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Natural numbers n for which $\lfloor n\alpha + s\rfloor \neq \lfloor n\beta + s\rfloor$

by

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1. Introduction. The sequence $\lfloor n\alpha\rfloor$, with $\alpha \in \mathbb{R}$, has been studied many times. A usual approach is to replace it with the simpler sequence $\lfloor na/b\rfloor$, where a/b is a rational approximation to α such as an approximant of the continued fraction of α. Here $\lfloor x\rfloor$ is the greatest integer less than or equal to x. Knowing how close a/b is to α, the natural question is how close the corresponding sequences $\lfloor n\alpha\rfloor$ and $\lfloor na/b\rfloor$ are to one another.

A typical example occurs when one studies the function $\sum_{n\ge 0} x^{\lfloor n\alpha\rfloor}$. This sum at first looks difficult to handle, but the key is to realize that it is exceptionally well approximated by the function $\sum_{n\ge 0} x^{\lfloor na/b\rfloor}$ when a/b is a good rational approximation to α. It is not difficult to show that this latter sum is in fact a rational function and one gets excellent information about the original function.

In [2] the question of exactly how well $\lfloor na/b\rfloor$ could approximate $\lfloor n\alpha\rfloor$ was thoroughly studied. This was motivated by the observation in [4] that

$$\left\lfloor n\frac{1+\sqrt{5}}{2}\right\rfloor = \left\lfloor n\frac{F_{i+1}}{F_i}\right\rfloor \tag{1}$$

for $0 \le n < F_{i+2}$. (Here, as usual, F_i is the ith Fibonacci number with $F_0 = 0$.) On the other hand, for the purpose of studying discrepancy, it was proved in [9] that

$$\lfloor n\alpha\rfloor = \left\lfloor n\frac{p_i}{q_i}\right\rfloor \tag{2}$$

for $0 \le n < q_{i+1}$. (Here p_i/q_i is the ith convergent of the regular continued of α.) It is not difficult to see that the interval of equality in (1) is optimal; at $n = F_{i+2}$ the two greatest integer functions are no longer equal. On the

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other hand, the interval of equality in (2) is not always maximal, (1) being an example of this.

This led one of the authors to consider in [2] the least natural number n for which the sequences $\lfloor n\alpha\rfloor$ and $\lfloor na/b\rfloor$ are not equal. In fact, in [2] the more general case of two *real numbers* α and β was considered, and this brought into consideration a whole new topological perspective. The least integer n for which $\lfloor n\alpha\rfloor \neq \lfloor n\beta\rfloor$ was denoted by $\Psi(\beta,\alpha)$. Furthermore, in [2], the more general problem was solved, of finding, for given real numbers α, β and s, the least natural number n for which $\lfloor n\alpha+s\rfloor \neq \lfloor n\beta+s\rfloor$. This number n was denoted by $\Psi(\beta,\alpha;s)$.

In this paper we move on to consider two basic problems which are motivated by the work in [2]. The first problem is to study the expected value and other statistical quantities related to $\Psi(\beta,\alpha;0)$. We give asymptotic formulas for the probability that $\Psi(\beta,\alpha;0)$ is greater than a positive integer Q and compute the expected value of $\Psi(\beta,\alpha;0)$. This last constant is found to have the value 4. Typically after the first value of n for which $\lfloor n\alpha+s\rfloor \neq \lfloor n\beta+s\rfloor$ there will be an interval of equality again, followed by a second point of inequality and so on, until they are eventually unequal for all large n. Our second problem is to investigate the successive points of inequality. We define $\Psi_k(\beta,\alpha;s)$ to be the kth natural number n for which $\lfloor n\alpha+s\rfloor \neq \lfloor n\beta+s\rfloor$. We give a theoretical formula for Ψ_k. We also denote by $\mathcal{N}(\beta,\alpha;s)$ the set of positive integers n for which $\lfloor n\alpha+s\rfloor \neq \lfloor n\beta+s\rfloor$. It is easy to see that $\mathcal{N}(\beta,\alpha;0)$ is a semigroup. We determine a set of generators (in a certain sense) for the semigroup and characterize it in various ways.

2. The probability $P(\Psi > Q)$. In this section we investigate how often $\Psi(\alpha,\beta)$ is large. More precisely, we want to establish an asymptotic formula for the probability, call it $P(\Psi > Q)$, that $\Psi(\alpha,\beta) > Q$, where Q is a large positive integer and α,β vary independently in $[0,1]$. We first look at the probability, call it $P(\Psi = N)$, that $\Psi(\alpha,\beta) = N$, where N is a given large positive integer and α,β vary independently in $[0,1]$. We could, of course, allow α,β to take also real values larger than 1, but by Proposition 3.1 of [2] we know that if $\Psi(\alpha,\beta) = N > 1$ then $\lfloor\alpha\rfloor = \lfloor\beta\rfloor$ and $\Psi(\{\alpha\},\{\beta\}) = N$. In view of this periodicity, in the following we will only consider the case $\alpha,\beta\in[0,1]$. Denote

$$\mathcal{M}_N = \{(\alpha,\beta)\in[0,1]\times[0,1] : \Psi(\alpha,\beta) = N\}. \tag{3}$$

If we denote by μ the Lebesgue measure on $[0,1]\times[0,1]$, then

$$P(\Psi = N) = \mu(\mathcal{M}_N). \tag{4}$$

Since the set of points $(\alpha,\beta)\in[0,1]\times[0,1]$ for which at least one of α,β is rational has measure zero, we may restrict in what follows to the case when both α and β are irrational.

Fix now an integer $N > 1$ and irrational numbers $0 < \alpha < \beta < 1$ such that $\Psi(\alpha,\beta) = N$. Then choose a large positive integer Q and consider the Farey sequence $\mathcal{F}_Q$ of order Q. The intersection of $\mathcal{F}_Q$ with the interval (α,β) consists of, say, M points, which we arrange in increasing order: $\gamma_1 < \cdots < \gamma_M$. By a repeated application of Proposition 3.1(d) of [2], we see that

$$N = \Psi(\alpha,\beta) = \min\{\Psi(\alpha,\gamma_1), \Psi(\gamma_1,\gamma_2), \ldots, \Psi(\gamma_{M-1},\gamma_M), \Psi(\gamma_M,\beta)\}. \tag{5}$$

Note that, since α and β are irrational, the numbers $n\alpha$ and $n\beta$ with $1 \leq n \leq N$ are not integers. Thus there exists an $\epsilon > 0$, depending on α, β and N, such that for any γ with $\alpha < \gamma < \alpha + \epsilon$ one has $\Psi(\alpha,\gamma) > N$, and similarly for any γ with $\beta - \epsilon < \gamma < \beta$ one has $\Psi(\gamma,\beta) > N$. We choose our Q to be large enough so that $\gamma_1 \in (\alpha, \alpha+\epsilon)$ and $\gamma_M \in (\beta-\epsilon, \beta)$. Then

$$\min\{\Psi(\alpha,\gamma_1), \Psi(\gamma_M,\beta)\} > N. \tag{6}$$

On combining (5) with (6) we find that

$$\Psi(\alpha,\beta) = \min\{\Psi(\gamma_1,\gamma_2), \ldots, \Psi(\gamma_{M-1},\gamma_M)\}. \tag{7}$$

Next, if we denote for $j = 1, \ldots, M$ by a_j and q_j the numerator and respectively the denominator of γ_j, written in its lowest terms, then by a fundamental property of Farey sequences we know that $a_{j+1}q_j - a_jq_{j+1} = 1$ for any $1 \leq j \leq M-1$. Therefore we may apply Corollary 4.1 of [2], which, in our case, states that

$$\Psi\left(\frac{a_j}{q_j}, \frac{a_{j+1}}{q_{j+1}}\right) = q_{j+1} \tag{8}$$

for $1 \leq j \leq M-1$. From (5), (7) and (8) it follows that

$$N = \min\{q_2, q_3, \ldots, q_M\}. \tag{9}$$

As a consequence, no element of the Farey sequence $\mathcal{F}_{N-1}$ of order $N-1$ lies inside the interval (α,β), otherwise one of the numbers q_j from the right side of (9) will have to be strictly less than N, contradicting (9). On the other hand, one of the numbers q_j from the right side of (9) equals N, thus the Farey sequence $\mathcal{F}_N$ does have at least one element inside the interval (α,β).

We may interpret the above results in the following way. Let $N > 1$ be an integer and consider the Farey sequences $\mathcal{F}_{N-1}$ and $\mathcal{F}_N$. Then, if α and β are irrational numbers with $0 < \alpha < \beta < 1$ and such that $\Psi(\alpha,\beta) = N$, the interval (α,β) will contain an element a/N, with $1 \leq a < N$ and $(a,N) = 1$, from $\mathcal{F}_N$, but it will contain no element from $\mathcal{F}_{N-1}$. If $\gamma' = a'/q'$ and $\gamma'' = a''/q''$ are the left and respectively right neighbors of a/N in $\mathcal{F}_N$, and if we denote the open intervals $(\gamma', a/N)$ and $(a/N, \gamma'')$ by $I_{a/N}$ and respectively $J_{a/N}$, then $\gamma', \gamma'' \in \mathcal{F}_{N-1}$, and one must have $\alpha \in I_{a/N}$ and

$\beta \in J_{a/N}$. By the basic properties of Farey fractions we know that the length $|I_{a/N}|$ of the interval $I_{a/N}$ satisfies

$$|I_{a/N}| = \frac{1}{q'N}, \tag{10}$$

and similarly,

$$|J_{a/N}| = \frac{1}{q''N}. \tag{11}$$

One also has the equality

$$q' + q'' = N. \tag{12}$$

Let us suppose, conversely, that the interval (α, β) contains an element of $\mathcal{F}_N$ but does not contain any element of $\mathcal{F}_{N-1}$. Then (9) holds, and from (7) and (8) one further obtains $\Psi(\alpha, \beta) = N$. In conclusion, if α, β are irrational numbers with $0 < \alpha < \beta < 1$, then the pair (α, β) belongs to $\mathcal{M}_N$ if and only if it belongs to the union

$$\mathcal{U}_N := \bigcup_{(a,N)=1} I_{a/N} \times J_{a/N}. \tag{13}$$

By (4) and (13) one finds that

$$P(\Psi = N) = 2\mu(\mathcal{U}_N) = 2 \sum_{\substack{1 \le a \le N \\ (a,N)=1}} |I_{a/N}| \cdot |J_{a/N}|. \tag{14}$$

On combining this with (10)–(12) we derive

$$P(\Psi = N) = \frac{2}{N^2} \sum_{\substack{1 \le a \le N \\ (a,N)=1}} \frac{1}{q'(N - q')}. \tag{15}$$

Now, for any $1 \le q \le N$ with $(q, N) = 1$, there exists a unique pair of consecutive elements of $\mathcal{F}_N$ with denominators q and N (in this order). It follows that

$$P(\Psi = N) = \frac{2}{N^2} \sum_{\substack{1 \le q \le N \\ (q,N)=1}} \frac{1}{q(N - q)} = \frac{4}{N^3} \sum_{\substack{1 \le q \le N \\ (q,N)=1}} \frac{1}{q}. \tag{16}$$

This gives an exact expression for the probability $P(\Psi = N)$. An asymptotic formula for the probability $P(\Psi > Q)$ that $\Psi(\alpha, \beta) > Q$, for a large positive integer Q, where as before α and β vary independently in $[0, 1]$, can be derived as follows. If we repeat the arguments from this section with the condition $\Psi(\alpha, \beta) = N$ replaced by the condition $\Psi(\alpha, \beta) > Q$, where α, β are irrational numbers satisfying $0 < \alpha < \beta < 1$, we see that $\Psi(\alpha, \beta) > Q$ if and only if the interval (α, β) does not contain any element of $\mathcal{F}_Q$. In other words, if $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{N(Q)} = 1$ denote the Farey fractions of order Q, then $\Psi(\alpha, \beta) > Q$ if and only if α and β belong to the same interval of the form (γ_j, γ_{j+1}). Therefore

$$P(\Psi > Q) = \sum_{j=1}^{N(Q)} (\gamma_{j+1} - \gamma_j)^2, \tag{17}$$

where we set $\gamma_{N(Q)+j} = \gamma_j + 1$ for $1 \le j < N(Q)$. A sharp estimate for the sum on the right side of (17) has been provided by Kanemitsu, Sitaramachandra Rao and Siva Rama Sarma in [8], the result being

$$\begin{aligned} S_0(Q) := \sum_{j=1}^{N(Q)} (\gamma_{j+1} - \gamma_j)^2 &= \frac{12}{\pi^2 Q^2}\left(\log Q + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{1}{2}\right) \\ &\quad + O_\epsilon\left(\frac{\log^{5/3} Q\,(\log\log Q)^{1+\epsilon}}{Q^3}\right), \end{aligned} \tag{18}$$

where γ is Euler's constant and ζ is the Riemann zeta function. We thus have the following result.

THEOREM 1.

(i) *For any positive integer* N,

$$P(\Psi = N) = \frac{4}{N^3} \sum_{\substack{1 \le q \le N \\ (q,N)=1}} \frac{1}{q}.$$

(ii) *For any* $\epsilon > 0$ *and any positive integer* Q,

$$P(\Psi > Q) = \frac{12}{\pi^2 Q^2}\left(\log Q + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{1}{2}\right) + O_\epsilon\left(\frac{\log^{5/3} Q\,(\log\log Q)^{1+\epsilon}}{Q^3}\right).$$

An estimate for a subsum of (18) of the form

$$S_0(Q, I) = \sum_{\gamma_j \in I} (\gamma_{j+1} - \gamma_j)^2,$$

where the summation is over Farey fractions from a fixed subinterval I of $[0, 1]$ with rational endpoints, has been obtained in Theorem 2 of [1]. The result is

$$S_0(Q, I) = |I| S_0(Q) + 2c_I Q^{-2} + O_\epsilon(Q^{-21/10+\epsilon}), \tag{19}$$

where the constant c_I is given by

$$c_I = \sum_{q \ge 1} \frac{\#\{a \in qI : \gcd(a, q) = 1\} - |I|\varphi(q)}{q^2}. \tag{20}$$

This provides us in turn with an asymptotic result for the probability, call it $P_I(\Psi > Q)$, that $\Psi(\alpha, \beta) > Q$, for a large positive integer Q, where α and β vary independently in a fixed subinterval I of $[0, 1]$. We state the result in the following theorem.

THEOREM 2. *For any subinterval I of $[0,1]$ with rational endpoints, any $\epsilon > 0$ and any positive integer Q,*

$$P_I(\Psi > Q) = \frac{12}{\pi^2 Q^2 |I|}\left(\log Q + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{1}{2}\right) + \frac{2c_I}{|I|^2 Q^2} + O_\epsilon\left(\frac{1}{|I|^2 Q^{21/10-\epsilon}}\right).$$

3. The expected value of Ψ. In this section we determine the expected value, call it $E(\Psi)$, of $\Psi(\alpha,\beta)$ as α,β vary independently in the interval $[0,1]$. We have

$$E(\Psi) = \sum_{N=1}^{\infty} N P(\Psi = N). \tag{21}$$

Note that $\Psi(\alpha,\beta) \geq 2$ for any $\alpha,\beta \in [0,1)$, so $P(\Psi = 1) = 0$. For $N > 1$ we may use (16), and (21) becomes

$$E(\Psi) = 4\sum_{N=2}^{\infty} \sum_{\substack{1\leq q\leq N\\(q,N)=1}} \frac{1}{qN^2} = -4 + 4\sum_{N=1}^{\infty} \sum_{\substack{1\leq q\leq N\\(q,N)=1}} \frac{1}{qN^2}. \tag{22}$$

On the right side of (22) we use Möbius summation to derive

$$E(\Psi) = -4 + 4\sum_{N=1}^{\infty} \sum_{1\leq q\leq N} \sum_{\substack{d|N\\d|q}} \frac{\mu(d)}{qN^2}. \tag{23}$$

Here we write $N = dm$, $q = dn$, then interchange the order of summation to obtain

$$E(\Psi) = -4 + 4\sum_{d=1}^{\infty} \frac{\mu(d)}{d^3} \sum_{m=1}^{\infty} \sum_{1\leq n\leq m} \frac{1}{m^2 n} = -4 + 4\zeta(3)^{-1} S, \tag{24}$$

where

$$S = \sum_{m\geq n\geq 1} \frac{1}{m^2 n} = \sum_{m>n\geq 1} \frac{1}{m^2 n} + \sum_{m\geq 1} \frac{1}{m^3} = \zeta(2,1) + \zeta(3) = 2\zeta(3). \tag{25}$$

Here $\zeta(2,1)$ is a multiple zeta value. The relation $\zeta(2,1) = \zeta(3)$ is an instance of the duality formula for multiple zeta values first conjectured by Hoffman [7] and proved by Zagier [10] (see also Section 3.1 of [3]). Hence we have the following theorem.

THEOREM 3. *The expected value of $\Psi(\alpha,\beta)$ as α,β vary independently in $[0,1]$ is $E(\Psi) = 4$.*

4. Determination of $\Psi_2(\beta,\alpha)$. Recall that $\Psi_k(\beta,\alpha)$ is the kth positive integer n such that $\lfloor n\alpha\rfloor \neq \lfloor n\beta\rfloor$.

In this section we are concerned with $\Psi_2(\beta,\alpha)$.

PROPOSITION 1. *Assume* $\beta \le \gamma \le \alpha$.

(i) *If* $\Psi_1(\beta,\gamma) = \Psi_1(\gamma,\alpha)$, *then*

$$\Psi_2(\beta,\alpha) = \min\{\Psi_2(\beta,\gamma), \Psi_2(\gamma,\alpha)\}. \tag{26}$$

(ii) *If* $\Psi_1(\beta,\gamma) \neq \Psi_1(\gamma,\alpha)$, *then*

$$\Psi_2(\beta,\alpha) = \min\{\max\{\Psi_1(\beta,\gamma), \Psi_1(\gamma,\alpha)\}, \Psi_2(\beta,\gamma), \Psi_2(\gamma,\alpha)\}. \tag{27}$$

Proof. Since $\beta \le \gamma \le \alpha$, it follows that $\lfloor n\beta\rfloor \le \lfloor n\gamma\rfloor \le \lfloor n\alpha\rfloor$. Thus if either $\lfloor n\beta\rfloor < \lfloor n\gamma\rfloor$ or $\lfloor n\gamma\rfloor < \lfloor n\alpha\rfloor$, then $\lfloor n\beta\rfloor < \lfloor n\alpha\rfloor$. We now proceed to prove (i). Because $\Psi_1(\beta,\gamma) = \Psi_1(\gamma,\alpha)$, the next n for which either $\lfloor n\beta\rfloor < \lfloor n\gamma\rfloor$ or $\lfloor n\gamma\rfloor < \lfloor n\alpha\rfloor$ will be $\Psi_2(\beta,\alpha)$. But this will occur at $\min\{\Psi_2(\beta,\gamma), \Psi_2(\gamma,\alpha)\}$, which proves (i).

In the following we will refer to the equality $\lfloor n\beta\rfloor = \lfloor n\gamma\rfloor$ as the "left equality", to $\lfloor n\gamma\rfloor = \lfloor n\alpha\rfloor$ as the "right equality", and to $\lfloor n\beta\rfloor = \lfloor n\alpha\rfloor$ as the "outer equality".

Next, we turn to the proof of (ii). If both $\Psi_1(\beta,\gamma)$ and $\Psi_1(\gamma,\alpha)$ are less than both of $\Psi_2(\beta,\gamma)$ and $\Psi_2(\gamma,\alpha)$, then $\Psi_2(\beta,\alpha)$ will equal the larger of $\Psi_1(\beta,\gamma)$ and $\Psi_1(\gamma,\alpha)$ since $\Psi_1(\beta,\gamma) \neq \Psi_1(\gamma,\alpha)$ and thus there will be two failures of equality at both left and right equalities before either $\Psi_2(\beta,\gamma)$ or $\Psi_2(\gamma,\alpha)$ are reached. The other possibility is that one of $\Psi_2(\beta,\gamma)$ or $\Psi_2(\gamma,\alpha)$ is less than one of $\Psi_1(\beta,\gamma)$ or $\Psi_1(\gamma,\alpha)$. In this case $\Psi_2(\beta,\alpha)$ will be the smaller of $\Psi_2(\beta,\gamma)$ or $\Psi_2(\gamma,\alpha)$, and this completes the proof of the proposition.

PROPOSITION 2. *If* $\frac{a}{b} + \frac{1}{bd} = \frac{c}{d}$, *then*

$$\Psi_2\left(\frac{a}{b}, \frac{c}{d}\right) = \begin{cases} b+d & \text{if } b < d, \\ 2d & \text{if } b \ge d. \end{cases} \tag{28}$$

Proof. Note that if $b = d$ then $b = d = 1$ and $\Psi_2\left(\frac{a}{b}, \frac{c}{d}\right) = 2 = 2d$. Now $\lfloor n\frac{a}{b}\rfloor = \lfloor n\frac{c}{d}\rfloor$ is equivalent to $\lfloor n\frac{c}{d} - \lfloor n\frac{a}{b}\rfloor\rfloor = 0$. This is the same as $\lfloor n\frac{c}{d} - n\frac{a}{b} + \{\frac{na}{b}\}\rfloor = 0$, which is equivalent to $0 \le \frac{n}{bd} + \{\frac{na}{b}\} < 1$. Hence we seek the second value of n such that

$$\frac{n}{bd} + \left\{\frac{na}{b}\right\} \ge 1. \tag{29}$$

It is easy to see that the first n for which (29) holds is $n = d$. Indeed, for $n < d$, $\frac{n}{bd} < \frac{1}{b}$. Also, $\{\frac{da}{b}\} = \{\frac{bc-1}{b}\} = \frac{b-1}{b}$. We now distinguish two cases.

CASE I: $d > b$. If $n = d + b$, then $\frac{1}{b} < \frac{n}{bd} < \frac{2}{b}$ and $\{\frac{na}{b}\} = \frac{b-1}{b}$. Thus $\Psi_2\left(\frac{a}{b}, \frac{c}{d}\right) = b+d$. Note that $n = d+b$ is the least $n > d$ for which $\{\frac{na}{b}\} = \frac{b-1}{b}$.

CASE II: $d < b$. If $d < n < 2d$, then $\frac{1}{b} < \frac{n}{bd} < \frac{2}{b}$ and accordingly, the only chance for (29) to hold for n in this range is for $\{\frac{na}{b}\} = \frac{b-1}{b}$. But after $n = d$, the next case of $\{\frac{na}{b}\} = \frac{b-1}{b}$ is $n = d + b > 2d$. The least n for which

$\frac{n}{bd} \geq \frac{2}{b}$ is $n = 2d$. At $n = 2d$, $\left\{\frac{na}{b}\right\} = \left\{\frac{2ad}{b}\right\} = \left\{\frac{2bc-2}{b}\right\} = \frac{b-2}{b}$. Accordingly $\Psi_2\left(\frac{a}{b}, \frac{c}{d}\right) = 2d$ in this case, which completes the proof of the proposition.

Recall from [2] the definition of the coconvergent and the semiconvergents of two real numbers β and α. (See also Section 6 below.) Choosing the semiconvergents between β and α and using Propositions 1 and 2 and simplifying the resulting minimums and maximums, we obtain the following result.

PROPOSITION 3. *Assume $\beta < \alpha$. Then*

$$\Psi_2(\beta, \alpha) = \min\{b_{-1}, 2b_0, b_1\}, \tag{30}$$

where a_0/b_0 is the coconvergent of β and α, a_{-1}/b_{-1} is the intermediate convergent of β just less than a_0/b_0, and a_1/b_1 is the intermediate convergent of α greater than a_0/b_0.

5. The probability $P(\Psi_2 > Q)$. Let Q be a large positive integer. Since $\Psi_2(\beta, \alpha)$ is larger than $\Psi_1(\beta, \alpha)$ for any α, β, we may write

$$P(\Psi_2 > Q) = P(\Psi_1 > Q) + P(\Psi_1 \leq Q, \Psi_2 > Q), \tag{31}$$

where $P(\Psi_1 \leq Q, \Psi_2 > Q)$ stands for the probability that one has simultaneously $\Psi_1(\beta, \alpha) \leq Q$ and $\Psi_2(\beta, \alpha) > Q$ as β and α vary independently in $[0, 1]$. For the probability $P(\Psi_1 > Q)$ we have the asymptotic result provided by Theorem 1. It remains to estimate $P(\Psi_1 \leq Q, \Psi_2 > Q)$. Let

$$\mathcal{V}_Q = \{(\alpha, \beta) \in [0, 1] \times [0, 1] : \Psi_1(\beta, \alpha) \leq Q,\ \Psi_2(\beta, \alpha) > Q\}. \tag{32}$$

Then

$$P(\Psi_1 \leq Q, \Psi_2 > Q) = \mu(\mathcal{V}_Q). \tag{33}$$

Let $(\alpha, \beta) \in \mathcal{V}_Q$ with $0 < \beta < \alpha < 1$. As before, we may assume that α and β are irrational. Denote $\Psi_1(\beta, \alpha) = N$. By the reasoning from Section 2 above we know that the interval (β, α) contains exactly one element of $\mathcal{F}_N$, and this element has the form a/N, with $1 \leq a < N$ and $(a, N) = 1$. Let a_1/q_1 be a rational number from the interval (β, α), with $a_1/q_1 \neq a/N$ and q_1 as small as possible. Thus $q_1 > N$.

We claim that $q_1 > Q$. Indeed, let us choose an irrational number γ between the numbers a_1/q_1 and a/N. Then one of the numbers a_1/q_1 or a/N lies inside the interval (β, γ), and the other lies inside the interval (γ, α). Again by the reasoning from Section 2 it follows that one of the numbers $\Psi_1(\beta, \gamma)$ or $\Psi_1(\gamma, \alpha)$ equals N, and the other equals q_1. Since $N < q_1$, one has $\Psi_1(\beta, \gamma) \neq \Psi_1(\gamma, \alpha)$. Thus Proposition 1(ii) is applicable, and we derive

$$\begin{aligned}\Psi_2(\beta, \alpha) &= \min\{\max\{\Psi_1(\beta, \gamma), \Psi_1(\gamma, \alpha)\}, \Psi_2(\beta, \gamma), \Psi_2(\gamma, \alpha)\}\\ &\leq \max\{\Psi_1(\beta, \gamma), \Psi_1(\gamma, \alpha)\} = \max\{N, q_1\} = q_1.\end{aligned}$$

But $\Psi_2(\beta, \alpha) > Q$, and so $q_1 > Q$, which proves the claim.

Our second claim is that $2N > Q$. Indeed, let $c/d \in (\beta, a/N)$ be such that

$$\frac{c}{d} + \frac{1}{dN} = \frac{a}{N}. \tag{34}$$

To find such a fraction c/d, one may simply choose a large M so that $\mathcal{F}_M$ has elements in the open interval $(\beta, a/N)$, and let c/d be the largest element of $\mathcal{F}_M$ in the interval $(\beta, a/N)$. Then c/d is the left neighbor of a/N in $\mathcal{F}_M$, and (34) follows from the basic properties of Farey fractions. Next, from (34) and Proposition 2 we find that

$$\Psi_2\left(\frac{c}{d}, \frac{a}{N}\right) = N + \min\{d, N\} = 2N. \tag{35}$$

By Proposition 1, applied to the numbers $\beta < c/d < a/N$, we see that

$$\Psi_2\left(\beta, \frac{a}{N}\right) \leq \min\left(\Psi_2\left(\beta, \frac{c}{d}\right), \Psi_2\left(\frac{c}{d}, \frac{a}{N}\right)\right). \tag{36}$$

Applying Proposition 1 to the numbers $\beta < a/N < \alpha$ we derive

$$\Psi_2(\beta, \alpha) \leq \min\left(\Psi_2\left(\beta, \frac{a}{N}\right), \Psi_2\left(\frac{a}{N}, \alpha\right)\right). \tag{37}$$

By (35)–(37) it follows that $\Psi_2(\beta, \alpha) \leq 2N$. On combining this with the inequality $\Psi_2(\beta, \alpha) > Q$, the claim follows.

Putting together the above two claims, we find that if $0 < \beta < \alpha < 1$, α, β irrational, $\Psi_1(\beta, \alpha) \leq Q \leq \Psi_2(\beta, \alpha)$, then the interval (β, α) contains exactly one element of $\mathcal{F}_Q$, and moreover the denominator N of this element is $> Q/2$.

Conversely, let $0 < \beta < \alpha < 1$, α, β irrational, such that the interval (β, α) contains exactly one element of $\mathcal{F}_Q$, call it a/N, where $1 \leq a \leq N$, $(a, N) = 1$, which furthermore satisfies the inequality $2N > Q$. We then claim that $\Psi_2(\beta, \alpha) > Q$. In order to prove the claim, let us first observe that since $2N > Q$, there is only one number $n \in \{1, \ldots, Q\}$ such that an/N is an integer, namely $n = N$. Hence there exists an $\epsilon > 0$ such that for any $\gamma \in (a/N - \epsilon, a/N)$ and any $\delta \in (a/N, a/N + \epsilon)$ one has $\lfloor n\gamma\rfloor = \lfloor n\delta\rfloor$ for all $n \in \{1, \ldots, Q\}$ with the exception of $n = N$. It then follows directly from the definition that $\Psi_2(\gamma, \delta) > Q$. Fix now some such irrational numbers γ and δ. We know from Section 2 that $\Psi_1(\delta, \alpha) > Q$ since $\mathcal{F}_Q$ does not have any elements in the interval (δ, α). Also, $\Psi_1(\gamma, \delta) = N$, since a/N is the only element of $\mathcal{F}_N$ inside the interval (γ, δ). Then Proposition 1(ii), applied to the numbers $\gamma < \delta < \alpha$, gives

$$\begin{aligned}\Psi_2(\gamma, \alpha) &= \min\{\max\{\Psi_1(\gamma, \delta), \Psi_1(\delta, \alpha)\}, \Psi_2(\gamma, \delta), \Psi_2(\delta, \alpha)\} \\ &= \min\{\Psi_1(\delta, \alpha), \Psi_2(\gamma, \delta), \Psi_2(\delta, \alpha)\} = \min\{\Psi_1(\delta, \alpha), \Psi_2(\gamma, \delta)\} > Q.\end{aligned} \tag{38}$$

Similarly, one has $\Psi_1(\beta,\gamma) > Q$ since $\mathcal{F}_Q$ does not have any element in the interval (β,γ), and $\Psi_1(\gamma,\alpha) = N$ since a/N is the only element of $\mathcal{F}_N$ inside the interval (γ,α). Using (38) in combination with Proposition 1(ii) applied to $\beta < \gamma < \alpha$, we deduce that

$$\begin{aligned}\Psi_2(\beta,\alpha) &= \min\{\max\{\Psi_1(\beta,\gamma),\Psi_1(\gamma,\alpha)\},\Psi_2(\beta,\gamma),\Psi_2(\gamma,\alpha)\} \\ &= \min\{\Psi_1(\beta,\gamma),\Psi_2(\beta,\gamma),\Psi_2(\gamma,\alpha)\} \\ &= \min\{\Psi_1(\beta,\gamma),\Psi_2(\gamma,\alpha)\} > Q.\end{aligned} \tag{39}$$

This proves the claim. In conclusion, the set $\mathcal{V}_Q$ has the following shape. For any N with $\lfloor Q/2\rfloor + 1 \le N \le Q$ and any $1 \le a \le N$ with $(a,N) = 1$, if γ' and γ'' denote the left and respectively the right neighbors of a/N in $\mathcal{F}_Q$, then β and α are allowed to vary independently inside the intervals $I'_{a/N} = (\gamma', a/N)$ and respectively $I''_{a/N} = (a/N, \gamma'')$. Putting together the contribution of all such fractions a/N, we see that

$$P(\Psi_1 \le Q, \Psi_2 > Q) = 2 \sum_{\lfloor Q/2\rfloor+1\le N\le Q}\ \sum_{(a,N)=1} |I'_{a/N}|\cdot|I''_{a/N}|. \tag{40}$$

If we denote the elements of $\mathcal{F}_Q$, in increasing order, by $\gamma_j = a_j/q_j$, $1 \le j \le N(Q)$, then (40) becomes

$$P(\Psi_1 \le Q, \Psi_2 > Q) = 2 \sum_{\substack{1\le j\le N(Q)\\ \lfloor Q/2\rfloor+1\le q_j}} (\gamma_{j+1}-\gamma_j)(\gamma_j-\gamma_{j-1}). \tag{41}$$

The sum on the right side of (41) is a subsum of

$$S_1(Q) := \sum_{j=1}^{N(Q)} (\gamma_{j+1}-\gamma_j)(\gamma_{j+2}-\gamma_{j+1}). \tag{42}$$

In [5], Hall provides the asymptotic formula

$$S_1(Q) = \frac{6}{\pi^2 Q^2}\left(\log Q + \gamma - \frac{\zeta'(2)}{\zeta(2)} + B\right) + O\left(\frac{\log Q}{Q^{5/2}}\right), \tag{43}$$

where γ is Euler's constant and

$$B = \frac{1}{2} + \log 2 + 2\sum_{k\ge1}\frac{\zeta(2k)-1}{2k-1} = 2.546277\ldots.$$

To evaluate the subsum on the right side of (41), we use the equalities $\gamma_{j+1}-\gamma_j = \frac{1}{q_jq_{j+1}}$ and $\gamma_j - \gamma_{j-1} = \frac{1}{q_{j-1}q_j}$ to obtain

$$P(\Psi_1 \le Q, \Psi_2 > Q) = 2 \sum_{\substack{1\le j\le N(Q)\\ \lfloor Q/2\rfloor+1\le q_j}} 1/q_{j-1}q_j^2q_{j+1}.$$

Next, recall (see [5], [6], [1]) the equality $q_{j+1} = \left\lfloor\frac{Q+q_{j-1}}{q_j}\right\rfloor q_j - q_{j-1}$, and the fact that the pairs (q_{j-1}, q_j) of consecutive denominators of $\mathcal{F}_Q$ are in one-to-one correspondence with the pairs of integers a, b with $1 \le a, b \le Q$, $(a, b) = 1$ and $a + b > Q$. It follows that

$$P(\Psi_1 \le Q, \Psi_2 > Q) = 2 \sum_{\substack{1\le a,b\le Q,\, a+b>Q \\ \lfloor Q/2\rfloor+1\le b,\, (a,b)=1}} f(a,b),$$

where the function f is defined by

$$f(x,y) = \frac{1}{xy^2\big(\lfloor\frac{Q+x}{y}\rfloor y - x\big)}.$$

We may now apply Lemma 2 of [1], which shows that

$$P(\Psi_1 \le Q, \Psi_2 > Q) \sim \frac{12}{\pi^2} \iint_{Q\Omega} f(x,y)\,dx\,dy, \tag{44}$$

where Ω is the polygon with vertices $(1/2, 1/2)$, $(1, 1/2)$, $(1, 1)$ and $(0, 1)$. As a side remark, note that $\lfloor\frac{Q+x}{y}\rfloor \in \{1, 2, 3\}$ for $(x, y) \in \Omega$, and these values 1, 2 and 3 are attained exactly when the point (x, y) lies in the corresponding regions T_1, T_2 and $T_3 \cap \Omega$ from Figure 1 of [1]. After making a linear change of variables in the integral from the right side of (44), one finds that

$$P(\Psi_1 \le Q, \Psi_2 > Q) \sim \frac{12C}{\pi^2 Q^2}, \tag{45}$$

where the constant C is given by

$$C = \iint_{\Omega} \frac{1}{xy^2\big(\lfloor\frac{1+x}{y}\rfloor y - x\big)}\,dx\,dy. \tag{46}$$

On combining (45), (31) and Theorem 1, and estimating the error terms as in Section 10 of [1], one obtains the following asymptotic result for $P(\Psi_2 > Q)$.

THEOREM 4. *For any positive integer* Q,

$$P(\Psi_2 > Q) = \frac{12}{\pi^2 Q^2}\bigg(\log Q + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{1}{2} + C\bigg) + O(Q^{-5/2}\log Q).$$

We remark that, as in Section 2, the entire discussion may be adapted to the case when α and β vary in some fixed subinterval of $[0, 1]$, and an analogue of Theorem 4 can be established in that case, following the method from Section 10 of [1].

6. Properties of $\mathcal{N}(\beta, \alpha)$. The goal of this section is to study the set $\mathcal{N}(\beta, \alpha)$. Obviously

$$\mathcal{N}(\beta, \alpha) = \{\Psi_k(\beta, \alpha) : k \in \mathbb{Z}^+\}.$$

It is clear from the definition of $\mathcal{N}(\beta, \alpha)$ that it contains all but finitely many natural numbers. We will in general deal with the inhomogeneous case $s \neq 0$ as it presents no greater difficulty, although some particular results will require $s = 0$. If we need to specify the value of s, we write $\mathcal{N}(\beta, \alpha; s)$.

PROPOSITION 4. *$\mathcal{N}(\beta, \alpha; 0)$ is a semigroup under addition of natural numbers.*

Proof. Without loss of generality, assume $\beta < \alpha$ and take $m, n \in \mathcal{N}(\beta, \alpha)$. Then there exist integers $P, Q \in \mathbb{Z}^+$ for which $m\beta < P \leq m\alpha$ and $n\beta < Q \leq n\alpha$. Adding these inequalities gives $(m+n)\beta < P + Q \leq (m+n)\alpha$. This implies that $m + n \in \mathcal{N}(\beta, \alpha)$ finishing the proof.

We will later determine the exact structure of this semigroup. In the inhomogeneous case, the above proposition clearly generalizes to the fact that $m \in \mathcal{N}(\beta, \alpha; s)$ and $n \in \mathcal{N}(\beta, \alpha; t)$ implies that $m + n \in \mathcal{N}(\beta, \alpha; s + t)$.

PROPOSITION 5. *For any $\beta \leq \gamma \leq \alpha$ and $s \geq 0$,*

$$\mathcal{N}(\beta, \alpha) = \mathcal{N}(\beta, \gamma) \cup \mathcal{N}(\gamma, \alpha). \tag{47}$$

Also, for any $\alpha_1 \leq \cdots \leq \alpha_m$ and $s \geq 0$,

$$\mathcal{N}(\alpha_1, \alpha_m) = \bigcup_{i=1}^{m-1} \mathcal{N}(\alpha_i, \alpha_{i+1}). \tag{48}$$

Proof. Since $\beta \leq \gamma \leq \alpha$, it follows that $\lfloor n\beta + s \rfloor \leq \lfloor n\gamma + s \rfloor \leq \lfloor n\alpha + s \rfloor$. If $\lfloor n\beta + s \rfloor \neq \lfloor n\alpha + s \rfloor$, then one of the above inequalities must be strict, hence we have the inclusion $\subseteq$. Conversely, if one of the above inequalities fails, then $\lfloor n\beta + s \rfloor \neq \lfloor n\alpha + s \rfloor$. The second statement immediately follows from the first. This proves the proposition.

PROPOSITION 6. *When $\beta \leq \alpha$,*

$$\begin{aligned} \mathcal{N}(\beta, \alpha) &= \left\{ n \in \mathbb{Z}^+ : \beta < \alpha - \frac{\{n\alpha + s\}}{n} \right\} \\ &= \left\{ n \in \mathbb{Z}^+ : \beta + \frac{1 - \{n\beta + s\}}{n} \leq \alpha \right\}. \end{aligned} \tag{49}$$

Proof. It is routine to show that $\lfloor n\beta + s \rfloor \neq \lfloor n\alpha + s \rfloor$ if and only if $0 \leq n(\beta - \alpha) + \{n\alpha + s\} < 1$. Since $\beta \leq \alpha$, this can fail if and only if n is in the first set above. The second part is similar.

PROPOSITION 7. *Assume $\frac{a}{b} + \frac{1}{bd} = \frac{c}{d}$. Then for $s \geq 0$,*

$$\mathcal{N}(a/b, c/d) = \{mb + nd : m \geq -\lfloor ds \rfloor,\ n > \lfloor bs \rfloor\}. \tag{50}$$

In particular, when $s = 0$, we have

$$\mathcal{N}(a/b, c/d) = \{mb + nd : m \geq 0, n \geq 1\}. \tag{51}$$

Proof. This follows directly from the proof of Lemma 4.2 of [2]. In particular, in [2] it was shown that, for $n \in \mathbb{Z}^+$,

$$\lfloor nc/d + s\rfloor \neq \lfloor na/b + s\rfloor$$

if and only if

$$\frac{n + d\{bs\}}{db} + \left\{\frac{na + \lfloor bs\rfloor}{b}\right\} \geq 1.$$

This last relation easily translates into the three relations $an \equiv -(M + \lfloor bs\rfloor) \pmod{b}$, $Kd - d\{bs\} \leq n$, and $0 < M \leq \min(b, K)$. It is easy to see that all solutions to $an \equiv r \pmod{b}$ with n satisfying $n \geq L$ are given by $n = \lceil (L + rd)/b\rceil + bN$ for all $N \geq 0$. (In [2], the least such n was sought, so the inequality $Kd - d\{bs\} \leq n < Kd - d\{bs\} + b$ was used and no N occurred in the formula.) Hence we have

$$\begin{aligned}(52)\qquad \mathcal{N}(a/b, c/d) &= \{dM + d\lfloor bs\rfloor + b\lceil (K - M)d/b - ds\rceil + bN : \\ &\qquad\qquad N \geq 0,\ K \geq M,\ 0 < M \leq b\} \\ &= \{d(M + \lfloor bs\rfloor) + b(N + \lceil -ds\rceil) : N \geq 0,\ M \geq 1\} \\ &= \{mb + nd : m \geq -\lfloor ds\rfloor,\ n > \lfloor bs\rfloor\}.\end{aligned}$$

The proof is complete.

When $s = 0$, it is convenient to write the conclusion of this proposition in the form $\mathcal{N}(a/b, c/d) = \{d + mb + nd : m, n \geq 0\}$. Since $(b, d) = 1$, we can rephrase the proposition by saying that the semigroup $(\mathcal{N}(a/b, c/d), +)$ is generated by the elements b and d, the multiplicity of d being at least one. It is a well-known theorem of Sylvester that when $(b, d) = 1$, the largest integer which is not a non-negative integer linear combination of b and d is exactly $bd - b - d$. Thus we have the following corollary.

Corollary 1. *If* $a/b + 1/bd = c/d$, *then the last integer* n *for which* $\lfloor na/b\rfloor = \lfloor nc/d\rfloor$ *is* $n = b(d - 1)$.

We are now ready to give for any positive integer k a theoretical formula for $\Psi_k(a/b, c/d)$.

Corollary 2. *Let* $s = 0$ *and assume* $a/b + 1/bd = c/d$. *Then, for each* k, $\Psi_k(a/b, c/d)$ *is equal to the* k*th element in the set* $\{d + mb + nd : m, n \geq 0\}$ *when its elements are arranged in increasing order. In particular,*

$$(53)\qquad \Psi_1(a/b, c/d) = d;$$

$$(54)\qquad \Psi_2(a/b, c/d) = d + \begin{cases} b, & b/d \in (0, 1], \\ d, & b/d \in [1, \infty);\end{cases}$$

$$\Psi_3(a/b,c/d) = d + \begin{cases} 2b, & b/d \in (0,1/2], \\ d, & b/d \in [1/2,1), \\ b, & b/d \in (1,2], \\ 2d, & b/d \in [2,\infty); \end{cases} \tag{55}$$

$$\Psi_4(a/b,c/d) = d + \begin{cases} 3b, & b/d \in (0,1/3], \\ d, & b/d \in [1/3,1/2), \\ 2b, & b/d \in (1/2,1), \\ 2d, & b/d \in (1,2), \\ b, & b/d \in (2,3], \\ 3d, & b/d \in [3,\infty); \end{cases} \tag{56}$$

$$\Psi_5(a/b,c/d) = d + \begin{cases} 4b, & b/d \in (0,1/4], \\ d, & b/d \in [1/4,1/3), \\ 3b, & b/d \in (1/3,1/2), \\ b+d, & b/d \in (1/2,1) \cup (1,2), \\ 3d, & b/d \in (2,3), \\ b, & b/d \in (3,4], \\ 4d, & b/d \in [4,\infty); \end{cases} \tag{57}$$

$$\Psi_6(a/b,c/d) = d + \begin{cases} 5b, & b/d \in (0,1/5], \\ d, & b/d \in [1/5,1/4), \\ 4b, & b/d \in (1/4,1/3), \\ b+d, & b/d \in (1/3,1/2), \\ 3b, & b/d \in (1/2,2/3], \\ 2d, & b/d \in [2/3,1), \\ 2b, & b/d \in (1,3/2], \\ 3d, & b/d \in [3/2,2), \\ b+d, & b/d \in (2,3), \\ 4d, & b/d \in (3,4), \\ b, & b/d \in (4,5], \\ 5d, & b/d \in [5,\infty). \end{cases} \tag{58}$$

In general, $\Psi_k(a/b,c/d)$ is only a function of the ratio b/d and the number k. The boundary cases in the above formulas are handled by the general evaluations

$$\Psi_k(a/b,c/d) = \begin{cases} k, & b/d \in \mathbb{Z}^+, \\ k+n-1, & d/b = n \in \mathbb{Z}^+. \end{cases} \tag{59}$$

Proof. The characterization of $\Psi_k(a/b, c/d)$ follows immediately from the previous corollary. The remaining parts are just a calculation based upon this characterization of $\Psi_k(a/b, c/d)$. This calculation is best facilitated by forming a tree structure where each vertex is labeled by a Farey fraction and a linear combination of b and d. The vertices of height k are labeled with the possible values of $\Psi_k - d$ written as linear combinations of b and d along with a maximal interval with rational endpoints over which Ψ_k achieves the assigned value. One starts with the root labeled 0 for the value and labeled with the interval $(0, \infty)$. The next larger linear combinations after 0 are b and d. The smaller of these two values is clearly b when $b/d \leq 1$ and is d when $b/d \geq 1$. These two possibilities give rise to the vertices of height 2 and the evaluation of Ψ_2 above. The tree and the evaluations follow by continuing this construction. For example consider the vertex of height 5 labeled with $b + d$ and the interval $(1/2, 1)$. The values already achieved by $\Psi_k - d$ with $b/d \in (1/2, 1)$ for $1 \leq k \leq 5$ are in increasing order: $0, b, d, 2b, b + d$. The next possible larger values are $2d$ and $3b$. Now $2d \leq 3b$ and $3b \leq 2d$ if and only if $b/d \in [2/3, 1)$ and $b/d \in (1/2, 2/3]$ respectively. Hence the vertex of height 5 considered gives rise to the two vertices of height 6 with the values $2d$ and $3b$ and the corresponding intervals just given.

That $\Psi_k(a/b, c/d)$ depends only on k and the value of b/d follows from the above construction and the fact that an inequality of the form $hb+jd < mb+nd$ holds if and only if either $b/d > (j-n)/(m-h)$ or $b/d < (j-n)/(m-h)$, according as $m - h$ is positive or negative. Thus the ordering of the linear combinations is completely determined by the value of the ratio b/d. Finally, the formulas at the end for Ψ_k for all k are easy consequences. This completes the proof.

We now consider the problem of finding $\mathcal{N}(\alpha, \beta)$ for general α and β. To this end we use the sequences of approximating fractions constructed in [2]. Specifically, assume $\beta < \alpha$ and $\lfloor\beta\rfloor = \lfloor\alpha\rfloor$. Put $\Psi = \Psi_1(\beta, \alpha; 0)$. Now arrange the union of the convergents and intermediate convergents of α less than α in increasing order. Do the same with β, except with the convergents and intermediate convergents *greater* than β. In the event that β is rational, choose the number of terms in the continued fraction for β to be odd. Then the last approximating fraction to β will be the least convergent or intermediate convergent greater than β. In this case, also adjoin the infinite sequence of mediants approaching β formed by taking the mediant of β and this last fraction and iterating the procedure. (The numerators and denominators of this sequence of special mediants will be in an arithmetic progression.) Together these sequences were termed *semiconvergents* in [2]. It was proved in [2] that these sequences of fractions approaching α and β have a common element. Moreover, this common convergent is exactly $\lfloor\Psi\alpha\rfloor/\Psi$. For this reason, $\lfloor\Psi\alpha\rfloor/\Psi$ is called the *coconvergent* of α and β.

Let $g_0 = a_0/b_0$ denote the coconvergent of β and α. For $i > 0$, let $g_i = a_i/b_i$ denote the ith convergent or intermediate convergent of α greater than g_0. For $i < 0$, let $g_i = a_i/b_i$ denote the $-i$th convergent or intermediate convergent (as constructed above) of β less than g_0. Thus we have a doubly infinite sequence of rational numbers $g_i = a_i/b_i$ satisfying

$$\lim_{i\to-\infty} g_i = \beta, \qquad \lim_{i\to\infty} g_i = \alpha,$$

and

$$\frac{a_i}{b_i} + \frac{1}{b_i b_{i+1}} = \frac{a_{i+1}}{b_{i+1}}.$$

We will require the following lemma.

Lemma 1. *Assume α is irrational. Suppose there exists a positive integer J such that $\{J\alpha + s\} = 0$. Then for any $N > 0$, there exists a $\beta < \alpha$ such that $\beta \le \gamma < \alpha$ implies that $\Psi_2(\gamma, \alpha) > N$.*

Proof. Let $J_1 = 1$ and define J_i for $i > 1$ inductively by the rule that J_{i+1} is the least positive integer greater than J_i such that $\{J_{i+1}\alpha+s\}/J_{i+1} < \{J_i\alpha + s\}/J_i$. Then the sequence J_i will be finite; let J_e be its last term. By Lemma 4.1 of [2], we know that $\alpha - \{J_{e-1}\alpha + s\}/J_{e-1} \le \gamma < \alpha$ implies that $\Psi_1(\gamma, \alpha) = J = J_e$. Now let K_1 be the least integer greater than J such that $\{K_1\alpha + s\}/K_1 < \{J_{e-1}\alpha + s\}/J_{e-1}$. Define the sequence K_i inductively by the rule that K_{i+1} is the least positive integer greater than K_i such that $\{K_{i+1}\alpha + s\}/K_{i+1} < \{K_i\alpha + s\}/K_i$. Then since α is irrational, the sequence K_i is infinite. Now by Proposition 6 it is clear that $\alpha - \{K_i\alpha + s\}/K_i \le \gamma < \alpha - \{K_{i+1}\alpha + s\}/K_{i+1} < \alpha$ implies that $\Psi_2(\gamma, \alpha) = K_{i+1}$ and the lemma is proved.

The next theorem characterizes the set $\mathcal{N}(\beta, \alpha)$. We assume α is irrational and s is some non-negative fixed real number. From [2] we know that if there exists a natural number L such that $\{L\alpha + s\} = 0$ then it is a value of Ψ_1. In [2] a formula for computing L was also provided.

Theorem 5. *Assume α is irrational and $\beta < \alpha$. Then for $s \ge 0$,*

$$\mathcal{N}(\beta, \alpha) = \{mb_i + nb_{i+1} : i \in \mathbb{Z},\ m > -\lfloor b_{i+1}s\rfloor,\ n \ge \lfloor b_i s\rfloor\} \cup \{L\}.$$

When $s = 0$ we have

$$\mathcal{N}(\beta, \alpha) = \{mb_i + nb_{i+1} : i \in \mathbb{Z},\ m > 0,\ n \ge 0\}.$$

In particular, when $s = 0$, $\mathcal{N}(\beta, \alpha)$ is generated by the set of numbers $\{b_k\}_{k\in\mathbb{Z}}$.

Proof. By Proposition 5,

$$\mathcal{N}(\beta, \alpha) = \mathcal{N}(\beta, g_{-j}) \cup \mathcal{N}(g_k, \alpha) \cup \bigcup_{-j\le i<k} \mathcal{N}(g_i, g_{i+1}). \tag{60}$$

If $\{n\alpha+s\} \ne 0$ for $n \in \mathbb{Z}^+$, then by Lemma 4.1 of [2], j and k can be chosen large enough that both $\Psi_1(\beta, g_{-j})$ and $\Psi_1(g_k, \alpha)$ are larger than $b_0(b_1-1)$. In this case we clearly have $\mathcal{N}(\beta, g_{-j}) \cup \mathcal{N}(g_k, \alpha) \subset \mathcal{N}(g_0, g_1)$. (For by Corollary 1, $n > b_0(b_1-1)$ implies that $n \in \mathcal{N}(g_0, g_1)$.)

If $\{L\alpha + s\} = 0$, then by Lemma 4.1 of [2], when k is sufficiently large, $\Psi_1(g_k, \alpha) = L$, so L is in the set. By Lemma 1 we can now choose k sufficiently large so that $\Psi_2(g_k, \alpha) > b_0(b_1-1)$ and hence $\mathcal{N}(g_k, \alpha) \subset \{L\} \cup \mathcal{N}(g_0, g_1)$. Thus for large enough j and k,

$$\mathcal{N}(\beta, \alpha) = \{L\} \cup \bigcup_{-j\le i<k} \mathcal{N}(g_i, g_{i+1}) \subseteq \{L\} \cup \bigcup_{-\infty\le i<\infty} \mathcal{N}(g_i, g_{i+1}).$$

But by Proposition 5 the last inclusion is actually an equality. Proposition 7 gives the first part of the theorem. The second part follows trivially from the first, and the third follows immediately from the second.

When $s = 0$ it is clear from the proof of this theorem how to choose a finite set of the b_i which will form a set of generators for the semigroup $(\mathcal{N}(\beta, \alpha), +)$.

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