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On a divisor problem related to the Epstein zeta-function, IV

by

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1. Introduction. In this paper, we continue our study of divisor problems related to the Epstein zeta-function [12–14]. Let $\ell \geq 2$, $\mathbf{y} := (y_1, \ldots, y_\ell) \in \mathbb{Z}^\ell$ and $\mathbf{A} = (a_{ij})$ be an integral matrix such that $a_{ii} \equiv 0 \pmod 2$ for $1 \leq i \leq \ell$. Then a positive definite quadratic form $Q(\mathbf{y})$ can be written as

$$Q(\mathbf{y}) = \frac{1}{2}\mathbf{y}^{\mathrm{t}}\mathbf{A}\mathbf{y} = \frac{1}{2}\sum_{1\leq i\leq \ell} a_{ii}y_i^2 + \sum_{1\leq i<j\leq \ell} a_{ij}y_iy_j,$$

where $\mathbf{y}^{\mathrm{t}}$ is the transpose of $\mathbf{y}$. The corresponding Epstein zeta-function is initially defined by the Dirichlet series

$$Z_Q(s) := \sum_{\mathbf{y}\in\mathbb{Z}^\ell\setminus\{\mathbf{0}\}} \frac{1}{Q(\mathbf{y})^s} = \sum_{n\geq 1} \frac{r(n,Q)}{n^s} \tag{1.1}$$

for $\Re e\, s > \ell/2$, where

$$r(n,Q) := |\{\mathbf{y} \in \mathbb{Z}^\ell : Q(\mathbf{y}) = n\}|.$$

According to [21], $Z_Q(s)$ has an analytic continuation to the whole complex plane $\mathbb{C}$ with only a simple pole at $s = \ell/2$, and satisfies a functional equation of Riemann type.

For each integer $k \geq 1$, we are interested in the mean value of the k-fold Dirichlet convolution of $r(n,Q)$ defined by

$$r_k(n,Q) := \sum_{n_1\cdots n_k=n} r(n_1,Q)\cdots r(n_k,Q). \tag{1.2}$$

The asymptotic behavior of the error term

$$\varDelta_k^*(x,Q) := \sum_{n\leq x} r_k(n,Q) - \operatorname*{Res}_{s=\ell/2}(Z_Q(s)^k x^s s^{-1}) \tag{1.3}$$

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has received much attention [11, 3, 21]. In particular Sankaranarayanan [21] showed, by the complex integration method, that for $k \geq 2$ and $\ell \geq 3$,

$$\Delta_k^*(x, Q) \ll x^{\ell/2-1/k+\varepsilon}; \tag{1.4}$$

here and throughout this paper, ε denotes an arbitrarily small positive constant.

Recently, inspired by Iwaniec's book [8, Chapter 11], Lü [12] noted that (1.4) can be improved for quadratic forms of level one. These quadratic forms satisfy the following supplementary conditions:

$$\ell \equiv 0 \pmod 8), \qquad \mathbf{A} \text{ is equivalent to } \mathbf{A}^{-1}, \qquad |\mathbf{A}| = 1.$$

For such forms, we have [8, (11.32)]

$$r(n, Q) = \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2)\Gamma(\ell/2)}\sigma_{\ell/2-1}(n) + a_f(n, Q) \quad (n \geq 1), \tag{1.5}$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, $\zeta(s)$ is the Riemann zeta-function, $\Gamma(s)$ is the Gamma function and $a_f(n, Q)$ is the nth Fourier coefficient of a cusp form $f(z, Q)$ of weight $\ell/2$ with respect to the full modular group $\mathrm{SL}(2, \mathbb{Z})$, satisfying Deligne's bound [4]

$$|a_f(n, Q)| \leq n^{(\ell/2-1)/2}\sigma_0(n) \quad (n \geq 1). \tag{1.6}$$

Thus

$$Z_Q(s) = \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2)\Gamma(\ell/2)}\zeta(s)\zeta(s - \ell/2 + 1) + L(s, f) \tag{1.7}$$

for $s \in \mathbb{C} \setminus \{\ell/2\}$, where $L(s, f)$ is the Hecke L-function associated with $f(z, Q)$. In view of basic properties of $\zeta(s)$ and $L(s, f)$, it is not difficult to see that $\zeta(s - \ell/2 + 1)$ is more dominant and we may view $\Delta_k^*(Q; x)$ as the classical k-dimensional divisor problem associated to the Riemann zeta-function. With the help of these ideas, Lü, Wu & Zhai [13] obtained, via a simple convolution argument,

$$\Delta_k^*(x, Q) \ll x^{\ell/2-1+\theta_k+\varepsilon} \quad (x \geq 2) \tag{1.8}$$

for $k = 2, 3$ ([1]), where θ_k is the exponent in the classical k-dimension divisor problem

$$\sum_{n \leq x} \tau_k(n) = \operatorname*{Res}_{s=1}(\zeta(s)^k x^s s^{-1}) + O(x^{\theta_k+\varepsilon}) \quad (x \geq 2). \tag{1.9}$$

Moreover, an Ω-result for $k = 2, 3$ and a mean value theorem for $\Delta_2^*(x, Q)$ have been established in [13] and [14], respectively.

In this paper we shall refine Sankaranarayanan's result (1.4) for general positive definite quadratic forms Q. In this case, it is known that [8, Theorem 11.2]

([1]) When $k \geq 4$, a similar result has been proved by Lü [12] using complex integration.

$$r(n,Q)=\frac{(2\pi)^{\ell/2}}{\Gamma(\ell/2)\sqrt{|\mathbf{A}|}}n^{\ell/2-1}\sigma(n,Q)+O(n^{\ell/4-\delta_\ell+\varepsilon}) \tag{1.10}$$

for $\ell\ge 4$, where with $\mathrm{e}(t):=\mathrm{e}^{2\pi\mathrm{i}t}$ $(t\in\mathbb{R})$,

$$S(Q):=\sum_{0\le y_1,\dots,y_\ell\le q-1}\mathrm{e}(Q(\mathbf{y})),$$

$$\sigma(n,Q):=\sum_{q=1}^{\infty}\frac{1}{q^\ell}\sum_{h=1}^{q}{}^{*}\,S\Big(\frac{hQ}{q}\Big)\mathrm{e}\Big(-\frac{hn}{q}\Big),$$

$$\delta_\ell:=\begin{cases}1/4 & \text{if } \ell \text{ is odd},\\ 1/2 & \text{if } \ell \text{ is even}.\end{cases}$$

Here and below, the symbol $\sum^{*}$ means $\sum_{(h,q)=1}$. We propose two methods to bound $\Delta_k^*(x,Q)$: the complex integration method and the convolution method. The former allows us to establish nontrivial estimates for $\Delta_k^*(x,Q)$ for all $k\ge 1$ and $\ell\ge 4$. But the convolution argument is more powerful for $k=1,2,3$ when $\ell\ge 6$.

Let

$$L_Q(s):=\sum_{n=1}^{\infty}\frac{\sigma(n,Q)}{n^s}\qquad(\Re e\, s>1). \tag{1.11}$$

In view of the bound (cf. [8, Lemma 10.5])

$$S(hQ/q)\ll q^{\ell/2}\qquad((h,q)=1), \tag{1.12}$$

the Dirichlet series $L_Q(s)$ is absolutely convergent for $\Re e\, s>1$ provided $\ell\ge 5$. In Section 2 we shall prove that $L_Q(s)$ can be analytically continued to a meromorphic function on the half-plane $\Re e\, s>0$, which has a simple pole at $s=1$ with residue 1 (see Lemma 2.1 below), and establish some individual and average subconvexity bounds for $L_Q(s)$ similar to $\zeta(s)$ (see Lemmas 2.2 and 2.3). With the help of these new tools, the standard complex integration method allows us to deduce the following result, which improves Sankaranarayanan's (1.4) when $k\ge 3$.

THEOREM 1. *Let $\ell\ge 4$ and $k\ge 1$. We have*

$$\Delta_k^*(x,Q)\ll x^{\ell/2-1+\vartheta_{k,\ell}+\varepsilon}\qquad(x\ge 2), \tag{1.13}$$

where

$$\vartheta_{k,\ell}=\begin{cases}1/2 & \text{if } 1\le k\le 4 \text{ and } \ell\ge 4,\\ k/(k+4) & \text{if } 5\le k\le 12 \text{ and } \ell=4 \text{ or } \ell\ge 6,\\ (13k-4)/(13k+44) & \text{if } 5\le k\le 12 \text{ and } \ell=5,\\ (k-3)/k & \text{if } 13\le k\le 49 \text{ and } \ell=4 \text{ or } \ell\ge 6,\\ (4k-11)/(4k+1) & \text{if } 13\le k\le 49 \text{ and } \ell=5,\\ 1-(2738k^2)^{-1/3} & \text{if } k\ge 50 \text{ and } \ell\ge 4.\end{cases}$$

The convolution argument of [13] can also be generalized to estimate $\Delta_k^*(x,Q)$. Though (1.10) is more complicated than (1.5), we can use it to establish a connection between $\Delta_k^*(x,Q)$ and the divisor problem with congruence conditions. We will discuss this in Section 4. For $\mathbf{q} := (q_1,\dots,q_k) \in \mathbb{N}^k$ and $\mathbf{r} := (r_1,\dots,r_r) \in \mathbb{N}^k$ such that $r_i \le q_i$ $(1 \le i \le k)$, define

$$\tau_k(n;\mathbf{q},\mathbf{r}) := \sum_{\substack{n_1\cdots n_k=n\\ n_i\equiv r_i \,(\mathrm{mod}\, q_i)\,(1\le i\le k)}} 1, \qquad D_k(x;\mathbf{q},\mathbf{r}) := \sum_{n\le x} \tau_k(n;\mathbf{q},\mathbf{r}).$$

The divisor problem with congruence conditions aims to bound the error term

$$\Delta_k(x;\mathbf{q},\mathbf{r}) := D_k(x;\mathbf{q},\mathbf{r}) - \operatorname*{Res}_{s=1}\bigl(\zeta(s,r_1/q_1)\cdots\zeta(s,r_k/q_k)x^s s^{-1}\bigr) \tag{1.14}$$

where $\zeta(s,\alpha)$ is the Hurwitz zeta-function defined by

$$\zeta(s,\alpha) := \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} \qquad (0<\alpha\le 1, \sigma>1). \tag{1.15}$$

With the help of a convolution argument, we can prove the following result, which offers better exponents than (1.4) for $k=1,2,3$ when $\ell \ge 6$.

Theorem 2. *Let $\ell \ge 6$ and $k = 1,2,3$. Assume that there is some $\vartheta_k \in (0,1)$ such that*

$$\Delta_k(x;\mathbf{q},\mathbf{r}) \ll_{k,\ell,\varepsilon} (x/(q_1\cdots q_k))^{\vartheta_k+\varepsilon}$$

uniformly for $1 \le r_i \le q_i$ $(1\le i\le k)$ and $q_1\cdots q_k \le x$. Then

$$\Delta_k^*(x,Q) \ll_{k,\ell,\varepsilon} x^{\ell/2-1+\vartheta_k+\varepsilon}.$$

In particular we can take

$$\vartheta_k = \begin{cases} 0 & \text{if } k=1,\\ 131/416 & \text{if } k=2,\\ 43/96 & \text{if } k=3. \end{cases} \tag{1.16}$$

Another interesting problem related to $r(n,Q)$ is to evaluate its kth power sum. In this direction, Landau [11] first showed that

$$\sum_{n\le x} r(n,Q) = \frac{(2\pi)^{\ell/2}}{\Gamma(\ell/2+1)\sqrt{|\det Q|}} x^{\ell/2} + O(x^{\ell/2-\ell/(\ell+1)}). \tag{1.17}$$

For $k=2$, Müller [16] proved that

$$\sum_{n\le x} r(n,Q)^2 = \begin{cases} A_Q x\log x + B_Q x + O(x^{3/5}\log x) & \text{if } \ell=2,\\ C_Q x^{\ell-1} + O(x^{\ell-1-2(\ell-1)/(4\ell-3)}) & \text{if } \ell\ge 3, \end{cases} \tag{1.18}$$

where A_Q, B_Q and C_Q are some constants depending on Q. In this paper we study a more general correlated sum of $r(n,Q)$, which contains the kth power sum as a special case.

Theorem 3. *Let $\ell \ge 5$, $k \ge 1$ and $a_1, \ldots, a_k$ be fixed nonnegative integers. Then*

$$\sum_{n\le x}\prod_{1\le i\le k} r(n+a_i,Q) = C_Q(a_1,\ldots,a_k)x^{(\ell/2-1)k+1} + O_{a_1,\ldots,a_k}(x^{(\ell/2-1)k+\eta_\ell(\varepsilon)}),$$

where $C_Q(a_1,\ldots,a_k)$ is a constant depending on Q and $(a_1,\ldots,a_k)$, and

$$\eta_\ell(\varepsilon) := \begin{cases} 1/2+\varepsilon & \text{if } \ell = 5,\\ \varepsilon & \text{if } \ell = 6,7,\\ 0 & \text{if } \ell \ge 8.\end{cases}$$

Obviously the two particular cases of Theorem 3:

$$\text{“}k=1,\ a_1=0\text{”} \quad \text{and} \quad \text{“}k=2,\ a_1=a_2=0\text{”}$$

improve (1.17) for $\ell \ge 6$ and (1.18) for $\ell \ge 5$, respectively. It is worth indicating that our method is different from Müller's [16] and simpler.

As an application of Theorem 3, we give the following asymptotic formula for the correlated sum involving the divisor sum function $\sigma_{\ell/2-1}(n)$.

Corollary 1.1. *Let $8 \mid \ell$, $k \ge 2$ and $a_1,\ldots,a_k$ be fixed nonnegative integers. Then*

$$\sum_{n\le x}\prod_{1\le i\le k}\sigma_{\ell/2-1}(n+a_i) = D_\ell(a_1,\ldots,a_k)x^{(\ell/2-1)k+1} + O_{a_1,\ldots,a_k}(x^{(\ell/2-1)k}),$$

where $D_\ell(a_1,\ldots,a_k)$ is a constant depending on ℓ and $a_1,\ldots,a_k$.

2. Study of $L_Q(s)$. This section is devoted to $L_Q(s)$, which is important in the proof of Theorem 1.

Lemma 2.1. *If $\ell \ge 5$, then $L_Q(s)$ can be analytically continued to a meromorphic function on the half-plane $\Re e\, s > 0$, which has a simple pole at $s = 1$ with residue 1.*

Proof. By using the definition of $\sigma(n,Q)$, a simple calculation shows that

$$\begin{aligned} L_Q(s) &= \sum_{q=1}^{\infty}\frac{1}{q^\ell}\sideset{}{^*}\sum_{h=1}^{q} S(hQ/q)F(s,-h/q) \\ &= \zeta(s) + \sum_{q=2}^{\infty}\frac{1}{q^\ell}\sideset{}{^*}\sum_{h=1}^{q} S(hQ/q)F(s,-h/q) \end{aligned} \tag{2.1}$$

for $\Re e\, s > 1$, where $F(s,a)$ is the periodic zeta-function defined by

$$F(s,a) := \sum_{n=1}^{\infty}\frac{\mathrm{e}(an)}{n^s} \qquad (\Re s > 1).$$

In view of well-known proprieties of $\zeta(s)$, it suffices to prove that the last double series in (2.1) can be continued analytically to the half-plane $\Re e\, s>0$.

Introducing the notation

$$M(u,\alpha) := \sum_{n\le u} \mathrm{e}(n\alpha) \ll \min\{u, \|\alpha\|^{-1}\}, \tag{2.2}$$

where $\|\alpha\| := \min_{t\in\mathbb{Z}} |\alpha - t|$, a simple integration by parts allows us to write, for $\Re e\, s > 1$, $q\ge 2$, and $(h,q)=1$, that

$$F(s,h/q) = \sum_{n\le|t|+1} \frac{\mathrm{e}(hn/q)}{n^s} - \frac{M(|t|+1,h/q)}{(|t|+1)^s} + s\int_{|t|+1}^{\infty} \frac{M(u,h/q)}{u^{s+1}}\,du.$$

This formula and (2.2) give an analytic continuation of $F(s,h/q)$ to the region $\Re e\, s > 0$, and the estimate

$$F(s,h/q) \ll \frac{|t|+1}{\|h/q\|}$$

holds uniformly for $\Re e\, s > 0$. From this and (1.12), we deduce that

$$\begin{aligned}\sum_{q=2}^{\infty} \frac{1}{q^{\ell}} \sideset{}{^*}\sum_{h=1}^{q} |S(hQ/q)F(s,-h/q)| &\ll \sum_{q=2}^{\infty} \frac{|t|+1}{q^{\ell/2}} \sum_{h=1}^{q/2} \frac{q}{h}\\ &\ll (|t|+1)\sum_{q=2}^{\infty} \frac{\log q}{q^{\ell/2-1}},\end{aligned}$$

which converges absolutely for $\Re e\, s > 0$ since $\ell \ge 5$. ∎

The next two lemmas give individual and average subconvexity bounds for $L_Q(s)$.

Lemma 2.2. *Let $\ell\ge 5$ and $\varepsilon > 0$. Then*

$$L_Q(\sigma+\mathrm{i}t) \ll \min\{|t|^{(1-\sigma)/3+\varepsilon}, |t|^{18.4974(1-\sigma)^{3/2}}(\log|t|)^{2/3}\} \tag{2.3}$$

uniformly for $1/2\le\sigma\le 1$ and $|t|\ge 2$.

Proof. According to [20, p. 127], we have, for $0<\alpha\le 1$,

$$\begin{aligned}F(s,\alpha) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{&\mathrm{e}^{\frac{\pi\mathrm{i}}{2}(1-s)}\zeta^*(1-s,\alpha) + \mathrm{e}^{\frac{\pi\mathrm{i}}{2}(1-s)}\alpha^{-(1-s)}\\ &+\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}(1-s)}\zeta^*(1-s,1-\alpha) + \mathrm{e}^{\frac{\pi\mathrm{i}}{2}(1-s)}(1-\alpha)^{-(1-s)}\},\end{aligned} \tag{2.4}$$

where $\zeta^*(s,\alpha) := \zeta(s,\alpha) - \alpha^{-s}$ and $\zeta(s,\alpha)$ is the Hurwitz zeta-function defined by (1.15). By combining (2.4) with Stirling's formula, we have, for $s = 1/2+\mathrm{i}t$ and $(h,q)=1$ with $q\ge 2$,

$$\begin{aligned}F(s,h/q) \ll\ &\zeta^*(1/2-\mathrm{i}t,h/q) + \zeta^*(1/2-\mathrm{i}t,1-h/q)\\ &+ q^{1/2}h^{-1/2} + q^{1/2}(q-h)^{-1/2}.\end{aligned} \tag{2.5}$$

Similar to the Riemann zeta-function, it is known that [2, Theorem]

$$\zeta^*(s,\alpha) \ll (|t|+1)^{(1-\sigma)/3+\varepsilon} \tag{2.6}$$

and

$$\zeta^*(s,\alpha) \ll |t|^{18.4974(1-\sigma)^{3/2}}(\log|t|)^{2/3} \tag{2.7}$$

uniformly for $0 < \alpha \le 1$, $1/2 \le \sigma \le 1$ and $|t| \ge 10$ (see e.g. [22] and [10], respectively). Now the required estimate (2.3) follows from (2.1) and (2.4)–(2.7), by noticing that

$$\sum_{q\ge 2}\frac{1}{q^\ell}\sum_{h=1}^{q}{}^{*}|S(hQ/q)|(q^{1/2}h^{-1/2}+q^{1/2}(q-h)^{-1/2}) \ll \sum_{q\ge 2}\frac{1}{q^{\ell/2-1}} \ll 1, \tag{2.8}$$

since $\ell \ge 5$. ∎

Lemma 2.3. *Let $\ell \ge 5$ and $k \ge 1$ be fixed integers. Then*

$$\int_1^T |L_Q(1/2+\mathrm{i}t)|^k\,dt \ll T^{1+\beta_{k,\ell}+\varepsilon}, \tag{2.9}$$

where

$$\beta_{k,\ell} := \begin{cases} 0 & \text{if } 1\le k\le 4 \text{ and } \ell \ge 5,\\ 13(k-4)/96 & \text{if } 5\le k\le 12 \text{ and } \ell = 5,\\ (k-4)/8 & \text{if } 5\le k\le 12 \text{ and } \ell \ge 6,\\ k/6-11/12 & \text{if } k>12 \text{ and } \ell=5,\\ k/6-1 & \text{if } k>12 \text{ and } \ell\ge 6.\end{cases}$$

Proof. Write $s = 1/2 + it$. It suffices to prove that

$$\int_1^T |L_Q(s)|^4\,dt \ll T^{1+\varepsilon}, \tag{2.10}$$

$$\int_1^T |L_Q(s)|^{12}\,dt \ll T^{2+\max\{(16-3\ell)/12,0\}+\varepsilon}. \tag{2.11}$$

Our key tools are the fourth mean value of Hurwitz' zeta-function [1, Theorem 4]

$$\int_1^T |\zeta^*(s,\alpha)|^4\,dt \ll T(\log T)^{10}, \tag{2.12}$$

which holds uniformly for $0 < \alpha \le 1$, $T \ge 2$, and the twelfth power moment of the Dirichlet L-function (see [15])

$$\sum_{\chi\,(\mathrm{mod}\,q)} \int_1^T |L(s,\chi)|^{12}\,dt \ll q^3 T^{2+\varepsilon}, \tag{2.13}$$

which holds uniformly for $q \geq 1$, $T \geq 2$.

From (2.1), (2.5) and (2.8), we deduce that

$$|L_Q(s)| \ll |\zeta(s)| + \sum_{q\geq 2} \frac{1}{q^{\ell/2}} \sum_{h\leq q/2} |\zeta^*(1/2-\mathrm{i}t, h/q)| + 1. \tag{2.14}$$

So by Hölder's inequality we have

$$\begin{aligned} |L_Q(s)|^4 \ll \bigg(\sum_{q\geq 2}\sum_{h\leq q/2} \frac{1}{q^{5/2}}\bigg)^3 \sum_{q\geq 2}\sum_{h\leq q/2} \frac{|\zeta^*(1/2-\mathrm{i}t,h/q)|^4}{q^{(4\ell-15)/2}} \\ + |\zeta(s)|^4 + 1, \end{aligned} \tag{2.15}$$

which combined with (2.12) leads to (2.10) since $\ell \geq 5$.

In order to prove (2.11), we write, by the orthogonality of Dirichlet characters,

$$F(s,h/q) = \sum_{a=1}^{q} \mathrm{e}(ah/q) \sum_{n\equiv a\,(\mathrm{mod}\,q)} \frac{1}{n^s} = \frac{1}{\varphi(q)} \sum_{\chi\,(\mathrm{mod}\,q)} G(h,\overline{\chi}) L(s,\chi),$$

where $\varphi(q)$ is the Euler function and $G(h,\chi)$ is the Gauss sum defined by

$$G(h,\chi) := \sum_{a=1}^{q} \chi(a)\mathrm{e}(ah/q).$$

By the well-known bound $|G(h,\chi)| \leq q^{1/2}$ ($(h,q)=1$), it follows that

$$F(s,h/q) \ll \frac{q^{1/2}}{\varphi(q)} \sum_{\chi\,(\mathrm{mod}\,q)} |L(s,\chi)|. \tag{2.16}$$

Let $\eta > 0$ be a parameter to be chosen later. We split the sum over q in (2.1) into two parts according to $q \leq T^\eta$ or $q > T^\eta$. Using (2.16) for $q \leq T^\eta$ and (2.5), (2.8) for $q > T^\eta$, we deduce that

$$|L_Q(s)| \ll L_{Q,1}(s) + L_{Q,2}(s) + 1 \tag{2.17}$$

where

$$\begin{aligned} L_{Q,1}(s) &:= \sum_{q\leq T^\eta} \frac{1}{q^{(\ell-1)/2}} \sum_{\chi\,(\mathrm{mod}\,q)} |L(s,\chi)|, \\ L_{Q,2}(s) &:= \sum_{q> T^\eta} \frac{1}{q^{\ell/2}} \sum_{h\leq q/2} |\zeta^*(1/2-\mathrm{i}t,h/q)|. \end{aligned}$$

By Hölder's inequality again we have

$$|L_{Q,1}(s)|^{12} \ll \bigg(\sum_{q\le T^\eta}\sum_{\chi\,(\mathrm{mod}\,q)}\frac{1}{q^2}\bigg)^{11}\sum_{q\le T^\eta}\sum_{\chi\,(\mathrm{mod}\,q)}\frac{|L(s,\chi)|^{12}}{q^{6\ell-28}}$$
$$\ll (\log T)^{11}\sum_{q\le T^\eta}\sum_{\chi\,(\mathrm{mod}\,q)}\frac{|L(s,\chi)|^{12}}{q^{6\ell-28}},$$

which combined with (2.13) gives

$$\int_1^T |L_{Q,1}(s)|^{12}\,dt \ll T^{2+\varepsilon}\sum_{q\le T^\eta} q^{-6\ell+31} \ll T^{2+\max\{\eta(32-6\ell),0\}+\varepsilon}. \tag{2.18}$$

The bound (2.6) implies trivially that

$$\sum_{q>T^\eta}\frac{1}{q^{\ell/2}}\sum_{h\le q/2}|\zeta^*(1/2-\mathrm{i}t,h/q)| \ll T^{1/6+\varepsilon}\sum_{q>T^\eta}\frac{1}{q^{\ell/2-1}} \ll T^{1/6-\eta(\ell/2-2)+\varepsilon}.$$

On the other hand, similarly to (2.15), we have

$$\bigg(\sum_{q>T^\eta}\frac{1}{q^{\ell/2}}\sum_{h\le q/2}|\zeta^*(1/2-\mathrm{i}t,h/q)|\bigg)^4 \ll T^{-3\eta/2}\sum_{q>T^\eta}\sum_{h\le q/2}\frac{|\zeta^*(1/2-\mathrm{i}t,h/q)|^4}{q^{(4\ell-15)/2}}.$$

Combining these with (2.12) yields

$$\int_1^T |L_{Q,2}(s)|^{12}\,dt \ll T^{8\{1/6-\eta(\ell/2-2)\}-3\eta/2-\eta(4\ell-19)/2+1+\varepsilon} \ll T^{7/3-(6\ell-24)\eta+\varepsilon}. \tag{2.19}$$

Now (2.11) follows from (2.18) and (2.19) with the choice of $\eta = \frac{1}{24}$. ∎

3. Proof of Theorem 1

3.1. The case $\ell \ge 5$ and $1 \le k \le 49$. By [21, Lemmas 3.1 and 3.2], it follows that

$$\sum_{n\le x} r_k(n,Q) = \frac{1}{2\pi\mathrm{i}}\int_{\ell/2+\varepsilon-\mathrm{i}T}^{\ell/2+\varepsilon+\mathrm{i}T} Z_Q(s)^k\frac{x^s}{s}\,ds + O\bigg(\frac{x^{\ell/2+\varepsilon}}{T}+x^\varepsilon\bigg). \tag{3.1}$$

In view of (1.10) and Lemma 2.1, we have

$$Z_Q(s) \ll |L_Q(s-\ell/2+1)|+1 \tag{3.2}$$

uniformly for $\Re e\, s \ge (\ell+3)/4+\varepsilon$ and $t \ne 0$. By noticing that $(\ell+3)/4 \le (\ell-1)/2$ (since $\ell \ge 5$), we can move the integration in (3.1) to the parallel segment with $\Re e\, s = (\ell-1)/2+\varepsilon$. By Lemma 2.1 and the residue theorem,

$$(3.3)\qquad \frac{1}{2\pi \mathrm{i}}\int_{\ell/2+\varepsilon-\mathrm{i}T}^{\ell/2+\varepsilon+\mathrm{i}T} Z_Q(s)^k\frac{x^s}{s}\,ds = \operatorname*{Res}_{s=\ell/2}(Z_Q(s)^k x^s s^{-1}) - \int_{\mathscr{L}} Z_Q(s)^k\frac{x^s}{s}\,ds,$$

where $\mathscr{L}$ is the contour joining $\ell/2+\mathrm{i}T$, $(\ell-1)/2+\varepsilon+\mathrm{i}T$, $(\ell-1)/2+\varepsilon-\mathrm{i}T$, $\ell/2-\mathrm{i}T$ with straight line segments. With the help of (3.2) and Lemmas 2.2–2.3, the contribution of the horizontal segments to the last integral of (3.3) is

$$(3.4)\qquad \ll x^{\ell/2+\varepsilon}T^{-1}$$

provided $T \le x^{3/k}$ $(2 \le k \le 49)$, and the contribution of the vertical segment is

$$(3.5)\qquad \ll x^{(\ell-1)/2+\varepsilon}\int_1^T \frac{|L_Q(1/2+\varepsilon+\mathrm{i}t)|^k}{t}\,dt \ll x^{(\ell-1)/2+\varepsilon}T^{\beta_{k,\ell}+\varepsilon}.$$

Combining (3.3)–(3.5) with (3.1) and taking $T = x^{1/(2+2\beta_{k,\ell})}$, we obtain the required estimate for $\ell \ge 5$ and $k \le 49$.

3.2. The case $\ell \ge 5$ and $k \ge 50$. In this case we apply Lemma 2.2. After applying Perron's formula, we move the integration to the parallel segment with $\Re e\, s = \sigma_0 = \ell/2 - 2Ak^{-2/3}$ and choose $T = x^{Ak^{-2/3}}$, where $A > 0$ is an absolute constant which will be determined later. By applying (3.2) and Lemma 2.2, the contribution of the vertical segment is

$$\begin{aligned}&\ll x^{\ell/2-2Ak^{-2/3}}T^{18.5k\{\ell/2-(\sigma_0-\ell/2+1)\}^{3/2}}(\log x)^{2k/3+1}\\ &= x^{\ell/2-(2A-18.5\times\sqrt{8}A^{5/2})k^{-2/3}}(\log x)^{2k/3+1},\end{aligned}$$

and the contribution of the horizontal segments is

$$\begin{aligned}&\ll x^{\ell/2+\varepsilon}T^{-1}(\log x)^{2k/3} + \max_{\sigma_0\le\sigma\le\ell/2} x^{\sigma}T^{18.5k\{1-(\sigma-\ell/2+1)\}^{3/2}-1}(\log x)^{2k/3}\\ &\ll (x^{\ell/2-(A-\varepsilon)k^{-2/3}} + x^{\ell/2-(2A-37\sqrt{2}A^{5/2})k^{-2/3}})(\log x)^{2k/3+1}.\end{aligned}$$

Now we choose A to satisfy $A = 2A - 37\sqrt{2}\,A^{5/2}$, which gives $A = 2738^{-1/3}$. Therefore for $k \ge 50$ we have

$$\Delta_k^*(x,Q) \ll x^{\ell/2-(2738k^2)^{-1/3}}(\log x)^{2k/3+1}.$$

3.3. The case $\ell = 4$. It is known that in this case

$$\theta(z,Q) := \sum_{n=0}^{\infty} r(n,Q)\mathrm{e}(nz)$$

is a modular form of weight 2 and level N (N is an integer such that $N\mathbf{A}^{-1}$ is also an integral matrix; see [8, Theorem 10.9]). Then by the standard theory of modular forms, $Z_Q(s)$ can be written as

$$Z_Q(s) = L_Q(s) + L(s, f),$$

where $L_Q(s)$ is a linear combination of series of the form

$$(t_1 t_2)^{-s} L(s, \chi_1) L(s - \ell/2 + 1, \chi_2),$$

and $L(s, f)$ is the Hecke L-function associated with a cusp form of weight 2 and level N. Here t_1, t_2 are positive divisors of N, and χ_1, χ_2 are Dirichlet characters modulo N/t_1, N/t_2 respectively.

According to (1.6) with $\ell = 4$, we learn that $|L(s, f)| \ll_\varepsilon 1$ for $\Re e\, s \geq 3/2 + \varepsilon$. When $\ell = 4$, we also have $\ell/2 - 1/2 = 3/2$. Therefore similar to (3.2), we have

$$|Z_Q(s)| \ll |L_Q(s)| + 1$$

for $\Re e\, s \geq 3/2 + \varepsilon$. On recalling the classical results [2]

$$L(1/2 + \mathrm{i}t, \chi) \ll (|t| + 1)^{1/6+\varepsilon}, \tag{3.6}$$

$$L(1/2 + \mathrm{i}t, \chi) \ll (|t| + 1)^{18.4974(1-\sigma)^{3/2}} (\log |t|)^{2/3}, \tag{3.7}$$

$$\int_1^T |L(1/2 + \mathrm{i}t, \chi)|^4 \, dt \ll T^{1+\varepsilon}, \tag{3.8}$$

$$\int_1^T |L(1/2 + \mathrm{i}t, \chi)|^{12} \, dt \ll T^{2+\varepsilon}, \tag{3.9}$$

it is easy to see that the estimates in Lemmas 2.2 and 2.3 also hold when $\ell = 4$. Thus we can follow the arguments of Section 3.1 to show that (1.13) is also true for $\ell = 4$. We omit the details.

4. The divisor problem with congruence conditions. The divisor problem with congruence conditions (1.14) was first studied by Nowak [18, 19] and Müller & Nowak [17]. They established very interesting Ω-type results for $\Delta_k(x; \mathbf{q}, \mathbf{r})$. As they indicated ([18, p. 456; p. 110], [17, Remarks]), it is straightforward to obtain the same O-results as in the classical divisor problem, since the theory of $\zeta(s)$ developed in the textbooks [22, 7] may be readily generalized to L-series. Here we state this O-result as a lemma, since it is important in the proof of Theorem 2.

Lemma 4.1. *Suppose $k = 1, 2, 3$. Then*

$$D_k(x; \mathbf{q}, \mathbf{r}) = \frac{x}{q_1 \cdots q_k} \mathcal{P}_{k-1}\Big(\log \frac{x}{q_1 \cdots q_k}\Big) + O_{k,\varepsilon}\Big(\Big(\frac{x}{q_1 \cdots q_k}\Big)^{\vartheta_k + \varepsilon}\Big)$$

[2] (3.6) is a special case of [5, Corollary 1]; (3.7) can be deduced easily from (2.7); (3.9) is a consequence of (2.13).

uniformly for $x \geq 3$, $1 \leq r_i \leq q_i$ $(1 \leq i \leq k)$ *and* $q_1 \cdots q_k \leq x$, *where* $\mathcal{P}_{k-1}(t)$ *is a polynomial of degree* $k-1$ *and* ϑ_k *is given by* (1.16). *Furthermore,*

$$\max|\text{coefficients of } \mathcal{P}_{k-1}| \ll \sum_{1\leq i_1<\cdots<i_{k-1}\leq k} \frac{q_{i_1}\cdots q_{i_{k-1}}}{r_{i_1}\cdots r_{i_{k-1}}}. \tag{4.1}$$

Proof. It is easy to see that

$$D_k(x;\mathbf{q},\mathbf{r}) = \sum_{\substack{1\leq n_1\cdots n_k\leq x\\ n_i\equiv r_i \,(\mathrm{mod}\, q_i)\,(1\leq i\leq k)}} 1 = \sum_{\substack{m_1\geq 0,\,\ldots,\,m_k\geq 0\\ (m_1+r_1/q_1)\cdots(m_k+r_k/q_k)\leq x/(q_1\cdots q_k)}} 1.$$

Thus the case of $k=1$ is trivial. When $k=2$, we can deduce from the above formula, by the well-known hyperbolic approach, that

$$D_2(x;\mathbf{q},\mathbf{r}) = (x/q_1q_2)\mathcal{P}_1(\log(x/(q_1q_2))) + \Delta_2(x;\mathbf{q},\mathbf{r}),$$

where $\psi(t) := \{t\} - 1/2$ ($\{t\}$ is the fractional part of t) and

$$\Delta_2(x;\mathbf{q},\mathbf{r}) = -\sum_{1\leq i\leq 2}\ \sum_{m_i\leq\sqrt{x/(q_1q_2)}-r_i/q_i} \psi\left(\frac{x/(q_1q_2)}{m_i+r_i/q_i}\right) + O(1).$$

Using Huxley's new result on exponential sums [6] we get

$$\Delta_2(x;\mathbf{q},\mathbf{r}) \ll (x/(q_1q_2))^{131/416+\varepsilon}.$$

For $k=3$, we could also follow Kolesnik's argument [9] to show $\vartheta_3 = 43/96$.

Next we prove (4.1). When s is near to 1, it is well known that (we suppose $0<\lambda\leq 1$)

$$\zeta(s,\lambda) = \frac{1}{s-1} - \frac{\Gamma'}{\Gamma}(\lambda) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\gamma_n(\lambda)(s-1)^n$$

where $\gamma_n(\lambda)$ is the nth Stieltjes constant. By the Cauchy formula, it is not difficult to see that $\gamma_n(\lambda) \ll_n 1$ uniformly for $0<\lambda\leq 1$. On the other hand, since $s=0$ is a pole of order 1 of $\Gamma(s)$, we have

$$\frac{\Gamma'}{\Gamma}(\lambda) \ll \frac{1}{\lambda}.$$

Finally we note that the polynomial $\mathcal{P}_{k-1}$ is determined by

$$\operatorname*{Res}_{s=1}\big(\zeta(s,\lambda_1)\cdots\zeta(s,\lambda_k)x^s s^{-1}\big) = \frac{x}{q_1\cdots q_k}\mathcal{P}_{k-1}\left(\log\frac{x}{q_1\cdots q_k}\right).$$

From all the above information, we can easily deduce (4.1). ∎

5. Proof of Theorem 2. In this section for any function $g(n)$ we define

$$g_j(n) := \sum_{n=n_1\cdots n_j} g(n_1)\cdots g(n_j),$$

which is similar to (1.2). Let

$$A := (2\pi)^{\ell/2}/(\Gamma(\ell/2)\sqrt{|\mathbf{A}|}), \quad \tilde{r}(n,Q) := A^{-1}n^{1-\ell/2}r(n,Q).$$

Since $r_k(n,Q) = A^k\tilde{r}_k(n,Q)n^{\ell/2-1}$, it is sufficient to prove that

$$\sum_{n\le x}\tilde{r}_k(n,Q) = x\tilde{P}_{k-1}(\log x) + O_{k,\varepsilon}(x^{\vartheta_k+\varepsilon}), \tag{5.1}$$

where $\tilde{P}_{k-1}(t)$ is a polynomial of degree $k-1$ and ϑ_k is defined by (1.16).

We first establish the following lemma.

LEMMA 5.1. *Suppose $\ell \ge 6$ and $k = 1,2,3$. Then for any $\varepsilon > 0$,*

$$\sum_{n\le x}\sigma_k(n,Q) = xP^*_{k-1}(\log x) + O_{k,\varepsilon}(x^{\vartheta_k+\varepsilon}), \tag{5.2}$$

where $P^*_{k-1}(t)$ is a polynomial of degree $k-1$ and ϑ_k is defined by (1.16).

Proof. Write

$$\sigma(n,Q) = \tilde{\sigma}(n,Q) + \hat{\sigma}(n,Q),$$

with

$$\tilde{\sigma}(n,Q) := \sum_{q\le x}\frac{1}{q^\ell}\sideset{}{^*}\sum_{h=1}^{q} S\Big(\frac{hQ}{q}\Big)\mathrm{e}\Big(-\frac{hn}{q}\Big),$$

$$\hat{\sigma}(n,Q) := \sum_{q> x}\frac{1}{q^\ell}\sideset{}{^*}\sum_{h=1}^{q} S\Big(\frac{hQ}{q}\Big)\mathrm{e}\Big(-\frac{hn}{q}\Big).$$

It is easy to see that $\tilde{\sigma}(n,Q) \ll 1$ and $\hat{\sigma}(n,Q) \ll x^{-1}$ (since $\ell \ge 6$). From these facts, we can deduce that

$$\tilde{\sigma}_j(n,Q) \ll \tau_j(n), \qquad \hat{\sigma}_j(n,Q) \ll x^{-j}\tau_j(n)$$

and

$$\begin{aligned}\sigma_k(n,Q) &= \sum_{j=0}^{k}\binom{k}{j}\sum_{dm=n}\tilde{\sigma}_{k-j}(d,Q)\hat{\sigma}_j(m,Q) \\ &= \tilde{\sigma}_k(n,Q) + O(x^{-1}\tau_{k-1}(n)).\end{aligned} \tag{5.3}$$

Thus in order to prove (5.2), it is sufficient to show that

$$\sum_{n\le x}\tilde{\sigma}_k(n,Q) = xP^*_{k-1}(\log x) + O(x^{\vartheta_k+\varepsilon}). \tag{5.4}$$

By using Lemma 4.1, it follows that

$$\begin{aligned}\sum_{n\le x}\tilde{\sigma}_k(n,Q) &= \prod_{i=1}^{k}\sum_{q_i\le x}\frac{1}{q_i^\ell}\sideset{}{^*}\sum_{h_i=1}^{q_i} S\Big(\frac{h_iQ}{q_i}\Big)\sum_{r_i=1}^{q_i}\mathrm{e}\Big(-\frac{h_ir_i}{q_i}\Big)D_k(x;\mathbf{q},\mathbf{r}) \\ &= xS_1(x) + S_2(x) + S_3(x),\end{aligned} \tag{5.5}$$

where

$$S_1(x) := \prod_{i=1}^{k} \sum_{\substack{q_i \le x \\ q_1 \cdots q_k \le x}} \frac{1}{q_i^{\ell+1}} \sum_{h_i=1}^{q_i}{}^{*} S\Big(\frac{h_i Q}{q_i}\Big) \sum_{r_i=1}^{q_i} \mathrm{e}\Big(-\frac{h_i r_i}{q_i}\Big) \mathcal{P}_{j-1}\Big(\log \frac{x}{q_1 \cdots q_k}\Big),$$

$$S_2(x) := \prod_{i=1}^{k} \sum_{\substack{q_i \le x \\ q_1 \cdots q_k > x}} \frac{1}{q_i^{\ell}} \sum_{h_i=1}^{q_i}{}^{*} S\Big(\frac{h_i Q}{q_i}\Big) \sum_{r_i=1}^{q_i} \mathrm{e}\Big(-\frac{h_i r_i}{q_i}\Big) D_k(x; \mathbf{q}, \mathbf{r}),$$

$$S_3(x) := \prod_{i=1}^{k} \sum_{q_i \le x} \frac{1}{q_i^{\ell}} \sum_{h_i=1}^{q_i}{}^{*} S\Big(\frac{h_i Q}{q_i}\Big) \sum_{r_i=1}^{q_i} \mathrm{e}\Big(-\frac{h_i r_i}{q_i}\Big) \varDelta_k(x; \mathbf{q}, \mathbf{r}).$$

It is easy to estimate

$$S_3(x) \ll x^{\vartheta_k+\varepsilon} \prod_{i=1}^{k} \sum_{q_i \le x} \frac{1}{q_i^{\ell/2-2+\vartheta_k+\varepsilon}} \ll x^{\vartheta_k+\varepsilon} \qquad (\text{since } \ell \ge 6). \tag{5.6}$$

When $q_1 \cdots q_k > x$, we use the trivial bound

$$D_k(x; \mathbf{q}, \mathbf{r}) \ll \frac{x}{r_1 \cdots r_k} + 1$$

to write

$$\begin{aligned} S_2(x) &\ll \prod_{i=1}^{k} \sum_{\substack{q_i \le x \\ q_1 \cdots q_k > x}} \frac{1}{q_i^{\ell/2}} \sum_{h_i=1}^{q_i}{}^{*} \sum_{r_i=1}^{q_i} \Big(\frac{x}{r_1 \cdots r_k} + 1\Big) \\ &\ll x \prod_{i=1}^{k} \sum_{\substack{q_i \le x \\ q_1 \cdots q_k > x}} \frac{\log q_i}{q_i^{\ell/2-1}} + \prod_{i=1}^{k} \sum_{\substack{q_i \le x \\ q_1 \cdots q_k > x}} \frac{1}{q_i^{\ell/2-2}} \\ &\ll x \sum_{n > x} \frac{\tau_k(n)(\log n)^k}{n^{\ell/2-1}} + \sum_{n > x} \frac{\tau_k(n)}{n^{\ell/2-2}} \\ &\ll x^{\varepsilon} \qquad (\text{since } \ell \ge 6). \end{aligned} \tag{5.7}$$

Obviously we can write

$$S_1(x) = x P_{k-1}^{*}(\log x) + O(R(x)) \tag{5.8}$$

where

$$R(x) := \prod_{i=1}^{k} \sum_{\substack{q_i \ge 1 \\ q_1 \cdots q_k > x}} \frac{1}{q_i^{\ell/2-1}} \Big| \mathcal{P}_{k-1}\Big(\log \frac{x}{q_1 \cdots q_k}\Big) \Big|.$$

By virtue of (4.1), we deduce that

$$(5.9)\quad R(x) \ll \prod_{i=1}^{k} \sum_{\substack{q_i \ge 1 \\ q_1\cdots q_k > x}} \frac{1}{q_i^{\ell/2}} \sum_{r_i=1}^{q_i} \sum_{1\le i_1<\cdots<i_{k-1}\le k} \frac{q_{i_1}\cdots q_{i_{k-1}}}{r_{i_1}\cdots r_{i_{k-1}}} \log^{k-1}(q_1\cdots q_k)$$

$$\ll \prod_{i=1}^{k} \sum_{\substack{q_i \le x \\ q_1\cdots q_k > x}} \frac{1}{q_i^{\ell/2-1}} \log^{2j-2}(q_1\cdots q_k) \ll \sum_{n>x} \frac{\tau_k(n)(\log n)^{2k-2}}{n^{\ell/2-1}}$$

$$\ll x^{-\ell/2+2+\varepsilon}.$$

Inserting (5.6)–(5.9) into (5.5), we obtain (5.4). ∎

Now we are ready to prove (5.1). By (1.10), we have

$$\tilde r(n,Q) = \sigma(n,Q) + \beta(n) \quad \text{with} \quad \beta(n) = O(n^{-1}).$$

Similar to (5.3), we have

$$\tilde r_k(n,Q) = \sum_{j=0}^{k} \binom{k}{j} \sum_{dm=n} \sigma_j(d,Q)\beta_{k-j}(m), \qquad \beta_j(n) \ll \tau_j(n)/n.$$

Thus Lemma 5.1 allows us to deduce

$$\sum_{n\le x} \tilde r_k(n,Q) = \sum_{j=0}^{k} \binom{k}{j} \sum_{m\le x} \beta_{k-j}(m) \sum_{d\le x/m} \sigma_j(d,Q)$$

$$= x\sum_{j=0}^{k} \binom{k}{j} \sum_{m\le x} \frac{\beta_{k-j}(m)}{m} P_{j-1}^{*}\Big(\log\frac{x}{m}\Big) + O(x^{\vartheta_j+\varepsilon}),$$

which implies (5.1) since

$$\sum_{m\le x} \frac{\beta_{k-j}(m)}{m} P_{j-1}^{*}\Big(\log\frac{x}{m}\Big) = \sum_{m\ge 1} \frac{\beta_{k-j}(m)}{m} P_{j-1}^{*}\Big(\log\frac{x}{m}\Big) + O(x^{-1+\varepsilon})$$

$$= P_{j-1}^{**}(\log x) + O(x^{-1+\varepsilon}),$$

where $P_{j-1}^{**}(t)$ is a polynomial of degree $j-1$.

6. Proof of Theorem 3. We reason by recurrence on k. The case of $k=1$ follows from Theorem 1 since a_1 is fixed. Assume that the required asymptotic formula holds for $1,\dots,k-1$. Then in view of (1.10) and the fact that $\ell/4-\delta_\ell \le \ell/2-1$, we can write

$$(6.1)\qquad \sum_{n\le x} \prod_{1\le i\le k} r(n+a_i,Q) = \bigg(\frac{\zeta(\ell/2)\Gamma(\ell/2)}{(2\pi)^{\ell/2}}\bigg)^{k} S + O(x^{(\ell/2-1)(k-1)+1+\ell/4-\delta_\ell+\varepsilon}),$$

where

$$S := \sum_{n\le x}\prod_{1\le i\le k}(n+a_i)^{\ell/2-1}\sigma(n+a_i,Q).$$

Inserting the series expansion for $\sigma(n,Q)$ and using the simple relation

$$(n+a_1)^{\ell/2-1}\cdots(n+a_k)^{\ell/2-1} = n^{(\ell/2-1)k} + O_{a_1,\dots,a_k}(n^{(\ell/2-1)k-1}),$$

it follows that

$$\begin{aligned} S = &\sum_{q_1=1}^{\infty}\cdots\sum_{q_k=1}^{\infty}\sum_{h_1=1}^{q_1}{}^{*}\cdots\sum_{h_k=1}^{q_k}{}^{*}\frac{S(h_1Q/q_1)\cdots S(h_kQ/q_k)}{(q_1\cdots q_k)^{\ell}}\\ &\times \mathrm{e}\Big(-\frac{h_1a_1}{q_1}-\cdots-\frac{h_ka_k}{q_k}\Big)\sum_{n\le x} n^{(\ell/2-1)k}\mathrm{e}\Big\{-n\Big(\frac{h_1}{q_1}+\cdots+\frac{h_k}{q_k}\Big)\Big\}\\ &+O(x^{(\ell/2-1)k}). \end{aligned}$$

By (1.12), the infinite series

$$\sum_{q_1=1}^{\infty}\cdots\sum_{q_k=1}^{\infty}\sum_{h_1=1}^{q_1}{}^{*}\cdots\sum_{h_k=1}^{q_k}{}^{*}\frac{S(h_1Q/q_1)\cdots S(h_kQ/q_k)}{(q_1\cdots q_k)^{\ell}}\mathrm{e}\Big(-\frac{h_1a_1}{q_1}-\cdots-\frac{h_ka_k}{q_k}\Big)$$

is absolutely convergent. Since

$$\sum_{n\le x} n^{(\ell/2-1)k} = \frac{x^{(\ell/2-1)k+1}}{(\ell/2-1)k+1} + O(x^{(\ell/2-1)k}),$$

the contribution of $(q_1,\dots,q_k,h_1,\dots,h_k)$ with $h_1/q_1+\cdots+h_k/q_k\in\mathbb{Z}$ to S is

$$C_Q(a_1,\dots,a_k)x^{(\ell/2-1)k+1} + O(x^{(\ell/2-1)k}). \tag{6.2}$$

By using (2.2), partial summation and the fact $\|h_1/q_1+\cdots+h_k/q_k\|\ge (q_1\cdots q_k)^{-1}$, the contribution of $(q_1,\dots,q_k,h_1,\dots,h_k)$ with $h_1/q_1+\cdots+h_k/q_k\notin\mathbb{Z}$ to S is

$$\ll x^{(\ell/2-1)k}\sum_{q_1=1}^{\infty}\cdots\sum_{q_k=1}^{\infty}\frac{\min\{x,q_1\cdots q_k\}}{(q_1\cdots q_k)^{\ell/2-1}} \ll x^{(\ell/2-1)k+\eta_\ell(\varepsilon)}, \tag{6.3}$$

where we have used the estimate

$$\min\{x,q_1\cdots q_k\}\le\begin{cases} x^{1/2+\varepsilon}(q_1\cdots q_k)^{1/2-\varepsilon} & \text{if } \ell=5,\\ x^{\varepsilon}(q_1\cdots q_k)^{1-\varepsilon} & \text{if } \ell=6,7,\\ q_1\cdots q_k & \text{if } \ell\ge 8.\end{cases}$$

Now Theorem 3 follows from (6.2) and (6.3), by noticing that

$$(\ell/2-1)(k-1)+1+\ell/4-\delta_\ell+\varepsilon \le (\ell/2-1)k+\eta_\ell(\varepsilon) \quad (\ell\ge 5).$$

7. Proof of Corollary 1.1. By (1.5) and (1.6), we have, for $n \le x$,

$$\prod_{i=1}^{k} \sigma_{\ell/2-1}(n+a_i) = \left(\frac{\zeta(\ell/2)\Gamma(\ell/2)}{(2\pi)^{\ell/2}}\right)^k \prod_{i=1}^{k} r(n+a_i, Q) + O\Big(x^{(k-d)(\ell/2-1)/2} \sum_{d=1}^{k-1} \sum_{\{i_1,\dots,i_d\}\subset\{1,\dots,k\}} \prod_{j=1}^{d} r(n+a_{i_j}, Q)\Big).$$

Now Theorem 3 implies the required result since

$$(k-d)(\ell/2-1)/2 + (\ell/2-1)d + 1 \le (\ell/2-1)(k-1/2) + 1 \le (\ell/2-1)k.$$

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