ACTA ARITHMETICA
151.1 (2012)

On lattice points in large convex bodies

by

JINGWEI GUO (Madison, WI)

1. Introduction. Let $\mathcal{B}$ denote a compact convex subset of $\mathbb{R}^d$ $(d \geq 3)$, which contains the origin as an interior point. Suppose that the boundary $\partial\mathcal{B}$ of $\mathcal{B}$ is a $(d-1)$-dimensional surface of class C^∞ with nonzero Gaussian curvature throughout. The remainder in the lattice point problem is defined to be

$$P_{\mathcal{B}}(t) = \#(t\mathcal{B} \cap \mathbb{Z}^d) - \mathrm{vol}(\mathcal{B})t^d \quad \text{for } t \geq 1.$$

We are interested in finding a number $\lambda(d)$ as small as possible such that

$$P_{\mathcal{B}}(t) = O(t^{d-2+\lambda(d)}).$$

It is conjectured that $\lambda(d) = 0$ for $d \geq 5$ and $\lambda(d) = \varepsilon$ for $d = 3$ and 4 where $\varepsilon > 0$ is arbitrary. For spheres this bound is sharp in dimension $d \geq 4$ (see Walfisz [23]) while open in dimension three. Bentkus and Götze [1] proved the conjecture for general ellipsoids in dimension $d \geq 9$, and their result was improved to $d \geq 5$ in Götze [3]. In Müller [17] even better bounds are proved for some high dimensional convex bodies with algebraic boundary, which may contain points with vanishing Gaussian curvature.

For general convex bodies the problem is still open. By a combination of the Poisson summation formula and (nowadays standard) oscillatory integral estimates, Hlawka [6] obtained $\lambda(d) = 2/(d+1)$.

Krätzel and Nowak [13, 14] used estimates for one- and two-dimensional exponential sums to improve the exponent. They obtained $\lambda(d) = 3/(2d)+\varepsilon$ for $d \geq 7$ among other results.

2010 *Mathematics Subject Classification*: Primary 11P21, 11L07.
Key words and phrases: lattice points, convex bodies, exponential sums, van der Corput's method.

DOI: 10.4064/aa151-1-6

Müller [16] significantly sharpened their result by extending their estimate to a d-dimensional version and he obtained

$$\lambda(d) = \begin{cases} \dfrac{d+4}{d^2+d+2} + \varepsilon & \text{for } d \geq 5, \\ 6/17 + \varepsilon & \text{for } d = 4, \\ 20/43 + \varepsilon & \text{for } d = 3, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is arbitrary.

We first observe that certain estimates of oscillatory integrals in Müller's paper can be obtained by using the method of stationary phase. This observation leads to our Proposition 2.4 which recovers Müller's [16, Theorem 2] without the ε there. This already leads to an improvement of (1.1) by removing the ε.

If we use asymptotic expansions of those oscillatory integrals, the leading terms form new exponential sums for which we can iterate Müller's d-dimensional estimate. This iteration leads to our new estimate of exponential sums in Theorem 2.6, which is the main result of this paper. As a consequence, we obtain the following new bound of $P_{\mathcal{B}}(t)$ for every dimension $d \geq 3$.

Theorem 1.1. *If $\mathcal{B}$ satisfies the conditions stated above, then $P_{\mathcal{B}}(t) = O(t^{d-2+\beta(d)})$ for*

$$\beta(d) = \begin{cases} \dfrac{d^2+3d+8}{d^3+d^2+5d+4} & \text{for } d \geq 4, \\ 73/158 & \text{for } d = 3. \end{cases}$$

The implicit constant may only depend on the body $\mathcal{B}$.

It is not hard to check that our estimate is indeed sharper than (1.1). In particular, for large d this is clear because $\beta(d) = 1/d + 2/d^2 + O(1/d^3)$ while $\lambda(d) = 1/d + 3/d^2 + O(1/d^3)$.

For more related results (for instance the average and lower bounds of the remainder) the reader could check [13]–[16] and [18]–[20].

In the case of planar domains, the sharpest known bound $P_{\mathcal{B}}(t) = O(t^{131/208}(\log t)^{2.26})$ is due to Huxley [9], who applied his refined variant of the discrete Hardy–Littlewood method originally due to Bombieri, Iwaniec, and Mozzochi. Huxley's method beats the classical theory of exponential sums, but it seems to be purely two-dimensional. In this paper, we focus on high dimensions and our main tools are still from the classical theory.

Notation. We use the usual Euclidean norm for a vector. $B(x,r) \subset \mathbb{R}^d$ represents the Euclidean ball centered at x with radius r. The norm of a matrix $A \in \mathbb{R}^{d\times d}$ is given by $\|A\| = \sup_{|x|=1} |Ax|$. We set $e(f(x)) = \exp(-2\pi i f(x))$ and $\mathbb{Z}^d_* = \mathbb{Z}^d \backslash \{0\}$. For a set $E \subset \mathbb{R}^d$ and a positive number a, we define $E_{(a)}$ to be the larger set

$$E_{(a)} = \{x \in \mathbb{R}^d : \operatorname{dist}(E, x) < a\}.$$

We use the differential operators

$$D_x^\nu = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \cdots \partial x_d^{\nu_d}} \qquad \Big(\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d,\ |\nu| = \sum_{i=1}^d \nu_i\Big)$$

and the gradient operator D_x. We often omit the subscript if no ambiguity occurs.

For functions f and g with g taking nonnegative real values, $f \lesssim g$ means $|f| \le Cg$ for some constant C. If f is also nonnegative, $f \gtrsim g$ means $g \lesssim f$. The Landau notation $f = O(g)$ is equivalent to $f \lesssim g$. The notation $f \asymp g$ means that $f \lesssim g$ and $g \lesssim f$.

A constant is said to be *admissible* if it only depends on the body $\mathcal{B}$. Throughout this paper except Section 2, all constants implied by the notation $\lesssim$, $\gtrsim$, $\asymp$, and $O(\cdot)$ are admissible. Wherever a variable occurs as a summation variable the reference is to integral values of the variable.

Structure of the paper. In §2 we prove three estimates of exponential sums which are the main results of this paper. We then turn to the lattice point problem and study the support function of $\mathcal{B}$ in §3. We show that certain matrices have nonvanishing determinants and their entries satisfy some special size estimates, which allow us to verify certain hypotheses needed in our estimates of exponential sums. In §4 we put these ingredients together to prove Theorem 1.1. The appendix contains a quantitative version of the inverse function theorem.

2. Estimates of exponential sums. The classical theory of exponential sums has two processes: the Weyl–van der Corput inequalities (*A-process*) and the Poisson summation formula followed by the method of stationary phase (*B-process*). Before we start the estimation of exponential sums, we first introduce two preliminary lemmas related to these two processes.

For integrals in the form

$$I(\lambda) = \int_{\mathbb{R}^d} w(x) e^{i\lambda f(x)}\, dx,$$

Hörmander [7, Theorem 7.7.5] gives an asymptotic formula for the case when the phase function f has a nondegenerate critical point. It is one of the expressions of the method of stationary phase and we only need it when f takes real values.

LEMMA 2.1. *Let $K \subset \mathbb{R}^d$ be a compact set, X an open neighborhood of K, and k a positive integer. If f is real and in $C^\infty(X)$, $w \in C_0^\infty(K)$,*

$Df(x_0)=0$, $\det(D^2f(x_0))\neq 0$, and $Df\neq 0$ in $K\setminus\{x_0\}$, then

$$\Big|I(\lambda)-(2\pi)^{d/2}e^{i(\frac{\pi}{4}\operatorname{sgn}(D^2f(x_0))+\lambda f(x_0))}|\det(D^2f(x_0))|^{-1/2}\lambda^{-d/2}\sum_{j=0}^{k-1}\lambda^{-j}L_jw\Big| \le C\lambda^{-k}\sum_{|\mu|\le 2k}\sup_x|D^\mu w(x)| \quad \textit{for } \lambda>1.$$

Here C is bounded when f stays in a bounded set in $C^{3k+1}(X)$ and $|x-x_0|/|Df(x)|$ has a uniform bound. With

$$g_{x_0}(x)=f(x)-f(x_0)-\langle D^2f(x_0)(x-x_0),x-x_0\rangle/2$$

which vanishes to third order at x_0 we have

$$L_jw=\sum_{\upsilon-\gamma=j}\sum_{2\upsilon\ge 3\gamma} i^{-j}2^{-\upsilon}\langle D^2f(x_0)^{-1}D,D\rangle^{\upsilon}(g_{x_0}^{\gamma}w)(x_0)/(\upsilon!\gamma!).$$

REMARKS. 1) L_j is a differential operator of order $2j$ acting on w at x_0. The sum has only a finite number of terms for each j.

2) The integral $I(\lambda)$ has the following asymptotic expansion:

$$I(\lambda)=\lambda^{-d/2}\sum_{j=0}^{N}a_j\lambda^{-j}+O(\lambda^{-d/2-N-1}) \quad \text{for any } N\in\mathbb{N}.$$

The constant implied in the error term depends on d, N, the size of K, upper bounds of finitely many derivatives of w and f in the support of w, and a lower bound of $|\det(D^2f(x_0))|$. Each coefficient a_j depends on d, j, values of finitely many derivatives of w and f at the point x_0, and the value of $|\det(D^2f(x_0))|$. These coefficients a_j have explicit formulas, in particular

$$a_0=(2\pi)^{d/2}w(x_0)e^{i(\frac{\pi}{4}\operatorname{sgn}(D^2f(x_0))+\lambda f(x_0))}/|\det(D^2f(x_0))|^{1/2}.$$

For the method of stationary phase the reader could also check Stein [22, Chapter 8].

Let $M>1$ and $T>0$ be parameters. We consider d-dimensional exponential sums of the form

$$S=S(T,M;G,F)=\sum_{m\in\mathbb{Z}^d}G(m/M)e(TF(m/M)), \tag{2.1}$$

where $G:\mathbb{R}^d\to\mathbb{R}$ is C^∞ smooth, compactly supported, and bounded above by a constant, and $F:\Omega\subset\mathbb{R}^d\to\mathbb{R}$ is C^∞ smooth on an open convex domain Ω such that

$$\operatorname{supp}(G)\subset\Omega\subset c_0B(0,1), \tag{2.2}$$

where $c_0>0$ is a fixed constant.

We are interested in finding upper bounds for S in terms of T and M. Exponential sums of the form (2.1) are essentially the same as those considered in Müller [16]. In lower dimensions Huxley studied sums of similar but more complicated forms in various papers (for example, see [9]).

The following lemma is Müller's [16, Lemma 1] (in a slightly different form), namely the so-called iterated one-dimensional Weyl–van der Corput inequality.

Lemma 2.2. *Let $q \in \mathbb{N}$, G, F, and S be as above, and $r_1, \ldots, r_q \in \mathbb{Z}^d$ be nonzero integral vectors with $|r_i| \lesssim 1$. Furthermore, let H be a real parameter which satisfies $1 < H \lesssim M$. Set $Q = 2^q$ and $H_l = H_{q,l} = H^{2^{l-q}}$ for $1 \le l \le q$. Then*

$$|S(T, M; G, F)|^Q \lesssim \frac{M^{Qd}}{H} + \frac{M^{(Q-1)d}}{H_1 \cdots H_q} \sum_{\substack{1 \le h_i < H_i \\ 1 \le i \le q}} |S(\mathscr{H} T M^{-q}, M; G_q, F_q)|,$$

where $\mathscr{H} = \prod_{l=1}^{q} h_l$ and the functions G_q, F_q are defined as follows:

$$G_q(x) = G_q(x, h_1, \ldots, h_q) = \prod_{\substack{u_i \in \{0,1\} \\ 1 \le i \le q}} G\Big(x + \sum_{l=1}^{q} \frac{h_l}{M} u_l r_l\Big)$$

and

$$\begin{aligned} F_q(x) &= F_q(x, h_1, \ldots, h_q) \\ &= \int_{(0,1)^q} (r_1 \cdot D) \cdots (r_q \cdot D) F\Big(x + \sum_{l=1}^{q} \frac{h_l}{M} u_l r_l\Big)\, du_1 \ldots du_q. \end{aligned}$$

The integral representation of F_q is well defined on the open convex set $\Omega_q = \Omega_q(h_1, \ldots, h_q) = \{x \in \Omega : x + \sum_{l=1}^{q} (h_l/M) u_l r_l \in \Omega$ for all $u_l \in \{0, 1\}$, $1 \le l \le q\}$. Moreover, $\operatorname{supp}(G_q) \subset \Omega_q \subset \Omega$.

We give, without proof, an easy but useful result concerning the distance between the boundary of $\operatorname{supp}(G_q)$ and Ω_q.

Lemma 2.3. *For fixed $(h_1, \ldots, h_q)$,*

$$\operatorname{dist}(\operatorname{supp}(G), \Omega^c) \ge c_1 \;\Rightarrow\; \operatorname{dist}(\operatorname{supp}(G_q), \Omega_q^c) \ge c_1.$$

The exponential sum S is bounded by CM^d trivially, but we lose cancelation by just putting absolute value on each term. We will prove three bounds of S, obtained by applying various combinations of A- and B-processes. In the statement of these results we will assume that derivatives of G and F up to certain orders are uniformly bounded. The orders may not be optimal but sufficient for the proof.

We first prove a bound of $S(T, M; G, F)$ by applying a B-process. For an analogous one-dimensional result, see Theorem 2.2 in [4].

Proposition 2.4 (Estimate by a B-process). *Let* $d \geq 2$. *Assume that* $\operatorname{dist}(\operatorname{supp}(G), \Omega^c) \geq c_1$ *for some constant* c_1, *and that for all* $x \in \Omega$ *and* $\nu \in \mathbb{N}_0^d$ *with* $|\nu| \leq 3\lceil d/2 \rceil + 1$ (1),

$$(D^\nu G)(x) \lesssim 1, \tag{2.3}$$

$$(D^\nu F)(x) \lesssim 1, \tag{2.4}$$

$$|\det(D^2 F(x))| \gtrsim 1. \tag{2.5}$$

Then

$$S(T, M; G, F) \lesssim T^{d/2} + M^d T^{-d/2}. \tag{2.6}$$

The implicit constant in (2.6) *depends on* d, c_0, c_1, *and the constants implied in* (2.3)–(2.5).

Proof. Applying to S the d-dimensional Poisson summation formula followed by a change of variables $y = Mx$ yields

$$\begin{aligned} S(T, M; G, F) &= \sum_{p \in \mathbb{Z}^d} \int_{\mathbb{R}^d} G(y/M) e(TF(y/M) - y \cdot p)\, dy \\ &= \sum_{p \in \mathbb{Z}^d} M^d \int G(x) e(TF(x) - Mx \cdot p)\, dx. \end{aligned} \tag{2.7}$$

By (2.4) there exists a constant A_0 such that

$$|DF(x)| \leq A_0/2.$$

We split the sum in (2.7) into two parts,

$$S(T, M; G, F) = \sum_{|p| \geq A_0 T/M} + \sum_{|p| < A_0 T/M} =: \mathrm{I} + \mathrm{II},$$

and separate the estimation into two cases.

Case 1: $|p| \geq A_0 T/M$. Let $\Psi(x, p) = (TF(x) - Mx \cdot p)/(M|p|)$. Then

$$\mathrm{I} = \sum_{|p| \geq A_0 T/M} M^d \int G(x) e(M|p|\Psi(x, p))\, dx.$$

Under the given assumptions, for all $x \in \Omega$ and $|\nu| \leq d + 2$ we have $D^\nu G(x) \lesssim 1$, $D_x^\nu \Psi(x, p) \lesssim 1$, and also

$$\big|D_x \Psi + p/|p|\big| = |T D_x F(x)/(M|p|)| \leq 1/2,$$

which ensures $|D_x \Psi| \geq 1/2$. By integration by parts (Hörmander [7, Theorem 7.7.1]) we get

$$\int G(x) e(M|p|\Psi(x, p))\, dx \lesssim (M|p|)^{-d-1},$$

(1) We use $\lceil x \rceil$ to represent the smallest integer not less than x.

which leads to

$$\mathrm{I} \lesssim M^{-1} \sum_{p \in \mathbb{Z}_*^d} |p|^{-d-1} \lesssim M^{-1}.$$

CASE 2: $|p| < A_0 T/M$. Let $\Phi(x,p) = F(x) - (M/T)x \cdot p$. Then

$$\mathrm{II} = \sum_{|p| < A_0 T/M} M^d \int G(x) e(T\Phi(x,p))\, dx.$$

If $T \le 1$, then $\mathrm{II} \lesssim M^d \le M^d T^{-d/2}$.

If $T > 1$, we claim that each integral in II is less than $CT^{-d/2}$. Assume this for a moment; then

$$\mathrm{II} \lesssim (1 + (T/M)^d) M^d T^{-d/2} = T^{d/2} + M^d T^{-d/2}.$$

Observe that the bound above is true for II no matter whether $T \le 1$ or $T > 1$. It follows that

$$S(T,M;G,F) \lesssim M^{-1} + T^{d/2} + M^d T^{-d/2} \lesssim T^{d/2} + M^d T^{-d/2},$$

which is the desired bound. The only thing left is to prove the claim.

Let us fix a $|p| < A_0 T/M$. For all $x \in \Omega$ and $|\nu| \le 3\lceil d/2 \rceil + 1$, the given assumptions imply $D_x^\nu \Phi(x,p) \lesssim 1$ and $|\det(D_{xx}^2 \Phi(x,p))| \gtrsim 1$. We first show that the number of critical points is bounded above by a constant independent of p, T, and M. Denote $f(x) = DF(x)$ and $\widetilde{p} = (M/T)p$. Then $D_x \Phi(x,p) = f(x) - \widetilde{p}$. The critical points are determined by the equation

$$f(x) = \widetilde{p}, \qquad x \in \Omega.$$

We know that $\mathrm{supp}(G)$ is strictly smaller than Ω and the distance between their boundaries is larger than c_1. Let $r_0 = c_1/2$. By Taylor's formula, there exists a uniform r_* ($< r_0$) such that if $\widetilde{x}$ is a critical point in $(\mathrm{supp}(G))_{(r_0)}$ (2) then $|D_x \Phi(x,p)| \gtrsim |x - \widetilde{x}|$ for any $x \in B(\widetilde{x}, r_*)$.

Applying Lemma A.1 (see Appendix) to f with r_0 as above yields two uniform positive numbers r_1, r_2 such that $2r_1 \le r_*$ and, for any $x \in (\mathrm{supp}(G))_{(r_0)}$, f is bijective from $B(x, 2r_1)$ to an open set containing $B(f(x), 2r_2)$.

If x_1, x_2 are two different critical points in $(\mathrm{supp}(G))_{(r_0)}$ (if any), then $B(x_1, r_1)$ and $B(x_2, r_1)$ are disjoint and still contained in Ω. It follows, simply by a size estimate, that the number of possible critical points in $(\mathrm{supp}(G))_{(r_0)}$ is bounded by a constant.

We will only consider critical points in $(\mathrm{supp}(G))_{(r_1)}$. Denote

$$S_p = \{x \in \mathrm{supp}(G) : |D_x \Phi(x,p)| < r_2\}.$$

If S_p is empty, which means $|D_x \Phi|$ has a lower bound r_2 on $\mathrm{supp}(G)$, by integration by parts the integral is of order $O(T^{-d/2})$.

(2) For this notation, see Section 1.

If S_p is not empty, there exists at least one critical point in $(\mathrm{supp}(G))_{(r_1)}$. To see this assume that $x \in S_p$, which implies $|f(x) - \widetilde{p}| < r_2$. Note that f is bijective from $B(x, r_1)$ to an open set containing $B(f(x), r_2)$, hence there exists a point $\widetilde{x} \in B(x, r_1)$ such that $f(\widetilde{x}) = \widetilde{p}$. This means $\widetilde{x}$ is a critical point and $x \in B(\widetilde{x}, r_1) \subset \Omega$. As a consequence, S_p is contained in the union of finitely many balls centered at critical points with radius r_1.

Assume $\widetilde{x}_i(p)$ $(1 \le i \le J(p))$ are all critical points in $(\mathrm{supp}(G))_{(r_1)}$. Let

$$\chi_i(x) = \chi((x - \widetilde{x}_i(p))/r_1),$$

where χ is a given smooth cut-off function whose value is 1 on $B(0, 1/2)$ and 0 on the complement of $B(0, 1)$. Let $\chi_0 = 1 - \sum_{i=1}^{J(p)} \chi_i$. Then

$$\int G(x) e(T\Phi(x,p))\,dx = \sum_{i=1}^{J(p)} \int \chi_i(x) G(x) e(T\Phi(x,p))\,dx + \int \chi_0(x) G(x) e(T\Phi(x,p))\,dx.$$

For each $1 \le i \le J(p)$, the integral in the summation above has its domain contained in $B(\widetilde{x}_i(p), r_1)$ and is of order $O(T^{-d/2})$ by Lemma 2.1.

If $x \in \mathrm{supp}(G) \setminus \bigcup_{i=1}^{J(p)} B(\widetilde{x}_i, r_1/2)$, there exists a uniform constant r_3 such that $|D_x \Phi(x,p)| \ge r_3$. Hence the last integral above is of order $O(T^{-d/2})$ by integration by parts. This finishes the proof. ■

Now we can prove another bound of $S(T, M; G, F)$ by applying the A-process q times (Lemma 2.2) followed by a B-process (Proposition 2.4). For analogous one-dimensional results, see Theorems 2.6, 2.8, 2.9 in [4].

Proposition 2.5 (Estimate by an A^{q}B-process). *Let $d \ge 3$. Assume that* $\mathrm{dist}(\mathrm{supp}(G), \Omega^c) \ge c_1$ *for some constant c_1, and for all $x \in \Omega$ and $\nu \in \mathbb{N}_0^d$ with $|\nu| \le 3\lceil d/2 \rceil + q + 1$,*

$$(D^\nu G)(x) \lesssim 1, \tag{2.8}$$

$$(D^\nu F)(x) \lesssim 1, \tag{2.9}$$

and for some fixed $\mu \in \mathbb{N}_0^d$ with $q = |\mu|$ and all $x \in \Omega$,

$$\left|\det\left(\frac{\partial^2 D^\mu F}{\partial x_i \partial x_j}(x)\right)_{1 \le i,j \le d}\right| \gtrsim 1. \tag{2.10}$$

If

$$T \ge M^{q - 2/d + 2/Q} \qquad (Q = 2^q), \tag{2.11}$$

then

$$S(T, M; G, F) \lesssim T^{w_{d,q}} M^{d - (q+2) w_{d,q}}, \tag{2.12}$$

where

$$w_{d,q} = \frac{d}{2d(Q-1)+2Q}.$$

The implicit constant in (2.12) *depends on* d, q, c_0, c_1, *and the constants implied in* (2.8)–(2.10).

Remarks. 1) If $T \gtrsim M^{q+2}$, the trivial bound $S \lesssim M^d$ is better than the estimate above.

2) This proposition recovers Müller's [16, Theorem 2] without ε.

Proof of Proposition 2.5. Let $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ be the standard orthonormal basis of $\mathbb{R}^d$. Then $\mu = \sum_{l=1}^{q} e_{k_l}$, where $1 \le k_l \le d$, $1 \le l \le q$. Assume that $1 < H \le c_2 M$ with a small constant c_2 (to be determined later) and that $M > c_2^{-1}$ (otherwise the trivial bound is better than (2.12)). By Lemma 2.2 with $r_l = e_{k_l}$, the estimation is reduced to that of $S(\mathscr{H} T M^{-q}, M; G_q, F_q)$. The G_q, F_q are as in that lemma, and so is the domain Ω_q. Note that $\mathrm{supp}(G_q) \subset \Omega_q \subset c_0 B(0,1)$ and $\mathrm{dist}(\mathrm{supp}(G_q), \Omega_q^c) \ge c_1$ by Lemma 2.3.

For all $x \in \Omega_q$ and $|\nu| \le 3\lceil d/2 \rceil + 1$, we have $D^\nu G_q(x) \lesssim 1$, $D^\nu F_q(x) \lesssim 1$, and $|\det(D^2 F_q(x))| \gtrsim 1$. The two upper bounds are easy to get. To prove the lower bound, we first have

$$\begin{aligned} \frac{\partial^2 F_q}{\partial x_i \partial x_j}(x) &= \int\limits_{(0,1)^q} \left(\frac{\partial^2 D^\mu F}{\partial x_i \partial x_j}\right)\left(x + \sum_{l=1}^{q} \frac{h_l}{M} u_l r_l\right) du_1 \ldots du_q \\ &= \frac{\partial^2 D^\mu F}{\partial x_i \partial x_j}(x) + O\left(\frac{H}{M}\right), \end{aligned}$$

thus

$$|\det(D^2 F_q(x))| = \left|\det\left(\frac{\partial^2 D^\mu F}{\partial x_i \partial x_j}(x)\right) + O\left(\frac{H}{M}\right)\right|.$$

If c_2 is sufficiently small, the desired lower bound follows from the lower bound (2.10) and $H \le c_2 M$.

Applying Proposition 2.4, we get

$$S(\mathscr{H} T M^{-q}, M; G_q, F_q) \lesssim (\mathscr{H} T M^{-q})^{d/2} + M^d (\mathscr{H} T M^{-q})^{-d/2}.$$

Since $H_1 \cdots H_q = H^{2-2/Q}$ and

$$\sum_{\substack{1 \le h_i < H_i \\ 1 \le i \le q}} \mathscr{H}^\alpha \lesssim \begin{cases} (H_1 \cdots H_q)^{\alpha+1} & \text{if } \alpha > -1, \\ 1 & \text{if } \alpha < -1, \end{cases}$$

Lemma 2.2 implies

$$\begin{aligned} |S(T, M; G, F)|^Q \lesssim M^{Qd} H^{-1} &+ M^{(Q-1-q/2)d} T^{d/2} (H^{2-2/Q})^{d/2} \\ &+ M^{(Q+q/2)d} T^{-d/2} H^{-2+2/Q}. \end{aligned}$$

Balancing the first two terms yields the optimal choice

$$H^{(1-1/Q)d+1} = B_1 T^{-d/2} M^{(q+2)d/2},$$

where B_1 can be chosen sufficiently small such that assumption (2.11) implies $H \le c_2 M$. Due to Remark 1 (following the statement of the proposition), we can assume $T \le B_2 M^{q+2}$ with a sufficiently small B_2, which implies $1 < H$. With this choice of H the third term is $\lesssim M^{(Q-1)d} H^{d-1} \lesssim M^{Qd} H^{-1}$. Hence we get

$$S(T, M; G, F) \lesssim M^d H^{-1/Q} \lesssim T^{w_{d,q}} M^{d-(q+2)w_{d,q}},$$

where $w_{d,q}$ is as defined in the statement of the proposition. ∎

Next we will estimate $S(T, \delta M; G, F)$ where $\delta > 0$ is a parameter. In the following theorem and its proof, if we write a δ in a subscript (for instance $\gtrsim_\delta$, $\lesssim_\delta$, $\asymp_\delta$, or O_δ), we emphasize that the implicit constant depends on δ; otherwise it does not.

The proof will proceed as follows. We first apply an A-process q times (Lemma 2.2) followed by a B-process, while in the latter process we use Lemma 2.1 to get the asymptotic expansions of certain oscillatory integrals. By looking at the leading terms, we obtain some new exponential sums to which we apply an AB-process (Proposition 2.5 with q there being 1).

Before we can apply Proposition 2.5, however, we need some preparation in the first B-process. For instance, we use partitions of unity to restrict certain domains to small balls on which a certain critical point function (if exists) is smooth; we distinguish the cases when we are allowed to use Lemma 2.1; we show that certain determinants needed in two B-processes are nonvanishing. One difficulty is to establish the nonvanishing determinants for the second B-process, and this is where we need the auxiliary condition (2.16) below. In the next section, we will show that this condition is indeed satisfied in the lattice point problem.

THEOREM 2.6 (Estimate by an A^{q}BAB-process). *Assume* $q \in \{1, 2\}$ *if* $d = 3$ *or* $q \in \mathbb{N}$ *if* $d \ge 4$. *Assume that* $M > \max(1, \delta^{-1})$, $\mathrm{dist}(\mathrm{supp}(G), \Omega^c) \ge c_1' \delta$ *for some constant* c_1', *and that for all* $x \in \Omega$ *and* $\nu \in \mathbb{N}_0^d$ *with* $|\nu| \le 3\lceil d/2\rceil + q + 3$,

$$(D^\nu G)(x) \lesssim \delta^{-|\nu|} \lesssim_\delta 1, \tag{2.13}$$

$$(D^\nu F)(x) \lesssim 1. \tag{2.14}$$

For $1 \le i, j \le d$ *denote*

$$a_{i,j}^{(k)}(x) = \frac{\partial^{k+2} F}{\partial x_1 \partial x_i \partial x_j \partial x_d^{k-1}}(x).$$

Further assume that for all $x \in \Omega$ and $k \in \{q, q+1\}$,

$$|\det(a_{i,j}^{(k)}(x))_{1\le i,j\le d}| \gtrsim 1 \tag{2.15}$$

and

$$\begin{cases} |a_{i,i}^{(q)}(x)| \asymp 1 & \text{for } 1 \le i \le d-1, \\ |a_{i,j}^{(q)}(x)| \lesssim 1 & \text{for } 2 \le i \le d-1,\ 1 \le j \le i-1, \\ |a_{d,1}^{(q)}(x)| \asymp 1, & \\ |a_{d,j}^{(q)}(x)| \lesssim \delta & \text{for } 2 \le j \le d. \end{cases} \tag{2.16}$$

If δ is sufficiently small (*only depending on d, q, and the constants implied in* (2.14)–(2.16)) *and*

$$M^{q+2/Q-1/d-4/d^2} \le T \le M^{q+2/Q+2/(d-2)} \qquad (Q = 2^q), \tag{2.17}$$

then

$$S(T, \delta M; G, F) \lesssim_\delta T^{\frac{d^2}{2(Q-1)d^2+2Qd+4Q}} M^{d-\frac{(q+2)d^2+d}{2(Q-1)d^2+2Qd+4Q}}. \tag{2.18}$$

Besides δ, the implicit constant in (2.18) *depends on d, q, c_0, c_1', and the constants implied in* (2.13)–(2.16).

Remarks. 1) If $T \gtrsim M^{q+2+1/d}$, the trivial bound $S \lesssim M^d$ is better than the estimate above.

2) When we apply this result to the lattice point problem, we will let $q = 1$ if $d \ge 4$ and 2 if $d = 3$; we will choose and fix a small δ and will not need it explicitly in the bound (2.18).

Proof of Theorem 2.6. Assume that $1 < H \le c_2\delta M$ with a small constant c_2 (to be determined later) and that $\delta M > c_2^{-1}$ (otherwise the trivial bound is better than (2.18)). Using Lemma 2.2 with $r_1 = e_1$ and $r_l = e_d$ ($2 \le l \le q$), we get

$$|S(T, \delta M; G, F)|^Q \lesssim \frac{(\delta M)^{Qd}}{H} + \frac{(\delta M)^{(Q-1)d}}{H_1 \cdots H_q} \cdot \sum_{\substack{1\le h_i<H_i \\ 1\le i\le q}} |S(\mathscr{H}T(\delta M)^{-q}, \delta M; G_q, F_q)|, \tag{2.19}$$

where G_q, F_q are as in Lemma 2.2, and so is the domain Ω_q. Applying to the innermost sums the d-dimensional Poisson summation formula followed by a change of variables yields

$$\begin{aligned} &S(\mathscr{H}T(\delta M)^{-q}, \delta M; G_q, F_q) \\ &\quad = \sum_{p\in\mathbb{Z}^d} (\delta M)^d \int G_q(x) e(\mathscr{H}T(\delta M)^{-q} F_q(x) - \delta M x \cdot p)\, dx. \end{aligned} \tag{2.20}$$

Lemmas 2.2 and 2.3 imply

$$\mathrm{supp}(G_q) \subset \Omega_q \subset c_0 B(0,1)$$

and

$$\mathrm{dist}(\mathrm{supp}(G_q), \Omega_q^c) \geq c_1' \delta.$$

By (2.14), there exists a constant A_0 (independent of δ) such that for any $x \in \Omega_q$,

$$|DF_q(x)| \leq A_0/2.$$

Define $\widetilde{M} = \mathscr{H}T(\delta M)^{-q-1}$. We split (2.20) into two parts,

$$S(\mathscr{H}T(\delta M)^{-q}, \delta M; G_q, F_q) = \sum_{|p| \geq A_0 \widetilde{M}} + \sum_{|p| < A_0 \widetilde{M}} =: \mathrm{I} + \mathrm{II},$$

and estimate them separately.

Case 1: $|p| \geq A_0 \widetilde{M}$. As in the proof of Proposition 2.4, it is easy to prove that $\mathrm{I} \lesssim_\delta M^{-1}$.

Case 2: $|p| < A_0 \widetilde{M}$. Let $\widetilde{T} = \delta M \widetilde{M}$ ([3]).

Subcase 2.1: $\widetilde{T} \geq 1$. Define

$$\Phi_q(x,z) = F_q(x) - x \cdot z, \qquad x \in \Omega_q,\ z \in \mathbb{R}^d.$$

Then

$$\mathrm{II} = (\delta M)^d \sum_{|p| < A_0 \widetilde{M}} \int G_q(x) e(\widetilde{T} \Phi_q(x, p/\widetilde{M}))\, dx.$$

For all $x \in \Omega_q$, $|z| < A_0$, and $|\nu| \leq 3\lceil d/2 \rceil + 3$, it is easy to see that

$$D_x^\nu G_q(x) \lesssim_\delta 1, \tag{2.21}$$

$$D_x^\nu \Phi_q(x,z) \lesssim 1, \tag{2.22}$$

and if c_2 is sufficiently small (depending on the constants implied in (2.14) and (2.15)) the condition (2.15) with $k = q$ implies

$$|\det(D_{xx}^2 \Phi_q(x,z))| \gtrsim 1. \tag{2.23}$$

Denote $f(x) = DF_q(x)$. Then $D_x \Phi_q(x,z) = f(x) - z$. For a particular z with $|z| < A_0$, if $x \in \Omega_q$ satisfies

$$f(x) = z$$

we call it a critical point of the function $x \mapsto \Phi_q(x,z)$.

We know that $\mathrm{supp}(G_q)$ is strictly smaller than Ω_q and the distance between their boundaries is larger than $c_1'\delta$. Let $r_0 = c_1'\delta/2$. By Taylor's formula, there exists $r_* \ (< r_0)$ such that if $\widetilde{x}$ is a critical point in $(\mathrm{supp}(G_q))_{(r_0)}$

([3]) The II will be reduced to a new exponential sum with T and M replaced by $\widetilde{T}$ and $\widetilde{M}$ respectively.

of the function $x \mapsto \Phi_q(x,z)$ then

$$|D_x\Phi_q(x,z)| \gtrsim |x-\widetilde{x}| \quad \text{for } x \in B(\widetilde{x}, r_*). \tag{2.24}$$

The implicit constant depends on d and the constants implied in (2.22), (2.23).

Applying Lemma A.1 to f with r_0 as above yields two positive numbers r_1, r_2 (in particular both depending on δ) such that $2r_1 \le r_*$ and, for any $x \in (\mathrm{supp}(G_q))_{(r_0)}$, f is bijective from $B(x, 2r_1)$ to an open set containing $B(f(x), 2r_2)$. Note that $r_1 < c_1'\delta/4$. If x_1, $x_2 \in (\mathrm{supp}(G_q))_{(r_0)}$ are two different critical points of the function $x \mapsto \Phi_q(x,z)$ (if any), then $B(x_1, r_1)$ and $B(x_2, r_1)$ are disjoint and contained in Ω_q.

Next we will use two partitions of unity to restrict the domains for both x and z to small balls. We can choose finitely many balls $\{X_k\}_{k=1}^K$ and $\{Z_s\}_{s=1}^S$ (from families $\{B(x, r_1/3) : x \in c_0B(0,1)\}$ and $\{B(z, r_2/3) : z \in (A_0/2)B(0,1)\}$, respectively) and two families of functions $\{\phi_k\}_{k=1}^K$ and $\{\eta_s\}_{s=1}^S$ such that

1. $c_0B(0,1) \subset \bigcup_{k=1}^K X_k$ and $(A_0/2)B(0,1) \subset \bigcup_{s=1}^S Z_s$;
2. K and S are both bounded above by some constants (depending on δ, but independent of p, T, and M);
3. $\sum_{k=1}^K \phi_k(x) \equiv 1$ if $x \in c_0B(0,1)$, and $\phi_k \in C_0^\infty(X_k)$;
4. $\sum_{s=1}^S \eta_s(z) \equiv 1$ if $z \in (A_0/2)B(0,1)$, and $\eta_s \in C_0^\infty(Z_s)$.

Denote $\eta_0 = 1 - \sum_{s=1}^S \eta_s$. Adding these cut-off functions, we get

$$\mathrm{II} = (\delta M)^d \sum_{k=1}^K \sum_{s=0}^S \mathrm{III}(k,s),$$

where

$$\mathrm{III}(k,s) = \sum_{|p|<A_0\widetilde{M}} \eta_s(p/\widetilde{M}) \int \phi_k(x) G_q(x) e(\widetilde{T}\Phi_q(x, p/\widetilde{M}))\,dx. \tag{2.25}$$

Let us fix arbitrarily $0 \le s \le S$, $1 \le k \le K$ and estimate the sum III. Denote $E_k = \mathrm{supp}(\phi_k) \cap \mathrm{supp}(G_q)$. We will only consider those k's such that $E_k \ne \emptyset$, otherwise the integrals above vanish.

For $|z| < A_0$ define

$$S_z = \{x \in E_k : |D_x\Phi_q(x,z)| < r_2/3\}.$$

If S_z is empty for a z, then $|D_x\Phi_q(x,z)|$ has a lower bound $r_2/3$ on E_k. As a consequence, for some p with empty $S_{p/\widetilde{M}}$ the integral in $\mathrm{III}(k,s)$ is of order $O_\delta(\widetilde{T}^{-\lceil d/2\rceil - 1})$ by integration by parts.

If S_z is not empty for a z, Lemma A.1 ensures that there exists a unique critical point $x(z)$ in $(E_k)_{(r_1/3)} \subset \Omega_q$.

If $s=0$, we actually sum over all integral p's such that $A_0\widetilde{M}/2<|p|<A_0\widetilde{M}$. For those p's, a straightforward computation yields $D_x\Phi_q(x,p/\widetilde{M})\neq 0$ for $x\in\Omega_q$. It follows that $S_{p/\widetilde{M}}$ is empty, hence each integral in (2.25) is of order $O_\delta(\widetilde{T}^{-\lceil d/2\rceil-1})$. Thus $\mathrm{III}(k,0)\lesssim_\delta(1+\widetilde{M}^d)\widetilde{T}^{-\lceil d/2\rceil-1}$.

Now let us assume $s\geq 1$. Since η_s is compactly supported we can replace the summation domain in III by $\{p\in\mathbb{Z}^d\}$. Assume that there exists a $p_1\in\mathbb{Z}^d$ such that $\eta_s(p_1/\widetilde{M})\neq 0$ and $S_{p_1/\widetilde{M}}$ is not empty. Hence the critical point $x(p_1/\widetilde{M})$ exists in $(E_k)_{(r_1/3)}$. It follows that for every $z\in B(p_1/\widetilde{M},r_2)$, the critical point $x(z)$ exists in $B(x(p_1/\widetilde{M}),r_1)$ and is smooth on $B(p_1/\widetilde{M},r_2)$. Since $\mathrm{supp}(\eta_s)\subset Z_s\subset B(p_1/\widetilde{M},2r_2/3)$, we have $\mathrm{dist}\{\mathrm{supp}(\eta_s),B(p_1/\widetilde{M},r_2)^c\}\geq r_2/3$.

We also have $E_k\subset B(x(z),2r_1)\subset\Omega_q$ for any $z\in B(p_1/\widetilde{M},r_2)$. Recalling (2.24) and applying Lemma 2.1 [4] yields

$$\mathrm{III}(k,s)=\widetilde{T}^{-d/2}S(\widetilde{T},\widetilde{M};\widetilde{G},\widetilde{F})+O_\delta\Big(\sum_{p\in\mathbb{Z}^d}\eta_s(p/\widetilde{M})\widetilde{T}^{-d/2-1}\Big),$$

where

$$\widetilde{G}(z)=\eta_s(z)\phi_k(x(z))G_q(x(z))|\det(Q(z))|^{-1/2},$$
$$\widetilde{F}(z)=\Phi_q(x(z),z)+\mathrm{sgn}(Q(z))/(8\widetilde{T}),$$

and $Q(z)=D^2_{xx}\Phi_q(x(z),z)$. Denote the domain of $\widetilde{F}$ by $\mathscr{D}$; a possible choice is $B(p_1/\widetilde{M},r_2)$. It satisfies $\mathrm{supp}(\widetilde{G})\subset\mathscr{D}\subset A_0B(0,1)$ and $\mathrm{dist}\{\mathrm{supp}(\widetilde{G}),\mathscr{D}^c\}\geq r_2/3$.

Now we need to estimate the new exponential sum $S(\widetilde{T},\widetilde{M};\widetilde{G},\widetilde{F})$. We first make the following claim.

Claim 2.7. *For all $z\in\mathscr{D}$ and $|\nu|\leq 3\lceil d/2\rceil+2$,*

$$(D^\nu\widetilde{G})(z)\lesssim_\delta 1,\qquad (D^\nu\widetilde{F})(z)\lesssim 1.$$

Furthermore, if δ and c_2 are sufficiently small (both depending on d and constants implied in (2.14)–(2.16)), then

$$|\det(D^3_{1,i,j}\widetilde{F}(z))_{1\leq i,j\leq d}|\gtrsim 1.$$

In particular, all three constants implied in these bounds are independent of the choice of the domain $\mathscr{D}$.

We defer the proof of this claim until later.

If $\widetilde{M}\geq 1$, we can pick c_2 to be sufficiently small (depending on d, q, and δ) such that the assumption $T\leq M^{q+2/Q+2/(d-2)}$ implies

$$\widetilde{T}\geq\widetilde{M}^{2-2/d}.$$

[4] The K, X in that lemma can be chosen to be E_k, $B(x(p/\widetilde{M}),2r_1)$ respectively.

Hence we can apply Proposition 2.5 (the q and μ there can be taken to be 1 and e_1, respectively) to get

$$S(\widetilde{T},\widetilde{M};\widetilde{G},\widetilde{F}) \lesssim_\delta \widetilde{T}^{\frac{d}{2(d+2)}}\widetilde{M}^{d-\frac{3d}{2(d+2)}}.$$

If $\widetilde{M} \le 1$, the trivial estimate gives

$$S(\widetilde{T},\widetilde{M};\widetilde{G},\widetilde{F}) \lesssim 1.$$

Combining these two bounds, we get

$$S(\widetilde{T},\widetilde{M};\widetilde{G},\widetilde{F}) \lesssim_\delta 1+\widetilde{T}^{\frac{d}{2(d+2)}}\widetilde{M}^{d-\frac{3d}{2(d+2)}}.$$

Finally we get the bound for II in Subcase 2.1:

(2.26)

$$\begin{aligned}\mathrm{II} &\lesssim_\delta (\delta M)^d[\widetilde{T}^{-d/2}(1+\widetilde{T}^{\frac{d}{2(d+2)}}\widetilde{M}^{d-\frac{3d}{2(d+2)}})+(1+\widetilde{M}^d)(\widetilde{T}^{-d/2-1}+\widetilde{T}^{-\lceil d/2\rceil-1})]\\ &\lesssim_\delta M^{\frac{d^2+3d}{2(d+2)}}\widetilde{M}^{\frac{d^2}{2(d+2)}}+M^{d/2-1}\widetilde{M}^{d/2-1}+M^{d/2}\widetilde{M}^{-d/2}\\ &\lesssim_\delta M^{\frac{d^2+3d}{2(d+2)}}\widetilde{M}^{\frac{d^2}{2(d+2)}}+M^{d/2}\widetilde{M}^{-d/2}.\end{aligned}$$

In the second inequality, we use $\widetilde{T}=\delta M\widetilde{M}\ge 1$. In the last inequality, we omit the second term since

$$M^{\frac{d^2+3d}{2(d+2)}}\widetilde{M}^{\frac{d^2}{2(d+2)}} = M^{\frac{3d}{2(d+2)}}(M\widetilde{M})^{\frac{d^2}{2(d+2)}} \gtrsim_\delta (M\widetilde{M})^{d/2-1}.$$

Subcase 2.2: $\widetilde{T}<1$. Then $\delta M<\widetilde{M}^{-1}$ and $\widetilde{M}<1$. Hence

$$\mathrm{II} \lesssim_\delta (\delta M)^d \lesssim_\delta M^{d/2}\widetilde{M}^{-d/2}.$$

Comparing this bound with (2.26), we conclude that (2.26) always holds for II.

Using the bounds for I and II, we get

$$\begin{aligned}S(\mathscr{H}T(\delta M)^{-q},\delta M;G_q,F_q) &\lesssim_\delta M^{-1}+M^{\frac{d^2+3d}{2(d+2)}}\widetilde{M}^{d^2/2(d+2)}+M^{d/2}\widetilde{M}^{-d/2}\\ &\lesssim_\delta \mathscr{H}^{\frac{d^2}{2(d+2)}}T^{\frac{d^2}{2(d+2)}}M^{\frac{-qd^2+3d}{2(d+2)}}+\mathscr{H}^{-d/2}T^{-d/2}M^{(q+2)d/2}.\end{aligned}$$

In the last step, we use the definition of $\widetilde{M}$ and omit M^{-1} since it is smaller than the sum of the other two, no matter whether $\widetilde{M}\ge 1$ or <1.

Plugging this bound into (2.19) yields

$$\begin{aligned}&|S(T,\delta M;G,F)|^Q\\ &\quad\lesssim_\delta M^{Qd}(H^{-1}+M^{-\frac{(q+2)d^2+d}{2(d+2)}}T^{\frac{d^2}{2(d+2)}}H^{\frac{(1-1/Q)d^2}{d+2}}+T^{-d/2}M^{qd/2}H^{-2+2/Q}).\end{aligned}$$

Balancing the first two terms yields the optimal choice

$$H=B_3T^{-\frac{d^2}{(2-2/Q)d^2+2d+4}}M^{\frac{(q+2)d^2+d}{(2-2/Q)d^2+2d+4}},$$

where B_3 can be chosen so small that $T \geq M^{q+2/Q-1/d-4/d^2}$ implies $H \leq c_2\delta M$. Due to Remark 1 (following the statement of the theorem), we can assume $T \leq B_4 M^{q+2+1/d}$ with a sufficiently small B_4, which implies $1 < H$.

For $q \in \{1,2\}$ if $d = 3$ or $q \in \mathbb{N}$ if $d \geq 4$, we have

$$T^{-d/2} M^{qd/2} H^{-2+2/Q} \lesssim_\delta M^{-d-1/2} H^{d-1/2} \lesssim_\delta H^{-1}.$$

Hence

$$S(T,\delta M; G, F) \lesssim_\delta T^{\frac{d^2}{2(Q-1)d^2+2Qd+4Q}} M^{d-\frac{(q+2)d^2+d}{2(Q-1)d^2+2Qd+4Q}}.$$

This finishes the proof of the theorem. ∎

Proof of Claim 2.7. The critical point function $x(z) = (x_1(z), \ldots, x_d(z))$ for $z \in \mathscr{D}$ satisfies the equation $D_x \Phi_q(x(z), z) = 0$, namely

$$D_x F_q(x(z)) - z = 0.$$

Differentiating this equation gives

$$D^2_{xx} F_q(x(z)) D_z x(z) - I_d = 0$$

where I_d is the unit matrix of size d, hence

$$D_z x(z) = (D^2_{xx} F_q(x(z)))^{-1}. \tag{2.27}$$

By differentiating this formula inductively and using bounds (2.14), (2.23) for F_q, we get

$$D^\nu_z x_i(z) \lesssim 1 \quad \text{for } 1 \leq i \leq d,\ z \in \mathscr{D}, \text{ and } |\nu| \leq 3\lceil d/2 \rceil + 2.$$

This bound together with the chain rule and product rule gives the two upper bounds in the claim. To prove the lower bound of $\det(D^3_{1,i,j}\widetilde{F})$, we first have

$$\begin{aligned} D_z \widetilde{F}(z) &= D_z(F_q(x(z)) - x(z) \cdot z + \operatorname{sgn}(Q(z))/(8\widetilde{T})) \\ &= -x(z) + D_z x(z)[D_x F_q(x(z)) - z] = -x(z). \end{aligned}$$

The derivative of $\operatorname{sgn}(Q(z))$ vanishes since it is a constant function, and the last equality follows from the defining equation of critical points. Thus

$$\begin{aligned} (D^3_{1,i,j}\widetilde{F}(z))_{1 \leq i,j \leq d} &= -\frac{\partial}{\partial z_1}(D_z x(z)) \\ &= (D^2_{xx} F_q(x(z)))^{-1} \frac{\partial}{\partial z_1}(D^2_{xx} F_q(x(z)))(D^2_{xx} F_q(x(z)))^{-1}. \end{aligned}$$

In the last step we use (2.27). Since $|\det(D^2_{xx} F_q(x(z)))| \asymp 1$ (due to (2.22) and (2.23)), we get the desired lower bound if we can prove

$$\left|\det\left(\frac{\partial}{\partial z_1}(D^2_{xx} F_q(x(z)))\right)\right| \gtrsim 1. \tag{2.28}$$

If δ is sufficiently small and $H \leq \delta^2 M$, the condition (2.16) ensures the following bounds for the entries of the symmetric matrix $D^2_{xx} F_q$: $|D^2_{i,i} F_q| \asymp 1$

for $1 \le i \le d-1$; $D^2_{i,j}F_q \lesssim 1$ for $2 \le i \le d-1$, $1 \le j \le i-1$; $|D^2_{d,1}F_q| \asymp 1$; $D^2_{d,j}F_q \lesssim \delta$ for $2 \le j \le d$. With these bounds we can then estimate the entries of

$$D_z x(z) = (\text{adjugate of } D^2_{xx}F_q(x(z)))/\det(D^2_{xx}F_q(x(z))).$$

Actually in the following computation we only need the sizes of the entries from the first column of $D_z x(z)$. It is easy to see that $\partial x_i/\partial z_1 \lesssim \delta$ for $1 \le i \le d-1$. Note that the formula above leads to

$$D^2_{d,1}F_q(x(z)) \cdot \det(D^2_{xx}F_q(x(z))) \cdot \frac{\partial x_d}{\partial z_1} + O(\delta) = \det(D^2_{xx}F_q(x(z))).$$

It follows that $|\partial x_d/\partial z_1| \asymp 1$ if δ is sufficiently small.

If $H \le c_3\delta M$ with a sufficiently small c_3 (depending on d and the constants implied in (2.14), (2.15)), the condition (2.15) with $k = q+1$ implies

$$|\det(D^3_{i,j,d}F_q)_{1\le i,j\le d}| \gtrsim 1. \tag{2.29}$$

Note that

$$\frac{\partial}{\partial z_1}(D^2_{xx}F_q(x(z))) = \sum_{l=1}^{d}(D^3_{i,j,l}F_q(x(z)))_{1\le i,j\le d}\frac{\partial x_l(z)}{\partial z_1}.$$

If δ is sufficiently small, then the terms $\partial x_l/\partial z_1$ $(1 \le l \le d-1)$ are much smaller than $\partial x_d/\partial z_1$. Thus (2.29) leads to (2.28).

The δ only depends on d and the constants implied in (2.14)–(2.16). We require c_2 to be smaller than δ and c_3, and it depends on the same quantities as δ does. From the argument we can see that all bounds are independent of the choice of $\mathscr{D}$. ∎

3. Nonvanishing of $d \times d$ determinants. In this section, we will establish certain results that allow us to verify (in the lattice point problem) auxiliary conditions such as (2.15) and (2.16). More precisely, we will give lower bounds of determinants of certain $d\times d$ matrices and a description of the sizes of their entries. These results are based on Müller's [16, Lemma 3 and its proof].

For $\xi \ne 0$, let $H(\xi) = \sup_{x\in\mathcal{B}}\langle \xi, x\rangle$ be the support function of $\mathcal{B}$. It is a real-valued function positively homogeneous of degree one, i.e. $H(k\xi) = kH(\xi)$ if $k > 0$. Due to the curvature condition imposed on $\partial\mathcal{B}$, H is smooth and the eigenvalues of the Hessian matrix of H at $\xi \ne 0$ are zero and $d-1$ real numbers comparable to $1/|\xi|$. This simple fact is not hard to prove (see also [2]).

Given any d vectors $v_1, \dots, v_d$, we write $V = (v_1, \dots, v_d)$ for the matrix with column vectors $v_1, \dots, v_d$. If $y \ne 0$ we define

$$F(u_1, \dots, u_d) = H\Big(y + \sum_{l=1}^{d} u_l v_l\Big), \qquad u_l \in \mathbb{R}\ (1 \le l \le d).$$

For $1 \le i, j \le d$ and $k \in \mathbb{N}$, define

$$g_{i,j}^{(k)}(y, v_1, \dots, v_d) = \frac{\partial^{k+2} F}{\partial u_1 \partial u_i \partial u_j \partial u_d^{k-1}}(0);$$

these form a symmetric matrix

$$G_k(y, v_1, \dots, v_d) = (g_{i,j}^{(k)}(y, v_1, \dots, v_d))_{1 \le i,j \le d}$$

with determinant $h_k(y, v_1, \dots, v_d) = \det(G_k(y, v_1, \dots, v_d))$.

Denote

$$\mathscr{C}_1 = \{x \in \mathbb{R}^d : 1/2 \le |x| \le 2\}, \quad \mathscr{C}_1^+ = \{x \in \mathbb{R}^d : 1/4 \le |x| \le 4\}.$$

Since H is smooth, its derivatives up to order $q+3$ on $\mathscr{C}_1^+$ are bounded by a constant (only depending on q and $\mathcal{B}$), namely

$$D^\nu H(\xi) \lesssim 1 \quad \text{for all } \xi \in \mathscr{C}_1^+ \text{ and } |\nu| \le q+3. \tag{3.1}$$

We will only consider points in $\mathscr{C}_1$ in the following lemma.

LEMMA 3.1. *Let q and N be positive integers. There exists $A_3 > 0$ (depending on q and $\mathcal{B}$) such that if $N \ge A_3$ then for every $\xi \in \mathscr{C}_1$ there exist d linearly independent vectors $v_1, \dots, v_d \in \mathbb{Z}^d$ (depending on ξ) such that $|v_l| \asymp N$ $(1 \le l \le d)$, $|\det(V)| \asymp N^d$, and for every $1 \le k \le q$ and $y \in B(\xi, 1/N)$,*

$$|h_k(y, v_1, \dots, v_d)| \gtrsim N^{(k+2)d}$$

and

$$\begin{cases} |g_{i,i}^{(k)}(y, v_1, \dots, v_d)| \asymp N^{k+2} & \text{for } 1 \le i \le d-1, \\ |g_{i,j}^{(k)}(y, v_1, \dots, v_d)| \lesssim N^{k+2} & \text{for } 2 \le i \le d-1,\ 1 \le j \le i-1, \\ |g_{d,1}^{(k)}(y, v_1, \dots, v_d)| \asymp N^{k+2}, & \\ |g_{d,j}^{(k)}(y, v_1, \dots, v_d)| \lesssim N^{k+1} & \text{for } 2 \le j \le d. \end{cases}$$

All implicit constants may depend on q and $\mathcal{B}$.

Proof. We will use the proof of Lemma 3 in Müller [16] (with some minor modification) and proceed in three steps for any fixed $\xi \in \mathscr{C}_1$.

STEP 1. We first choose d vectors $P_l \in \mathbb{R}^d$ $(1 \le l \le d)$, in particular $P_1 = \xi$, such that $|P_l| = |\xi|$ and the vectors $P_l/|\xi|$ form an orthogonal matrix. Let $P = (P_1, \dots, P_d)$ and $\widetilde{H}(y) = H(Py)$. Then $\widetilde{H}$ is positively homogeneous of degree one and the eigenvalues of the Hessian matrix of $\widetilde{H}$ at e_1 are zero and $d-1$ real numbers comparable to 1 since $D^2\widetilde{H}(e_1)$ is similar to $D^2H(\xi)$ up to a number $|\xi|^2$ and $|\xi| \asymp 1$.

Set $A = (\widetilde{H}_{ij}(e_1))$. Then A is a symmetric matrix of rank $d-1$ with vanishing first row and column (due to homogeneity, see the proof of Lemma 3 in Müller [16]). Choose a system of orthonormal eigenvectors $w_1', \dots, w_{d-1}'$

of A, whose first components all vanish. Denote the eigenvalue of w_i' by λ_i (comparable to 1). Note that for every $\alpha > 1$ the vector $w_1 = w_1' + \alpha e_1$ is orthogonal to w_j' for $2 \le j \le d-1$ and satisfies $Aw_1 = \lambda_1 w_1'$. Denote

$$w_i = \begin{cases} w_1' + \alpha e_1 & \text{if } i = 1, \\ w_i' & \text{if } 2 \le i \le d-1, \\ e_1 & \text{if } i = d. \end{cases}$$

Then $|w_1| \asymp \alpha$, $|w_l| = 1$ $(2 \le l \le d)$, and $\det(W) = 1$ where $W = (w_1, \dots, w_d)$. Denote $w_i = (w_{i,1}, \dots, w_{i,d})^t$, $F(u_1, \dots, u_d) = \widetilde{H}(e_1 + \sum_{l=1}^d u_l w_l)$, and

$$b_{i,j}^{(k)}(\alpha) = \frac{\partial^{k+2} F}{\partial u_1 \partial u_i \partial u_j \partial u_d^{k-1}}(0).$$

Define $v_l^* = P w_l$. Then $|v_1^*| \asymp \alpha$, $|v_l^*| \asymp 1$ $(2 \le l \le d)$, and $|\det(V^*)| \asymp 1$ where $V^* = (v_1^*, \dots, v_d^*)$. Note that $F(u_1, \dots, u_d) = H(\xi + \sum_{l=1}^d u_l v_l^*)$ and $b_{i,j}^{(k)}(\alpha) = g_{i,j}^{(k)}(\xi, v_1^*, \dots, v_d^*)$.

If $1 \le i, j \le d-1$, then

$$b_{i,j}^{(k)}(0) = \sum_{m,n,s=1}^{d} \frac{\partial^{k+2} \widetilde{H}}{\partial y_1^{k-1} \partial y_m \partial y_n \partial y_s}(e_1) w_{1,m}' w_{i,n}' w_{j,s}' \lesssim 1.$$

The last inequality is due to (3.1).

If $i = 1$, $1 \le j \le d-1$, then

$$b_{1,j}^{(k)}(\alpha) = b_{1,j}^{(k)}(0) + 3\alpha(-1)^k k! \lambda_1 \delta_{1j},$$

where δ_{ij} is the Kronecker symbol.

If $2 \le i, j \le d-1$, then

$$b_{i,j}^{(k)}(\alpha) = b_{i,j}^{(k)}(0) + \alpha(-1)^k k! \lambda_j \delta_{ij}.$$

If $1 \le i \le d$, $j = d$, then

$$b_{i,d}^{(k)}(\alpha) = (-1)^k k! \lambda_1 \delta_{1i}.$$

Using these formulas, we get

$$\det(b_{i,j}^{(k)}(\alpha))_{1 \le i,j \le d} = -(k!\lambda_1)^2 \det(b_{i,j}^{(k)}(\alpha))_{2 \le i,j \le d-1},$$
$$\det(b_{i,j}^{(k)}(\alpha))_{2 \le i,j \le d-1} = \det(b_{i,j}^{(k)}(0) + \alpha(-1)^k k! \lambda_j \delta_{ij})_{2 \le i,j \le d-1}.$$

The last determinant is a polynomial in α of degree $d-2$ with leading coefficient comparable to 1. If we fix α to be a sufficiently large constant (only depending on q and $\mathcal{B}$), then

$$|h_k(\xi, v_1^*, \dots, v_d^*)| = |\det(b_{i,j}^{(k)}(\alpha))_{1 \le i,j \le d}| \gtrsim 1 \quad \text{for } 1 \le k \le q$$

and

$$\begin{cases} |g_{i,i}^{(k)}(\xi, v_1^*, \ldots, v_d^*)| \asymp 1 & \text{for } 1 \le i \le d-1, \\ |g_{i,j}^{(k)}(\xi, v_1^*, \ldots, v_d^*)| \lesssim 1 & \text{for } 2 \le i \le d-1,\ 1 \le j \le i-1, \\ |g_{d,1}^{(k)}(\xi, v_1^*, \ldots, v_d^*)| \asymp 1, & \\ |g_{d,j}^{(k)}(\xi, v_1^*, \ldots, v_d^*)| = 0 & \text{for } 2 \le j \le d, \end{cases}$$

where the implicit constants only depend on q and $\mathcal{B}$.

Step 2. There exist vectors $v_l^{**} \in \mathbb{Q}^d$ $(1 \le l \le d)$ each of whose components is the ratio of an integer to N, and $|v_l^{**} - v_l^*| \le \sqrt{d}/N$. There exists a large number A_1 (only depending on q and $\mathcal{B}$) such that if $N \ge A_1$ then $|v_l^{**}| \asymp 1$ $(1 \le l \le d)$ and $|\det(V^{**})| \asymp 1$ where $V^{**} = (v_1^{**}, \ldots, v_d^{**})$. Since

$$|g_{i,j}^{(k)}(\xi, v_1^{**}, \ldots, v_d^{**}) - g_{i,j}^{(k)}(\xi, v_1^*, \ldots, v_d^*)| \lesssim 1/N,$$

there exists a large number $A_2 \ge A_1$ (only depending on q and $\mathcal{B}$) such that if $N \ge A_2$ then

$$|h_k(\xi, v_1^{**}, \ldots, v_d^{**})| \gtrsim 1 \quad \text{for } 1 \le k \le q$$

and

$$\begin{cases} |g_{i,i}^{(k)}(\xi, v_1^{**}, \ldots, v_d^{**})| \asymp 1 & \text{for } 1 \le i \le d-1, \\ |g_{i,j}^{(k)}(\xi, v_1^{**}, \ldots, v_d^{**})| \lesssim 1 & \text{for } 2 \le i \le d-1,\ 1 \le j \le i-1, \\ |g_{d,1}^{(k)}(\xi, v_1^{**}, \ldots, v_d^{**})| \asymp 1, & \\ |g_{d,j}^{(k)}(\xi, v_1^{**}, \ldots, v_d^{**})| \lesssim 1/N & \text{for } 2 \le j \le d, \end{cases}$$

where the implicit constants only depend on q and $\mathcal{B}$.

Step 3. Let $v_l = N v_l^{**}$. Then $v_l \in \mathbb{Z}^d \setminus \{0\}$, $|v_l| \asymp N$ $(1 \le l \le d)$, and $|\det(V)| \asymp N^d$. Note that

$$g_{i,j}^{(k)}(\xi, v_1, \ldots, v_d) = N^{k+2} g_{i,j}^{(k)}(\xi, v_1^{**}, \ldots, v_d^{**}).$$

Applying the mean value theorem, we have, for $y \in \mathscr{C}_1^+$,

$$|g_{i,j}^{(k)}(y, v_1^{**}, \ldots, v_d^{**}) - g_{i,j}^{(k)}(\xi, v_1^{**}, \ldots, v_d^{**})| \lesssim |y - \xi|.$$

Thus there exists a large number $A_3 \ge A_2$ (only depending on q and $\mathcal{B}$) such that if $N \ge A_3$ and $y \in B(\xi, 1/N)$ then the desired bounds for determinants and entries are both true and the implicit constants only depend on q and $\mathcal{B}$. This finishes the proof. ∎

4. Proof of Theorem 1.1. By a standard procedure, we can replace the combinatorial problem of counting lattice points in a dilated domain by an analytical problem. The essential issue is the estimation of an exponential sum. In order to apply the results of Theorem 2.6, we need to introduce a dyadic decomposition and a partition of unity.

Proof of Theorem 1.1. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} \rho(y)\,dy = 1$, $\varepsilon > 0$, $\rho_\varepsilon(y) = \varepsilon^{-d}\rho(\varepsilon^{-1}y)$, and

$$N_\varepsilon(t) = \sum_{k\in\mathbb{Z}^d} \chi_{t\mathcal{B}} * \rho_\varepsilon(k),$$

where $\chi_{t\mathcal{B}}$ denotes the characteristic function of $t\mathcal{B}$. By the Poisson summation formula,

$$N_\varepsilon(t) = t^d \sum_{k\in\mathbb{Z}^d} \hat{\chi}_{\mathcal{B}}(tk)\hat{\rho}(\varepsilon k) = \mathrm{vol}(\mathcal{B})t^d + R_\varepsilon(t),$$

where

$$R_\varepsilon(t) = t^d \sum_{k\in\mathbb{Z}^d_*} \hat{\chi}_{\mathcal{B}}(tk)\hat{\rho}(\varepsilon k).$$

Müller proved in [15] that there exists a constant C_1 such that

$$N_\varepsilon(t - C_1\varepsilon) \le \#(t\mathcal{B}\cap\mathbb{Z}^d) = \sum_{k\in\mathbb{Z}^d} \chi_{t\mathcal{B}}(k) \le N_\varepsilon(t + C_1\varepsilon),$$

which implies

$$P_{\mathcal{B}}(t) \lesssim |R_\varepsilon(t + C_1\varepsilon)| + |R_\varepsilon(t - C_1\varepsilon)| + t^{d-1}\varepsilon. \tag{4.1}$$

It suffices to estimate $R_\varepsilon(t)$ for any large t. By Hörmander [7, Corollary 7.7.15], we have the asymptotic expansion

$$\hat{\chi}_{\mathcal{B}}(\xi) = [CK_\xi^{-1/2}e^{-2\pi iH(\xi)} + C'K_{-\xi}^{-1/2}e^{2\pi iH(-\xi)}]|\xi|^{-(d+1)/2} + O(|\xi|^{-(d+3)/2}),$$

where C, C' are constants, $H(\xi) = \sup_{x\in\mathcal{B}}\langle\xi, x\rangle$, and K_ξ is the curvature at the point on $\partial\mathcal{B}$ where the exterior normal is along ξ. We know that K_ξ is smooth on $\mathbb{R}^d\setminus\{0\}$ and positively homogeneous of degree zero. Applying this formula gives

$$R_\varepsilon(t) = CS_1 + C'\widetilde{S}_1 + \mathrm{Error},$$

where

$$S_1 = t^{(d-1)/2} \sum_{k\in\mathbb{Z}^d_*} |k|^{-(d+1)/2}K_k^{-1/2}\hat{\rho}(\varepsilon k)e(tH(k)), \tag{4.2}$$

$$\widetilde{S}_1 = t^{(d-1)/2} \sum_{k\in\mathbb{Z}^d_*} |k|^{-(d+1)/2}K_{-k}^{-1/2}\hat{\rho}(\varepsilon k)e(-tH(-k)),$$

and

$$\mathrm{Error} \lesssim t^{(d-3)/2} \sum_{k\in\mathbb{Z}^d_*} |k|^{-(d+3)/2}\hat{\rho}(\varepsilon k) \lesssim t^{(d-3)/2}\varepsilon^{-(d-3)/2}. \tag{4.3}$$

Since the first two sums are similar, it suffices to estimate S_1. With $\mathscr{C}_1$ as defined in Section 3, we can find a real radial function $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}(\psi) \subset \mathscr{C}_1$, $0 \le \psi \le 1$, and

$$\sum_{j=-\infty}^{\infty} \psi(y/2^j) = 1 \qquad \text{for } y \in \mathbb{R}^d \setminus \{0\}.$$

Denote

$$S_{1,M} = t^{(d-1)/2} \sum_{k \in \mathbb{Z}_*^d} \psi(M^{-1}k)|k|^{-(d+1)/2} K_k^{-1/2} \hat{\rho}(\varepsilon k) e(tH(k)).$$

Then $S_1 = \sum_{j=0}^{\infty} S_{1,2^j}$. It suffices to estimate $S_{1,M}$ for a fixed $M = 2^j$, $j \in \mathbb{N}_0$.

With the notation of Section 3, Lemma 3.1 ensures that there exists an admissible constant $A_3 > 0$ such that if N is an integer not less than A_3 then for every $\xi \in \mathscr{C}_1$ there exist linearly independent vectors $v_1(\xi), \dots, v_d(\xi)$ in $\mathbb{Z}^d$ such that $|v_l| \asymp N$ $(1 \le l \le d)$, $|\det(V)| \asymp N^d$, and

$$|h_k(y, v_1(\xi), \dots, v_d(\xi))| \gtrsim N^{(k+2)d} \qquad \text{for } 1 \le k \le 3,\ y \in B(\xi, 2r),$$

where $r = 1/(2N)$. The entries of $G_k(y, v_1(\xi), \dots, v_d(\xi))$ satisfy the size estimates in Lemma 3.1.

Since $\mathscr{C}_1$ is compact, we can find finitely many balls $\{B(\xi_i, r)\}_{i=1}^{I}$ ($\xi_i \in \mathscr{C}_1$ and $I \lesssim N^d$) and a partition of unity $\{\psi_i\}_{i=1}^{I}$ such that

1. the balls have the bounded overlap property;
2. $\mathscr{C}_1 \subset \bigcup_{i=1}^{I} B(\xi_i, r)$;
3. $\sum_i \psi_i(y) \equiv 1$ if $y \in \mathscr{C}_1$, and $\psi_i \in C_0^{\infty}(B_i)$;
4. $D^{\nu}\psi_i \lesssim N^{|\nu|}$.

Here we denote $B_i = B(\xi_i, r)$ and $B_i^* = B(\xi_i, 2r)$.

Denote

$$S_{1,M}^{(i)} = t^{(d-1)/2} \sum_{k \in \mathbb{Z}_*^d} U(k) e(tH(k))$$

where

$$U(k) = \psi_i(M^{-1}k)\psi(M^{-1}k)|k|^{-(d+1)/2} K_k^{-1/2} \hat{\rho}(\varepsilon k).$$

Then

$$S_{1,M} = \sum_{i=1}^{I} S_{1,M}^{(i)}.$$

It suffices to estimate $S_{1,M}^{(i)}$ for a fixed i. Denote by L the index of the lattice spanned by $v_1(\xi_i), \dots, v_d(\xi_i)$ in the lattice $\mathbb{Z}^d$. Then $L = |\det(V)| \asymp N^d$ and there exist vectors $b_l \in \mathbb{Z}^d$ $(1 \le l \le L)$ such that

$$\mathbb{Z}^d = \biguplus_{l=1}^{L} (\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_d + b_l).$$

Let N_1 be an arbitrary integer $\geq \lceil d/2 \rceil$. Applying the decomposition above, for any $k \in \mathbb{Z}^d$ we can write $k = \sum_{s=1}^{d} m_s v_s + b_l$ where $m_s \in \mathbb{Z}$ $(1 \leq s \leq d)$. Hence

$$\begin{aligned} S_{1,M}^{(i)} &= t^{(d-1)/2} \sum_{l=1}^{L} \sum_{m \in \mathbb{Z}^d} U\Big(\sum_{s=1}^{d} v_s m_s + b_l\Big) e\Big(tH\Big(\sum_{s=1}^{d} v_s m_s + b_l\Big)\Big) \\ &= t^{(d-1)/2} M^{-(d+1)/2} (1 + |M\varepsilon|)^{-N_1} \sum_{l=1}^{L} S_l(T, \delta M; G, F), \end{aligned}$$

where $T = tM$, $\delta = N^{-1}$, and

$$\begin{aligned} G(x) &= M^{(d+1)/2} (1 + |M\varepsilon|)^{N_1} U\Big(M \sum_{s=1}^{d} \delta v_s x_s + b_l\Big), \\ F(x) &= H\Big(\sum_{s=1}^{d} \delta v_s x_s + b_l/M\Big). \end{aligned}$$

We consider F restricted to the convex domain

$$\Omega = \Big\{x \in \mathbb{R}^d : \sum_{s=1}^{d} \delta v_s x_s + b_l/M \in B_i^*\Big\}. \tag{4.4}$$

If $\delta^{-1} < M$, then $\Omega \subset c_0 B(0,1)$ for an admissible constant c_0. We also have

$$\operatorname{supp}(G) \subset \Big\{x \in \mathbb{R}^d : \sum_{s=1}^{d} \delta v_s x_s + b_l/M \in \overline{B_i} \cap \mathscr{C}_1\Big\} \subset \Omega, \tag{4.5}$$

and

$$\operatorname{dist}(\operatorname{supp}(G), \Omega^c) \geq c_1' \delta,$$

where c_1' is an admissible constant. Note that

$$D^\nu U \lesssim \delta^{-|\nu|} M^{-(d+1)/2-|\nu|} (1 + |M\varepsilon|)^{-N_1},$$

and for all $x \in \Omega$, $1 \leq i, j \leq d$, and $1 \leq k \leq 3$,

$$\frac{\partial^{k+2} F}{\partial x_1 \partial x_i \partial x_j \partial x_d^{k-1}}(x) = \delta^{k+2} g_{i,j}^{(k)}\Big(\sum_{s=1}^{d} \delta v_s x_s + b_l/M, v_1(\xi_i), \ldots, v_d(\xi_i)\Big),$$

where the functions $g_{i,j}^{(k)}$ are as defined in Section 3. It is not hard to check that the assumptions of Theorem 2.6 are satisfied.

If $d \geq 4$, we apply to $S_l(T, \delta M; G, F)$ Theorem 2.6 with $q = 1$, which determines the size of δ, hence that of N. Note that δ is admissible; we will not write it explicitly in various bounds below. If $t \geq M \geq t^{1-2/d}$, the inequality $M > \delta^{-1}$ and the restrictions of Theorem 2.6 are both satisfied, thus

$$S_l(T,\delta M;G,F)\lesssim t^{\frac{d^2}{2(d^2+2d+4)}}M^{d-\frac{2d^2+d}{2(d^2+2d+4)}},$$

which leads to

$$S_{1,M}=\sum S_{1,M}^{(i)}\lesssim t^{\frac{d-1}{2}+\frac{d^2}{2(d^2+2d+4)}}M^{\frac{d-1}{2}-\frac{2d^2+d}{2(d^2+2d+4)}}(1+|M\varepsilon|)^{-N_1}.$$

We split S_1 into three parts as follows:

$$S_1=\sum_{j=0}^{\infty}S_{1,2^j}=\Big(\sum_{2^j<t^{1-2/d}}+\sum_{t^{1-2/d}\le 2^j\le t}+\sum_{2^j>t}\Big)S_{1,2^j}.$$

With the choice of ε below, the second sum is bounded by

$$\sum_{t^{1-2/d}\le 2^j\le t}t^{\frac{d-1}{2}+\frac{d^2}{2(d^2+2d+4)}}(2^j)^{\frac{d-1}{2}-\frac{2d^2+d}{2(d^2+2d+4)}}(1+|2^j\varepsilon|)^{-N_1}\lesssim t^{\frac{d-1}{2}+\frac{d^2}{2(d^2+2d+4)}}\varepsilon^{-\frac{d-1}{2}+\frac{2d^2+d}{2(d^2+2d+4)}},\tag{4.6}$$

while the first and third, by the trivial estimate, are bounded by $t^{d-2+1/d}$ and 1, respectively. This finishes the estimate of S_1.

Note that the bound (4.3) for the Error term is smaller than (4.6), hence we get the bound for $R_\varepsilon(t)$. Since $t\pm C_1\varepsilon\asymp t$, we get the bound for $R_\varepsilon(t\pm C_1\varepsilon)$. Plugging these bounds in (4.1) yields

$$P_{\mathcal{B}}(t)\lesssim t^{d-2+1/d}+t^{\frac{d-1}{2}+\frac{d^2}{2(d^2+2d+4)}}\varepsilon^{-\frac{d-1}{2}+\frac{2d^2+d}{2(d^2+2d+4)}}+t^{d-1}\varepsilon.$$

Balancing the second and third terms yields

$$\varepsilon=t^{-\frac{d^3+2d-4}{d^3+d^2+5d+4}}.$$

With this choice of ε, the first term is smaller than the third one. Hence for $d\ge 4$,

$$P_{\mathcal{B}}(t)\lesssim t^{d-2+\beta(d)},$$

where $\beta(d)=(d^2+3d+8)/(d^3+d^2+5d+4)$.

If $d=3$, applying Theorem 2.6 with $q=2$ yields $\beta(3)=73/158$. We omit the calculation since it is similar to the argument above. ∎

Remark. To prove our exponent $\beta(d)$ for large d, we use the estimate of exponential sums obtained by using an ABAB-process (see Theorem 2.6). If we use more A- and B-processes we may further improve it at the cost of more technical difficulties. For example, the application of an ABABAB-process may improve the exponent $\beta(d)$ by $1/d^3$.

Appendix. Inverse function theorem. Here we give a quantitative version of the inverse function theorem. It is routine to prove it by following the proof in Rudin [21].

LEMMA A.1. *Suppose f is a $C^{(k)}$ ($k \geq 2$) mapping from an open set $\Omega \subset \mathbb{R}^d$ into $\mathbb{R}^d$ and $b = f(a)$ for some $a \in \Omega$. Assume $|\det(Df(a))| \geq c$ and for any $x \in \Omega$,*

$$|D^\alpha f_i(x)| \leq C \quad \text{for } |\alpha| \leq 2,\ 1 \leq i \leq d.$$

If $r_0 \leq \sup\{r > 0 : B(a,r) \subset \Omega\}$, then f is bijective from $B(a,r_1)$ to an open set containing $B(b,r_2)$ where

$$r_1 = \min\left\{\frac{c}{2d^2 d! C^d}, r_0\right\}, \qquad r_2 = \frac{c}{4d! C^{d-1}} r_1.$$

The inverse mapping f^{-1} is also in $C^{(k)}$.

REMARK. Note that r_2 is linear in r_1. If f is bijective from $B(a,r_1)$ to an open set containing $B(b,r_2)$, then for any $r_1' \leq r_1$ we can find the corresponding r_2' such that f is bijective from $B(a,r_1')$ to an open set containing $B(b,r_2')$.

Acknowledgments. The subject of this paper was suggested by Professor Andreas Seeger. I would like to express my gratitude to him for his valuable advice and great help during the work. I also want to thank Zhenisbek Assylbekov and the referee for useful comments and additional references.

References

[1] V. Bentkus and F. Götze, *On the lattice point problem for ellipsoids*, Acta Arith. 80 (1997), 101–125.
[2] T. Bonnesen and W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, ID, 1987.
[3] F. Götze, *Lattice point problems and values of quadratic forms*, Invent. Math. 157 (2004), 195–226.
[4] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge Univ. Press, Cambridge, 1991.
[5] D. R. Heath-Brown, *The growth rate of the Dedekind Zeta-function on the critical line*, Acta Arith. 49 (1988), 323–339.
[6] E. Hlawka, *Über Integrale auf konvexen Körpern I*, Monatsh. Math. 54 (1950), 1–36.
[7] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, Berlin, 1983.
[8] M. N. Huxley, *Area, Lattice Points, and Exponential Sums*, Clarendon Press, Oxford Univ. Press, New York, 1996.
[9] —, *Exponential sums and lattice points III*, Proc. London Math. Soc. 87 (2003), 591–609.
[10] A. Iosevich, *Lattice points and generalized Diophantine conditions*, J. Number Theory 90 (2001), 19–30.
[11] A. Iosevich, E. Sawyer, and A. Seeger, *Mean square discrepancy bounds for the number of lattice points in large convex bodies*, J. Anal. Math. 87 (2002), 209–230.
[12] E. Krätzel, *Lattice Points*, Kluwer, Dordrecht, 1988.

[13] E. Krätzel and W. G. Nowak, *Lattice points in large convex bodies*, Monatsh. Math. 112 (1991), 61–72.
[14] —, —, *Lattice points in large convex bodies, II*, Acta Arith. 62 (1992), 285–295.
[15] W. Müller, *On the average order of the lattice rest of a convex body*, ibid. 80 (1997), 89–100.
[16] —, *Lattice points in large convex bodies*, Monatsh. Math. 128 (1999), 315–330.
[17] —, *Lattice points in bodies with algebraic boundary*, Acta Arith. 108 (2003), 9–24.
[18] W. G. Nowak, *On the lattice rest of a convex body in* $\mathbb{R}^s$, Arch. Math. (Basel) 45 (1985), 284–288.
[19] —, *On the lattice rest of a convex body in* $\mathbb{R}^s$ *II*, ibid. 47 (1986), 232–237.
[20] —, *On the lattice rest of a convex body in* $\mathbb{R}^s$ *III*, Czechoslovak Math. J. 41 (116) (1991), 359–367.
[21] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1953.
[22] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.
[23] A. Walfisz, *Gitterpunkte in mehrdimensionalen Kugeln*, Polish Sci. Publ., Warszawa, 1957.

Jingwei Guo
Department of Mathematics
University of Wisconsin – Madison
Madison, WI 53706, U.S.A.
E-mail: guo@math.wisc.edu

Current address:
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green Street (M/C 382)
Urbana, IL 61801, U.S.A.
E-mail: jwguo@illinois.edu

Received on 23.7.2010
and in revised form on 15.2.2011 (6453)