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Nonvanishing of automorphic L-functions at special points

by

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1. Introduction. The nonvanishing of automorphic L-functions at its critical points has received considerable attention. One reason for this is its connection with topics such as the conjecture of Birch and Swinnerton-Dyer, and the theory of liftings of automorphic forms. There are a lot of nonvanishing results for the L-functions attached to the family $S_k^*(q)$ of weight k primitive forms for $\Gamma_0(q)$ (see [23, 11, 12, 13, 9, 18, 20]). In particular, Iwaniec and Sarnak established, in their important paper [9] on the Landau–Siegel zero problem, a positive proportion nonvanishing result for the central values $L(1/2, f)$ of holomorphic newforms f with respect to large weights k or large squarefree levels q, and a similar result was obtained by Rouymi [20] when q is a power of a fixed prime.

It is natural to consider the nonvanishing of Maass cusp forms on $\mathrm{GL}(2)$. Actually, for the Maass forms, Luo [17] got a positive proportion nonvanishing result for the special values of $L(s, Q \otimes u_j)$, where Q is a holomorphic form cusp form of weight 4 for $\Gamma_0(p)$ (p is a prime), and u_j is a Maass cusp form with Laplace eigenvalue $1/4 + t_j^2$ for $\mathrm{SL}(2,\mathbb{Z})$. That is, roughly speaking, based on the pioneering work of Phillips and Sarnak [19], he finally showed that, under standard multiplicity assumptions, the Weyl law is false for generic hyperbolic surfaces, by establishing a positive proportion nonvanishing result for the special values of $L(s, Q \otimes u_j)$:

$$\sharp\{t_j \le T : L(1/2 + \mathrm{i}t_j, Q \otimes u_j) \ne 0\} \gg T^2 \tag{1.1}$$

for sufficiently large T.

Motivated by the above works, we deal with the nonvanishing of the $\mathrm{GL}(2)$ Maass L-functions at special points in short intervals. Before stating our result, let us fix our notation. Let $\{u_j\}$ be an orthonormal basis of the space of Maass cusp forms for the full modular group $\mathrm{SL}(2,\mathbb{Z})$ such that $\Delta u_j(z) = \lambda_j u_j(z)$ with $\lambda_j = 1/4+t_j^2$ ($t_j > 0$), and each u_j is an eigenfunction

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of all the Hecke operators and T_{-1} as well. $\{u_j\}$ consists of even Maass forms and odd forms according to $T_{-1}u_j(z) = u_j(z)$ or $T_{-1}u_j(z) = -u_j(z)$. Each $u_j(z)$ has the Fourier expansion

$$u_j(z) = \cosh^{1/2}(\pi t_j) y^{1/2} \sum_{n\neq 0} v_j(n) K_{\mathrm{i}t_j}(2\pi|n|y)\mathrm{e}(nx), \tag{1.2}$$

where $\mathrm{e}(x) := \mathrm{e}^{2\pi \mathrm{i}x}$ and $K_\nu(x)$ is the K-Bessel function. The Fourier coefficients $v_j(n)$ are proportional to the eigenvalues $\lambda_j(n)$ of the nth Hecke operator T_n, i.e., $v_j(n) = v_1(n)\lambda_j(n)$ $(n \geq 1)$. Also, according to [7, 5], for any $\varepsilon > 0$ we have

$$t_j^{-\varepsilon} \ll_\varepsilon v_j(1) \ll_\varepsilon t_j^{\varepsilon} \tag{1.3}$$

uniformly for j. For the numbers of u_j of height $t_j \leq T$, one has the Weyl law [4, 24]:

$$\sharp\{j : t_j \leq T\} = \frac{T^2}{12} - \frac{T\log T}{2\pi} + CT + O\left(\frac{T}{\log T}\right), \tag{1.4}$$

where C is a constant. The eigenvalues $\lambda_j(n)$ enjoy the multiplicative property

$$\lambda_j(m)\lambda_j(n) = \sum_{d|(m,n)} \lambda_j(mn/d^2) \tag{1.5}$$

and satisfy the following bound [10, Appendix 2]:

$$|\lambda_j(n)| \leq n^\theta d(n) \quad (n \in \mathbb{N}), \tag{1.6}$$

where $\theta = 7/64$ and $d(n)$ is the divisor function. The Ramanujan–Petersson conjecture predicts $\theta = 0$. Rankin–Selberg theory implies that the Ramanujan–Petersson conjecture bound holds on average: one has, for any $\varepsilon > 0$,

$$\sum_{n\leq X} |\lambda_j(n)|^2 \ll_\varepsilon (t_j X)^\varepsilon X \tag{1.7}$$

uniformly for all $X \geq 1$ and j.

The automorphic L-function associated to any even cusp form $u_j(z)$ is given by the absolutely convergent Dirichlet series

$$L(s, u_j) := \sum_{n\geq 1} \lambda_j(n) n^{-s}$$

for $\Re e\, s > 1$, which has analytic continuation to an entire function and satisfies the functional equation on $\mathbb{C}$:

$$\Lambda(s, u_j) := \frac{1}{\pi^s} \Gamma\left(\frac{s + \mathrm{i}t_j}{2}\right) \Gamma\left(\frac{s - \mathrm{i}t_j}{2}\right) L(s, u_j) = \Lambda(1 - s, u_j). \tag{1.8}$$

The aim of this paper is to prove the following nonvanishing result in short intervals.

THEOREM 1.1. *Let $\{u_j\}$ be an orthonormal basis of even Hecke–Maass forms for* $\mathrm{SL}(2,\mathbb{Z})$ *with Laplace eigenvalues $1/4+t_j^2$ with $t_j>0$. Then there exists an absolute large constant A_0 such that, for sufficiently large T and $A_0 \log T \le U \le T$,*

$$\sharp\{T-U \le t_j \le T+U : L(1/2+\mathrm{i}t_j, u_j) \ne 0\} \gg TU, \tag{1.9}$$

where the implied constant is absolute.

By Weyl's law (1.4), we actually get a positive proportion nonvanishing result in short intervals. The implied constant may be small. However, getting a good constant is not our aim. Instead, we want the short interval U to be as small as possible. And thus there is an important part which has no counterpart in Luo's work [17]. In order to reduce the short interval, we use the Poisson summation formula (see the comments after Proposition 1.2 below).

As in previous works [23, 11, 12, 13, 9, 18, 17, 20], we shall apply the moment method with a mollifier. Here we choose a similar mollifier to the one in [17]:

$$\mathfrak{m}_j := \sum_{n\ge 1} \frac{a_n \mu(n)}{n^{1/2+\mathrm{i}t_j}} \lambda_j(n), \tag{1.10}$$

where $\mu(n)$ is the Möbius function and

$$\begin{aligned} a_n &:= \frac{1}{2\pi \mathrm{i}} \int_{(3)} \frac{(\xi^2/n)^s - (\xi/n)^s}{s^2} \frac{\mathrm{d}s}{\log \xi} \\ &= \begin{cases} 1 & \text{if } 1 \le n \le \xi, \\ \dfrac{\log(\xi^2/n)}{\log \xi} & \text{if } \xi < n \le \xi^2, \\ 0 & \text{if } n > \xi^2. \end{cases} \end{aligned} \tag{1.11}$$

We shall choose $\xi := T^a$ with some suitably small positive constant a.

To prove Theorem 1.1, we need to consider

$$\mathcal{M}_1 := \sum_j{}' \, |v_j(1)|^2 L(1/2+\mathrm{i}t_j, u_j) \mathfrak{m}_j \mathrm{e}^{-(t_j-T)^2/U^2}, \tag{1.12}$$

$$\mathcal{M}_2 := \sum_j{}' \, |v_j(1)|^2 |L(1/2+\mathrm{i}t_j, u_j)|^2 |\mathfrak{m}_j|^2 \mathrm{e}^{-(t_j-T)^2/U^2}, \tag{1.13}$$

$$\mathcal{J} := \sum_j |v_j(1)|^4 \mathrm{e}^{-(t_j-T)^2/U^2}, \tag{1.14}$$

where $\sum'$ restricts to the even Maass forms. In Section 6, we shall see that Theorem 1.1 is an immediate consequence of the following proposition.

Proposition 1.2. *For any* $\varepsilon > 0$, *let* $a \in (0, 1/20 - \varepsilon)$. *Then, for sufficiently large* T *and* $\log T \le U \le T^{1-\varepsilon}$, *we have*

$$\mathcal{M}_1 = \pi^{-3/2} TU + O(T^{1/2+3a+\varepsilon} U), \tag{1.15}$$

$$\mathcal{M}_2 \ll TU, \tag{1.16}$$

$$\mathcal{J} \ll TU. \tag{1.17}$$

Here the implied constants in (1.15) *and* (1.16) *depend on* ε *and* a, *respectively, and the implied constant in* (1.17) *is absolute.*

The most part of the present work is to prove Proposition 1.2. We first represent $L(1/2 + \mathrm{i}t_j, u_j)$ and $|L(1/2 + \mathrm{i}t_j, u_j)|^2$ as approximate functional equations (see (3.1) and (4.1) below). Then we use the Kuznetsov trace formula (see Lemmas 2.1 and 2.2). The part dealing with the nondiagonal term has no counterpart in [17]. We use the technique in Li's work [15] and [16] to treat the Bessel functions. After doing this, we can get the short interval U to be of size $T^{1/2+\varepsilon}$. However, we can open the Kloosterman sum and use the classical Poisson summation formula to save more. Before explaining how to deal with the diagonal terms, we remark that in the process of proving (1.15), one may usually choose the parameter T_0 of size $T^{1/2}$ so that the essential sums of the two terms in (3.1) are both $m \le T^{1/2+\varepsilon}$ (see the beginning of Section 3). However, in that case, the Poisson summation formula does not work for the second term in (3.1) after using the Kuznetsov trace formula because of the factor $\Gamma(1/4 + s/2 - t_j)/\Gamma(1/4 - s/2 + t_j)$. Therefore we choose $T_0 = T^{1+\varepsilon}$ so that the second term in (3.2) is small.

For the diagonal term, the difficult part comes from the process of proving (1.16). As in Luo [17], we apply the mollification technique. The power of the mollifier lies in that it behaves like the inverse of $L(1/2 + \mathrm{i}t_j, u_j)$ on average, and thus we can save a $\log T$ factor when using Cauchy's inequality. This technique has its origin in Bohr–Landau's work [1] on zeros of $\zeta(s)$, and more profoundly in the work of Selberg [21] who used it to show that a positive proportion of the nontrivial zeros of $\zeta(s)$ lie on the critical line. To deal with these special values seems easier than dealing with the central values since $\mathrm{i}t_j$ can remove half the gamma factors in the functional equation. This "explains" why U can be taken so small. But the reason that U cannot be smaller is that, in Subsection 4.1, there are several terms like (4.12) and (4.13) which contribute $O(TU + T \log T)$ to the left-hand side in (1.16). So we have to let $U \ge \log T$.

Throughout the paper, ε is an arbitrarily small positive number and A is a sufficiently large positive number which may not be the same at each occurrence.

2. Preliminaries. In this section we state some useful lemmas.

Let $h(t)$ be a test function satisfying

$$\begin{aligned} &h(t) = h(-t), \qquad h(t) \ll (|t|+1)^{-\vartheta}, \\ &h(t) \text{ is holomorphic in } |\Im m\, t| \le \varsigma, \end{aligned} \tag{2.1}$$

for some constants $\vartheta > 2$ and $\varsigma > 1/2$.

We have the Kuznetsov trace formula (see [14, 2]):

LEMMA 2.1. *Under the previous notation, we have*

$$\sum_j |v_j(1)|^2 h(t_j)\lambda_j(m)\lambda_j(n) = \frac{2\delta_{m,n}}{\pi^2} H - \frac{1}{\pi}\int_{\mathbb{R}} \frac{h(t)d_{\mathrm{i}t}(m)d_{\mathrm{i}t}(n)}{|\zeta(1+2\mathrm{i}t)|^2}\,\mathrm{d}t + \sum_{c\ge 1}\frac{S(m,n;c)}{c} H^{\pm}\Big(\frac{2\sqrt{|mn|}}{c}\Big) \tag{2.2}$$

for all integers m and n, where $\delta_{m,n}$ is the Kronecker symbol, $d_\nu(n) := \sum_{ab=|n|}(a/b)^\nu$, $\pm$ is the sign of mn and

$$H := \int_0^\infty t h(t)\tanh(\pi t)\,\mathrm{d}t, \tag{2.3}$$

$$H^+(x) := \frac{2\mathrm{i}}{\pi}\int_{\mathbb{R}} t h(t)\frac{J_{2\mathrm{i}t}(2\pi x)}{\cosh(\pi t)}\,\mathrm{d}t, \tag{2.4}$$

$$H^-(x) := \frac{4}{\pi^2}\int_{\mathbb{R}} t h(t) K_{2\mathrm{i}t}(2\pi x)\sinh(\pi t)\,\mathrm{d}t, \tag{2.5}$$

$$S(m,n;c) := \sum_{d\bar{d}\equiv 1 \,(\mathrm{mod}\, c)} \mathrm{e}\Big(\frac{md+n\bar{d}}{c}\Big). \tag{2.6}$$

In the above, $J_\nu(x)$ and $K_\nu(x)$ are the standard J-Bessel function and K-Bessel function, respectively. We have Weil's bound

$$|S(m,n;c)| \le (m,n,c)^{1/2}c^{1/2}d(c). \tag{2.7}$$

To simplify the presentation we restrict the spectral sum in (2.2) to the even forms; these can be selected by adding (2.2) for m, n to that for $-m$, n.

LEMMA 2.2. *Under the previous notation, we have*

$$\sideset{}{'}\sum_j |v_j(1)|^2 h(t_j)\lambda_j(m)\lambda_j(n) = \frac{\delta_{m,n}}{\pi^2} H - \frac{1}{\pi}\int_{\mathbb{R}} \frac{h(t)d_{\mathrm{i}t}(m)d_{\mathrm{i}t}(n)}{|\zeta(1+2\mathrm{i}t)|^2}\,\mathrm{d}t + \sum_{\eta=\pm}\sum_{c\ge 1}\frac{S(\eta m,n;c)}{2c} H^{\eta}\Big(\frac{2\sqrt{mn}}{c}\Big)$$

for all integers $m\ge 1$ and $n \ge 1$, where $\sum'$ restricts to the even Maass forms.

We also need the following result [6, Theorem 5.2].

LEMMA 2.3. *Let $a_1, \dots, a_N$ be arbitrary complex numbers. Then*

$$\int_0^T \Big| \sum_{n\le N} a_n n^{\mathrm{i}t}\Big|^2 \mathrm{d}t = T\sum_{n\le N} |a_n|^2 + O\Big(\sum_{n\le N} n|a_n|^2\Big), \tag{2.8}$$

and the above formula remains also valid if $N=\infty$, provided that the series on the right-hand side of (2.8) *converges.*

Let

$$G(s) := \Big(\cos\frac{\pi s}{A}\Big)^{-A}. \tag{2.9}$$

Moreover let $\sigma_0 > 2$, $y > 0$, and $|\Im m\, t| < \sigma_0/2$, and define

$$V_1(y) := \frac{1}{2\pi\mathrm{i}}\int_{(\sigma_0)} G(s)\frac{\Gamma\big(\frac14+\frac s2\big)}{\Gamma\big(\frac14\big)}\cdot\frac{y^{-s}}{s}\,\mathrm{d}s, \tag{2.10}$$

$$V_2(y,t) := \frac{1}{2\pi\mathrm{i}}\int_{(\sigma_0)} G(s)\frac{\Gamma\big(\frac14+\frac s2\big)^2\Gamma\big(\frac14+\frac s2+\mathrm{i}t\big)\Gamma\big(\frac14+\frac s2-\mathrm{i}t\big)}{\Gamma\big(\frac14\big)^2\Gamma\big(\frac14+\mathrm{i}t\big)\Gamma\big(\frac14-\mathrm{i}t\big)}\cdot\frac{y^{-s}}{s}\,\mathrm{d}s. \tag{2.11}$$

For $y > 0$ and $t > 0$, define

$$W_1(y,t) := \frac{1}{2\pi\mathrm{i}}\int_{(\sigma_0)} G(s)\frac{\Gamma\big(\frac14+\frac s2\big)\Gamma\big(\frac14+\frac s2-\mathrm{i}t\big)}{\Gamma\big(\frac14\big)\Gamma\big(\frac14-\frac s2+\mathrm{i}t\big)}\cdot\frac{y^{-s}}{s}\,\mathrm{d}s, \tag{2.12}$$

$$W_2(y,t) := \frac{1}{2\pi\mathrm{i}}\int_{(1/2)} G(s)\frac{\Gamma(\frac14+\frac s2)^2}{\Gamma(\frac14)^2}\cdot\frac{(y/t)^{-s}}{s}\,\mathrm{d}s. \tag{2.13}$$

The next lemma will be useful in the proof of (1.15) and (1.16).

LEMMA 2.4. *We have*

(2.14)	$y^i\partial^i V_1(y)/\partial y^i = \delta_{0,i} + O(y^{1/2-\varepsilon})$	$(0 < y \le 1)$,
(2.15)	$y^i\partial^i V_1(y)/\partial y^i \ll y^{-A}$	$(y \ge 1)$,
(2.16)	$W_1(y,t) \ll (y/t)^{-A}$	$(y > 1,\ t \ge 1)$,
(2.17)	$V_2(y,t) \ll (y/\lvert t\rvert)^{-A}$	$(y > 1,\ \lvert t\rvert \ge 1,\ A > 2\lvert\Im m\, t\rvert)$,
(2.18)	$y^i\partial^i W_2(y,t)/\partial y^i \ll y^{-i}$	$(i \ge 0,\ y \ge 1)$,
(2.19)	$t^i\partial^i W_2(y,t)/\partial t^i \ll t^{-i}$	$(i \ge 0,\ t \ge 1)$,
(2.20)	$V_2(y,t) = W_2(y,t) + O(y^{-1/2}t^{-1/2+\varepsilon})$	$(1 \le y \le t^{1+\varepsilon})$.

Proof. To prove (2.14)–(2.19) we use the strategy in [8, p. 100]. For $z = u + \mathrm{i}v$ with $|v| \ge 1$, the Stirling formula states that

(2.21)
$$\Gamma(z) = \sqrt{2\pi}\,\mathrm{e}^{-(\pi/2)|v|}|v|^{u-1/2}\mathrm{e}^{\mathrm{i}v(\log|v|-1)}\mathrm{e}^{\mathrm{sign}(v)\mathrm{i}(\pi/2)(u-1/2)}\{1 + O_u(|v|^{-1})\}.$$

In order to prove (2.14) and (2.15), it is sufficient to differentiate $V_1(y)$ and move the integration line to $\Re e\, s = 1/2 - \varepsilon$ and $\Re e\, s = A$, respectively. In the same way, we can get (2.18) and (2.19).

We use the Stirling formula to obtain, for $\Re e\, s = \sigma_0 > 2$,
$$\frac{\Gamma\big(\frac14 + \frac s2 - \mathrm{i}t\big)}{\Gamma\big(\frac14 - \frac s2 + \mathrm{i}t\big)} \ll (|t| + |s|)^{\Re e\, s} \qquad (t \ge 1),$$
$$\frac{\Gamma\big(\frac14 + \frac s2 + \mathrm{i}t\big)\Gamma\big(\frac14 + \frac s2 - \mathrm{i}t\big)}{\Gamma\big(\frac14 + \mathrm{i}t\big)\Gamma\big(\frac14 - \mathrm{i}t\big)} \ll (|t| + |s|)^{\Re e\, s}(1 + |s|)^{1/4} \qquad (|t| \ge 1, |\Im m\, t| < \sigma_0/2).$$
By shifting the line of integration of $W_1(y,t)$ and $V_2(y,t)$ to $\Re e\, s = A$, we derive (2.16) and (2.17).

In order to prove (2.20), we move the line of integration in (2.11) to $\Re e\, s = 1/2$. With the help of (2.21), a simple calculation shows that for $s = 1/2 + \mathrm{i}v$ and $t \ge 1$,

(2.22)
$$\frac{\Gamma\big(\frac s2 + \frac14 + \mathrm{i}t\big)\Gamma\big(\frac s2 + \frac14 - \mathrm{i}t\big)}{\Gamma\big(\frac14 + \mathrm{i}t\big)\Gamma\big(\frac14 - \mathrm{i}t\big)} \begin{cases} \ll (t + |v|)^{1/2} & \text{if } |\Im m\, s| > t^{\varepsilon}, \\ = t^s\{1 + O_\varepsilon(t^{-1+\varepsilon})\} & \text{if } |\Im m\, s| \le t^{\varepsilon}. \end{cases}$$
Thus the contribution from $|\Im m\, s| \ge t^{\varepsilon}$ is $\ll_\varepsilon y^{-1/2}t^{-1/2+\varepsilon}$, and we can write
$$V_2(y,t) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\substack{(1/2)\\ |\Im m\, s| \le t^{\varepsilon}}} G(s)\frac{\Gamma\big(\frac s2 + \frac14\big)^2}{\Gamma\big(\frac14\big)^2}\,\frac{(y/t)^{-s}}{s}\,\mathrm{d}s + O(y^{-1/2}t^{-1/2+\varepsilon}).$$
This implies the required asymptotic formula (2.20) by extending the domain $|\Im m\, s| \le t^{\varepsilon}$ to $\mathbb{R}$ with an error $O(y^{-1/2}t^{-1/2+\varepsilon})$. ■

Let $L(s, \mathrm{sym}^2 u_j)$ be the symmetric square L-function of u_j. The next two lemmas (due to Luo [17]) will be needed in the proof of (1.17).

LEMMA 2.5. *Denote by $N(\alpha, X, \mathrm{sym}^2 u_j)$ the number of zeros of the function $L(s, \mathrm{sym}^2 u_j)$ in the region $\Re e\, s \ge \alpha$, $|\Im m\, s| \le X$. There is an absolute constant b such that for any $\varepsilon > 0$,*

(2.23)
$$\sum_{t_j \le T} N(\alpha, (\log T)^3, \mathrm{sym}^2 u_j) \ll_\varepsilon T^{b(1-\alpha)+\varepsilon}$$

holds uniformly for $\alpha \ge 1/2$ and $T \ge 2$.

LEMMA 2.6. *Let $0 < \varepsilon_0 < 1/2$. If $L(s, \mathrm{sym}^2 u_j)$ $(t_j \le T)$ has no zero in the domain $1 - 10\varepsilon_0 \le \Re e\, s \le 1$ and $|\Im m\, s| \le (\log T)^3$, then for any $\varepsilon > 0$, we have $L^{-1}(s, \mathrm{sym}^2 u_j) \ll_\varepsilon T^\varepsilon$ for $1 - \varepsilon_0/2 \le \Re e\, s \le 1$ and $|\Im m\, s| \le (\log T)^2$.*

3. Proof of (1.15). Let $G(s)$ be defined as in (2.9), $\log T \le U \le T^{1-\varepsilon}$ and $T_0 = T^{1+\varepsilon}$. Consider the integral

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{(\sigma_0)} G(s) L(s + 1/2 + \mathrm{i}t_j, u_j) \frac{\Gamma\left(\frac{1}{4} + \frac{s}{2}\right)}{\Gamma\left(\frac{1}{4}\right)} \frac{T_0^s}{s}\, \mathrm{d}s.$$

By moving the line of integration to $(-\sigma_0)$ and by using (1.8), we get

$$\begin{aligned}(3.1)\qquad L(1/2 + \mathrm{i}t_j, u_j) = &\sum_{m\ge1} \frac{\lambda_j(m)}{m^{1/2+\mathrm{i}t_j}} V_1\left(\frac{m}{T_0}\right)\\ &+ \pi^{\mathrm{i}2t_j} \sum_{m\ge1} \frac{\lambda_j(m)}{m^{1/2-\mathrm{i}t_j}} W_1(\pi^2 T_0 m, t_j),\end{aligned}$$

where $V_1(y)$ and $W_1(y,t)$ are defined as in (2.10) and (2.12). With the help of (1.4), (1.7), (2.14), (2.15) and (2.16), it is easy to see that the j-sum in $\mathcal{M}_1$ (see (1.12)) is over $T - T^\varepsilon U \le t_j \le T + T^\varepsilon U$. Thus the first sum in (3.1) is essentially supported on $m \le T_0^{1+\varepsilon}$, while the second sum is $\ll T^{-A}$. Inserting the formula obtained for $L(1/2 + \mathrm{i}t_j, u_j)$ into the definition of $\mathcal{M}_1$ and estimating the contribution of T^{-A} by using (1.4) and (1.7), we can find that

$$(3.2)\qquad \mathcal{M}_1 = \sum_{m\ge1}\sum_{n\ge1} V_1\left(\frac{m}{T_0}\right) \frac{a_n \mu(n)}{(mn)^{1/2}} \Xi_1(T; m, n) + O(T^{-A}),$$

provided A and a are suitably large and small, respectively, where

$$\Xi_1(T; m, n) := \sum_j{}' \, |v_j(1)|^2 \frac{\mathrm{e}^{-(t_j - T)^2/U^2}}{(mn)^{\mathrm{i}t_j}} \lambda_j(m)\lambda_j(n).$$

To apply Lemma 2.2, we let

$$(3.3)\qquad h_1(t) = h_1(t, mn) := \mathrm{e}^{-(t-T)^2/U^2}(mn)^{-\mathrm{i}t} + \mathrm{e}^{-(t+T)^2/U^2}(mn)^{\mathrm{i}t}$$

be the test function. It is not difficult to check that $h_1(t)$ satisfies (2.1). Observing that (1.3), (1.4) and (1.6) imply that

$$(3.4)\qquad \sum_j{}' \, |v_j(1)|^2 \frac{\mathrm{e}^{-(t_j + T)^2/U^2}}{(mn)^{-\mathrm{i}t_j}} \lambda_j(m)\lambda_j(n) \ll T^{-A},$$

we can write

$$\begin{aligned}\Xi_1(T;m,n) &= \sum_j{}' |v_j(1)|^2 h_1(t_j)\lambda_j(m)\lambda_j(n) + O(T^{-A})\\ &= \frac{\delta_{m,n}}{\pi^2}H_1 - \frac{1}{\pi}\int_{\mathbb{R}} \frac{d_{\mathrm{i}t}(m)d_{\mathrm{i}t}(n)}{|\zeta(1+2\mathrm{i}t)|^2}h_1(t)\,\mathrm{d}t + O(T^{-A})\\ &\quad + \sum_{c\ge 1}\frac{1}{2c}\bigg\{S(m,n;c)H_1^+\bigg(\frac{2\sqrt{mn}}{c}\bigg) + S(-m,n;c)H_1^-\bigg(\frac{2\sqrt{mn}}{c}\bigg)\bigg\},\end{aligned}$$

where H_1, $H_1^+(x)$ and $H_1^-(x)$ are defined as in (2.3), (2.4) and (2.5) with $h_1(t)$ given in (3.3) in place of $h(t)$, respectively.

For the term involving $d_{\mathrm{i}t}(m)d_{\mathrm{i}t}(n)$, we recall that $|\zeta(1+2\mathrm{i}t)| \gg (|t|+1)^{-\varepsilon}$. So by a trivial estimate, one can see easily that its contribution to $\mathcal{M}_1$ is $O(T^{1/2+a+\varepsilon}U)$.

3.1. The contribution of H_1. Note $m=n$, so the contribution of H_1 to (3.2) is

$$\frac{1}{\pi^2}\sum_{n\ge 1}\frac{a_n\mu(n)}{n}V_1\bigg(\frac{n}{T_0}\bigg)H_1.$$

Since $\tanh(\pi t) = 1 + O(\mathrm{e}^{-2\pi|t|})$, by the change of variable $(t-T)/U = x$ and by the results of [3, 3.896.4 and 3.952.1] we have

$$\begin{aligned}H_1 &= \int_0^\infty t\mathrm{e}^{-(t-T)^2/U^2}n^{-2\mathrm{i}t}\,\mathrm{d}t + O(T^{-A})\\ &= \frac{2U}{n^{2\mathrm{i}T}}\int_0^\infty \mathrm{e}^{-x^2}\big(T\cos(2Ux\log n) + \mathrm{i}Ux\sin(2Ux\log n)\big)\,\mathrm{d}x + O(T^{-A})\\ &= \frac{\pi^{1/2}}{n^{2\mathrm{i}T}}(TU + \mathrm{i}U^3\log n)\mathrm{e}^{-(U\log n)^2} + O(T^{-A}).\end{aligned}\tag{3.5}$$

Combining this with (2.14), we get

$$\frac{1}{\pi^2}\sum_{n\ge 1}\frac{a_n\mu(n)}{n}V_1\bigg(\frac{n}{T_0}\bigg)H_1 = \pi^{-3/2}TU + O\big(T^{1/2+\varepsilon}U\big).\tag{3.6}$$

3.2. The contribution of $H_1^+(2\sqrt{mn}/c)$. We partition the m-sum in (3.2) using a smooth function $\eta(x)$ which is zero for $x\le 1/2$, one for $x\ge 1$, and partition further into smooth functions by

$$\eta(x) = \sum_{N\ge 1}\eta_N(x)\tag{3.7}$$

with η_N compactly supported in $[N/2,2N]$ such that $x^i\eta_N^{(i)}(x) \ll_i 1$ for any $i\ge 0$. We also require that $\sum_{N\le X}1 \ll \log X$. Therefore, we are led to estimate

$$\sum_{N\ge1}\sum_{m\ge1}\sum_{n\ge1}\eta_N(m)V_1\Big(\frac{m}{T_0}\Big)\frac{a_n\mu(n)}{(mn)^{1/2}}\sum_{c\ge1}\frac{S(m,n;c)}{c}H_1^+\Big(\frac{2\sqrt{mn}}{c}\Big). \tag{3.8}$$

By using the integral representation of the J-Bessel function (see [3, 8.411-5])

$$J_\nu(z)=\frac{\big(\frac{z}{2}\big)^\nu}{\Gamma\big(\nu+\frac12\big)\Gamma\big(\frac12\big)}\int_0^\pi\cos(z\cos\theta)(\sin\theta)^{2\nu}\,\mathrm{d}\theta\qquad(\Re e\,\nu>-1/2), \tag{3.9}$$

it is easy to deduce

$$J_{2\sigma+2\mathrm{i}u}(2\pi x)\ll\Big(\frac{x}{|u|+1}\Big)^{2\sigma}\mathrm{e}^{\pi|u|}. \tag{3.10}$$

By moving the integration line of $H_1^+(x)$ to $\Im m\,t=-\sigma$, we see that

$$H_1^+(x)=\frac{2\mathrm{i}}{\pi}\int_{\mathbb{R}}(u-\sigma\mathrm{i})h_1(u-\sigma\mathrm{i})\frac{J_{2\sigma+2\mathrm{i}u}(2\pi x)}{\cosh(\pi u-\pi\sigma\mathrm{i})}\,\mathrm{d}u+O(T^{-A}),$$

where the error term comes from the residues $t=-(k+1/2)\mathrm{i}$ for $0\le k<\sigma-1/2$. Combining it with (3.10), we have

$$H_1^+(x)\ll x^{2\sigma}(mn)^\sigma T^{1-2\sigma}U. \tag{3.11}$$

With the help of (1.11), the trivial bound $|S(m,n;c)|\le c$, (2.15) and (3.11) with $\sigma=3/4$, the contribution from $N>T_0^{1+\varepsilon}$ to (3.8) is

$$\begin{aligned}&\ll\sum_{N>T_0^{1+\varepsilon}}\sum_{n\le T^{2a}}\frac{N}{(nN)^{1/2}}\Big(\frac{N}{T_0}\Big)^{-A}\sum_{c\ge1}c^{-3/2}(nN)^{3/2}T^{-1/2}U\\&\ll T^{-(1+\varepsilon)\{\varepsilon(A-3)-3\}+4a-1/2}U\ll1\end{aligned} \tag{3.12}$$

provided $A=A(\varepsilon)$ is suitably large.

It remains to bound the contribution from $N\le T_0^{1+\varepsilon}$ to (3.8). Similarly to (3.12), the contribution from $c>\sqrt{nN}$ to (3.8) is

$$\ll T^{1-2\sigma}U\sum_{N\le T_0^{1+\varepsilon}}\sum_{m\ge1}\sum_{n\le T^{2a}}\frac{\eta_N(m)}{(mn)^{1/2-2\sigma}}\sum_{c>\sqrt{nN}}\frac{1}{c^{2\sigma}}\ll\quad1 \tag{3.13}$$

provided $\sigma=\sigma(\varepsilon)<\sigma_0/2$ is suitably large.

Now, we use the representation (see [3, 8.411-11])

$$\frac{J_{2\mathrm{i}t}(2\pi x)-J_{-2\mathrm{i}t}(2\pi x)}{\cosh(\pi t)}=-\frac{2\mathrm{i}}{\pi}\tanh(\pi t)\int_{\mathbb{R}}\cos(2\pi x\cosh u)\mathrm{e}\Big(\frac{tu}{\pi}\Big)\,\mathrm{d}u.$$

For $c \le \sqrt{nN}$ and $|t| \le T^{1+\varepsilon}$, partial integration gives

$$\tag{3.14} \frac{J_{2\mathrm{i}t}(2\pi x) - J_{-2\mathrm{i}t}(2\pi x)}{\cosh(\pi t)} = -\frac{2\mathrm{i}}{\pi}\tanh(\pi t)\int_{-T^\varepsilon}^{T^\varepsilon} \cos(2\pi x\cosh u)\mathrm{e}\Big(\frac{tu}{\pi}\Big)\,\mathrm{d}u + O(T^{-A}).$$

Using this together with the definition of $h_1(t)$ and $2\cos z = \mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}$, we get

$$\tag{3.15} \begin{aligned} &H_1^+(2\sqrt{mn}/c) \\ &= \frac{2}{\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} th_1(t)\tanh(\pi t)\cos\Big(\frac{4\pi\sqrt{mn}}{c}\cosh u\Big)\mathrm{e}\Big(\frac{tu}{\pi}\Big)\,\mathrm{d}u\,\mathrm{d}t \\ &= \frac{4}{\pi^2}\int_{|t-T|\le UT^\varepsilon}\int_{-T^\varepsilon}^{T^\varepsilon} \frac{t\tanh(\pi t)}{(mn)^{\mathrm{i}t}\mathrm{e}^{(t-T)^2/U^2}}\cos\Big(\frac{4\pi\sqrt{mn}}{c}\cosh u\Big)\mathrm{e}\Big(\frac{tu}{\pi}\Big)\,\mathrm{d}u\,\mathrm{d}t \\ &\quad + O(T^{-A}) \\ &= \frac{2}{\pi^2}\sum_{\eta=\pm1}\int_{|t-T|\le UT^\varepsilon}\int_{-T^\varepsilon}^{T^\varepsilon} \frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}}\mathrm{e}\Big(\eta\frac{\mathrm{e}^u mn + \mathrm{e}^{-u}}{c} + \frac{tu}{\pi}\Big)\,\mathrm{d}u\,\mathrm{d}t + O(T^{-A}), \end{aligned}$$

where in the last step, we changed $u - \frac{1}{2}\log(mn) \mapsto u$. As in (3.5), the above t-integral is

$$\tag{3.16} \begin{aligned} &\int_{\mathbb{R}} t\mathrm{e}^{-(t-T)^2/U^2}\mathrm{e}(tu/\pi)\,\mathrm{d}t + O(T^{-A}) \\ &\quad = U\mathrm{e}(Tu/\pi)\int_{\mathbb{R}} \mathrm{e}^{-x^2}\big(T\cos(2Uux) + \mathrm{i}Ux\sin(2Uux)\big)\,\mathrm{d}x + O(T^{-A}) \\ &\quad = \sqrt{\pi}(TU + \mathrm{i}U^3u)\mathrm{e}(Tu/\pi)\mathrm{e}^{-(Uu)^2} + O(T^{-A}) = O(T^{-A}) \end{aligned}$$

provided $|u| \ge U^{-1/2+\varepsilon}$. Introducing a smooth partition $w_1(x) + w_2(x) \equiv 1$, where $w_1(x)$ is compactly supported on $[-2,2]$ and equals one on $[-1,1]$, inserting $w_1(u/U^{-1/2+\varepsilon}) + w_2(u/U^{-1/2+\varepsilon}) \equiv 1$ to the u-integral in (3.15) and using the above argument, one sees that

$$\tag{3.17} \begin{aligned} H_1^+(2\sqrt{mn}/c) = \frac{2}{\pi^2}\sum_{\eta=\pm1}\int_{|t-T|\le UT^\varepsilon}\int_{-T^\varepsilon}^{T^\varepsilon} & w_1\Big(\frac{u}{U^{-1/2+\varepsilon}}\Big) \\ &\times \frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}}\mathrm{e}(\varphi(u))\,\mathrm{d}u\,\mathrm{d}t + O(T^{-A}), \end{aligned}$$

where $\varphi(u) = \eta(\mathrm{e}^u mn + \mathrm{e}^{-u})/c + tu/\pi$. We consider the u-integral

$$\int_{-T^\varepsilon}^{T^\varepsilon} w_1\Big(\frac{u}{U^{-1/2+\varepsilon}}\Big)\mathrm{e}(\varphi(u))\,\mathrm{d}u.$$

For $100nN/T < c \le \sqrt{nN}$ or $c < nN/(100T)$, we have $|\varphi'(u)| \gg T$. And for $r \ge 2$, we have $\varphi^{(r)}(u) \ll T^{1+2a+\varepsilon}$. The derivative of the integral without the factor $\mathrm{e}(\varphi(u))$ is $O(U^{1/2-\varepsilon})$. Hence, by multiple partial integration, the contribution from these c is $O(T^{-A})$. Therefore, we only need to evaluate

$$\sum_{N\le T_0^{1+\varepsilon}} \sum_{n\ge 1} \frac{a_n\mu(n)}{n^{1/2}} \sum_{nN/(100T)\le c\le 100nN/T} \frac{1}{c} \sum_{\substack{d\,(\mathrm{mod}\, c)\\ (c,d)=1}} \mathrm{e}\Big(\frac{\bar{d}n}{c}\Big) \\ \times \int\limits_{|t-T|\le UT^{\varepsilon}} \int\limits_{-T^{\varepsilon}}^{T^{\varepsilon}} w_1\Big(\frac{u}{U^{-1/2+\varepsilon}}\Big) \frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}} \mathrm{e}\Big(\frac{tu}{\pi} + \eta\frac{e^{-u}}{c}\Big) g_+(u)\,\mathrm{d}u\,\mathrm{d}t, \tag{3.18}$$

where

$$g_{\pm}(u) := \sum_{m\ge 1} \frac{\eta_N(m)}{m^{1/2}} V_1\Big(\frac{m}{T_0}\Big) \mathrm{e}\Big(\frac{\pm(d+\eta n\mathrm{e}^u)m}{c}\Big). \tag{3.19}$$

Note that N cannot be very small now. In fact, $N \gg n^{-1}T$. We use the Poisson summation formula for $g_{\pm}(u)$ to get

$$g_{\pm}(u) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \frac{\eta_N(x)}{x^{1/2}} V_1\Big(\frac{x}{T_0}\Big) \mathrm{e}\Big(\frac{(kc \pm d \pm \eta n\mathrm{e}^u)x}{c}\Big)\,\mathrm{d}x.$$

For $|\mathrm{e}^u - 1|nN/c > N^{\varepsilon}$, every integration by parts in the above integral produces a saving of $O(N^{-\varepsilon})$. So doing this many times, we see that

$$g_{\pm}(u) \ll \sum_{k\in\mathbb{Z}} \Big(\frac{c}{|kc \pm d \pm \eta n\mathrm{e}^u|N}\Big)^A \ll T^{-A}.$$

Thus the contribution from these u to (3.18) is $O(T^{-A})$.

Whenever $|\mathrm{e}^u - 1|nN/c \le N^{\varepsilon}$, we have $\log(1 - c/(nN^{1-\varepsilon})) \le u \le \log(1 + c/(nN^{1-\varepsilon}))$. Thus, by a trivial estimate, its contribution to (3.18) is

$$\sum_{N\le T_0^{1+\varepsilon}} \sum_{n\ge 1} \frac{a_n\mu(n)}{n^{1/2}} \sum_{nN/(100T)\le c\le 100nN/T} \frac{1}{c} \sum_{\substack{d\,(\mathrm{mod}\, c)\\ (c,d)=1}} \mathrm{e}\Big(\frac{\bar{d}n}{c}\Big) \\ \times \int\limits_{|t-T|\le UT^{\varepsilon}} \int\limits_{\log(1-c/(nN^{1-\varepsilon}))}^{\log(1+c/(nN^{1-\varepsilon}))} w_1\Big(\frac{u}{U^{-1/2+\varepsilon}}\Big) \frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}} \mathrm{e}\Big(\frac{tu}{\pi} + \eta\frac{e^{-u}}{c}\Big) g_+(u)\,\mathrm{d}u \\ \ll T^{1/2+3a+\varepsilon}U.$$

Combining these with (3.12) and (3.13), one can see that the contribution from $H_1^+(2\sqrt{mn}/c)$ to $\mathcal{M}_1$ is $O(T^{1/2+3a+\varepsilon}U)$.

3.3. The contribution of $H_1^-(2\sqrt{mn}/c)$. The treatment is similar to that in Subsection 3.2. We need to estimate

$$\sum_{N\geq 1}\sum_{m\geq 1}\sum_{n\geq 1}\eta_N(m)V_1\left(\frac{m}{T_0}\right)\frac{a_n\mu(n)}{(mn)^{1/2}}\sum_{c\geq 1}\frac{S(-m,n;c)}{c}H_1^-\left(\frac{2\sqrt{mn}}{c}\right). \tag{3.20}$$

By the representation (see [25, p. 78])

$$K_\nu(z)=\frac{\pi}{2}\,\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\pi\nu)}, \tag{3.21}$$

where $I_\nu(z)$ is the I-Bessel function [3, 8.431-3]

$$I_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac12\right)\Gamma\left(\frac12\right)}\int_0^\pi e^{z\cos\theta}(\sin\theta)^{2\nu}\,d\theta\qquad(\Re e\,\nu>-1/2), \tag{3.22}$$

we have

$$H_1^-(x)=-\frac{4}{\pi}\int_{\mathbb{R}}\frac{I_{2it}(2\pi x)}{\sin(2\pi it)}th_1(t)\sinh(\pi t)\,dt. \tag{3.23}$$

Moving the integration line in $H_1^-(2\sqrt{mn}/c)$ to $\Im m\,t=-\sigma$, we have

$$H_1^-(2\sqrt{mn}/c)\ll(mn)^{2\sigma}c^{-2\sigma}T^{1-2\sigma}U \tag{3.24}$$

provided $c>\sqrt{nN}$.

On the other hand, for $c\leq\sqrt{nN}$, we use the integral representation [3, 8.432-4]

$$K_{2it}(2\pi x)=\frac{1}{2\cosh(\pi t)}\int_{-\infty}^{\infty}\cos(2\pi x\sinh u)e\left(-\frac{tu}{\pi}\right)du.$$

Integrating by parts once, we get, for $c\leq\sqrt{nN}$ and $|t|\leq T^{1+\varepsilon}$,

$$K_{2it}(2\pi x)=\frac{1}{2\cosh(\pi t)}\int_{-T^\varepsilon}^{T^\varepsilon}\cos(2\pi x\sinh u)e\left(-\frac{tu}{\pi}\right)du+O(T^{-A}). \tag{3.25}$$

Therefore, similarly to (3.15), we have

$$\begin{aligned}
&H_1^-(2\sqrt{mn}/c)\\
&=\frac{4}{\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{t\tanh(\pi t)}{(mn)^{it}e^{(t-T)^2/U^2}}\cos\left(\frac{4\pi\sqrt{mn}}{c}\sinh u\right)e\left(-\frac{tu}{\pi}\right)du\,dt\\
&=\frac{2}{\pi^2}\sum_{\eta=\pm1}\int_{|t-T|\leq UT^\varepsilon}\int_{-T^\varepsilon}^{T^\varepsilon}\frac{t\tanh(\pi t)}{e^{(t-T)^2/U^2}}e\left(\eta\frac{e^u-e^{-u}mn}{c}-\frac{tu}{\pi}\right)du\,dt+O(T^{-A}).
\end{aligned}$$

Using the above together with (3.24) and (2.15), one can see that the contribution from $N_0>T_0^{1+\varepsilon}$ or $c>\sqrt{nN}$ to (3.20) is $O(1)$ by choosing suitable σ.

Now we consider the contribution from $N \le T_0^{1+\varepsilon}$ and $c \le \sqrt{nN}$. As in (3.16), we calculate the t-integral to get

$$\int_{-T^\varepsilon}^{T^\varepsilon} \frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}} \mathrm{e}\Big(-\frac{tu}{\pi}\Big)\,\mathrm{d}t = \sqrt{\pi}\frac{TU - \mathrm{i}U^3u}{\mathrm{e}^{(Uu)^2}}\mathrm{e}\Big(-\frac{Tu}{\pi}\Big) + O(T^{-A}) = O(T^{-A})$$

if $|u| > U^{-1/2+\varepsilon}$.

As in Subsection 3.2, we insert the same partition $w_1(u/U^{-1/2+\varepsilon}) + w_2(u/U^{-1/2+\varepsilon}) \equiv 1$ into the u-integral, and get

$$\begin{aligned}(3.26)\qquad H_1^-(2\sqrt{mn}/c) = \frac{2}{\pi^2}\sum_{\eta=\pm1}\int_{|t-T|\le UT^\varepsilon}\int_{-T^\varepsilon}^{T^\varepsilon} w_1\Big(\frac{u}{U^{-1/2+\varepsilon}}\Big)\\ \times \frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}}\mathrm{e}\Big(\eta\frac{\mathrm{e}^u - \mathrm{e}^{-u}mn}{c} - \frac{tu}{\pi}\Big)\,\mathrm{d}u\,\mathrm{d}t + O(T^{-A}).\end{aligned}$$

For the u-integral in (3.26), one sees that the contribution from $100nN/T < c \le \sqrt{nN}$ or $c < nN/(100T)$ is $O(T^{-A})$ by partial integration. Thus, we only need to estimate

$$\begin{aligned}(3.27)\qquad &\sum_{N\le T_0^{1+\varepsilon}}\sum_{n\ge1}\frac{a_n\mu(n)}{n^{1/2}}\sum_{nN/(100T)\le c\le 100nN/T}\frac{1}{c}\sum_{\substack{d\,(\mathrm{mod}\,c)\\(c,d)=1}}\mathrm{e}\Big(\frac{\bar{d}n}{c}\Big)\\ &\times\int_{|t-T|\le UT^\varepsilon}\int_{-T^\varepsilon}^{T^\varepsilon} w_1\Big(\frac{u}{U^{-1/2+\varepsilon}}\Big)\frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}}\mathrm{e}\Big(-\frac{tu}{\pi}+\eta\frac{e^u}{c}\Big)g_-(-u)\,\mathrm{d}u\,\mathrm{d}t\\ &= \sum_{N\le T_0^{1+\varepsilon}}\sum_{n\ge1}\frac{a_n\mu(n)}{n^{1/2}}\sum_{nN/(100T)\le c\le 100nN/T}\frac{1}{c}\sum_{\substack{d\,(\mathrm{mod}\,c)\\(c,d)=1}}\mathrm{e}\Big(\frac{\bar{d}n}{c}\Big)\int_{|t-T|\le UT^\varepsilon}\\ &\times\int_{-T^\varepsilon}^{T^\varepsilon} w_1\Big(\frac{-u}{U^{-1/2+\varepsilon}}\Big)\frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}}\mathrm{e}\Big(\frac{tu}{\pi}+\eta\frac{e^{-u}}{c}\Big)g_-(u)\,\mathrm{d}u\,\mathrm{d}t,\end{aligned}$$

where $g_-(u)$ is defined as in (3.19).

It is almost the same as (3.18) except for some signs. So by the same argument, (3.27) contributes to $\mathcal{M}_1$ at most $O(T^{1/2+3a+\varepsilon}U)$. This completes the proof of (1.15).

4. Proof of (1.16). Let $G(s)$ be the function as in (2.9) and $\log T \le M \le T^{1-\varepsilon}$. Consider the integral

$$\frac{1}{2\pi\mathrm{i}}\int_{(\sigma_0)} G(s)\frac{\Lambda(s+\frac12+\mathrm{i}t_j,u_j)\Lambda(s+\frac12-\mathrm{i}t_j,u_j)}{\pi\Gamma(\frac14)^2\Gamma(\frac14+\mathrm{i}t_j)\Gamma(\frac14-\mathrm{i}t_j)}\,\frac{\mathrm{d}s}{s}.$$

By shifting the line of integration to $(-\sigma_0)$ and by applying the functional equation (1.8), we infer that

$$|L(1/2+\mathrm{i}t_j,u_j)|^2 = 2\sum_{m_1\geq 1}\sum_{m_2\geq 1}\frac{\lambda_j(m_1)\lambda_j(m_2)}{(m_1m_2)^{1/2}}\left(\frac{m_2}{m_1}\right)^{\mathrm{i}t_j}V_2(\pi^2 m_1m_2,t_j), \tag{4.1}$$

where $V_2(y,t)$ is defined as in (2.12). Inserting this and (1.10) into (1.13) and using the Hecke relation (1.5)

$$\lambda_j(m_1)\lambda_j(m_2) = \sum_{m|(m_1,m_2)}\lambda_j\left(\frac{m_1m_2}{m^2}\right),$$

$$\lambda_j(n_1)\lambda_j(n_2) = \sum_{n|(n_1,n_2)}\lambda_j\left(\frac{n_1n_2}{n^2}\right),$$

we can deduce, writing $m_1 = mm_3$, $m_2 = mm_4$, $n_1 = nn_3$ and $n_2 = nn_4$,

$$\mathcal{M}_2 = 2\sum_{m\geq 1}\sum_{n\geq 1}\sum_{m_3\geq 1}\sum_{m_4\geq 1}\sum_{n_3\geq 1}\sum_{n_4\geq 1}\frac{a_{nn_3}\mu(nn_3)a_{nn_4}\mu(nn_4)}{mn(m_3m_4n_3n_4)^{1/2}}\Xi_2(T;m,\mathbf{m},\mathbf{n}), \tag{4.2}$$

where $\mathbf{m} := (m_3,m_4)$, $\mathbf{n} := (n_3,n_4)$ and

$$\Xi_2(T;m,\mathbf{m},\mathbf{n}) := \sum_j{}' |v_j(1)|^2\frac{\mathrm{e}^{-(t_j-T)^2/U^2}}{(m_3n_3/(m_4n_4))^{\mathrm{i}t_j}}V_2(\pi^2m^2m_3m_4,t_j)\lambda_j(m_3m_4)\lambda_j(n_3n_4).$$

In order to evaluate $\Xi_2(T;m,\mathbf{m},\mathbf{n})$, we use Lemma 2.2 with the test function

$$h_2(t) := \left\{\frac{\mathrm{e}^{-(t-T)^2/U^2}}{(m_3n_3/(m_4n_4))^{\mathrm{i}t}}+\frac{\mathrm{e}^{-(t+T)^2/2}}{(m_4n_4/(m_3n_3))^{\mathrm{i}t}}\right\}V_2(\pi^2m^2m_3m_4,t). \tag{4.3}$$

We can check that $h_2(t)$ satisfies (2.1) in the region $|\Im m\, t| < \sigma_0/2$. As before we obtain

$$\begin{aligned}\Xi_2(T;m,\mathbf{m},\mathbf{n}) &= \sum_j{}' |v_j(1)|^2 h_2(t_j)\lambda_j(m_3m_4)\lambda_j(n_3n_4) + O(T^{-A}) \\ &= \frac{\delta_{|\mathbf{m}|,|\mathbf{n}|}}{\pi^2}H_2 - \frac{1}{\pi}\int_{\mathbb{R}}\frac{d_{\mathrm{i}t}(|\mathbf{m}|)d_{\mathrm{i}t}(|\mathbf{n}|)}{|\zeta(1+2\mathrm{i}t)|^2}h_2(t)\,\mathrm{d}t + O(T^{-A}) \\ &\quad + \sum_{\eta=\pm}\sum_{c\geq 1}\frac{S(\eta|\mathbf{m}|,|\mathbf{n}|;c)}{2c}H_2^{\eta}\left(\frac{2\sqrt{|\mathbf{m}|\,|\mathbf{n}|}}{c}\right),\end{aligned} \tag{4.4}$$

where $|\mathbf{m}| := m_3m_4$, $|\mathbf{n}| := n_3n_4$, and H_2, $H_2^+(x)$, $H_2^-(x)$ are defined as in (2.3), (2.4), (2.5) with the choice of $h_2(t)$ given in (4.3), respectively.

For the term involving $d_{\mathrm{i}t}(|\mathbf{m}|)d_{\mathrm{i}t}(|\mathbf{n}|)$, we see that its contribution to (4.2) is $O_\varepsilon(T^{1/2+2a+\varepsilon}U)$ by using $|\zeta(1+2\mathrm{i}t)| \gg_\varepsilon (|t|+2)^{-\varepsilon}$.

4.1. The contribution of H_2. Note that $|\mathbf{m}| = |\mathbf{n}|$. So we have $m_3n_3/(m_4n_4) = (n_3/m_4)^2$. Combining this and $\tanh(\pi t) = 1 + O(\mathrm{e}^{-\pi|t|})$, we infer that

$$\begin{aligned} H_2 &= \int\limits_{|t-T|\le T^\varepsilon U} \frac{t}{\mathrm{e}^{(t-T)^2/U^2}(n_3/m_4)^{2\mathrm{i}t}} V_2(\pi^2m^2n_3n_4,t)\,\mathrm{d}t + O(T^{-A}) \\ &= \int\limits_{|t-T|\le T^\varepsilon U} \frac{t}{\mathrm{e}^{(t-T)^2/U^2}(n_3/m_4)^{2\mathrm{i}t}} W_2(\pi^2m^2n_3n_4,t)\,\mathrm{d}t + O\biggl(\frac{t^{1/2+\varepsilon}U}{m(n_3n_4)^{1/2}}\biggr) \\ &= \int_0^\infty \frac{t}{\mathrm{e}^{(t-T)^2/U^2}(n_3/m_4)^{2\mathrm{i}t}} W_2(\pi^2m^2n_3n_4,t)\,\mathrm{d}t + O\biggl(\frac{t^{1/2+\varepsilon}U}{m(n_3n_4)^{1/2}}\biggr). \end{aligned}$$

Obviously, the contribution from the error term to (4.2) is $O_\varepsilon(T^{1/2+\varepsilon}U)$.

If we write $\sigma_a(n) := \sum_{d|n} d^a$, the contribution of the last integral in H_2 to (4.2) is

$$S_0 := \frac{2}{\pi^2}\int_0^\infty t\mathrm{e}^{-(t-T)^2/U^2} S(t)\,\mathrm{d}t, \tag{4.5}$$

where

$$S(t) = \sum_{(n,n_3,n_4)\in\mathbb{N}^3} \frac{a_{nn_3}\mu(nn_3)a_{nn_4}\mu(nn_4)}{nn_3^{1+2\mathrm{i}t}n_4}\sigma_{2\mathrm{i}t}(n_3n_4)\mathscr{W}_t(n_3n_4)$$

with

$$\begin{aligned} \mathscr{W}_t(\ell) &= \sum_{m\ge1}\frac{W_2(\pi^2m^2\ell,t)}{m} \\ &= \frac{1}{2\pi\mathrm{i}}\int\limits_{(1/2)} G(s)\frac{\Gamma\bigl(s+\frac12\bigr)^2}{\Gamma\bigl(\frac14\bigr)^2}\zeta(1+2s)\frac{(\pi^2\ell/t)^{-s}}{s}\,\mathrm{d}s. \end{aligned}$$

Further writing $n_3 = d_1n_5$ and $n_4 = d_1n_6$ with $(n_5,n_6)=1$ and using the Möbius formula for $\sum_{d_2|(n_5,n_6)}\mu(d_2)$ to remove $(n_5,n_6)=1$, we get

$$\begin{aligned} S(t) = {}&\sum_{d_1\ge1}\frac{\sigma_{2\mathrm{i}t}(d_1^2)}{d_1^{2+2\mathrm{i}t}}\sum_{d_2\ge1}\frac{\sigma_{2\mathrm{i}t}(d_2)^2\mu(d_2)}{d_2^{2+2\mathrm{i}t}}\sum_{n\ge1}\frac{\mu(d_1d_2n)^2}{n} \\ &\times \sum_{\substack{n_7\ge1\ n_8\ge1 \\ (n_7n_8,d_1d_2n)=1}} \frac{a_{d_1d_2nn_7}\mu(n_7)\sigma_{2\mathrm{i}t}(n_7)a_{d_1d_2nn_8}\mu(n_8)\sigma_{2\mathrm{i}t}(n_8)}{n_7^{1+2\mathrm{i}t}n_8}\mathscr{W}_t(d_1^2d_2^2n_7n_8). \end{aligned}$$

Expanding $\sigma_{2\mathrm{i}t}(n_i)$ $(i=7,8)$ shows that

$$\begin{aligned}S(t) = &\sum_{d_1\ge 1}\frac{\sigma_{2\mathrm{i}t}(d_1^2)}{d_1^{2+2\mathrm{i}t}}\sum_{d_2\ge 1}\frac{\sigma_{2\mathrm{i}t}(d_2)^2\mu(d_2)}{d_2^{2+2\mathrm{i}t}}\sum_{n\ge 1}\frac{\mu(d_1d_2n)^2}{n}\\ &\times \sum_{\substack{r_1\ge 1\\(r_1,d_1d_2n)=1}}\frac{\mu(r_1)}{r_1^{1+2\mathrm{i}t}}\sum_{\substack{n_9\ge 1\\(n_9,d_1d_2r_1n)=1}}\frac{a_{d_1d_2r_1nn_9}\mu(n_9)}{n_9}\\ &\times \sum_{\substack{r_2\ge 1\\(r_2,d_1d_2n)=1}}\frac{\mu(r_2)}{r_2^{1-2\mathrm{i}t}}\sum_{\substack{n_{10}\ge 1\\(n_{10},d_1d_2r_2n)=1}}\frac{a_{d_1d_2r_2nn_{10}}\mu(n_{10})}{n_{10}}\\ &\times \mathscr{W}_t(d_1^2d_2^2r_1r_2n_9n_{10}).\end{aligned}$$

To remove $(r_1,n_9)=1$ and $(r_2,n_{10})=1$, we use the Möbius formula (for $\sum_{d_3\mid(r_1,n_9)}\mu(d_3)$ and $\sum_{d_4\mid(r_2,n_{10})}\mu(d_4)$) again to find that

$$\begin{aligned}S(t) = &\sum_{d_1\ge 1}\frac{\sigma_{2\mathrm{i}t}(d_1^2)}{d_1^{2+2\mathrm{i}t}}\sum_{d_2\ge 1}\frac{\sigma_{2\mathrm{i}t}(d_2)^2\mu(d_2)}{d_2^{2+2\mathrm{i}t}}\sum_{n\ge 1}\frac{\mu(d_1d_2n)^2}{n}\\ &\times \sum_{\substack{d_3\ge 1\\(d_3,d_1d_2n)=1}}\frac{\mu(d_3)}{d_3^{2+2\mathrm{i}t}}\sum_{\substack{r_3\ge 1\\(r_3,d_1d_2d_3n)=1}}\frac{\mu(r_3)}{r_3^{1+2\mathrm{i}t}}\sum_{\substack{n_{11}\ge 1\\(n_{11},d_1d_2d_3n)=1}}\frac{a_{d_1d_2d_3^2r_3nn_{11}}\mu(n_{11})}{n_{11}}\\ &\times \sum_{\substack{d_4\ge 1\\(d_4,d_1d_2n)=1}}\frac{\mu(d_4)}{d_4^{2-2\mathrm{i}t}}\sum_{\substack{r_4\ge 1\\(r_4,d_1d_2d_4n)=1}}\frac{\mu(r_4)}{r_4^{1-2\mathrm{i}t}}\sum_{\substack{n_{12}\ge 1\\(n_{12},d_1d_2d_4n)=1}}\frac{a_{d_1d_2d_4^2r_4nn_{12}}\mu(n_{12})}{n_{12}}\\ &\times \mathscr{W}_t(d_1^2d_2^2d_3^2d_4^2r_3r_4n_{11}n_{12}).\end{aligned}$$

On the other hand, moving the line of integration to $\Re e\, s=-1/2+\varepsilon$ in $\mathscr{W}_t(\ell)$, we pass the double pole of the integrand at $s=0$. By the residue theorem, we infer that

$$\mathscr{W}_t(\ell)=c_1\log t+c_2\log\ell+c_3+O(\ell^{1/2-\varepsilon}t^{-1/2+\varepsilon}), \tag{4.6}$$

where c_1, c_2, and c_3 are constants. In our case, $\ell=(d_1d_2d_3d_4)^2r_3r_4n_{11}n_{12}=n_3n_4$. Since $n_3n_4\le T^{4a}$, the above error term contributes to S_0 at most $O(T^{1/2+2a+\varepsilon}U)$.

To consider the contribution from $\log t$ to S_0, our goal is to prove

$$\sum_{d_1,d_2,d_3,d_4\ge 1}\sum_{n\le\xi^2}\frac{\sigma_0(d_1^2)\sigma_0(d_2)^2|\mu(d_1d_2n)\mu(d_3)\mu(d_4)|}{(d_1d_2d_3d_4)^2n}I_{\mathbf{d},n}\ll_a TU, \tag{4.7}$$

where $\mathbf{d}:=(d_1,d_2,d_3,d_4)$ and

$$I_{\mathbf{d},n}:=\int_0^\infty\frac{t\log t}{\mathrm{e}^{(t-T)^2/U^2}}|S_1(t)|\,\mathrm{d}t \tag{4.8}$$

with

$$(4.9)\quad S_1(t) := \sum_{r_3\ge 1}^{*} \frac{\mu(r_3)}{r_3^{1+2it}} \sum_{n_{11}\ge 1}^{*} \frac{a_{d_1d_2d_3^2r_3nn_{11}}\mu(n_{11})}{n_{11}} \sum_{r_4\ge 1}^{**} \frac{\mu(r_4)}{r_4^{1-2it}} \sum_{n_{12}\ge 1}^{**} \frac{a_{d_1d_2d_4^2r_4nn_{12}}\mu(n_{12})}{n_{12}}.$$

Here $*$ and $**$ mean the condition $(\cdot, d_1d_2d_3n) = 1$ and $(\cdot, d_1d_2d_4n) = 1$, respectively.

We first deal with the n_{11}-sum, which is denoted by Σ_{11}. Note that $nr_3 \le T^{2a} = \xi^2$. We distinguish the cases $r_3 \le \xi/n$ and $\xi/n < r_3 \le \xi^2/n$. In the first case, by (1.11), the n_{11}-sum is equal to

$$(4.10)\quad \Sigma_{11} = \frac{1}{2\pi i}\int_{(3)} \frac{(\xi/(d_1d_2d_3^2nr_3))^s(\xi^s-1)}{s^2\zeta(1+s)} \prod_{p\mid d_1d_2d_3n}\left(1-\frac{1}{p^{1+s}}\right)^{-1} \frac{ds}{\log\xi}.$$

Recall (see [22, (3.11.7) and (3.11.8)]) that in the region $\Re e\, s \ge 1 - c/\log(|\Im m\, s|+3)$ (c is a positive constant), $\zeta(s)$ is analytic except for a single pole at $s=1$, has no zeros and satisfies $\zeta(s)^{-1} \ll \log(|\Im m\, s|+3)$ and $\zeta'(s)/\zeta(s) \ll \log(|\Im m\, s|+3)$. We move the line of integration in (4.10) to

$$(4.11)\quad \Gamma_\varepsilon := \{ix : |x|\ge\varepsilon\} \cup \{\varepsilon e^{i\vartheta} : \pi/2 \le \vartheta \le 3\pi/2\}$$

where ε is sufficiently small. There is no pole when doing this. So we have

$$(4.12)\quad \Sigma_{11} = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \prod_{p\mid d_1d_2d_3n}\left(1-\frac{1}{p^{s+1}}\right)^{-1} \frac{(\xi/(d_1d_2d_3^2nr_3))^s(\xi^s-1)}{s^2\zeta(1+s)\log\xi}\,ds.$$

If $\xi/n < r_3 \le \xi^2/n$, by noticing that

$$a_n = \frac{1}{2\pi i}\int_{(3)} \frac{(\xi^2/n)^s}{s^2}\frac{ds}{\log\xi}$$

and by moving the integration line in (4.10) to Γ_ε, it follows that

$$(4.13)\quad \Sigma_{11} = \frac{c_4}{\log\xi}\prod_{p\mid d_1d_2d_3n}\frac{p}{p-1} + \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \prod_{p\mid d_1d_2d_3n}\left(1-\frac{1}{p^{s+1}}\right)^{-1} \frac{(\xi^2/(d_1d_2d_3^2nr_3))^s}{s^2\zeta(1+s)\log\xi}\,ds,$$

where c_4 is a constant. Here the reason that we do not use (1.11) is that

$(\xi/(nr_3))^s$ may be large when $s \in \Gamma_\varepsilon$. We have similar expressions for the n_{12}-sum by the same argument. Inserting these into (4.9), we find that

$$\begin{aligned} S_1(t) = &\frac{1}{(\log \xi)^2} \\ &\times \biggl\{ \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_\varepsilon} \biggl(\frac{\xi^{2s_1} - \xi^{s_1}}{n^{s_1}} \sideset{}{^*}\sum_{r_3 \le \xi/n} \frac{\mu(r_3)}{r_3^{1+2\mathrm{i}t+s_1}} + \frac{\xi^{2s_1}}{n^{s_1}} \sideset{}{^*}\sum_{\xi/n < r_3 \le \xi^2/n} \frac{\mu(r_3)}{r_3^{1+2\mathrm{i}t+s_1}} \biggr) \\ &\times \prod_{p \mid d_1 d_2 d_3 n} \biggl(1 - \frac{1}{p^{s_1+1}} \biggr)^{-1} \frac{(d_1 d_2 d_3^2)^{-s_1}}{s_1^2 \zeta(1+s_1)} \, \mathrm{d}s_1 \\ &+ c_4 \prod_{p \mid d_1 d_2 d_3 n} \frac{p}{p-1} \sideset{}{^*}\sum_{\xi/n < r_3 \le \xi^2/n} \frac{\mu(r_3)}{r_3^{1+2\mathrm{i}t}} \biggr\} \\ &\times \biggl\{ \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_\varepsilon} \biggl(\frac{\xi^{2s_2} - \xi^{s_2}}{n^{s_2}} \sideset{}{^{**}}\sum_{r_4 \le \xi/n} \frac{\mu(r_4)}{r_4^{1-2\mathrm{i}t+s_2}} + \frac{\xi^{2s_2}}{n^{s_2}} \sideset{}{^{**}}\sum_{\xi/n < r_4 \le \xi^2/n} \frac{\mu(r_4)}{r_4^{1-2\mathrm{i}t+s_2}} \biggr) \\ &\times \prod_{p \mid d_1 d_2 d_4 n} \biggl(1 - \frac{1}{p^{s_2+1}} \biggr)^{-1} \frac{(d_1 d_2 d_4^2)^{-s_2}}{s_2^2 \zeta(1+s_2)} \, \mathrm{d}s_2 \\ &+ c_4 \prod_{p \mid d_1 d_2 d_4 n} \frac{p}{p-1} \sideset{}{^{**}}\sum_{\xi/n < r_4 \le \xi^2/n} \frac{\mu(r_4)}{r_4^{1-2\mathrm{i}t}} \biggr\}. \end{aligned}$$

After inserting this into (4.8), consider the resulting t-integral. Clearly a typical term of this t-integral is

$$\begin{aligned} I_{\text{typical}} &:= \int_0^\infty \frac{|t \log t|}{\mathrm{e}^{(t-T)^2/U^2}} \biggl| \sideset{}{^*}\sum_{r_3 \le \xi/n} \frac{\mu(r_3)}{r_3^{1+2\mathrm{i}t+s_1}} \biggr| \biggl| \sideset{}{^{**}}\sum_{r_4 \le \xi/n} \frac{\mu(r_4)}{r_4^{1-2\mathrm{i}t+s_2}} \biggr| \, \mathrm{d}t \\ &\le (I^*_{\text{typical}} I^{**}_{\text{typical}})^{1/2}, \end{aligned}$$

where

$$I^*_{\text{typical}} := \int_0^\infty \frac{|t \log t|}{\mathrm{e}^{(t-T)^2/U^2}} \biggl| \sideset{}{^*}\sum_{r_3 \le \xi/n} \frac{\mu(r_3)}{r_3^{1+2\mathrm{i}t+s_1}} \biggr|^2 \mathrm{d}t$$

and I^{**}_{typical} is defined similarly. By Lemma 2.3, we easily see that

$$\begin{aligned} I^*_{\text{typical}} &\ll T(\log T) \sum_{k \ge 0} \mathrm{e}^{-k^2} \log(k+2) \int_{T+kU}^{T+(k+1)U} \biggl| \sideset{}{^*}\sum_{r_3 \le \xi/n} \frac{\mu(r_3)}{r_3^{1+2\mathrm{i}t+s_1}} \biggr|^2 \mathrm{d}t \\ &\ll \frac{TU \log T}{(\xi/n)^{2\Re e\, s_1}} \end{aligned}$$

uniformly for $s_1 \in \Gamma_\varepsilon$, since $U \ge \log T$. Inserting these into (4.8), we obtain

$$I_{\mathbf{d},n} \ll_a \frac{TU}{\log \xi} \prod_{p \mid d_1 d_2 d_3 n} \left(1 - \frac{1}{p^{1-\varepsilon}}\right)^{-2},$$

which implies (4.7).

Similarly, we can deduce that the contribution from $c_2 \log(d_1^2 d_2^2 d_3^2 d_4^2 r_3 r_4) + c_3$ in (4.6) is $O_a(TU)$. For the term $\log(n_{11}n_{12}) = \log n_{11} + \log n_{12}$, we have

$$\sum_{n_{11}\ge 1}^{*} \frac{\mu(n_{11}) \log n_{11}}{n_{11}^{1+s}} = -\left\{ \frac{1}{\zeta(1+s)} \prod_{p \mid d_1 d_2 d_3 n} \left(1 - \frac{1}{p^{s+1}}\right)^{-1} \right\}'.$$

We can also use a similar argument to prove that its contribution to S_0 is at most $O_a(TU)$.

4.2. The contribution of $H_2^+(2\sqrt{|\mathbf{m}|\,|\mathbf{n}|}/c)$. The treatment is similar to that in the last section, so a sketch proof is enough. We partition the m_3-sum and the m_4-sum using smooth functions $\eta(x)$ as in (3.7). Therefore, we are led to estimate

$$\sum_{\mathbf{N}\in\mathbb{N}^2} \sum_{\mathfrak{k}\in\mathbb{N}^6} \frac{a_{nn_3}\mu(nn_3) a_{nn_4}\mu(nn_4)}{mn(m_3m_4n_3n_4)^{1/2}} \eta_{N_1}(m_3)\eta_{N_2}(m_4) \times \sum_{c\ge 1} \frac{S(|\mathbf{m}|, |\mathbf{n}|; c)}{c} H_2^+\left(\frac{2\sqrt{|\mathbf{m}|\,|\mathbf{n}|}}{c}\right),$$

where $\mathbf{N} := (N_1, N_2)$ and $\mathfrak{k} := (m, n, m_3, m_4, n_3, n_4)$. Without loss of generality, we suppose $N_2 \le N_1$. Moving the integration line of $H_2^+(x)$ to $\Im m\, t = -\sigma$, and using $V_2(\pi^2 m^2 m_3 m_4, -\sigma \mathrm{i} + y) \ll (m^2 m_3 m_4/y)^{-2\sigma-\varepsilon}$ by (2.17), we can see that the contribution from $N_1N_2 > T^{1+\varepsilon}$ or $c > \sqrt{n_3n_4N_1N_2}$ is $O(1)$. Thus, we can assume $N_1N_2 \le T^{1+\varepsilon}$ and $c \le \sqrt{n_3n_4N_1N_2}$.

By (3.14) and by a similar treatment to that in the last section, we obtain

$$H_2^+\left(\frac{2\sqrt{|\mathbf{m}|\,|\mathbf{n}|}}{c}\right) = \frac{2}{\pi^2} \sum_{\eta=\pm 1} \int_{|t-T|\le T^\varepsilon U} \int_{-T^\varepsilon}^{T^\varepsilon} \frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}} W_2(\pi^2 m^2 m_3 m_4, t) \times \mathrm{e}\left(\frac{tu}{\pi} + \eta \frac{\mathrm{e}^u m_3 n_3 + \mathrm{e}^{-u} m_4 n_4}{c}\right) \mathrm{d}u\, \mathrm{d}t + O_\varepsilon\left(\frac{T^{1/2+\varepsilon}U^{1+\varepsilon}}{m(m_3m_4)^{1/2}}\right). \tag{4.14}$$

By Weil's bound (2.7), the contribution from the error term to (4.2) is

$$\sum_{N_1N_2\le T^{1+\varepsilon}}\sum_{\mathfrak{k}\in\mathbb{N}^6}\frac{a_{nn_3}\eta_{N_1}(m_3)a_{nn_4}\eta_{N_2}(m_4)}{mn(m_3m_4n_3n_4)^{1/2}}\sum_{c\le\sqrt{n_3n_4N_1N_2}}c^{-1/2+\varepsilon} \times (m_3m_4,n_3n_4,c)^{1/2}\frac{T^{1/2+\varepsilon}U^{1+\varepsilon}}{m(m_3m_4)^{1/2}}\ll T^{3/4+5a+\varepsilon}U. \tag{4.15}$$

Now, we prove the contribution from $|u|\ge U^{-1/2+\varepsilon}$ is acceptable by considering the t-integral in (4.14)

$$\int\limits_{|t-T|\le T^\varepsilon U}\frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}}W_2(\pi^2m^2m_3m_4,t)\mathrm{e}\biggl(\frac{tu}{\pi}\biggr)\,\mathrm{d}t. \tag{4.16}$$

We distinguish the cases $T^\varepsilon<U\le T^{1-\varepsilon}$ and $\log T\le U\le T^\varepsilon$.

If $T^\varepsilon<U\le T^{1-\varepsilon}$ and $|u|\ge U^{-1+\varepsilon}$, by a single partial integration with respect to t, we can save $O((|u|U)^{-1})=O(U^{-\varepsilon})$. So after many integrations, we get

$$(4.16)=O(T^{-A}).$$

If $\log T\le U\le T^\varepsilon$, by using the differential mean value theorem, we have

$$(4.16)=W_2(\pi^2m^2m_3m_4,T)\int\limits_{|t-T|\le T^\varepsilon U}\frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}}\mathrm{e}\biggl(\frac{tu}{\pi}\biggr)\,\mathrm{d}t+O\bigl(T^\varepsilon U^2\bigr).$$

We use Weil's bound (2.7) again, and see that the contribution from the above error term to (4.2) is

$$\sum_{N_1N_2\le T^{1+\varepsilon}}\sum_{\mathfrak{k}\in\mathbb{N}^6}\frac{a_{nn_3}\eta_{N_1}(m_3)a_{nn_4}\eta_{N_2}(m_4)}{mn(m_3m_4n_3n_4)^{1/2}}\sum_{c\le\sqrt{n_3n_4N_1N_2}}c^{-1/2+\varepsilon} \times (m_3m_4,n_3n_4,c)^{1/2}T^\varepsilon U^2\ll T^{3/4+5a+\varepsilon}. \tag{4.17}$$

For the other integral, it is rapidly decreasing when $|u|\ge U^{-1/2+\varepsilon}$ by (3.16).

We repeat the argument of the last section. That is, inserting the same partition $w_1(u/U^{-1/2+\varepsilon})+w_2(u/U^{-1/2+\varepsilon})\equiv 1$ in the u-integral of (4.14) and using the above argument, one sees that we only need to consider the contribution from

$$\int\limits_{-T^\varepsilon}^{T^\varepsilon}w_1\biggl(\frac{u}{U^{-1/2+\varepsilon}}\biggr)\mathrm{e}(\phi(u))\,\mathrm{d}u,$$

where $\phi(u)=tu/\pi+\eta(\mathrm{e}^u m_3n_3+\mathrm{e}^{-u}m_4n_4)/c$. For $100n_3N_1/T<c\le n_3T^\varepsilon$ or $c\le n_3N_1/(100T)$, we have $|\phi'(u)|\gg T$. And for $r\ge 2$, we have $\phi^{(r)}(u)\ll T^{1+2a+\varepsilon}$. The derivative of the integral without the factor $\mathrm{e}(\phi(u))$

is $O(U^{1/2-\varepsilon})$. Hence, by multiple partial integration, the contribution from these c is $O(T^{-A})$. So we are led to estimate

$$\begin{aligned}(4.18)\quad & \sum_{m\ge1}\sum_{n\ge1}\sum_{m_4\ge1}\sum_{n_3\ge1}\sum_{n_4\ge1}\eta_{N_2}(m_4)\frac{a_{nn_3}\mu(nn_3)a_{nn_4}\mu(nn_4)}{mn(m_4n_3n_4)^{1/2}}\\ &\times\sum_{n_3N_1/(100T)\le c\le 100n_3N_1/T}\frac{1}{c}\sum_{\substack{d\,(\mathrm{mod}\,c)\\(c,d)=1}}\mathrm{e}\left(\frac{\bar{d}n_3n_4}{c}\right)\\ &\times\int_{|t-T|\le UT^{\varepsilon}}\int_{-T^{\varepsilon}}^{T^{\varepsilon}} w_1\left(\frac{u}{U^{-1/2+\varepsilon}}\right)\frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}}\mathrm{e}\left(\frac{tu}{\pi}+\eta\frac{\mathrm{e}^{-u}m_4n_4}{c}\right)\\ &\times\sum_{m_3\ge1}\frac{\eta_{N_1}(m_3)}{m_3^{1/2}}W_2(\pi^2m^2m_3m_4,t)\mathrm{e}\left(\frac{dm_3m_4+\eta\mathrm{e}^um_3n_3}{c}\right)\mathrm{d}u\,\mathrm{d}t.\end{aligned}$$

We have $N_1\gg n_3^{-1}T$ and $N_2\ll n_3T^{\varepsilon}$ now. For the m_3-sum, we use the Poisson summation formula to obtain

$$\begin{aligned}(4.19)\quad & \sum_{m_3\ge1}\frac{\eta_{N_1}(m_3)}{m_3^{1/2}}W_2(\pi^2m^2m_3m_4,t)\mathrm{e}\left(\frac{dm_3m_4+\eta\mathrm{e}^um_3n_3}{c}\right)\\ &=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\frac{\eta_{N_1}(x)}{x^{1/2}}W_2(\pi^2m^2m_4x,t)\mathrm{e}\left(\frac{(kc+dm_4+\eta\mathrm{e}^un_3)x}{c}\right)\mathrm{d}x.\end{aligned}$$

If $|\mathrm{e}^u-1|n_3N_1/c>N_1^{\varepsilon}$, we can integrate by parts many times to see that (4.19) is negligible. For $|\mathrm{e}^u-1|n_3N_1/c\le N_1^{\varepsilon}$ ($|u|\ll c/(n_3N_1^{1-\varepsilon})$), one can find that the contribution from these u in (4.18) to (4.2) is $O_\varepsilon(T^{1/2+4a+\varepsilon}U)$ by a trivial estimate.

Based on the above argument, we infer that $H_2^+(2\sqrt{|\mathbf{m}|\,|\mathbf{n}|}/c)$ contributes to (4.2) at most $O_\varepsilon(T^{3/4+5a+4\varepsilon}U)$.

4.3. The contribution of $H_2^-(2\sqrt{|\mathbf{m}|\,|\mathbf{n}|}/c)$. As in Subsection 4.2, we use the smooth functions η_{N_i} ($i=1,2$) to partition the m_3-sum and m_4-sum, and suppose $N_2\le N_1$. And we are led to estimate

$$\begin{aligned}&\sum_{\mathbf{N}\in\mathbb{N}^2}\sum_{\mathfrak{k}\in\mathbb{N}^6}\eta_{N_1}(m_3)\eta_{N_2}(m_4)\frac{a_{nn_3}\mu(nn_3)a_{nn_4}\mu(nn_4)}{mn(m_3m_4n_3n_4)^{1/2}}\\ &\qquad\times\sum_{c\ge1}\frac{S(-|\mathbf{m}|,|\mathbf{n}|;c)}{c}H_2^-\left(\frac{2\sqrt{|\mathbf{m}|\,|\mathbf{n}|}}{c}\right),\end{aligned}$$

where $\mathbf{N}:=(N_1,N_2)$ and $\mathfrak{k}:=(m,n,m_3,m_4,n_3,n_4)$. As before, the main contribution comes from $N_1N_2\le T^{1+\varepsilon}$ and $c\le\sqrt{n_3n_4N_1N_2}$.

After a similar argument to that in the last section, we get

$$H_2^-\left(\frac{2\sqrt{|\mathbf{m}|\,|\mathbf{n}|}}{c}\right) = \frac{2}{\pi^2}\sum_{\eta=\pm 1}\int_{|t-T|\le T^\varepsilon U}\int_{-T^\varepsilon}^{T^\varepsilon}\frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}}W_2(\pi^2 m^2 m_3 m_4, t) \times \mathrm{e}\left(\eta\frac{\mathrm{e}^u m_4 n_4 - \mathrm{e}^{-u} m_3 n_3}{c} - \frac{tu}{\pi}\right)\mathrm{d}u\,\mathrm{d}t + O(T^{-A}).$$

We can see that the term we need to estimate is very similar to the one involving the J-Bessel function. They only differ by the sign of some parameters. We can use the same treatment to prove that the contribution from $H_2^-(2\sqrt{|\mathbf{m}|\,|\mathbf{n}|}/c)$ to (4.2) is $O_\varepsilon(T^{3/4+5a+\varepsilon}U)$.

Since $a \in (0, 1/20-\varepsilon)$, we have completed the proof of (1.16).

5. Proof of (1.17). We will prove (1.17) by following the idea of [17, Lemma 5]. There are differences in some details, so, for completeness, we will give the whole proof. It is known that

$$2|v_j(1)|^2 = L(1, \operatorname{sym}^2 u_j)^{-1}. \tag{5.1}$$

For $\Re e\, s > 1$, we have

$$L(s, \operatorname{sym}^2 u_j)^{-1} = \prod_p \left(1 - \frac{\lambda_j(p^2)}{p^s} + \frac{\lambda_j(p^2)}{p^{2s}} - \frac{1}{p^{3s}}\right) = A_j(s)B_j(s), \tag{5.2}$$

where

$$A_j(s) := \prod_p \left(1 - \frac{\lambda_j(p^2)}{p^s}\right), \quad B_j(s) := \prod_p \left(1 + \frac{\lambda_j(p^2)p^{-2s} - p^{-3s}}{1 - \lambda_j(p^2)p^{-s}}\right).$$

We know that $B_j(s)$ is analytic and has no zero for $\Re e\, s > 9/10$, and both $B_j(s)$ and $B_j^{-1}(s)$ are uniformly bounded in this region. Therefore,

$$L(1, \operatorname{sym}^2 u_j)^{-1} \le C_0 A_j(1) \tag{5.3}$$

with some absolute constant $C_0 > 0$. On the other hand, by [22, Lemma, §7.9], we have

$$\sum_{n\ge 1}\frac{\lambda_j(n^2)\mu(n)}{n}\mathrm{e}^{-n/T} = \frac{1}{2\pi \mathrm{i}}\int_{(2)} A_j(s+1)\Gamma(s)T^s\,\mathrm{d}s.$$

Moving the line of integration to the path Γ_ε defined as in (4.11), we pass a simple pole at $s = 0$ with residue $A_j(1)$. Combining the resulting expression for $A_j(1)$ with (5.1) and (5.3), we find that

$$|v_j(1)|^2 \le \frac{C_0}{2}\left(\sum_{n\ge 1}\frac{\lambda_j(n^2)\mu(n)}{n}\mathrm{e}^{-n/T} - K_j\right),$$

where

$$K_j := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\varepsilon} A_j(s+1)\Gamma(s)T^s \,\mathrm{d}s.$$

Thus

$$\begin{aligned}
(5.4)\qquad \mathcal{J} &\le \frac{C_0}{2}\bigg(\sum_{n\ge 1}\frac{\mu(n)}{n}\mathrm{e}^{-n/T}\sum_j \frac{|v_j(1)|^2\lambda_j(n^2)}{\mathrm{e}^{(t_j-T)^2/U^2}} - \sum_j \frac{|v_j(1)|^2K_j}{\mathrm{e}^{(t_j-T)^2/U^2}}\bigg)\\
&= \frac{C_0}{2}\bigg(\sum_{n\le T(\log T)^{5/4}}\frac{\mu(n)}{n}\mathrm{e}^{-n/T}\sum_j \frac{|v_j(1)|^2\lambda_j(n^2)}{\mathrm{e}^{(t_j-T)^2/U^2}} - \sum_j \frac{|v_j(1)|^2K_j}{\mathrm{e}^{(t_j-T)^2/U^2}}\bigg)\\
&\quad + O(T^{-A}).
\end{aligned}$$

By Lemma 2.1 with $h_0(t) := \mathrm{e}^{-(t-T)^2/U^2} + \mathrm{e}^{-(t+T)^2/U^2}$, we have

$$\begin{aligned}
\sum_j \frac{|v_j(1)|^2\lambda_j(n^2)}{\mathrm{e}^{(t_j-T)^2/U^2}} &= \frac{2\delta_{n^2,1}}{\pi^{3/2}}TU + \frac{2\mathrm{i}}{\pi}\sum_{c\ge 1}\frac{S(n^2,1;c)}{c}\int_{\mathbb{R}}\frac{J_{2\mathrm{i}t}(4\pi n/c)}{\cosh(\pi t)}th_0(t)\,\mathrm{d}t\\
&\quad + O(T^\varepsilon U).
\end{aligned}$$

If $c > n(\log T)^{1/4}/T$, by moving the integration line in the last integral to $\Im m\, t = -A$ and by using (3.10), we get

$$\int_{\mathbb{R}}\frac{J_{2\mathrm{i}t}(4\pi n/c)}{\cosh(\pi t)}th_0(t)\,\mathrm{d}t \ll TU\bigg(\frac{n}{cT}\bigg)^{2A},$$

which implies that its contribution to (5.4) is $O(TU(\log T)^{-A})$.

If $c \le n(\log T)^{1/4}/T$ (note that $n \ge T/(\log T)^{1/4}$ now), by using (3.14) and (3.16), we have

$$\begin{aligned}
&\int_{\mathbb{R}}\frac{J_{2\mathrm{i}t}(4\pi n/c)}{\cosh(\pi t)}th_0(t)\,\mathrm{d}t\\
&\qquad = -\frac{2\mathrm{i}}{\pi}\int_{\mathbb{R}}\bigg\{\cos\bigg(\frac{4\pi n}{c}\cosh u\bigg)\int_{\mathbb{R}}\frac{t\tanh(\pi t)}{\mathrm{e}^{(t-T)^2/U^2}}\mathrm{e}\bigg(\frac{tu}{\pi}\bigg)\,\mathrm{d}t\bigg\}\,\mathrm{d}u\\
&\qquad \ll \int_0^\infty (TU + U^3u)\mathrm{e}^{-(Uu)^2}\,\mathrm{d}u = \int_0^\infty (T+Uu)\mathrm{e}^{-u^2}\,\mathrm{d}u \ll T.
\end{aligned}$$

By combining these estimates with Weil's bound (2.7), the contribution from $c \le n(\log T)^{1/4}/T$ to (5.4) is no more than

$$\sum_{T/(\log T)^{1/4}\le n\le T(\log T)^{5/4}}\frac{T\mathrm{e}^{-n/T}}{n}\sum_{c\le n(\log T)^{1/4}/T}\frac{|S(n^2,1;c)|}{c} \ll T(\log T)^{4/5}.$$

Consequently,

$$\sum_{n\geq 1}\frac{\mu(n)}{n}\mathrm{e}^{-n/T}\sum_j \frac{|v_j(1)|^2\lambda_j(n^2)}{\mathrm{e}^{(t_j-T)^2/U^2}} = 2\pi^{-3/2}TU + o(TU). \tag{5.5}$$

Next we shall treat the second j-sum in (5.4) by using Lemmas 2.5 and 2.6. According to whether or not $L(s,\mathrm{sym}^2 u_j)$ is zero free in the domain

$$1-10\varepsilon_0\leq\sigma\leq 1,\qquad |t|\leq(\log T)^3,$$

we divide the set $\{j : |t_j - T| \leq T^{\varepsilon_0/10}U\}$ into two subsets J_1 and J_2. If $j\in J_1$, we shift the line of integration in K_j to

$$\{-\varepsilon_0/2+\mathrm{i}t : |t|\leq(\log T)^2\}\cup\{\sigma\pm \mathrm{i}(\log T)^2 : -\varepsilon_0/2\leq\sigma\leq 1\}$$
$$\cup\,\{1+\mathrm{i}t : |t|\geq(\log T)^2\}.$$

Then, from Lemma 2.6, we know that $L^{-1}(s,\mathrm{sym}^2 u_j)\ll_{\varepsilon_0} T^{\varepsilon_0/20}$ for $\sigma\geq 1-\varepsilon_0/2$ and $|t|\leq(\log T)^2$, while for $\sigma=2$ and $|t|\geq(\log T)^2$, this inequality is trivial. Thus, by Stirling's formula and the factorization (5.2), we have $K_j\ll_{\varepsilon_0} T^{-2\varepsilon_0/5}$. By the Weyl law, we have $|J_1|\ll T^{1+\varepsilon_0/10}U$, which implies that

$$\sum_{j\in J_1}|K_j|\ll T^{1-3\varepsilon_0/10}U. \tag{5.6}$$

If $j\in J_2$, we shift the integration line to $\Re e\, s=\varepsilon_0/20$ to get

$$K_j=-A_j(1)+\frac{1}{2\pi\mathrm{i}}\int_{(\varepsilon_0/20)}A_j(s+1)\Gamma(s)T^s\,\mathrm{d}s\ll(t_jT)^{\varepsilon_0/10},$$

in view of the bound (see [5]) $A_j(1)\ll t_j^{\varepsilon_0/20}$. On the other hand, Lemma 2.5 implies that $|J_2|\ll T^{1/5}$ if ε_0 is sufficiently small. Combining these with (1.3) and (5.6), we find that

$$\begin{aligned}\sum_j K_j|v_j(1)|^2\mathrm{e}^{-(t_j-T)^2/U^2} &\ll\Big(\sum_{j\in J_1}|K_j|+\sum_{j\in J_2}|K_j|\Big)T^{\varepsilon_0/10}\\ &= o(TU).\end{aligned} \tag{5.7}$$

Now (1.17) follows from (5.5) and (5.7).

6. Proof of Theorem 1.1. First by the Hölder inequality, we deduce

$$|\mathcal{M}_1|\leq\Big(\mathcal{M}_2^2\mathcal{J}\sum_{j:\,L(1/2+\mathrm{i}t_j,u_j)\neq 0}{}' \mathrm{e}^{-(t_j-T)^2/U^2}\Big)^{1/4}.$$

This implies, via Proposition 1.2, that

$$\sum_{j:\,L(1/2+\mathrm{i}t_j,u_j)\neq 0}{}' \mathrm{e}^{-(t_j-T)^2/U^2}\geq C_1TU$$

for $\log T \le U \le T^{1-\varepsilon}$, where $C_1 > 0$ is an absolute constant. On the other hand, the Weyl law (1.4) allows us to write, for any constant $A_0 > 0$,

$$\sum_{j:\,|t_j-T|\ge A_0U}{}' \mathrm{e}^{-(t_j-T)^2/U^2} \le C_2 \mathrm{e}^{-A_0^2/2} TU,$$

where $C_2 > 0$ is an absolute constant. Thus, we deduce that

$$(C_1 - C_2\mathrm{e}^{-A_0^2/2})TU \le \sum_{\substack{|t_j-T|\le A_0U\\ L(1/2+\mathrm{i}t_j,u_j)\neq 0}}{}' \mathrm{e}^{-(t_j-T)^2/U^2} \le \sum_{\substack{|t_j-T|\le A_0U\\ L(1/2+\mathrm{i}t_j,u_j)\neq 0}}{}' 1,$$

which implies (1.9) for $\log T \le U \le T^{1-\varepsilon}$, provided $A_0 > \sqrt{2\log(C_2/C_1)}$. When $T^{1-\varepsilon} \le U \le T$, we divide $[T-U, T+U]$ into $O(U/T^{1-\varepsilon})$ subintervals of length $T^{1-\varepsilon}$ and apply the previous result to every subinterval. Thus we get (1.9) for $T^{1-\varepsilon} \le U \le T$. This completes the proof of Theorem 1.1.

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