforms it in the Jacobian of a rational genus two curve. Notice that modular tools will not be used in this construction.

Let C be the elliptic curve $Y^2 = X^3 + aX + b$, where $a = -9(767 + 212\sqrt{13})$ and $b = -18(17225 + 4778\sqrt{13})$. We denote by σ the nontrivial Galois conjugation of $K = \mathbb{Q}(\sqrt{13})$ over $\mathbb{Q}$. The points on C with x-coordinate equal to $3(-13 + 4\sqrt{13})$ generate a subgroup G of 3-torsion points, and the elliptic curve C/G is isomorphic over K to ${}^\sigma C$. More precisely, there is a cyclic isogeny $\mu : C \to {}^\sigma C$ of degree 3 defined over K such that $\mu^*(\omega_\sigma) = \lambda\omega$, where $\lambda = 4 + \sqrt{13}$ (see [5]) and ω_σ and ω are the invariant differential forms of ${}^\sigma C$ and C respectively. In particular, ${}^\sigma\mu^*(\omega) = {}^\sigma\lambda\omega_\sigma = 3/\lambda\omega_\sigma$. We note that the

Weil restriction of C is of real GL_2-type, since $\mathbb{Q}\otimes\mathrm{End}(\mathrm{Res}_{K/\mathbb{Q}}\,C) = \mathbb{Q}(\sqrt{3})$. Moreover, as there are no units of negative norm in this algebra, Corollary 3.4 ensures the existence of a second polarization on $\mathrm{Res}_{K/\mathbb{Q}}\,C$. We will now build it.

Consider the period lattices

$$\Lambda = \langle w_1 = 0.220377\ldots, w_2 = 0.428744\ldots i\rangle,$$
$$\Lambda_\sigma = \langle w_{\sigma,1} = -\lambda w_1, w_{\sigma,2} = -\lambda/3w_2\rangle,$$

of the curves C, ${}^{\sigma}C$ respectively. Let γ_1, γ_2 (resp. $\gamma_{\sigma,1}$, $\gamma_{\sigma,2}$) be a basis of $H_1(C,\mathbb{Z})$ (resp. $H_1({}^{\sigma}C,\mathbb{Z})$) such that $\int_{\gamma_i}\omega = w_i$ (resp. $\int_{\gamma_{\sigma,i}}\omega_\sigma = w_{\sigma,i}$). Then $(\gamma_1,\gamma_{\sigma,1},\gamma_2,\gamma_{\sigma,2})$ is a symplectic basis of $H_1(C\times{}^{\sigma}C,\mathbb{Z})$ for the canonical polarization $\mathcal{L}$ attached to $C\times{}^{\sigma}C$.

The action of the endomorphisms $T=\sqrt{3}$ on $H_1(C\times{}^{\sigma}C,\mathbb{Z})$ is obtained from its action on $\Lambda\times\Lambda_\sigma$:

$$T:\Lambda\to\Lambda_\sigma,\qquad w\mapsto\lambda\cdot w,$$
$$T:\Lambda_\sigma\to\Lambda,\qquad w\mapsto{}^{\sigma}\lambda\cdot w = 3/\lambda\cdot w.$$

From this, we compute the matrix of the action of the fundamental unit $u = 2+\sqrt{3}$ of $\mathrm{End}(\mathrm{Res}_{K/\mathbb{Q}}\,C)$ on $H_1(C\times{}^{\sigma}C,\mathbb{Z})$ (with respect to the basis $\gamma_1,\gamma_{\sigma,1},\gamma_2,\gamma_{\sigma,2}$):

$$M = \begin{pmatrix} 2 & -3 & 0 & 0\\ -1 & 2 & 0 & 0\\ 0 & 0 & 2 & -1\\ 0 & 0 & -3 & 2\end{pmatrix}.$$

Hence, the Riemann form attached to the polarization $\mathcal{L}^{(u)}$ is given by

$$E_u = \begin{pmatrix} 0 & 0 & 2 & -1\\ 0 & 0 & -3 & 2\\ -2 & 3 & 0 & 0\\ 1 & -2 & 0 & 0\end{pmatrix}.$$

Taking into account that $\omega_1 = \omega+\omega_\sigma$, $\omega_2 = (\omega-\omega_\sigma)/\sqrt{13}$ form a rational basis of $H^0(\mathrm{Res}_{K/\mathbb{Q}}\,C,\Omega^1_{\mathbb{Q}})$, we compute the period matrix of these differential forms with respect to an E_u-symplectic basis of $H_1(C\times{}^{\sigma}C,\mathbb{Z})$, and apply the procedure of [3]. We find that $(\mathrm{Res}_{K/\mathbb{Q}}\,C,\mathcal{L}^{(u)})$ is the Jacobian of one of the two curves

$$Y^2 = \pm(12909572X^6+17307966X^5+8746257X^4$$
$$+2170636X^3+278850X^2+17238X+377).$$

Both curves have good reduction at $p=23$. Only for the curve corresponding to the $+$ sign, the characteristic polynomial of Frob_p acting on the Tate module of its Jacobian $\pmod p$ equals the square of the characteristic polynomial of Frob_p acting on the Tate module of $C/\mathbb{F}_p$. In conclusion, the right sign is $+$.

4.5. *Nonprincipally polarized abelian surfaces.* Coming back to the modular examples, we now illustrate what can be done to describe briefly those abelian surfaces A_f which are nonprincipally polarized. While our ideas do not provide a systematic method to treat any abelian surface, since we use the fact that they are of GL_2-type, it covers many interesting cases. This will be apparent in the following subsection, where we will build a number of genus two curves with isogenous Jacobian.

We will work with the abelian surface A_f which is the unique two-dimensional factor of $J_0(35)$. It corresponds to the newform

$$f = q + aq^2 + (-a-1)q^3 + (-a+2)q^4 + q^5 - 4q^6 + \cdots + (a+3)q^{13} + \cdots$$

where $a = (-1+\sqrt{17})/2$. We see that $\mathrm{End}_{\mathbb{Q}}(A_f) = \mathbb{Z}[(1+\sqrt{17})/2] =: \mathcal{O}$ is the integer ring of $K = \mathbb{Q}(a)$. Since f does not have any extra-twist, A_f is simple in its $\overline{\mathbb{Q}}$-isogeny class and, in particular, every principal polarization on A_f determines a curve whose Jacobian is isomorphic to A_f. In addition, by Corollary 3.4, if such a curve exists, it must be unique up to $\overline{\mathbb{Q}}$-isomorphism.

Let σ denote the Galois conjugation of $K/\mathbb{Q}$, and take $\alpha = (17+\sqrt{17})/34$. The cuspidal forms $h_1 = \alpha f + \sigma(\alpha)\,{}^{\sigma}\!f$, $h_2 = (f - {}^{\sigma}\!f)/\sqrt{17}$ provide an integral basis of $H^0(A_f, \Omega^1_{\mathbb{Q}})$. A basis of the $\mathbb{Z}$-module $H_1(A_f,\mathbb{Z})$ is given by the modular symbols

$$\begin{aligned}
\gamma_1 &= \{-1/10, 0\} - \{-1/25, 0\},\\
\gamma_2 &= \{-1/21, 0\} - \{-1/28, 0\},\\
\gamma_3 &= \{-1/7, 0\} - \{-1/15, 0\} + \{2/5, 3/7\},\\
\gamma_4 &= \{-1/10, 0\} - \{-1/15, 0\} + \{-1/25, 0\} - \{-1/30, 0\}.
\end{aligned}$$

In this basis, the matrix of the alternating Riemann form attached to the canonical polarization $\mathcal{L}$ on A_f is

$$M_E = (E(\gamma_i,\gamma_j))_{i,j} = \begin{pmatrix} 0 & 1 & -1 & 0\\ -1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & -1 & -1 & 0 \end{pmatrix}.$$

This polarization over $\mathbb{Q}$ is not principal, since it is of type $(1,2)$. Thus, we cannot describe $(A_f,\mathcal{L})$ as a Jacobian or as a Weil restriction. We will look for a different polarization $\mathcal{L}_0$ on A_f allowing this explicit description for A_f. Of course, we will require $\mathcal{L}_0$ to be principal and defined over $\mathbb{Q}$.

The existence of this polarization is guaranteed by Proposition 3.11. By Theorem 2.10 and Corollary 2.12, we must check the polarizations $\mathcal{L}^{(u)}$ for those endomorphisms $u \in \mathrm{End}_{\mathbb{Q}}(A_f)$ of norm 2. We take $u = T_{13} = (5+\sqrt{17})/2$. The rational representation of u with respect to the basis

$(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is

$$M_u = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 2 & 0 & 3 & -2 \\ 0 & 1 & -1 & 3 \end{pmatrix}.$$

The symplectic product E_u of the polarization $\mathcal{L}^{(u)}$ is given by the matrix $M_{E_u} = M_E \cdot M_u$; it is of type $(2, 2)$, so that there is a principal polarization $\mathcal{L}_0$ on A_f over $\mathbb{Q}$ such that $\mathcal{L}_0^{\otimes 2} = \mathcal{L}^{(u)}$. A symplectic basis for $H_1(A_f, \mathbb{Z})$ with respect to this principal polarization is

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}.$$

We now compute the periods of the differential forms $h_1 dq/q, h_2 dq/q$ along these paths to represent $(A_f, \mathcal{L}_0)$ as the complex torus $\mathbb{C}/\Lambda$, where Λ is the lattice spanned by the columns of the matrix $\Omega = (\Omega_1 \mid \Omega_2)$, with

$$\Omega_1 = \begin{pmatrix} 3.429722\ldots i & 1.265864\ldots i \\ -0.224497\ldots i & 1.714858\ldots i \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} -4.737944\ldots & 3.044837\ldots + 1.898796\ldots i \\ 2.706904\ldots & -2.368972\ldots + 2.572287\ldots i \end{pmatrix}.$$

We finally apply the method of [3] to see that this torus is the Jacobian of the curve

$$C_{35,A} : Y^2 = -(X+1)(8X+3)(10X^3 + 14X^2 + 6X + 1),$$

whose absolute Igusa invariants are

$$\{i_1, i_2, i_3\} = \left\{ -\frac{2^{23} 29^5}{5^5 7^5}, \frac{2^{17} 29^3 37 \cdot 83}{5^5 7^5}, \frac{2^8 29^2 83^2 1913}{5^5 7^5} \right\}.$$

As we mentioned before, this is the only rational curve whose Jacobian is isomorphic to A_f.

4.6. *Distinct genus two curves with isogenous Jacobians.* We now describe a method to find many nonisomorphic genus two curves with isogenous Jacobians. As before, we will work with the only two-dimensional factor A_f of $J_0(35)$. We shall consider abelian surfaces obtained as quotients A_f/G, where G is a finite rational torsion subgroup of A_f.

The existence of a nontrivial rational 2-torsion point on the Jacobian $J(C_{35,A})$ is evident from the equation of the curve $C_{35,A}$. Rational torsion points on A_f can also be found by means of cuspidal divisors on $X_0(35)$. We have the rational cusps $0, 1/5, 1/7, \infty$ of $X_0(35)$ at our disposal. Consider the cuspidal divisors $D_5 = (0)-(1/5)$, $D_7 = (0)-(1/7)$ and $D_\infty = (0)-(\infty)$ on $X_0(35)$. We denote by G the group generated by their projections on A_f.

The integrals $\int_\infty^0 h_j dq/q$ provide the projection of D_∞ on A_f: it is the point $(-0.677587\ldots, -0.084914\ldots)$, which corresponds to the *path* in $H_1(A_f,\mathbb{Q})$ given by $\frac{1}{8}\gamma_2 - \frac{1}{4}\gamma_3 + \frac{1}{8}\gamma_4 = -\frac{9}{8}\delta_2 + \frac{5}{8}\delta_3 + \frac{3}{4}\delta_4$. Similarly, we obtain

$$D_5 \leftrightarrow \tfrac{1}{8}\gamma_2 + \tfrac{1}{4}\gamma_3 + \tfrac{1}{8}\gamma_4 = -\tfrac{1}{8}\delta_2 + \tfrac{1}{8}\delta_3 + \tfrac{1}{4}\delta_4,$$
$$D_7 \leftrightarrow \tfrac{1}{2}\gamma_2 = -\tfrac{1}{2}\delta_2 + \tfrac{1}{2}\delta_3 + \tfrac{1}{2}\delta_4.$$

Thus $G \simeq \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is generated by the images of the divisors $D_5, D_7, D_5 - D_\infty$ (we shall denote these images by the same letters, since there is no risk of confusion). Note that all points in G are rational torsion points on A_f. For every subgroup G' of G the quotient abelian variety A_f/G' could admit, in principle, a principal polarization over $\mathbb{Q}$ since the degree of the polarization induced by $\mathcal{L}$ on A/G' is $2\cdot \#G' \in \operatorname{Norm}(K_f/\mathbb{Q})$. We will only examine the cyclic subgroups of G.

The first step to check whether A_f/G' is principally polarized is the determination of $\operatorname{End}_\mathbb{Q}(A_f/G')$. Consider the Hecke operator $v = 2u-1 = T_{11} + T_{18}$ corresponding to the fundamental unit of negative norm $4+\sqrt{17}$ in K_f. We have

$$v(D_\infty) = 3D_\infty, \quad v(D_5) = 3D_5, \quad v(D_7) = 3D_7,$$

i.e., v acts on G as the multiplication by 3 and hence leaves every subgroup of G stable. This implies that the order $\mathbb{Z}[\sqrt{17}]$ is contained in $\operatorname{End}_\mathbb{Q}(A/G')$ for all subgroups G' of G. Nevertheless, the action of u on G only leaves the following cyclic subgroups stable:

$G' = \langle P\rangle$	Order of P	$u(P)$
$\langle 4D_\infty\rangle$	2	0
$\langle 2D_\infty\rangle$	4	$4D_\infty$
$\langle D_5 + D_7\rangle$	8	$-2(D_5+D_7)$

For these three subgroups G', we can ensure that $\operatorname{End}_\mathbb{Q}(A/G') = \mathcal{O}$, and then Proposition 3.11 tells us that A/G' is principally polarized; we will show how to build the principal polarization for $A_f/\langle 4D_\infty\rangle$.

In order to do so, we consider the lattice

$$\Lambda' = \left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \tfrac{1}{2}\gamma_2 - \gamma_3 + \tfrac{1}{2}\gamma_4\right\rangle \subset H_1(A_f,\mathbb{Q}),$$

with basis $\gamma_1' = \frac{1}{2}(\gamma_2+\gamma_4)$, $\gamma_2' = \frac{1}{2}(\gamma_2 - \gamma_4)$, $\gamma_3' = \gamma_1'$, $\gamma_4' = \gamma_3$. The canonical polarization E on $J_0(35)$ provides a natural symplectic form on this lattice, which we shall denote by $E_{/\Lambda'}$. Its matrix with respect to the basis

$(\gamma_1', \gamma_2', \gamma_3', \gamma_4')$ is

$$M_{E_{/\Lambda'}} = \begin{pmatrix} 0 & -1/2 & -1/2 & -1/2 \\ 1/2 & 0 & -1/2 & 1/2 \\ 1/2 & 1/2 & 0 & -1 \\ 1/2 & -1/2 & 1 & 0 \end{pmatrix}.$$

The type of the corresponding primitive polarization on $A/\langle 4D_\infty\rangle$ is $(1,4)$. We now proceed as in the previous subsection to derive a principal polarization from $E_{/\Lambda'}$: we use the operator u^2 (of norm 4) to define the Riemann form E' determined by the matrix $A/\langle 4D_\infty\rangle$ given by the symplectic form

$$M_{E'} := M_{E_{/\Lambda'}} M_u^2 = \begin{pmatrix} 0 & -4 & -14 & -14 \\ 4 & 0 & 6 & 4 \\ 14 & -6 & 0 & -8 \\ 14 & -4 & 8 & 0 \end{pmatrix},$$

which is of type $(2,2)$, and hence it is the square of a principal polarization $\mathcal{L}_0'$. Computing periods and Jacobian Thetanullwerte, we find that $(A_f/\langle 4D_\infty\rangle, \mathcal{L}_0')$ is the Jacobian of the curve

$$C_{35,B} : Y^2 = -10X(4X+5)(5X+8)(25X^3+110X^2+156X+70),$$

with absolute Igusa invariants

$$\{i_1, i_2, i_3\} = \Big\{ -\frac{2^{13}43^5 359^5}{5^{17}7^5}, -\frac{2^{10}43^3 359^3 8933}{5^{13}7^5}, \\ -\frac{2^4 37 \cdot 43^2 359^2 571 \cdot 126949}{5^{13}7^5} \Big\}.$$

Applying the same procedure to the quotient $A/\langle 2D_5\rangle$, we arrive at the curve

$$C_{35,C} : Y^2 = -2(11X+16)(5X+8)(3X+5)(127X^3+598X^2+938X+490),$$

with absolute Igusa invariants

$$\{i_1, i_2, i_3\} = \Big\{ \frac{2^{28}89^5}{5^3 7^7}, -\frac{2^{19}11 \cdot 23 \cdot 89^3 1489}{5^3 7^7}, -\frac{2^{10}43 \cdot 89^2 2683 \cdot 11239}{5^3 7^7} \Big\}.$$

Finally, the quotient $A/\langle D_5 + D_7\rangle$ is $\mathbb{Q}$-isomorphic to the Jacobian of the curve

$$C_{35,D} : Y^2 = -142(8X+13)(X^2+4744X+3776) \\ \times (2173X^3+10154X^2+15820X+8218),$$

with absolute Igusa invariants

$$\{i_1, i_2, i_3\} = \Big\{ \frac{2^{13}109^5 214063^5}{5^2 7^5 71^{12}}, \frac{2^{12}11 \cdot 17 \cdot 109^3 5171 \cdot 214063^3}{5^2 7^5 71^8}, \\ \frac{2^4 3^5 109^2 33871 \cdot 214063^2 271175273}{5^2 7^5 71^8} \Big\}.$$

In conclusion, we have found four nonisomorphic curves $C_{35,A}$, $C_{35,B}$, $C_{35,C}$, $C_{35,D}$, whose Jacobians are pairwise isogenous.

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