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Generalizations of Apostol–Vu and Mordell–Tornheim multiple zeta functions

by

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1. Introduction. It was Matsumoto [2] who first introduced Euler–Zagier–Barnes multiple zeta functions

$$\begin{aligned}\zeta_{\mathrm{EZB},r}&((s_1,\dots,s_r);(\alpha_1,\dots,\alpha_r),(w_1,\dots,w_r))\\ &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1}(\alpha_2+m_1w_1+m_2w_2)^{-s_2}\\ &\quad\times\cdots\times(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s_r},\end{aligned}$$

where $s_1,\dots,s_r$ are complex variables and $\alpha_1,\dots,\alpha_r$, $w_1,\dots,w_r$ are complex parameters, including both Barnes multiple zeta functions

$$\zeta_{\mathrm{B},r}(s;\alpha,(w_1,\dots,w_r)) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha+m_1w_1+\cdots+m_rw_r)^{-s},$$

where s is a complex variable and $\alpha, w_1,\dots,w_r$ are complex parameters, and Euler–Zagier sums

$$\zeta_{\mathrm{EZ},r}(s_1,\dots,s_r) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}m_1^{-s_1}(m_1+m_2)^{-s_2}\times\cdots\times(m_1+\cdots+m_r)^{-s_r},$$

where $s_1,\dots,s_r$ are complex variables, as special cases. He proved the meromorphic continuation to the whole space, asymptotic expansions and upper bound estimates of Euler–Zagier–Barnes multiple zeta functions by using the Mellin–Barnes formula. He also obtained a recursive structure in the family of those multiple zeta functions by using the Mellin–Barnes integral.

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Apostol and Vu [1] first treated the following sum:

$$\sum_{m_1=1}^{\infty}\sum_{m_2>m_1} m_1^{-s_1}m_2^{-s_2}(m_1+m_2)^{-1}.$$

Matsumoto [3] introduced Apostol–Vu multiple zeta functions

$$\begin{aligned}
&\zeta_{\mathrm{AV},r}(s_1,\ldots,s_r;s_{r+1})\\
&\quad=\sum_{m_r>\cdots>m_1} m_1^{-s_1}m_2^{-s_2}\cdots m_r^{-s_r}(m_1+m_2+\cdots+m_r)^{-s_{r+1}}\\
&\quad=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} m_1^{-s_1}(m_1+m_2)^{-s_2}\times\cdots\times(m_1+\cdots+m_r)^{-s_r}\\
&\qquad\quad\times(rm_1+(r-1)m_2+\cdots+m_r)^{-s_{r+1}},
\end{aligned}$$

where $s_1,\ldots,s_{r+1}$ are complex variables, which generalizes the above sum. He also proved the recursive structure

$$\begin{gathered}
\zeta_{\mathrm{AV},r}=\varphi_{r,r}\to\varphi_{r-1,r}\to\cdots\to\varphi_{1,r}=\zeta_{\mathrm{EZ},r},\\
\zeta_{\mathrm{EZ},r}\to\zeta_{\mathrm{EZ},r-1}\to\cdots\to\zeta_{\mathrm{EZ},2}\to\zeta
\end{gathered}$$

in [3]. Hence the r-ple Apostol–Vu multiple zeta function can be reduced to the r-ple Euler–Zagier multiple zeta function. Here $\zeta_1\to\zeta_2$ means that ζ_1 can be expressed as a Mellin–Barnes integral involving ζ_2, ζ is the Riemann zeta function, and $\varphi_{i,r-1}$ is the auxiliary multiple series

$$\begin{aligned}
&\varphi_{i,r}(s_1,\ldots,s_i;s_{i+1},\ldots,s_r;s_{r+1})\\
&\quad=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} m_1^{-s_1}(m_1+m_2)^{-s_2}\times\cdots\times(m_1+\cdots+m_r)^{-s_r}\\
&\qquad\quad\times(rm_1+(r-1)m_2+\cdots+(r-i+1)m_i)^{-s_{r+1}},
\end{aligned}$$

where $s_1,\ldots,s_{r+1}$ are complex variables.

Tornheim [6] introduced the series

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} m_1^{-s_1}m_2^{-s_2}(m_1+m_2)^{-s_3},$$

where s_1,s_2,s_3 are complex variables, and studied its values when these complex variables are integers in the region of absolute convergence. Also Mordell [5] independently considered the case $s_1=s_2=s_3$ of the above sum, and studied the values of the following multiple sum:

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} m_1^{-1}\cdots m_r^{-1}(m_1+\cdots+m_r+a)^{-1},$$

where $s_1, \dots, s_r$ are complex variables and $a > -r$. Matsumoto [4] introduced Mordell–Tornheim multiple zeta functions

$$\zeta_{\mathrm{MT},r}(s_1, \dots, s_r; s_{r+1}) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} m_1^{-s_1} m_2^{-s_2} \cdots m_r^{-s_r} (m_1 + m_2 + \cdots + m_r)^{-s_{r+1}},$$

where $s_1, \dots, s_r$ are complex variables, which generalize the above sums. He also proved the recursive structure

$$\zeta_{\mathrm{MT},r} \to \zeta_{\mathrm{MT},r-1} \to \cdots \to \zeta_{\mathrm{MT},2} \to \zeta$$

in [3]. We can prove the meromorphic continuation of these functions to the whole space by going upstream these arrows.

The purpose of the present paper is to introduce generalizations of Apostol–Vu and Mordell–Tornheim multiple zeta functions, and discuss their analytic properties. Our proofs are analogous to those in Matsumoto [2], [3]. However, the recursive structure of Apostol–Vu multiple zeta functions given in our Theorem 5 is different from that of Matsumoto mentioned above.

2. Definitions and results on a generalization of Apostol–Vu multiple zeta functions

Definition 1. Let r be a positive integer, $s_1, \dots, s_{r+1}$ be complex variables, $\alpha_1, \dots, \alpha_{r+1}$, $w_1, \dots, w_r$ be complex parameters, and define

$$\begin{aligned} \tilde{\zeta}_{\mathrm{AV},r}((s_1, \dots, s_r; s_{r+1}); (\alpha_1, \dots, \alpha_r; \alpha_{r+1}), (w_1, \dots, w_r)) \\ = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} (\alpha_1 + m_1 w_1)^{-s_1} (\alpha_2 + m_1 w_1 + m_2 w_2)^{-s_2} \times \cdots \\ \times (\alpha_r + m_1 w_1 + \cdots + m_r w_r)^{-s_r} \\ \times (\alpha_{r+1} + r m_1 w_1 + (r-1) m_2 w_2 + \cdots + m_r w_r)^{-s_{r+1}}. \end{aligned} \tag{2.1}$$

If $\alpha_j + m_1 w_1 + \cdots + m_j w_j = 0$ for some j and some $(m_1, \dots, m_j)$ or $\alpha_{r+1} + r m_1 w_1 + \cdots + m_r w_r = 0$ for some $(m_1, \dots, m_r)$, then the corresponding terms are to be removed from the sum.

The multiple zeta functions of the form (2.1) generalize both Apostol–Vu multiple zeta functions (the case $w_i = 1$ for $1 \le i \le r$, $\alpha_j = j$ for $1 \le j \le r$, $\alpha_{r+1} = r(r+1)/2$) and Euler–Zagier–Barnes multiple zeta functions (the case $s_{r+1} = 0$).

Next let ℓ be a fixed line on the complex plane $\mathbb{C}$ through the origin. Then ℓ divides $\mathbb{C}$ into three parts; two open half-planes and ℓ itself. Let $H(\ell)$ be one of those half-planes, and assume that

$$w_j \in H(\ell) \quad (1 \le j \le r). \tag{2.2}$$

Theorem 1. *If condition* (2.2) *holds*, *then the series* (2.1) *is absolutely convergent in the region*

$$A_r = \{(s_1,\dots,s_r,s_{r+1}) \in \mathbb{C}^{r+1} \mid \Re(s_{r-k+1}+\cdots+s_{r+1}) > k\ (0\le k\le r)\},$$

uniformly in any compact subset of A_r.

This theorem is analogous to Theorem 3 of [4].

We assume

$$\alpha_j \in H(\ell) \quad (1\le j\le r) \tag{2.3}$$

and

$$\alpha_{j+1}-\alpha_j \in H(\ell) \quad (1\le j\le r). \tag{2.4}$$

By (2.2) and (2.3) we see that always $\alpha_j + m_1w_1+\cdots+m_jw_j \in H(\ell)$ and $\alpha_{r+1}+rm_1w_1+\cdots+m_rw_r \in H(\ell)$. Let $\theta\in(-\pi,\pi]$ be the argument of the vector contained in $H(\ell)$ and orthogonal to ℓ. Then the line ℓ consists of the points whose arguments are $\theta\pm\pi/2$, and

$$H(\ell) = \{w\in\mathbb{C}\setminus\{0\} \mid \theta-\pi/2 < \arg w < \theta+\pi/2\}.$$

The branch of the logarithm in each factor

$$(\alpha_j+m_1w_1+\cdots+m_jw_j)^{-s_j} = \exp(-s_j\log(\alpha_j+m_1w_1+\cdots+m_jw_j))$$

and

$$(\alpha_{r+1}+rm_1w_1+\cdots+m_rw_r)^{-s_{r+1}} \\ = \exp(-s_{r+1}\log(\alpha_{r+1}+rm_1w_1+\cdots+m_rw_r))$$

on the right-hand side of (2.1) is to be chosen by the condition

$$\theta-\pi/2 < \arg(\alpha_j+m_1w_1+\cdots+m_jw_j) < \theta+\pi/2$$

and

$$\theta-\pi/2 < \arg(\alpha_{r+1}+rm_1w_1+\cdots+m_rw_r) < \theta+\pi/2.$$

Definition 2. For any complex numbers α and β we use the notation

$$\rho(\alpha,\beta) = \max\{|\arg\alpha|, |\arg\beta|\}.$$

Put $a_0 = 0$, $\Re s_i = \sigma_i$ and $\Im s_i = t_i$. Also denote by $\mathbb{N}_0$ the set of all nonnegative integers and by $d_r(\,,)$ the Euclidean metric on $\mathbb{C}^{r+1}$. The letter ϵ denotes an arbitrarily small positive number, not necessarily the same at each occurrence.

Theorem 2. *Let r be a positive integer. If $r\ge 2$, we assume* (2.2), (2.3), (2.4),

$$\rho(\alpha_i-\alpha_{i-1}, w_i) < \pi/2 \quad (1\le i\le r) \tag{2.5}$$

and

$$\alpha_{r+1} - \alpha_r - \cdots - \alpha_1 = 0. \tag{2.6}$$

Also, *if* $r = 1$, *we assume* (2.2), (2.3), (2.5) *and*

$$\alpha_2 - \alpha_1 = 0. \tag{2.7}$$

Then the following assertions hold:

(1) *The series* (2.1) *can be continued meromorphically*, *as a function of* $s_1, \ldots, s_r, s_{r+1}$, *to the whole* $\mathbb{C}^{r+1}$.

(2) *The function* $\tilde{\zeta}_{\mathrm{AV},r}$ *is holomorphic except for possible singularities in*

$$\begin{aligned} &\mathrm{Sing}(r) \\ &\quad = \bigcup_{n\in\mathbb{N}_0} \bigcup_{i=1}^{r} \{(s_1, \ldots, s_r, s_{r+1}) \in \mathbb{C}^{r+1} \mid s_i + \cdots + s_{r+1} = r + 1 - i - n\}. \end{aligned} \tag{2.8}$$

(3) *Fix* $s_1, \ldots, s_{r-1}$, *let* σ', σ'', λ' *and* λ'' *be real numbers with* $\sigma' < \sigma''$, $\lambda' < \lambda''$, *and* η *be a small positive number. Then*

$$\begin{aligned} &\tilde{\zeta}_{\mathrm{AV},r}((s_1, \ldots, s_r; s_{r+1}); (\alpha_1, \ldots, \alpha_r; \alpha_{r+1}), (w_1, \ldots, w_r)) \\ &\qquad = O\Big(\sum_{j=1}^{r} (|t_{r+1}| + 1)^{f(j,r,\sigma_1,\ldots,\sigma_{r+1})} \exp(|t_{r+1}|\rho(\alpha_j - \alpha_{j-1}, w_j)) \\ &\qquad\qquad \times \sum_{i=1}^{r} (|t_r| + 1)^{g(i,r,\sigma_1,\ldots,\sigma_{r+1})} \exp(|t_r|\rho(\alpha_i - \alpha_{i-1}, w_i))\Big) \end{aligned}$$

for any s_{r+1}, s_r *with* $d_r((s_1, \ldots, s_r, s_{r+1}), \mathrm{Sing}(r)) \geq \eta$, $\sigma' \leq \sigma_{r+1} \leq \sigma''$ *and* $\lambda' \leq \sigma_r \leq \lambda''$, *where the implied constant depends only on* $\alpha_j - \alpha_{j-1}$ $(1 \leq j \leq r)$, w_j $(1 \leq j \leq r)$, r, $\sigma_1, \ldots, \sigma_{r-1}$, $t_1, \ldots, t_{r-1}$, $\sigma', \sigma'', \eta, \epsilon, \lambda'$ *and* λ''. *The quantities* $f(j, r, \sigma_1, \ldots, \sigma_{r+1})$ *and* $g(i, r, \sigma_1, \ldots, \sigma_{r+1})$ *can be written down explicitly.*

Our proof of Theorem 2 will be presented in Sections 5 and 6.

Remark. In the above theorem we assume (2.6) and (2.7), but actually we can weaken the assumption to

$$\alpha_{r+1} - \alpha_r - \cdots - \alpha_1 \in H(\ell) \cup \{0\}. \tag{2.9}$$

However, we then also assume

$$|\arg(\alpha_{r+1} - \alpha_r - \cdots - \alpha_1)| < \pi/2. \tag{2.10}$$

Hence if $r \geq 2$, we assume (2.2)–(2.5), (2.9) and (2.10), while if $r = 1$, we assume (2.2), (2.3), (2.5), (2.9) and (2.10). Then (3) of Theorem 2 is to be changed to

(3′) *Fix* $s_1,\dots,s_{r-1}$ *and let* $\sigma',\sigma'',\lambda'$ *and* λ'' *be real numbers with* $\sigma'<\sigma''$, $\lambda'<\lambda''$, *and* η *be a small positive number. Then*

$$\begin{aligned}
&\tilde{\zeta}_{\mathrm{AV},r}((s_1,\dots,s_r;s_{r+1});(\alpha_1,\dots,\alpha_r;\alpha_{r+1}),(w_1,\dots,w_r))\\
&\quad= O\Big(\Big\{\sum_{j=1}^{r}(|t_{r+1}|+1)^{f(j,r,\sigma_1,\dots,\sigma_{r+1})}\exp(|t_{r+1}|\rho(\alpha_j-\alpha_{j-1},w_j))\\
&\qquad\quad+(|t_{r+1}|+1)^{f(r+1,r,\sigma_1,\dots,\sigma_{r+1})}\exp(|t_{r+1}|\,|\arg(\alpha_{r+1}-\alpha_r-\cdots-\alpha_1)|)\Big\}\\
&\qquad\quad\times\sum_{i=1}^{r}(|t_r|+1)^{g(i,r,\sigma_1,\dots,\sigma_{r+1})}\exp(|t_r|\rho(\alpha_i-\alpha_{i-1},w_i))\Big)
\end{aligned}$$

for any s_{r+1}, s_r *with* $d_r((s_1,\dots,s_r,s_{r+1}),\mathrm{Sing}(r))\ge\eta$, $\sigma'\le\sigma_{r+1}\le\sigma''$ *and* $\lambda'\le\sigma_r\le\lambda''$, *where the implied constant depends only on* $\alpha_j-\alpha_{j-1}$ $(1\le j\le r)$, w_j $(1\le j\le r)$, $\alpha_{r+1}-\alpha_r-\cdots-\alpha_1, r, \sigma_1,\dots,\sigma_{r-1},t_1,\dots,t_{r-1},\sigma',\sigma'',\eta,\epsilon,\lambda'$ *and* λ''. *The quantities* $f(j,r,\sigma_1,\dots,\sigma_{r+1})$, $f(r+1,r,\sigma_1,\dots,\sigma_{r+1})$ *and* $g(i,r,\sigma_1,\dots,\sigma_{r+1})$ *can be written down explicitly.*

The proof of (3′) is similar to that of (3) of Theorem 2.

THEOREM 3. *Let* r *be a positive integer. Then the assertions* (1) *and* (2) *of Theorem* 2 *hold without the assumption* (2.5).

This theorem is analogous to Theorem 1 of [2].

THEOREM 4. *Let* $r\ge 2$ *be a positive integer and assume* (2.2)–(2.6) *and*

$$\alpha_r-\alpha_{r-1}=bw_r \tag{2.11}$$

with a constant b *satisfying* $|\arg b|<\pi$. *Then, for any positive integers* M *and* N *satisfying* $M\le N$, *the asymptotic expansion*

$$\begin{aligned}
&\tilde{\zeta}_{\mathrm{AV},r}((s_1,\dots,s_r;s_{r+1});(\alpha_1,\dots,\alpha_r;\alpha_{r+1}),(w_1,\dots,w_r))\\
&\qquad=\sum_{j=0}^{M-1}\binom{-s_{r+1}}{j}\Big\{-\frac{1}{1-s_{r+1}-s_r-j}\\
&\qquad\quad\times\tilde{\zeta}_{\mathrm{AV},r-1}((s_1,\dots,s_{r-2},s_{r-1}+s_r+s_{r+1}+j-1;-j);\\
&\qquad\qquad\qquad(\alpha_1,\dots,\alpha_{r-1};\alpha_{r+1}-\alpha_r),(w_1,\dots,w_r))w_r^{-1}\\
&\qquad\quad+\sum_{k=0}^{N-1}\binom{-s_{r+1}-s_r-j}{k}\\
&\qquad\quad\times\tilde{\zeta}_{\mathrm{AV},r-1}((s_1,\dots,s_{r-2},s_{r-1}+s_r+s_{r+1}+j+k;-j);\\
&\qquad\qquad\qquad(\alpha_1,\dots,\alpha_{r-1};\alpha_{r+1}-\alpha_r),(w_1,\dots,w_r))\\
&\qquad\quad\times\zeta(-k,b)w_r^k+O(|w_r|^N)\Big\}+O(|w_r|^{-1})
\end{aligned}$$

holds for $|w_r|\le 1$ *in the whole* $\mathbb{C}^{r+1}$ *except for the singularities.*

Our proof of Theorem 4 will be presented in Section 5.

Remark 1. Let s, α be complex numbers, $\alpha \neq -l$ ($l \in \mathbb{N}_0$). Then the Hurwitz zeta function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} (\alpha + m)^{-s},$$

where

$$(\alpha + m)^{-s} = \exp(-s \log(\alpha + m))$$

with the branch $-\pi < \arg(\alpha + m) \leq \pi$, is well defined and is absolutely convergent if $\Re s > 1$.

Remark 2. Let v be a complex number and n be a nonpositive integer. Then we put

$$\binom{v}{n} = \begin{cases} v(v-1)\cdots(v-n+1)/n! & \text{if } n \text{ is a positive integer,} \\ 1 & \text{if } n = 0. \end{cases}$$

Also we obtain the following theorem from the proof of Theorem 2(1), (2) which will be presented in Section 5.

Theorem 5. *The family of the functions* $\tilde{\zeta}_{\mathrm{AV},r}$ *has the following recursive structure*:

$$\tilde{\zeta}_{\mathrm{AV},r} \to \tilde{\zeta}_{\mathrm{AV},r-1} \to \cdots \to \tilde{\zeta}_{\mathrm{AV},2} \to \tilde{\zeta}_{\mathrm{AV},1}.$$

In particular, *Apostol–Vu multiple zeta functions have the following recursive structure which only features Apostol–Vu multiple zeta functions*:

$$\zeta_{\mathrm{AV},r} \to \zeta_{\mathrm{AV},r-1} \to \cdots \to \zeta_{\mathrm{AV},2} \to \zeta.$$

In this notation, $\zeta_1 \to \zeta_2$ means that ζ_1 can be expressed as a Mellin–Barnes integral involving ζ_2.

3. Definitions and results on generalization of Mordell–Tornheim multiple zeta functions. In this section we consider general multiple zeta functions of multi-variables, including Mordell–Tornheim multiple zeta functions.

Definition 3. Let r be a positive integer, $s_1, \ldots, s_{r+1}$ be complex variables, $\alpha_1, \ldots, \alpha_{r+1}$, $w_1, \ldots, w_r$ be complex parameters, and define

$$\begin{aligned} (3.1)\quad & \tilde{\zeta}_{\mathrm{MT},r}((s_1, \ldots, s_r; s_{r+1}); (\alpha_1, \ldots, \alpha_r; \alpha_{r+1}), (w_1, \ldots, w_r)) \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} (\alpha_1 + m_1 w_1)^{-s_1} (\alpha_2 + m_2 w_2)^{-s_2} \times \cdots \times (\alpha_r + m_r w_r)^{-s_r} \\ &\quad \times (\alpha_{r+1} + m_1 w_1 + m_2 w_2 + \cdots + m_r w_r)^{-s_{r+1}}. \end{aligned}$$

The class of multiple zeta functions of the form (3.1) includes a well-known class of multiple zeta functions, Mordell–Tornheim multiple zeta functions (the case $w_i = 1$ for $1 \le i \le r$, $\alpha_j = 1$ for $1 \le j \le r$, $\alpha_{r+1} = r$). The Mordell–Tornheim multiple zeta functions were introduced in [2].

The following Theorems 6–8 are analogous to Theorems 1–3. The proofs are similar to those of Theorem 3 of [4], Theorem 2 of [3] and Theorem 1 of [2] and we omit them.

THEOREM 6. *If condition* (2.2) *holds*, *then the series* (3.1) *is absolutely convergent in the region*

$$B_r = \{(s_1, \dots, s_r, s_{r+1}) \in \mathbb{C}^{r+1} \mid \Re s_k > 1\,(1 \le k \le r),\ \Re s_{r+1} > 0\},$$

uniformly in any compact subset of B_r.

THEOREM 7. *Let* r *be a positive integer. If* $r \ge 2$, *assume* (2.2)–(2.6), *and if* $r = 1$, *assume* (2.2), (2.3), (2.5) *and* (2.7). *Then the following assertions hold*:

(1) *The series* (3.1) *can be continued meromorphically*, *as a function of* $s_1, \dots, s_r, s_{r+1}$, *to the whole* $\mathbb{C}^{r+1}$.

(2) *The function* $\tilde{\zeta}_{\mathrm{MT},r}$ *is holomorphic except for possible singularities in*

$$\begin{aligned}
&s_j + s_{r+1} = 1 - l \ (1 \le j \le r,\ l \in \mathbb{N}_0),\\
&s_{j_1} + s_{j_2} + s_{r+1} = 2 - l \ (1 \le j_1 < j_2 \le r,\ l \in \mathbb{N}_0),\\
&\qquad\qquad \cdots\\
&s_{j_1} + \cdots + s_{j_{r-1}} + s_{r+1} = r - 1 - l \ (1 \le j_1 < \cdots < j_{r-1} \le r,\ l \in \mathbb{N}_0),\\
&s_1 + \cdots + s_r + s_{r+1} = r.
\end{aligned}$$

(3) *Fix* $s_1, \dots, s_r$, *let* σ', σ'' *be real numbers with* $\sigma' < \sigma''$, η *be a small positive number*, *and* $\mathrm{Sing}(r)$ *be the above set of possible singularities. Then*

$$\begin{aligned}
&\tilde{\zeta}_{\mathrm{MT},r}((s_1, \dots, s_r; s_{r+1}); (\alpha_1, \dots, \alpha_r; \alpha_{r+1}), (w_1, \dots, w_r))\\
&\qquad\qquad = O\Big(\sum_{j=1}^{r} (|t_{r+1}| + 1)^{f(j,r,\sigma_1,\dots,\sigma_{r+1})} \exp(|t_{r+1}|\rho(\alpha_j, w_j))\Big)
\end{aligned}$$

for any s_{r+1} *with* $\sigma' \le \sigma_{r+1} \le \sigma''$ *and* $d_r((s_1, \dots, s_r, s_{r+1}), \mathrm{Sing}(r)) \ge \eta$, *where the implied constant depends only on* $\alpha_j - \alpha_{j-1}$ $(1 \le j \le r)$, w_j $(1 \le j \le r)$, r, $\sigma_1, \dots, \sigma_r$, $t_1, \dots, t_r$, σ', σ'', η *and* ϵ. *The quantities* $f(j, r, \sigma_1, \dots, \sigma_{r+1})$ *can be written down explicitly.*

REMARK. In the above theorem we assume (2.6) and (2.7), but actually we can weaken the assumption to (2.9). Hence if $r \ge 2$, we assume (2.2)–(2.5), (2.9) and (2.10), while if $r = 1$, we assume (2.2), (2.3), (2.5), (2.9) and (2.10). Then (3) of Theorem 7 is to be changed to

(3′) *Fix* $s_1,\dots,s_r$, *let* σ',σ'' *be real numbers with* $\sigma'<\sigma''$, *and* η *be a small positive number. Then*

$$\begin{aligned}&\tilde{\zeta}_{\mathrm{MT},r}((s_1,\dots,s_r;s_{r+1});(\alpha_1,\dots,\alpha_r;\alpha_{r+1}),(w_1,\dots,w_r))\\&\quad=O\Big\{\sum_{j=1}^{r}(|t_{r+1}|+1)^{f(j,r,\sigma_1,\dots,\sigma_{r+1})}\exp(|t_{r+1}|\rho(\alpha_j,w_j))\\&\qquad+(|t_{r+1}|+1)^{f(r+1,r,\sigma_1,\dots,\sigma_{r+1})}\exp(|t_{r+1}|\,|\arg(\alpha_{r+1}-\alpha_r-\dots-\alpha_1)|)\Big\}\end{aligned}$$

for any s_{r+1} *with* $\sigma'\le\sigma_{r+1}\le\sigma''$ *and* $d_r((s_1,\dots,s_r,s_{r+1}),\mathrm{Sing}(r))\ge\eta$, *where the implied constant depends only on* $\alpha_j-\alpha_{j-1}$ $(1\le j\le r)$, w_j $(1\le j\le r)$, $\alpha_{r+1}-\alpha_r-\dots-\alpha_1$, r, $\sigma_1,\dots,\sigma_r$, $t_1,\dots,t_r$, σ',σ'',η *and* ϵ. *The quantities* $f(j,r,\sigma_1,\dots,\sigma_{r+1})$ *and* $f(r+1,r,\sigma_1,\dots,\sigma_{r+1})$ *can be written down explicitly.*

THEOREM 8. *Let* r *be a positive integer. Then the assertions* (1) *and* (2) *of Theorem* 7 *hold without the assumption* (2.5).

4. Lemmas. First of all we introduce lemmas on Hurwitz zeta functions. The following Lemmas 1 and 2 will be used to prove Theorems 2 and 4.

LEMMA 1. *Let* $\alpha\in\mathbb{C}\setminus\{-l\}$ $(l\in\mathbb{N}_0)$. *The function* $\zeta(s,\alpha)$ *can be continued meromorphically to the whole complex plane, and the only pole is at* $s=1$.

LEMMA 2. *Let* σ',σ'' *be real numbers satisfying* $\sigma'<\sigma''$, η *be a small fixed positive number, and* $\alpha,w\in H(\ell)$ *so* $|\arg(\alpha/w)|<\pi$. *Then*

$$\zeta(s,\alpha/w)w^{-s}=O(|w|^{-\sigma}(|t|+1)^{\max\{0,1-\sigma\}+\epsilon}\exp(|t|\rho(\alpha,w)))\tag{4.1}$$

for any $s=\sigma+it$ *with* $\sigma'\le\sigma\le\sigma''$ *and* $|s-1|\ge\eta$, *where the implied constant depends only on* $\alpha/w,\sigma',\sigma'',\eta$ *and* ϵ.

Next we give lemmas on the order estimate. We will use the following lemmas to prove Theorem 2(3).

Let p,q,A,B,t be real numbers with $A+B<0$, and

$$I=\int_{-\infty}^{\infty}(|t+y|+1)^p(|y|+1)^q\exp(A|t+y|+B|y|)\,dy.$$

LEMMA 3. *We have*

$$I=O((1+\tau^p)\tau^{q+\delta}e^{B|t|}+(1+\tau^p)e^{A|t|}),\tag{4.2}$$

where $\tau=|t|+1$, $\delta=1$ *or* 0 *according as* $A=B$ *or* $A\ne B$, *and the implied constant depends only on* p,q,A *and* B.

Lemmas 1–3 are proved in [2].

Next we define

$$J=\int_{-\infty}^{\infty}(|t_2+t_1+y|+1)^p(|t_2+y|+1)^q(|y|+1)^r \times \exp(A|t_2+t_1+y|+B|t_2+y|+C|y|)\,dy,$$

where t_1,t_2,p,q,r,A,B,C are real numbers with $A\geq 0$ and $A+B+C<0$.

LEMMA 4. *We have*

$$J=O(e^{A|t_1|}\max\{1,\tau^p\}(1+\nu^{p+q}+\nu^q)\{\nu^{r+\delta'}e^{C|t_2|}+e^{(A+B)|t_2|}\}), \tag{4.3}$$

where $\nu=|t_2|+1$, $\tau=|t_1|+1$, $\delta'=1$ *or* 0 *according as* $A+B=C$ *or* $A+B\neq C$, *and the implied constant depends only on* p,q,r,A,B *and* C.

The proof of Lemma 4 is straightforward and is left to the reader (see Section 2 of [2]). Lemma 4 yields

COROLLARY 1. *We have*

$$J=O((1+\nu^{p+q}+\nu^q)\{\nu^{r+\delta'}e^{C|t_2|}+e^{(A+C)|t_2|}\}), \tag{4.4}$$

where $\nu=|t_2|+1$, $\delta'=1$ *or* 0 *according as* $A+B=C$ *or* $A+B\neq C$, *and the implied constant depends only on* t_1,p,q,r,A,B *and* C.

Lastly we give the Mellin–Barnes integral formula.

LEMMA 5 (The Mellin–Barnes integral formula). *Let* s,λ *be complex numbers*, $\Re s>0$, $|\arg\lambda|<\pi$ *and* $\lambda\neq 0$. *We have*

$$\Gamma(s)(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\Gamma(s+z)\Gamma(-z)\lambda^z\,dz, \tag{4.5}$$

where $-\Re s<c<0$ *and the path of integration is the vertical line* $\Re z=c$.

For the proof we refer to [7, Section 14.51, p. 289, Corollary].

5. Proof of Theorem 2(1), (2) and Theorem 4. The most novel part of the present paper is to obtain a new recursive structure in the family of Apostol–Vu multiple zeta functions. Hence we give all the details of how to use the Mellin–Barnes integral formula for this proof, and for the rest of the proof we refer to [2]. In this section we prove (1) and (2) of Theorem 2 by induction and Theorem 4. First assume that $\Re s_j=\sigma_j>1$ $(1\leq j\leq r)$ and $\Re s_{r+1}=\sigma_{r+1}>0$.

When $r=1$, we only assume (2.2), (2.3), (2.5) and (2.7). Then we have

$$\begin{aligned}(5.1)\qquad &\tilde{\zeta}_{\mathrm{AV},1}((s_1;s_2);(\alpha_1;\alpha_2),w_1)\\ &=\sum_{m_1=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1}(\alpha_2+m_1w_1)^{-s_2}=\sum_{m_1=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1-s_2}\\ &=\sum_{m_1=0}^{\infty}(m_1+\alpha_1/w_1)^{-s_1-s_2}w_1^{-s_1-s_2}.\end{aligned}$$

Hence we have proved (1) and (2) of Theorem 2 for $r=1$ by Lemma 1.

Next we assume the validity of Theorem 2(1), (2) for $\tilde{\zeta}_{\mathrm{AV},r-1}$, and we will prove these assertions and Theorem 4 for $\tilde{\zeta}_{\mathrm{AV},r}$. The validity of Theorem 2(3), will be proved inductively in the next section.

We have

$$\begin{aligned}&\tilde{\zeta}_{\mathrm{AV},r}((s_1,\dots,s_r;s_{r+1});(\alpha_1,\dots,\alpha_r;\alpha_{r+1}),(w_1,\dots,w_r))\\ &=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1}(\alpha_2+m_1w_1+m_2w_2)^{-s_2}\times\cdots\\ &\quad\times(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s_r}(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s_{r+1}}\\ &\quad\times\left(1+\frac{\alpha_{r+1}-\alpha_r+(r-1)m_1w_1+(r-2)m_2w_2+\cdots+m_{r-1}w_{r-1}}{\alpha_r+m_1w_1+m_2w_2+\cdots+m_rw_r}\right)^{-s_{r+1}}.\end{aligned}$$

Put

$$\lambda=\frac{\alpha_{r+1}-\alpha_r+(r-1)m_1w_1+(r-2)m_2w_2+\cdots+m_{r-1}w_{r-1}}{\alpha_r+m_1w_1+m_2w_2+\cdots+m_rw_r}.$$

By (2.2)–(2.4), both the numerator and the denominator of λ are elements of $H(\ell)$, hence $\lambda\neq 0$ and $|\arg\lambda|<\pi$. Therefore we can use Lemma 5 to get

$$\begin{aligned}(5.2)\qquad &\tilde{\zeta}_{\mathrm{AV},r}((s_1,\dots,s_r;s_{r+1});(\alpha_1,\dots,\alpha_r;\alpha_{r+1}),(w_1,\dots,w_r))\\ &=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1}\\ &\quad\times(\alpha_2+m_1w_1+m_2w_2)^{-s_2}\times\cdots\times(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s_r-s_{r+1}-z}\\ &\quad\times(\alpha_{r+1}-\alpha_r+(r-1)m_1w_1+(r-2)m_2w_2+\cdots+m_{r-1}w_{r-1})^{z}\,dz.\end{aligned}$$

We may assume

$$(5.3)\qquad \max_{0\le i\le r}\{i-\Re(s_{r+1}+s_r+\cdots+s_{r+1-i})\}<c<0.$$

We put

$$\begin{aligned}
&\tilde{\zeta}^{*}_{\mathrm{AV},r}((s_1,\ldots,s_{r-1},-z;s_r+s_{r+1}+z);(\alpha_1,\ldots,\alpha_{r-1},\alpha_{r+1}-\alpha_r;\alpha_r),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(w_1,\ldots,w_r))\\
&\quad=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1}(\alpha_2+m_1w_1+m_2w_2)^{-s_2}\times\cdots\\
&\qquad\times(\alpha_{r-1}+m_1w_1+\cdots+m_{r-1}w_{r-1})^{-s_{r-1}}\\
&\qquad\times(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s_r-s_{r+1}-z}\\
&\qquad\times(\alpha_{r+1}-\alpha_r+(r-1)m_1w_1+(r-2)m_2w_2+\cdots+m_{r-1}w_{r-1})^{z}.
\end{aligned}$$

From now on we study the analytic properties of the function $\tilde{\zeta}^{*}_{\mathrm{AV},r}$ as a function in z. First we have

$$\begin{aligned}
\alpha_r+m_1w_1+\cdots+m_rw_r&=(\alpha_{r-1}+m_1w_1+\cdots+m_{r-1}w_{r-1})\\
&\qquad\times\left(1+\frac{\alpha_r-\alpha_{r-1}+m_rw_r}{\alpha_{r-1}+m_1w_1+\cdots+m_{r-1}w_{r-1}}\right).
\end{aligned}$$

We put

$$\lambda'=\frac{\alpha_r-\alpha_{r-1}+m_rw_r}{\alpha_{r-1}+m_1w_1+\cdots+m_{r-1}w_{r-1}}.$$

By (2.2)–(2.4), both the numerator and the denominator of λ' are elements of $H(\ell)$, hence $\lambda'\neq 0$ and $|\arg\lambda'|<\pi$. Therefore we can use Lemma 5 again to get

$$\begin{aligned}
&\tilde{\zeta}^{*}_{\mathrm{AV},r}((s_1,\ldots,s_{r-1},-z;s_r+s_{r+1}+z);(\alpha_1,\ldots,\alpha_{r-1},\alpha_{r+1}-\alpha_r;\alpha_r),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(w_1,\ldots,w_r))\\
&\quad=\frac{1}{2\pi i}\int_{(c')}\frac{\Gamma(s_r+s_{r+1}+z+z')\Gamma(-z')}{\Gamma(s_r+s_{r+1}+z)}\,\zeta\left(-z',\frac{\alpha_r-\alpha_{r-1}}{w_r}\right)w_r^{z'}\\
&\qquad\times\tilde{\zeta}_{\mathrm{AV},r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+s_{r+1}+z+z';-z);\\
&\qquad\qquad\qquad(\alpha_1,\ldots,\alpha_{r-1};\alpha_{r+1}-\alpha_r),(w_1,\ldots,w_{r-1}))\,dz'.
\end{aligned}$$

We may assume

$$-\Re(s_r+s_{r+1}+z)<c'<-1.$$

Now we shift the path to $\Re z'=N-\epsilon$, where N is a positive integer and ϵ is a small positive number. The legitimacy of this shifting is easily shown by using the Stirling formula and Lemma 2. Counting the residues of the poles, we get

$$\tag{5.4} \begin{aligned} &\tilde{\zeta}^*_{\mathrm{AV},r}((s_1,\ldots,s_{r-1},-z;s_r+s_{r+1}+z);\\ &\qquad\qquad\qquad(\alpha_1,\ldots,\alpha_{r-1},\alpha_{r+1}-\alpha_r;\alpha_r)(w_1,\ldots,w_r))\\ &= -\frac{1}{1-s_r-s_{r+1}-z}\\ &\quad\times\tilde{\zeta}_{\mathrm{AV},r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+s_{r+1}+z-1;-z);\\ &\qquad\qquad\qquad(\alpha_1,\ldots,\alpha_{r-1};\alpha_{r+1}-\alpha_r),(w_1,\ldots,w_{r-1}))w_r^{-1}\\ &\quad+\sum_{k=0}^{N-1}\binom{-s_r-s_{r+1}-z}{k}\zeta\Big(-k,\frac{\alpha_r-\alpha_{r-1}}{w_r}\Big)w_r^k\\ &\quad\times\tilde{\zeta}_{\mathrm{AV},r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+s_{r+1}+z+k;-z);\\ &\qquad\qquad\qquad(\alpha_1,\ldots,\alpha_{r-1};\alpha_{r+1}-\alpha_r),(w_1,\ldots,w_{r-1}))\\ &\quad+S_{r-1,N} \end{aligned}$$

where

$$\begin{aligned} S_{r-1,N}=\frac{1}{2\pi i}\int\limits_{(N-\epsilon)}&\frac{\Gamma(s_r+s_{r+1}+z+z')\Gamma(-z')}{\Gamma(s_r+s_{r+1}+z)}\zeta\Big(-z',\frac{\alpha_r-\alpha_{r-1}}{w_r}\Big)w_r^{z'}\\ &\times\tilde{\zeta}_{\mathrm{AV},r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+s_{r+1}+z+z';-z);\\ &\qquad\qquad(\alpha_1,\ldots,\alpha_{r-1};\alpha_{r+1}-\alpha_r),(w_1,\ldots,w_{r-1}))\,dz', \end{aligned}$$

under the conditions

$$\tag{5.5} \Re(s_{r-k}+\cdots+s_{r-1}+s_r+s_{r+1})>k-N+\epsilon\quad(1\le k\le r-1)$$

and

$$\tag{5.6} x=\Re z>-\Re(s_r+s_{r+1})-N+\epsilon.$$

Moreover, we assume

$$\tag{5.7} s_i+\cdots+s_{r-1}+s_r+s_{r+1}\ne r+1-i-l\quad(1\le i\le r-1,\ l\in\mathbb{N}_0)$$

because if $s_i+\cdots+s_{r-1}+s_r+s_{r+1}=r+1-i-l$ then the right-hand side of (5.4) is singular. Since $S_{r-1,N}$ is holomorphic in z under condition (5.7), we can see from (5.5) that the only pole of $\tilde{\zeta}^*_{\mathrm{AV},r}$, as a function of z, in the region (5.7) is at $z=1-s_r-s_{r-1}$.

Next we estimate $S_{r-1,N}$. We put

$$g(i,r-1,\sigma_1,\ldots,\sigma_{r-1}+\sigma_r+\sigma_{r+1}+h,x)=g(i,r-1,h,x),$$
$$f(j,r-1,\sigma_1,\ldots,\sigma_{r-1}+\sigma_r+\sigma_{r+1}+h,x)=f(j,r-1,h,x),$$
$$\mu=|y+t_r+t_{r+1}|+1,\ p_i=g(i,r-1,x+N-\epsilon,x)\ (1\le i\le r-1),$$

$q=\sigma_r+\sigma_{r+1}+x+N-\epsilon-1/2$, $r=-N+\epsilon-1/2+\max\{\epsilon,1-N+2\epsilon\}$, $z=x+iy$ and $z'=x'+iy'$, where $x,y,x',y'\in\mathbb{R}$, and we assume $x_0\le x'\le x_1$, where x_0, x_1 are (arbitrarily) fixed real numbers. Similarly to Sections 4 and 5 of [2], we obtain

$$
\begin{aligned}
(5.8)\qquad & S_{r-1,N} \\
&= O\Big(|w_r|^{N-\epsilon}\sum_{j=1}^{r-1}(|y|+1)^{f(j,r-1,x+N-\epsilon,x)}\exp(|y|\rho(\alpha_j-\alpha_{j-1},w_j)) \\
&\qquad \times\sum_{i=1}^{r}\mu^{h(i,r-1,x)}\exp(|t_r+t_{r+1}+y|\rho(\alpha_i-\alpha_{i-1},w_i))\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
h(i,r-1,x) = &-\sigma_r-\sigma_{r+1}-x+1/2 \\
&+\max_{1\le i\le r-1}\{0,q,r+\delta',p_i+q,p_i+q+r+\delta',q+r+\delta'\}.
\end{aligned}
$$

We note that the implied constant is independent of y.

We evaluate (5.4) with Lemma 2, (5.8) and assumption (3) of Theorem 2 for $\tilde{\zeta}_{\mathrm{AV},r-1}$ and, by (5.4) and the Stirling formula, we obtain

$$
\begin{aligned}
(5.9)\qquad & \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\tilde{\zeta}^{*}_{\mathrm{AV},r}((s_1,\ldots,s_{r-1},-z;s_r+s_{r+1}+z); \\
&\qquad\qquad (\alpha_1,\ldots,\alpha_{r-1},\alpha_{r+1}-\alpha_r;\alpha_r),(w_1,\ldots,w_r)) \\
&\ll \exp\Big(-\frac{\pi}{2}(|y+t_{r+1}|+|y|)\Big)(|y+t_{r+1}|+1)^{x+\sigma_{r+1}-1/2}(|y|+1)^{-x-1/2} \\
&\quad\times\Big\{\sum_{k=-1}^{N-1}|w_r|^k\mu^k\sum_{j=1}^{r-1}(|y|+1)^{f(j,r-1,x+k,-x)}\exp(|y|\rho(\alpha_j-\alpha_{j-1},w_j)) \\
&\quad\times\sum_{i=1}^{r-1}\mu^{g(i,r-1,x+k,-x)}\exp(|t_r+t_{r+1}+y|\rho(\alpha_i-\alpha_{i-1},w_i)) \\
&\quad\times\sum_{i=1}^{r-1}\mu^{g(i,r-1,x+k,-x)}\exp(|t_r+t_{r+1}+y|\rho(\alpha_i-\alpha_{i-1},w_i)) \\
&\quad+|w_r|^{N-\epsilon}\sum_{j=1}^{r-1}(|y|+1)^{f(j,r-1,x+N-\epsilon,x)}\exp(|y|\rho(\alpha_j-\alpha_{j-1},w_j)) \\
&\quad\times\sum_{i=1}^{r}\mu^{h(i,r-1,x)}\exp(|t_r+t_{r+1}+y|\rho(\alpha_i-\alpha_{i-1},w_i))\Big\},
\end{aligned}
$$

which is valid under conditions (5.5)–(5.7), and the implied constant is independent of y.

We return to (5.2), which is valid for

$$\Re(s_{r-k+1}+\cdots+s_r+s_{r+1})>k \qquad (0\le k\le r).$$

Now we shift the path to $\Re z = M - \epsilon$, where M is a positive integer and ϵ is a small positive number. From (5.3) we see that (5.6) holds for $\Re z \geq c$, hence we can use estimate (5.9) in the strip $c \leq \Re z \leq M - \epsilon$. The legitimacy of this shifting is easily shown by using the Stirling formula and Lemma 1. By (5.3), the only pole $z = 1 - s_r - s_{r+1}$ of $\tilde{\zeta}^*_{\mathrm{AV},r}$ is irrelevant to this shift. Counting the residues at $z = 0, 1, \ldots, M-1$, we get

$$\begin{aligned}(5.10)\quad &\tilde{\zeta}_{\mathrm{AV},r}((s_1, \ldots, s_r; s_{r+1}); (\alpha_1, \ldots, \alpha_r; \alpha_{r+1}), (w_1, \ldots, w_r)) \\ &\quad = \sum_{j=0}^{M-1} \binom{-s_{r+1}}{j} \tilde{\zeta}^*_{\mathrm{AV},r}((s_1, \ldots, s_{r-1}, -j; s_r + s_{r+1} + j); \\ &\qquad\qquad (\alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1} - \alpha_r; \alpha_r), (w_1, \ldots, w_r)) \\ &\qquad + T_{r,M},\end{aligned}$$

where

$$\begin{aligned}(5.11)\quad &T_{r,M} \\ &= \frac{1}{2\pi i} \int_{(M-\epsilon)} \frac{\Gamma(s_{r+1} + z)\Gamma(-z)}{\Gamma(s_{r+1})} \tilde{\zeta}^*_{\mathrm{AV},r}((s_1, \ldots, s_{r-1}, -z; s_r + s_{r+1} + z); \\ &\qquad\qquad (\alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1} - \alpha_r; \alpha_r), (w_1, \ldots, w_r))\, dz.\end{aligned}$$

Now we assume $N \geq M$ and put

$$\begin{aligned}F_r(M, \epsilon) = \{&(s_1, \ldots, s_r; s_{r+1}) \in \mathbb{C}^{r+1} \mid \\ &\Re(s_{r-k+1} + \cdots + s_{r+1}) > k - M + \epsilon\ (0 \leq k \leq r)\}.\end{aligned}$$

If $(s_1, \ldots, s_r, s_{r+1}) \in F_r(M, \epsilon)$, we see that

$$\Re(s_{r-k+1} + \cdots + s_{r+1}) > k - M + \epsilon \geq k - N + \epsilon,$$

so (5.5) holds. Also (5.6) obviously holds for $x = M - \epsilon$. Hence, if $(s_1, \ldots, s_r, s_{r+1}) \in F_r(M, \epsilon)$ and $(s_1, \ldots, s_r, s_{r+1})$ satisfies (5.7), we can apply (5.9) to the right-hand side of (5.11), and get the absolute convergence of the integal in the region from (2.5) and (2.6). The poles of the integrand are only at $z = -s_{r+1} - l$, $z = l$ ($l \in \mathbb{N}_0$) and $z = 1 - s_r - s_{r+1}$. Hence by (5.11) we find that $T_{r,M}$ can be continued meromorphically to

$$\begin{aligned}F'_r(M, \epsilon) = \{&(s_1, \ldots, s_r, s_{r+1}) \in F_r(M, \epsilon) \mid \\ &s_i + \cdots + s_{r-1} + s_r + s_{r+1} \neq r + 1 - i - l\ (1 \leq i \leq r-1,\ l \in \mathbb{N}_0)\}.\end{aligned}$$

Therefore (5.10) gives the analytic continuation of $\tilde{\zeta}_{\mathrm{AV},r}$ to $F'_r(M, \epsilon)$.

By using (5.9), we obtain

$$T_{r,M} \ll \Big(\sum_{k=-1}^{N-1} |w_r|^k + |w_r|^{N-\epsilon} \Big) \ll |w_r|^{-1}$$

if $(s_1, \ldots, s_r, s_{r+1}) \in F'_r(M, \epsilon)$ and $|w_r| \leq 1$. Also, since (5.6) holds if $z =$

$j \geq 0$, we can substitute (5.4) with $z = j$ into (5.10). The error term $S_{r-1,N}$ is $O(|w_r|^{N-\epsilon})$ by (5.8). Similarly to Section 4 of [2], we obtain

$$S_{r-1,N} \ll |w_r|^N$$

if $|w_r| \leq 1$. Hence we obtain Theorem 4. Since $N \geq M$ can be arbitrarily large we substitute (5.4) into (5.10), and find that

$$s_r + s_{r+1} = 1 - l \qquad (l \in \mathbb{N}_0) \tag{5.12}$$

are possible singularities of $\tilde{\zeta}_{\mathrm{AV},r}$. From (5.7) and (5.12), we obtain (2) of Theorem 2. Also since N and M are arbitrary with $N \geq M$, by using (5.10) and Theorem 2(1) for $\tilde{\zeta}_{\mathrm{AV},r-1}$, we can continue

$$\tilde{\zeta}_{\mathrm{AV},r}((s_1, \ldots, s_r; s_{r+1}); (\alpha_1, \ldots, \alpha_r; \alpha_{r+1}), (w_1, \ldots, w_r))$$

meromorphically to the whole $\mathbb{C}^{r+1}$, hence we obtain Theorem 2(1).

To end this section we note that, in the above proof, we have found the recursive structure

$$\tilde{\zeta}_{\mathrm{AV},r} \to \tilde{\zeta}^*_{\mathrm{AV},r} \to \tilde{\zeta}_{\mathrm{AV},r-1},$$

which can be written in terms of Mellin–Barnes integrals. This observation gives the assertion of Theorem 5.

6. Proof of Theorem 2(3). In this section we prove Theorem 2(3) by induction. This proof is similar to Section 5 of [2].

When $r = 1$, we estimate the order of $\tilde{\zeta}_{\mathrm{AV},1}$ at (s_1, s_2) satisfying $\sigma' \leq \sigma_2 \leq \sigma''$, $\lambda' \leq \sigma_1 \leq \lambda''$ and $d_1((s_1, s_2), \mathrm{Sing}(1)) \geq \eta$.

First, from (5.1), we have

$$\tilde{\zeta}_{\mathrm{AV},1}((s_1; s_2); (\alpha_1; \alpha_2), w_1) = w_1^{-s_1-s_2} \sum_{m_1=0}^{\infty} (m_1 + \alpha_1/w_1)^{-s_1-s_2}.$$

By Lemma 2, we have

$$\begin{aligned}
&\zeta(s_1 + s_2, \alpha/w_1) w_1^{-s_1-s_2} \\
&\qquad \ll (|t_1 + t_2| + 1)^{\max\{0,1-\sigma_1-\sigma_2\}+\epsilon} \exp(|t_1 + t_2| \rho(\alpha_1, w_1)) \\
&\qquad \ll (|t_1| + 1)^{\max\{0,1-\sigma_1-\sigma_2\}+\epsilon} \exp(|t_1| \rho(\alpha_1, w_1)) \\
&\qquad\quad \times (|t_2| + 1)^{\max\{0,1-\sigma_1-\sigma_2\}+\epsilon} \exp(|t_2| \rho(\alpha_1, w_1)),
\end{aligned}$$

where

$$g(1, 1, \sigma_1, \sigma_2) = f(1, 1, \sigma_1, \sigma_2) = \max\{0, 1 - \sigma_1 - \sigma_2\} + \epsilon.$$

The implied constant here depends only on $w_1, \alpha_1, \sigma', \sigma'', \lambda', \lambda'', \eta, \epsilon$. Hence we have proved Theorem 2(3) for $r = 1$.

Next we assume the validity of Theorem 2(3) for $\tilde{\zeta}_{\mathrm{AV},r-1}$, and we will prove it for $\tilde{\zeta}_{\mathrm{AV},r}$.

We put $z = x + iy$ $(x, y \in \mathbb{R})$, assume that $x_2 \le x \le x_3$, where x_2, x_3 are fixed real numbers, fix $s_1, \dots, s_{r-1}$, and estimate the order of $\tilde{\zeta}_{\mathrm{AV},r}$ at $(s_1, \dots, s_r, s_{r+1})$ satisfying $\sigma' \le \sigma_{r+1} \le \sigma''$, $\lambda' \le \sigma_r \le \lambda''$ and $d_r((s_1, \dots, s_r, s_{r+1}), \mathrm{Sing}(r)) \ge \eta$. Also we put

$$V_r = \max\{j - (\sigma_{r-j+1} + \cdots + \sigma_r + \sigma_{r+1}) \mid 0 \le j \le r\}.$$

Then we choose a small $\epsilon > 0$ such that $V_r + \epsilon$ is not an integer, and choose an integer $N > V_r + 2\epsilon$. Then $(s_1, \dots, s_r, s_{r+1}) \in F'_r(N, \epsilon)$. We shift the path $\Re z = N - \epsilon$ of the integral in (5.4) to $\Re z = V_r + \epsilon$. Then we obtain

$$\begin{aligned}
(6.1)\quad & \tilde{\zeta}^*_{\mathrm{AV},r}((s_1, \dots, s_{r-1}, -z; s_r + s_{r+1} + z); (\alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1} - \alpha_r; \alpha_r), \\
&\qquad (w_1, \dots, w_r)) \\
&= -\frac{1}{1 - s_r - s_{r+1} - z}\,\tilde{\zeta}_{\mathrm{AV},r-1}((s_1, \dots, s_{r-2}, s_{r-1} + s_r + s_{r+1} + z - 1; -z); \\
&\qquad (\alpha_1, \dots, \alpha_{r-1}; \alpha_{r+1} - \alpha_r), (w_1, \dots, w_{r-1}))w_r^{-1} \\
&\quad + \sum_{k=0}^{K-1} \binom{-s_r - s_{r+1} - z}{k} \zeta\Big(-k, \frac{\alpha_r - \alpha_{r-1}}{w_r}\Big) w_r^k \\
&\quad \times \tilde{\zeta}_{\mathrm{AV},r-1}((s_1, \dots, s_{r-2}, s_{r-1} + s_r + s_{r+1} + z + k; -z); \\
&\qquad (\alpha_1, \dots, \alpha_{r-1}; \alpha_{r+1} - \alpha_r), (w_1, \dots, w_{r-1})) \\
&\quad + \frac{1}{2\pi i} \int_{(V_r+\epsilon)} \frac{\Gamma(s_r + s_{r+1} + z + z')\Gamma(-z')}{\Gamma(s_r + s_{r+1} + z)}\, \zeta\Big(-z', \frac{\alpha_r - \alpha_{r-1}}{w_r}\Big) w_r^{z'} \\
&\quad \times \tilde{\zeta}_{\mathrm{AV},r-1}((s_1, \dots, s_{r-2}, s_{r-1} + s_r + s_{r+1} + z + z'; -z); \\
&\qquad (\alpha_1, \dots, \alpha_{r-1}; \alpha_{r+1} - \alpha_r), (w_1, \dots, w_{r-1}))\, dz',
\end{aligned}$$

where K is the integer satisfying $K - 1 < V_r + \epsilon < K$. Note that if $K \le 0$ then the sum on the right-hand side of (6.1) is empty, and if $K \le -1$ then the first term on the right-hand side does not appear.

Using the notation introduced in Section 5, we obtain

$$\begin{aligned}
(6.2)\quad & \tilde{\zeta}^*_{\mathrm{AV},r}((s_1, \dots, s_{r-1}, -z; s_r + s_{r+1} + z); (\alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1} - \alpha_r; \alpha_r), \\
&\qquad (w_1, \dots, w_r)) \\
&\ll \sum_{i=1}^{r} \mu^{a(i,r-1,x)} \exp(|t_r + t_{r+1} + y|\rho(\alpha_i - \alpha_{i-1}, w_i)) \\
&\quad \times \sum_{j=1}^{r-1} (|y| + 1)^{b(j,r-1,x)} \exp(|y|\rho(\alpha_j - \alpha_{j-1}, w_j)),
\end{aligned}$$

where

$$a(i, r-1, x) = \max\{K - 1, \max_{-1 \le k \le K-1}\{g(i, r-1, x + k, -x)\}, h(i, r-1, x)\},$$

$$b(j, r-1, x) = \max\{\max_{-1 \le k \le K-1}\{f(j, r-1, x + k, -x)\}, f(j, r-1, x + N - \epsilon, x)\}.$$

The implied constant here depends only on $x_2, x_3, (\alpha_r - \alpha_{r-1})/w_r = b$ and $\alpha_j - \alpha_{j-1}$ $(1 \le j \le r)$, w_j $(1 \le j \le r)$, r, $\sigma_1, \dots, \sigma_{r-1}$, $t_1, \dots, t_{r-1}$, $\sigma', \sigma'', \eta, \epsilon, \lambda', \lambda''$. Similarly we shift the path $\Re z = M - \epsilon$ of the integral in (5.10) to $\Re z = V_r + \epsilon$ and obtain

$$\begin{aligned}(6.3)\quad & \tilde{\zeta}_{\mathrm{AV},r}((s_1, \dots, s_r; s_{r+1}); (\alpha_1, \dots, \alpha_r; \alpha_{r+1}), (w_1, \dots, w_r)) \\ &= \sum_{d=0}^{K-1} \binom{-s_{r+1}}{d} \tilde{\zeta}^*_{\mathrm{AV},r}((s_1, \dots, s_{r-1}, -d; s_r + s_{r+1} + d); \\ &\qquad\qquad (\alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1} - \alpha_r; \alpha_r), (w_1, \dots, w_r)) \\ &\quad + \frac{1}{2\pi i} \int\limits_{(V_r+\epsilon)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})} \tilde{\zeta}^*_{\mathrm{AV},r}((s_1, \dots, s_{r-1}, -z; s_r + s_{r+1} + z); \\ &\qquad\qquad (\alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1} - \alpha_r; \alpha_r), (w_1, \dots, w_r))\, dz.\end{aligned}$$

Applying (6.2) to the sum on the right-hand side of (6.3), we obtain

$$\begin{aligned}(6.4)\quad & \binom{-s_{r+1}}{d} \tilde{\zeta}^*_{\mathrm{AV},r}((s_1, \dots, s_{r-1}, -d; s_r + s_{r+1} + d); \\ &\qquad (\alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1} - \alpha_r; \alpha_r), (w_1, \dots, w_r)) \\ &\qquad \ll \sum_{i=1}^{r} (|t_r| + 1)^{a(i,r-1,d)} (|t_{r+1}| + 1)^{a(i,r-1,d)+d} \\ &\qquad\quad \times \exp((|t_r| + |t_{r+1}|)\rho(\alpha_i - \alpha_{i-1}, w_i)).\end{aligned}$$

Also, applying (6.2) and Lemma 3 to the integral on the right-hand side of (6.3), we obtain

$$\begin{aligned}(6.5)\quad & \frac{1}{2\pi i} \int\limits_{(V_r+\epsilon)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})} \tilde{\zeta}^*_{\mathrm{AV},r}((s_1, \dots, s_{r-1}, -z; s_r + s_{r+1} + z); \\ &\qquad (\alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1} - \alpha_r; \alpha_r), (w_1, \dots, w_r))\, dz \\ &\ll \sum_{i=1}^{r} (|t_r| + 1)^{g(i,r,\sigma_1,\dots,\sigma_{r+1})} \exp(|t_r|\rho(\alpha_i - \alpha_{i-1}, w_i)) \\ &\quad \times \sum_{j=1}^{r} (|t_{r+1}| + 1)^{f(j,r,\sigma_1,\dots,\sigma_{r+1})} \exp(|t_{r+1}|\rho(\alpha_j - \alpha_{j-1}, w_j)),\end{aligned}$$

where

$$\begin{aligned}f(j, r, \sigma_1, \dots, \sigma_{r+1}) = \max\{&0, 1/2 - \sigma_{r+1}, V_r + \epsilon, \\ &\max_{1 \le i \le r}\{a(i, r-1, V_r + \epsilon) + 1/2 + b(j, r-1, V_r + \epsilon)\}, \\ &-\sigma_{r+1} - V_r - \epsilon + 1 + b(j, r-1, V_r + \epsilon), 1/2 + b(j, r-1, V_r + \epsilon)\}\end{aligned}$$

and

$$g(i, r, \sigma_1, \ldots, \sigma_{r+1}) = a(i, r-1, V_r + \epsilon).$$

The implied constant here depends only on $x_2, x_3, (\alpha_r - \alpha_{r-1})/w_r = b$ and $\alpha_j - \alpha_{j-1}$ $(1 \leq j \leq r)$, w_j $(1 \leq j \leq r)$, r, $\sigma_1, \ldots, \sigma_{r-1}$, $t_1, \ldots, t_{r-1}$, $\sigma', \sigma'', \eta, \epsilon, \lambda', \lambda''$. Hence, from (6.4) and (6.5), we have proved Theorem 2(3).

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