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On Hecke L-functions associated with cusp forms

by

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1. Introduction. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a holomorphic cusp form of even integral weight $k > 0$ with respect to the modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and define the associated Hecke L-function by

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1.1}$$

for $\Re s > (k+1)/2$. Throughout this paper we assume that $f(z)$ is a Hecke eigenform with $a_1 = 1$. It is known (see [7]) that $L_f(s)$ admits analytic continuation to $\mathbb{C}$ as an entire function and it satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L_f(s) = (-1)^{k/2}(2\pi)^{-(k-s)}\Gamma(k-s)L_f(k-s). \tag{1.2}$$

$L_f(s)$ has an Euler product representation (for $\Re s > (k+1)/2$)

$$L_f(s) = \prod_p (1 - a_p p^{-s} + p^{k-1}p^{-2s})^{-1}. \tag{1.3}$$

The non-trivial zeros of $L_f(s)$ lie within the critical strip $(k-1)/2 < \Re s < (k+1)/2$, symmetrically to the real axis and also to the line $\Re s = k/2$. The Riemann hypothesis in this situation asserts that all non-trivial zeros are on the critical line $\Re s = k/2$. From Deligne's proof of Ramanujan–Peterson's conjecture (see [2] and [3]), we have the bound for the coefficients

$$|a_n| \le d(n) n^{(k-1)/2}. \tag{1.4}$$

We denote by $N_f(T)$ the number of zeros $\beta + i\gamma$ of $L_f(s)$ for which $0 < \gamma < T$, for T not equal to any γ; otherwise we put

$$N_f(T) = \lim_{\varepsilon \to 0} \frac{1}{2}\{N_f(T+\varepsilon) + N_f(T-\varepsilon)\}. \tag{1.5}$$

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Then one can show that (following Theorem 9.3 of [14])

$$N_f(T) = \frac{T}{\pi}\log\frac{T}{\pi} - \frac{T}{\pi} + 1 + S_f(T) + O(1/T) \tag{1.6}$$

where

$$S_f(t) = \frac{1}{\pi}\arg L_f(k/2+it). \tag{1.7}$$

The amplitude is obtained by a continuous variation along the straight lines joining the points $k/2+1, k/2+1+iT$ and $k/2+iT$, starting with the value zero. Hence the variation of $S_f(t)$ is closely connected with the distribution of the imaginary parts of the zeros of $L_f(s)$.

We now define, for $\sigma \geq k/2$, $T \geq 1$ and $H \leq T$,

$$N_f(\sigma, T, T+H) = |\{\beta+i\gamma : L_f(\beta+i\gamma) = 0,\ \beta \geq \sigma,\ T \leq \gamma \leq T+H\}|. \tag{1.8}$$

2. Notation and preliminaries

- $A_1, A_2, \ldots$ denote effective absolute constants, sometimes positive.
- $f(x) \ll g(x)$ and $f(x) = O(g(x))$ will mean that there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$.
- ε denotes any small positive constant.
- As usual, $s = \sigma + it$, $w = u + iv$.

When k is even, it is known that a_n's are real and in fact they are totally real algebraic numbers. Hence a_p is real from (1.1) and (1.3). By Deligne's estimate, we also have $|a_p| \leq 2p^{(k-1)/2}$. We define a real number A'_p such that $a_p = 2A'_p p^{(k-1)/2}$ and clearly $|A'_p| \leq 1$. Let α'_p and $\overline{\alpha}'_p$ be the roots of the equation $x^2 - 2A'_p x + 1 = 0$; note that $|\alpha'_p| = 1$. Therefore, from the Euler product of $L_f(s)$, we can write

$$L_f(s) = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\overline{\alpha}_p p^{-s})^{-1} \tag{2.1}$$

with $|\alpha_p| = p^{(k-1)/2}$ and $a_p = \alpha_p + \overline{\alpha}_p$. Taking logarithms and differentiating both sides with respect to s we find that

$$-\frac{L'_f}{L_f}(s) = \sum_{m\geq 1,p} (\alpha_p^m + \overline{\alpha}_p^m)p^{-ms}(\log p). \tag{2.2}$$

Now we define

$$\Lambda_f(n) = \begin{cases} (\alpha_p^m + \overline{\alpha}_p^m)(\log p) & \text{if } n = p^m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Hence we obtain

$$-\frac{L'_f}{L_f}(s) = \sum_{n=2}^{\infty} \Lambda_f(n) n^{-s} \quad (\text{in } \Re s > (k+1)/2). \tag{2.4}$$

Note that

$$\Lambda_f(n) \le 2(\log n) n^{(k-1)/2}. \tag{2.5}$$

Let $x > 1$ and write

$$\Lambda_{x,f}(n) = \begin{cases} \Lambda_f(n) & \text{if } 1 \le n \le x, \\ \Lambda_f(n) \dfrac{\{(\log(x^3/n))^2 - 2(\log(x^2/n))^2\}}{2(\log x)^2} & \text{if } x \le n \le x^2, \\ \Lambda_f(n) \dfrac{(\log(x^3/n))^2}{2(\log x)^2} & \text{if } x^2 \le n \le x^3. \end{cases} \tag{2.6}$$

We define a non-negative smooth C^∞ function $\Psi_U(t)$ as follows. For $H \le T$,

$$\Psi_U(t) = \begin{cases} 0 & \text{if } t < 1 + 1/U \text{ or } t > 1 + H/T - 1/U, \\ 1 & \text{if } 1 + 1/U \le t \le 1 + H/T - 1/U. \end{cases} \tag{2.7}$$

Also assume that Ψ_U is chosen in such a way that

$$\Psi_U^{(p)}(t) \ll U^p \tag{2.8}$$

where U is a positive parameter to be fixed later. Let $\phi(\xi), \phi^*(\xi)$ be suitable smooth ($\mathcal{C}^\infty$) functions satisfying $\phi^*(\xi) = 1 - \phi(1/\xi)$ and

$$\phi(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 2/3, \\ 0 & \text{if } |\xi| \ge 3/2. \end{cases} \tag{2.9}$$

Define

$$L_f^{-1}(s) = \sum_{n=1}^{\infty} \mu_f(n) n^{-s} \quad \text{in } \Re s > (k+1)/2, \tag{2.10}$$

so that from the Euler product for $L_f(s)$, we have

$$\mu_f(p^r) = \begin{cases} 1 & \text{if } r = 0, \\ -a_p & \text{if } r = 1, \\ p^{k-1} & \text{if } r = 2, \\ 0 & \text{if } r \ge 3. \end{cases} \tag{2.11}$$

Now, we define

$$g_\xi(n) = \begin{cases} 1 & \text{if } 1 \le n \le \xi, \\ \dfrac{\log(\xi^2/n)}{\log \xi} & \text{if } \xi \le n \le \xi^2, \\ 0 & \text{if } n \ge \xi^2, \end{cases} \tag{2.12}$$

and define $\lambda_n = \mu_f(n) g_\xi(n)$. Here $\xi = T^\theta$ with $0 < \theta < 1/4$ to be chosen appropriately later. We introduce a Dirichlet polynomial as in [9],

$$M_{\xi^2}(s) = \sum_{v=1}^{\infty} \lambda_v v^{-s}. \tag{2.13}$$

In this paper we prove the following two theorems.

THEOREM 1. *For $t \geq 2, 2 \leq x \leq t^2$, we have*

$$S_f(t) = -\frac{1}{\pi} \sum_{n<x^3} \frac{\Lambda_{x,f}(n) \sin(t \log n)}{n^{\sigma_{x,t}} \log n} + O\bigg((\sigma_{x,t} - k/2) \bigg| \sum_{n<x^3} \frac{\Lambda_{x,f}(n)}{n^{\sigma_{x,t}+it}} \bigg| \bigg) + O((\sigma_{x,t} - k/2) \log t),$$

where

$$\sigma_{x,t} = k/2 + 2 \max(\beta - k/2, 2/\log x)$$

with $\varrho = \beta + i\gamma$ running over those zeros for which

$$|t - \gamma| \leq x^{3|\beta - k/2|} (\log x)^{-1},$$

and $\Lambda_{x,f}(n)$ is as in (2.6).

As corollaries, by choosing $x = \sqrt{\log t}$ we obtain

$$S_f(t) = O(\log t)$$

unconditionally, and assuming Riemann hypothesis, we get

$$S_f(t) = O\bigg(\frac{\log t}{\log \log t} \bigg).$$

THEOREM 2. *Let B be any fixed small positive constant. Let*

$$B' = \frac{19}{20} + \frac{13.505}{5} B \quad \textit{and} \quad B' < \alpha \leq 1.$$

Then for $T^\alpha \leq H \leq T$, we have

$$N_f(\sigma, T, T+H) \ll H \bigg(\frac{H}{T^{B'}} \bigg)^{-\frac{B}{1-B'}(\sigma - k/2)} \log T$$

uniformly for $k/2 \leq \sigma \leq (k+1)/2$.

REMARK 1. Theorems 1 and 2 (with $T^{1/2+\varepsilon} \leq H \leq T$ and $B' = 1/2$) in the case of the Riemann zeta-function $\zeta(s)$ are due to Selberg [13]. The importance of Theorem 2 is in the exponent of the log factor when $|\sigma - k/2| \ll (\log T)^{-1}$. In fact later developments in the theory allow us to take even a much shorter interval in the case of $\zeta(s)$ in Theorem 2. Theorem 2 (with $H = T$) in the case of $L_f(s)$ is due to Luo [9]. Here we prove an analogue of a result of Selberg for $L_f(s)$ (Theorem 1) and the density estimate for $L_f(s)$ over shorter intervals (Theorem 2). We follow closely the papers [13] and [9].

REMARK 2. It should be pointed out here that some more important results have recently been proved in [1] assuming certain hypotheses (which are true in this situation) for a class of Dirichlet series which are linear combinations of Euler products. We also suggest some basic references related to our paper: [6], [8], [11], [12].

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3. Some lemmas

LEMMA 3.1. *We have* $\xi(s) = (2\pi)^{-s}\Gamma(s)L_f(s)$ *is an integral function of order* 1.

Proof. It is standard. ∎

LEMMA 3.2. *If* $s \neq \varrho,\ t \geq 2,$ *then*

$$\frac{L_f'}{L_f}(s) = \sum_{\varrho}((s-\varrho)^{-1} + \varrho^{-1}) + O(\log t)$$

uniformly for $k/2 \leq \sigma \leq k/2 + 10$.

Proof. Since $\xi(s)$ is an integral function of order 1 it has the Weierstrass product representation

$$\xi(s) = e^{b_0 + b_1 s} \prod_{\varrho} \left\{ \left(1 - \frac{s}{\varrho}\right) e^{s/\varrho} \right\} \tag{3.2.1}$$

where b_0, b_1 are certain constants. Also we have

$$\xi(s) = (2\pi)^{-s}\Gamma(s)L_f(s). \tag{3.2.2}$$

Taking logarithms and differentiating (3.2.1) and (3.2.2) with respect to s, and using (for $a \leq \Re z \leq b$)

$$\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} + O(|z|^{-2}), \tag{3.2.3}$$

we obtain the lemma. ∎

LEMMA 3.3. *In the region defined by* $\sigma \leq 1/4,\ |s-n| \geq 1/2\ (n = 0, -1, -2, \ldots),$ *we have*

$$\left|\frac{L_f'}{L_f}(s)\right| < A_1(\log|s| + 1).$$

Proof. From the functional equation $\xi(s) = e^{i\pi k/2}\xi(k-s)$, we have

$$\frac{L_f'}{L_f}(s) = 2\log(2\pi) - \frac{\Gamma'}{\Gamma}(s) - \frac{\Gamma'}{\Gamma}(k-s) - \frac{L_f'}{L_f}(k-s). \tag{3.3.1}$$

Note that, for $\sigma \leq 1/4,\ k - \sigma \geq k - 1/4$ and hence

$$\left|\frac{L_f'}{L_f}(k-s)\right| \leq \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^{k-\sigma}} \ll 1. \tag{3.3.2}$$

Now the lemma follows on using (3.2.3). ∎

LEMMA 3.4. *There exists a sequence of numbers $T_2, T_3, \ldots$ such that $m < T_m < m+1$ $(m = 2, 3, \ldots)$ and*

$$\left|\frac{L_f'}{L_f}(s)\right| < A_2 \log^2 m$$

for $k/2+1 \geq \sigma \geq 1/4$, $t = \pm T_m$.

Proof. From the Weierstrass product representation of $\xi(s)$, we obtain

$$\frac{L_f'}{L_f}(s) = b_1 + \log(2\pi) - \frac{\Gamma'}{\Gamma}(s) + \sum_{\varrho}((s-\varrho)^{-1} + \varrho^{-1}) \tag{3.4.1}$$
$$= g(s) + \Sigma(s) \quad \text{(say)}$$

where

$$g(s) = b_1 + \log(2\pi) - \frac{\Gamma'}{\Gamma}(s), \qquad \Sigma(s) = \sum_{\varrho}((s-\varrho)^{-1} + \varrho^{-1}).$$

Let $s = \sigma + it$, $s_0 = k/2 + 2 + it$ where $1/4 \leq \sigma \leq k/2+2$, $t > 2$ and t is not equal to any γ. Let δ_0 be the distance of t from the nearest γ and let

$$\delta = \delta(t) = \min(\delta_0, 1). \tag{3.4.2}$$

Then for every zero $\varrho = \beta + i\gamma$ with $0 \leq \beta \leq (k+1)/2$, we have

$$|s-\varrho|^2 \geq (t-\gamma)^2 \geq \delta^2/2 + (t-\gamma)^2/2 \geq \frac{\delta^2}{2}\{1 + (t-\gamma)^2\} \tag{3.4.3}$$

and

$$|s_0-\varrho|^2 = (k/2+2-\beta)^2 + (t-\gamma)^2 \geq 1 + (t-\gamma)^2. \tag{3.4.4}$$

Therefore from (3.4.3) and (3.4.4), we get

$$|\Sigma(s) - \Sigma(s_0)| = \left|\sum_{\varrho} \frac{s_0 - s}{(s-\varrho)(s_0-\varrho)}\right| \tag{3.4.5}$$
$$\leq \sum_{\varrho} \frac{k/2+2-1/4}{(\delta^2/2)^{1/2}\{1+(t-\gamma)^2\}}.$$

On the other hand,

$$\Re\Sigma(s_0) = \sum_{\varrho}\left(\frac{k/2+2-\beta}{|s_0-\varrho|^2} + \frac{\beta}{|\varrho|^2}\right) \tag{3.4.6}$$
$$\geq \sum_{\varrho}\{(k/2+2-\beta)^2 + (t-\gamma)^2\}^{-1}$$
$$\geq (k/2+2)^{-2}\sum_{\varrho}\{1+(t-\gamma)^2\}^{-1}.$$

Hence from (3.4.5) and (3.4.6), we get

$$|\Sigma(s) - \Sigma(s_0)| < 2\delta^{-1}(k/2+2)^2(k/2+2-1/4)\Re\Sigma(s_0). \tag{3.4.7}$$

This implies that

$$\begin{aligned}\left|\frac{L_f'}{L_f}(s) - g(s)\right| &= |\Sigma(s)| \le \delta^{-1}\{2(k/2+2)^3+1\}|\Sigma(s_0)| \\ &\le A_3(k)\delta^{-1}\left|\frac{L_f'}{L_f}(s_0) - g(s_0)\right|.\end{aligned} \tag{3.4.8}$$

Note that $\left|\frac{L_f'}{L_f}(s_0)\right| \ll 1$. Now applying the asymptotic expression (3.2.3) for $\frac{\Gamma'}{\Gamma}(s)$ to $g(s)$ and $g(s_0)$, we get

$$\left|\frac{L_f'}{L_f}(s)\right| < A_4(k)\delta^{-1}\log t. \tag{3.4.9}$$

Now let m be any integer greater than 1, and ν_m the number of ϱ for which $m < \gamma < m+1$ so that $\nu_m = N(m+1) - N(m) < A_5 \log m$. If we divide the interval $(m, m+1)$ into $\nu_m + 1$ equal parts, then one subinterval at least will contain no γ in its interior. We choose T_m to be the midpoint of such an interval. If $t = T_m$, then $\delta \ge 1/(2(\nu_m+1))$ and hence

$$\left|\frac{L_f'}{L_f}(s)\right| < A_6 \log^2 m,$$

which proves the lemma. ∎

Lemma 3.5. *There exists a sequence of numbers $T_2, T_3, \ldots$ such that $m < T_m < m+1$ $(m = 2, 3, \ldots)$ and*

$$\left|\frac{L_f'}{L_f}(s)\right| < A_7 \log^2 m$$

for $\sigma \ge -m - 1/2$, $t = \pm T_m$ or $\sigma = -m - 1/2$, $|t| < T_m$.

Proof. This follows from Lemmas 3.3 and 3.4. ∎

Lemma 3.6. *For $s \ne \varrho$, $s \ne -q$ $(q = 0, 1, 2, \ldots)$, we have*

$$\begin{aligned}\frac{L_f'}{L_f}(s) = &-\sum_{n<x^3}\frac{\Lambda_{x,f}(n)}{n^s} + (\log x)^{-2}\sum_{q=0}^{\infty}\frac{x^{-q-s}(1-x^{-q-s})^2}{(q+s)^3} \\ &+ (\log x)^{-2}\sum_{\varrho}\frac{x^{\varrho-s}(1-x^{\varrho-s})^2}{(s-\varrho)^3}.\end{aligned}$$

Proof. First, we notice that for $\alpha_1, y > 0$,

$$\frac{1}{2\pi i}\int_{\alpha_1-i\infty}^{\alpha_1+i\infty}\frac{y^w}{w^3}\,dw = \begin{cases}(\log y)^2/2 & \text{if } y \ge 1,\\ 0 & \text{if } 0 < y \le 1.\end{cases} \tag{3.6.1}$$

Fix $\alpha_1 = \max(k/2+1, (k+1)/2+\sigma)$. From (3.6.1), we obtain

$$\sum_{n<x^3} \frac{\Lambda_{x,f}(n)}{n^s} = \frac{1}{2\pi i (\log x)^2} \int_{\alpha_1 - i\infty}^{\alpha_1 + i\infty} \frac{x^w(1-x^w)^2}{w^3} \left(-\frac{L_f'}{L_f}(s+w) \right) dw. \tag{3.6.2}$$

After taking into account the residues at singularities between the contours, we replace the interval $(\alpha_1 - iT_m, \alpha_1 + iT_m)$ of the integration line by the straight lines joining the points $\alpha_1 - iT_m, -m-1/2-\sigma-iT_m, -m-1/2-\sigma+iT_m, \alpha_1+iT_m$ where $m \geq 2$ is an integer and T_m is the number defined in Lemma 3.5. We note that whenever $u+\sigma \geq (k+1)/2+\varepsilon$,

$$\left| \frac{L_f'}{L_f}(s+w) \right| \ll 1$$

and also from Lemma 3.5, the contributions from the horizontal portion and the vertical portion tend to zero as $m \to \infty$. Therefore the integral on the right hand side of (3.6.2) is equal to $2\pi i$ times the sum of the residues of the integrand in the half plane $\Re w < \alpha_1$. The singularities are $w=0$, $w=-q-s$ $(q=0,1,2,\ldots)$ and $w=\varrho - s$, and the corresponding residues are

$$-(\log x)^2 \frac{L_f'}{L_f}(s), \qquad \frac{x^{-q-s}(1-x^{-q-s})^2}{(q+s)^3}, \qquad \frac{x^{\varrho-s}(1-x^{\varrho-s})^2}{(s-\varrho)^3}$$

respectively. Since the series of the residues is absolutely convergent we get the lemma. ∎

LEMMA 3.7. *Let $w = u+iv$. Then for $|v| \geq 10$, we have*

$$J_1 = \int_{-\infty}^{\infty} \Psi_U(t/T) t^w \, dt \ll \frac{T^{u+1} U^3 (1+U^{-1})^{u+4}}{|(w+1)(w+2)(w+3)v|}.$$

Proof. Integration by parts and the properties of Ψ_U give

$$|J_1| \leq \frac{T^{u+1} \left| \int_{-\infty}^{\infty} \Psi_U^{(3)}(t) t^{u+3+iv} \, dt \right|}{|(w+1)(w+2)(w+3)|}. \tag{3.7.1}$$

Let $G(t) = \Psi_U^{(3)}(t) t^{u+3}$, $F(t) = v \log t$. Then

$$\frac{G(t)}{F'(t)} = v^{-1} \Psi_U^{(3)}(t) t^{u+4}$$

is monotonic in t in the interval $1+U^{-1} \leq t \leq 1+HT^{-1}-U^{-1}$. Also, for any $v > 0$,

$$\frac{F'(t)}{G(t)} \geq vU^{-3}(1+U^{-1})^{-u-4} > 0$$

since $\Psi_U^{(3)}(t) \ll U^3$. Hence by Lemma 4.3 of [14], we have

$$\Big| \int_{-\infty}^{\infty} \Psi_U^{(3)}(t) t^{u+3+iv}\, dt \Big| \le 4v^{-1} U^3 (1+U^{-1})^{u+4}. \tag{3.7.2}$$

This proves the lemma for $v > 0$. For $v < 0$ the proof is similar. ∎

Lemma 3.8. *For $|v| \le 10$, we have*

$$J_2 = \int_{-\infty}^{\infty} \Psi_U(t/T) t^w\, dt \ll \frac{T^{u+1}}{u+1}\{(2-U^{-1})^{u+1} - (1+U^{-1})^{u+1}\}.$$

Proof. Using the properties of Ψ_U and taking the absolute value inside the integral, a trivial estimation gives the lemma. ∎

Lemma 3.9. *If $U = 4T/H$, then for $k/2 \le \sigma \le k/2 + \varepsilon$, we have*

$$|J_3| = \Big| \int_{-\infty}^{\infty} \Psi_U(t/T) \Big(\frac{t}{2\pi}\Big)^{2k-4\sigma} dt \Big| \ll_\varepsilon H.$$

Proof. By a suitable change of variable, we have

$$\begin{aligned} |J_3| &= \Big| \Big(\frac{T}{2\pi}\Big)^{2k-4\sigma} T \int_{-\infty}^{\infty} \Psi_U(t) t^{2k-4\sigma}\, dt \Big| \\ &\le \frac{T^{2k-4\sigma+1}}{(2\pi)^{2k-4\sigma}} \int_{1+U^{-1}}^{1+HT^{-1}-U^{-1}} t^{2k-4\sigma}\, dt \\ &\ll_\varepsilon T^{2k-4\sigma+1}(H/T - 2/U)(1+U^{-1})^{2k-4\sigma} \ll H \end{aligned}$$

since $\sigma \ge k/2$. ∎

Lemma 3.10. *For $b > 0$, $\sigma > k/2$, we have*

$$\sum_{v_1, v_2 \le \xi^2} \frac{\lambda_{v_1}\lambda_{v_2}}{(v_1 v_2)^\sigma} \frac{(v_1 v_2)^b}{(v_1, v_2)^{2b}} \ll \xi^{4b+2} (\log \xi)^2.$$

Proof. Since $|\lambda_v| = |\mu_f(v) g_\xi(v)| \le d(v) v^{(k-1)/2}$, we have

$$\begin{aligned} \sum_{v_1, v_2 \le \xi^2} \frac{\lambda_{v_1}\lambda_{v_2}}{(v_1 v_2)^\sigma} \frac{(v_1 v_2)^b}{(v_1, v_2)^{2b}} &\ll \sum_{v_1, v_2 \le \xi^2} d(v_1) d(v_2) (v_1 v_2)^{b-1/2} \\ &\ll \xi^{4b-2} \Big(\sum_{v \le \xi^2} d(v) \Big)^2 \ll \xi^{4b+2} (\log \xi)^2 \end{aligned}$$

because $\sigma > k/2$, and this proves the lemma. ∎

Lemma 3.11. *We have*

$$\int_T^{T+H} |L_f(k/2+1+it)M_{\xi^2}(k/2+1+it)-1|^2\,dt \ll H/\xi^{2-\varepsilon}.$$

Proof. We write

$$L_f(s)M_{\xi^2}(s)-1=\sum_{n=1}^{\infty} c_n n^{-s}. \tag{3.11.1}$$

We note that $a_n * \mu_f(n) = I(n) = [1/n]$ (the Dirichlet convolution) and hence $c_n = 0$ for $2 \le n \le \xi$. Also we notice that, by definition, $c_1 = 1$ and for $n \ge \xi$,

$$c_n=\sum_{d\mid n} a_d\mu_f(n/d)g_\xi(n/d). \tag{3.11.2}$$

Therefore

$$|c_n| \le \sum_{m\mid n} d(m)m^{(k-1)/2}d(n/m)(n/m)^{(k-1)/2} \le d_4(n)n^{(k-1)/2} \tag{3.11.3}$$

since $|\mu_f(n/d)| \le a_{n/d}$, $|g_\xi(n/d)| \le 1$ and $|a_m| \le d(m)m^{(k-1)/2}$. Hence we obtain

$$|c_n|^2 \le n^{k-1}d_{16}(n). \tag{3.11.4}$$

From the Montgomery–Vaughan theorem (see [10]), on using (3.11.4) we get

$$\begin{aligned} J_4 &:= \int_T^{T+H} |L_f(k/2+1+it)M_{\xi^2}(k/2+1+it)-1|^2\,dt \\ &= \sum_{n\ge\xi} |c_n|^2 n^{-k-2}(H+O(n)) \\ &\ll H\sum_{\xi\le n\le H} n^{-3}d_{16}(n) + \sum_{n\ge H} n^{-2}d_{16}(n) \\ &\ll \frac{H(\log T)^{15}}{\xi^2} + \frac{(\log T)^{15}}{H} \ll \frac{H}{\xi^{2-\varepsilon}}, \end{aligned}$$

which proves the lemma. ■

Lemma 3.12. *If $k/2 < \sigma < k/2 + 1/1000$ and μ,ν are co-prime positive integers $\le T$, then*

$$J_5 := \int_{-\infty}^{\infty} \Psi_U(t/T)|L_f(\sigma+it)|^2(\mu/\nu)^{it}\,dt$$

$$= (\mu\nu)^{-\sigma} D_{\mu\nu}(2\sigma) \int_{-\infty}^{\infty} \Psi_U(t/T)\,dt$$
$$+ (\mu\nu)^{-(k-\sigma)} D_{\mu\nu}(2(k-\sigma)) \int_{-\infty}^{\infty} \Psi_U(t/T)(t/T)^{2k-4\sigma}\,dt$$
$$+ O\biggl(\frac{(\mu\nu)^{2876/1000} U^4 T^{3/4+\varepsilon}}{2\sigma - k}\biggr)$$

where

$$D_{\mu\nu}(s) = \sum_{l=1}^{\infty} \frac{a_{\mu l} a_{\nu l}}{l^s}.$$

Proof. First consider the following expression:

(3.12.1) $$E = (\mu\nu)^{\sigma} \times \sum_{0 \le l \le \sqrt{\mu\nu}\,UT^{\varepsilon}} \frac{1}{2\pi i} \int_{(2)} \biggl(\frac{\sqrt{\mu\nu}}{2\pi}\biggr)^{s} H_l(s) D_{\mu\nu}(s+2\sigma, l) \int_{-\infty}^{\infty} \Psi_U(t/T) t^s\,dt$$

where

(3.12.2) $$H_l(w) = \int_0^{\infty} \phi(\xi) e^{2\pi i (l/\sqrt{\mu\nu})\xi^{-1}} \xi^{w-1}\,d\xi$$

and

(3.12.3) $$D_{\mu\nu}(w+2\sigma, l) = \sum_{n=1}^{\infty} \frac{a_n a_{(n\nu+l)/\mu}}{(n\nu + l/2)^{w+2\sigma}}.$$

We move the line of integration in (3.12.1) to $\Re s = -1/4$. From Lemma 5 of [5], we have

(3.12.4) $$D_{\mu\nu}(w+2\sigma, l) = O\biggl(\frac{l|w+2\sigma|^{1+\varepsilon}}{(\mu\nu)^{k/2-2}(u+2\sigma-k+1/4)}\biggr)$$

uniformly in $\mu, \nu, l \ge 1, u + 2\sigma \ge k - 1/4$. Note that here $u = -1/4$. Using integration by parts and from the properties of $\phi(\xi)$, it follows that

(3.12.5) $$H_l(w) \ll \frac{\sqrt{\mu\nu}}{l} |w|.$$

From Lemmas 3.7, 3.8 and the inequalities (3.12.4), (3.12.5) we obtain

(3.12.6) $$E \ll \frac{(\mu\nu)^{\sigma+u/2+1/2+1/2}}{(2\sigma-k)(\mu\nu)^{k/2-2}} U^4 T^{3/4+\varepsilon} \ll \frac{(\mu\nu)^{\sigma+23/8-k/2}}{2\sigma-k} U^4 T^{3/4+\varepsilon} \ll \frac{(\mu\nu)^{2876/1000}}{2\sigma-k} U^4 T^{3/4+\varepsilon}.$$

Now the lemma follows from Lemma 2.1 of [9]. ∎

LEMMA 3.13. *For $\sigma = k/2 + A_8/\log T$, with $U = 4T/H$ we have*

$$\int_T^{T+H} |L_f(\sigma+it)M_{\xi^2}(\sigma+it) - 1|^2\,dt \ll H + O\bigg(\frac{T^{19/4+\varepsilon}\xi^{13.504}(\log\xi)^3}{H^4}\bigg).$$

Proof. From Lemmas 3.10 and 3.12, with $b = 2876/1000$, we have

$$\begin{aligned} (3.13.1)\qquad J_6 &:= \int_{-\infty}^{\infty} \Psi_U(t/T)|L_f(\sigma+it)M_{\xi^2}(\sigma+it) - 1|^2\,dt \\ &= \sum_{v_1,v_2\le\xi^2} \frac{\lambda_{v_1}\lambda_{v_2}}{(v_1v_2)^{2\sigma}}\,(v_1,v_2)^{2\sigma} J_7 \\ &\quad + \sum_{v_1,v_2\le\xi^2} \frac{\lambda_{v_1}\lambda_{v_2}}{(v_1v_2)^{k}}\,(v_1,v_2)^{2(k-\sigma)} J_8 \\ &\quad + O(T^{3/4+\varepsilon}U^4\xi^{13.504}(\log\xi)^3), \end{aligned}$$

where

$$J_7 := D_{v_1/(v_1,v_2),v_2/(v_1,v_2)}(2\sigma)\int_{-\infty}^{\infty}\Psi_U(t/T)\,dt,$$

$$J_8 := D_{v_1/(v_1,v_2),v_2/(v_1,v_2)}(2(k-\sigma))\int_{-\infty}^{\infty}\Psi_U(t/T)(t/T)^{2k-4\sigma}\,dt.$$

Also note that

$$\begin{aligned} (3.13.2)\qquad \int_{-\infty}^{\infty}\Psi_U(t/T)\,dt &= T\int_{-\infty}^{\infty}\Psi_U(t)\,dt \\ &= T\int_{1+U^{-1}}^{1+HT^{-1}-U^{-1}} dt = T\{HT^{-1} - 2U^{-1}\} \ll H. \end{aligned}$$

Now the lemma follows from the arguments of Section 3 of [9]. ∎

LEMMA 3.14. *Let B be any small positive constant. For $T^{19/20+13.505B/5} \ll H \le T$, we have*

$$\int_T^{T+H} |L_f(\sigma+it)M_{\xi^2}(\sigma+it) - 1|^2\,dt \ll_\varepsilon HT^{-(2-\varepsilon)(\sigma-k/2)B}$$

uniformly for $k/2 + A_9/\log T \le \sigma \le k/2 + 1$.

Proof. We fix $\xi = T^B$ so that the error in Lemma 3.13 is

$$(3.14.1)\qquad \ll H^{-4}T^{19/4+13.504B+\varepsilon}(\log T)^3 \ll H^{-4}T^{19/4+13.504B+\varepsilon_1} \ll H$$

for some other small positive constant ε_1 ($< 0.001B$), since $T^{19/20+13.505B/5} \ll H$. Hence from Lemma 3.13 we have

$$\int_T^{T+H} |L_f(k/2+A_9/\log T+it)M_{\xi^2}(k/2+A_9/\log T+it)-1|^2\,dt \ll H. \tag{3.14.2}$$

Also from Lemma 3.11, we have

$$\int_T^{T+H} |L_f(k/2+1+it)M_{\xi^2}(k/2+1+it)-1|^2\,dt \ll HT^{-B(2-\varepsilon)}. \tag{3.14.3}$$

Now we use the two-variable Gabriel convexity theorem (see [4]):

Convexity Theorem. *Let $g(s)$ be an analytic function in a specified region and for any positive λ, let*

$$G(\sigma,\lambda) = \Big(\int_T^{T+H} |g(\sigma+it)|^{1/\lambda}\,dt\Big)^{\lambda}. \tag{3.14.4}$$

Then, for $\alpha<\sigma<\beta$ and any positive numbers λ,μ, with $p=(\beta-\sigma)/(\beta-\alpha)$ and $q=(\sigma-\alpha)/(\beta-\alpha)$, we have

$$G(\sigma,\lambda p+\mu q) \ll (G(\alpha,\lambda))^p(G(\beta,\mu))^q. \tag{3.14.5}$$

In the above convexity theorem, we take $\lambda=\mu=1/2, \alpha=k/2+A_9/\log T$, $\beta=k/2+1$ and $g(s)=L_f(s)M_{\xi^2}(s)-1$. This implies that

$$p=\frac{k/2+1-\sigma}{1-A_9/\log T},\qquad q=\frac{\sigma-k/2-A_9/\log T}{1-A_9/\log T}.$$

Note that $p+q=1$. From (3.14.2), (3.14.3) and (3.14.5), we obtain

$$\int_T^{T+H} |L_f(\sigma+it)M_{\xi^2}(\sigma+it)-1|^2\,dt \ll H^p(HT^{-B(2-\varepsilon)})^q \ll HT^{-Bq(2-\varepsilon)} \ll HT^{-B(2-\varepsilon)(\sigma-k/2)} \tag{3.14.6}$$

and hence the lemma. ∎

Remark. Let $B'=19/20+13.505B/5$ where B is any fixed small positive constant. Let $T^\alpha\le H\le T$, $B'<\alpha\le 1$. We notice that

$$\Big(\frac{H}{T^{B'}}\Big)\le T^{1-B'}\quad\text{and hence}\quad\Big(\frac{H}{T^{B'}}\Big)^{1/(1-B')}\le T.$$

Therefore, we have

$$T^{-(2-\varepsilon)B(\sigma-k/2)}\le T^{-B(\sigma-k/2)}\le\Big(\frac{H}{T^{B'}}\Big)^{-\frac{B}{1-B'}(\sigma-k/2)}.$$

This implies that, for $T^\alpha \le H \le T$ with $B' < \alpha \le 1$,

$$\int_T^{T+H} |L_f(\sigma+it)M_{\xi^2}(\sigma+it)-1|^2\,dt \ll H\left(\frac{H}{T^{B'}}\right)^{-\frac{B}{1-B'}(\sigma-k/2)},$$

which holds uniformly for $k/2 + A_{10}/\log\xi \le \sigma \le k/2+1/2$.

4. Proof of Theorem 1. We follow closely the arguments of Selberg (see [13]). For $x \ge 2$, $t > 0$, we define a number

$$\sigma_{x,t} = \frac{k}{2} + 2\max_{\varrho}\left(\beta - \frac{k}{2}, \frac{2}{\log x}\right) \tag{4.1}$$

where ϱ runs through all zeros $\varrho = \beta + i\gamma$ for which

$$|t-\gamma| \le \frac{x^{3|\beta-k/2|}}{\log x}. \tag{4.2}$$

We notice that

$$\sum_{\varrho} \frac{\beta}{\beta^2+\gamma^2} = O(\log t)$$

and hence from Lemma 3.2, for $t \ge 2$, taking real parts on both sides, we obtain

$$S := \sum_{\varrho} \frac{\sigma_{x,t}-\beta}{(\sigma_{x,t}-\beta)^2+(t-\gamma)^2} = \Re\frac{L_f'}{L_f}(\sigma_{x,t}+it) + O(\log t). \tag{4.3}$$

Since zeros lie symmetrically with respect to the line $\sigma = k/2$, we have

$$\begin{aligned} S &= (\sigma_{x,t}-k/2) \\ &\quad\times\sum_{\varrho} \frac{\{(\sigma_{x,t}-k/2)^2-(\beta-k/2)^2+(t-\gamma)^2\}}{\{(\sigma_{x,t}-\beta)^2+(t-\gamma)^2\}\{(\sigma_{x,t}-k+\beta)^2+(t-\gamma)^2\}}. \end{aligned} \tag{4.4}$$

Arguing as in [13], we find that

$$\begin{aligned} S_1 &:= \left(\sigma_{x,t}-\frac{k}{2}\right)^2 - \left(\beta-\frac{k}{2}\right)^2 + (t-\gamma)^2 \\ &\ge \frac{3}{10}\{(\sigma_{x,t}-\beta)^2+(\sigma_{x,t}-k+\beta)^2+2(t-\gamma)^2\}. \end{aligned} \tag{4.5}$$

Therefore from (4.4) and (4.5), we get

$$\begin{aligned} S &\ge \frac{3}{10}(\sigma_{x,t}-k/2) \\ &\quad\times\sum_{\varrho}\left\{\frac{1}{(\sigma_{x,t}-\beta)^2+(t-\gamma)^2} + \frac{1}{(\sigma_{x,t}-k+\beta)^2+(t-\gamma)^2}\right\} \\ &= \frac{3}{5}(\sigma_{x,t}-k/2)\sum_{\varrho}\frac{1}{(\sigma_{x,t}-\beta)^2+(t-\gamma)^2}. \end{aligned} \tag{4.6}$$

From (4.3) and (4.6) we have

$$\sum_{\varrho} \frac{1}{(\sigma_{x,t}-\beta)^2+(t-\gamma)^2} < \frac{5}{3}\,\frac{1}{\sigma_{x,t}-k/2}\left|\frac{L_f'}{L_f}(\sigma_{x,t}+it)\right| + O\left(\frac{\log t}{\sigma_{x,t}-k/2}\right). \tag{4.7}$$

For $t\ge 2,\ 2\le x\le t^2,\ \sigma\ge\sigma_{x,t}$, Lemma 3.6 yields

$$\frac{L_f'}{L_f}(s) = -\sum_{n<x^3}\frac{\Lambda_{x,f}(n)}{n^s} + O(x^{-\sigma}(\log x)^{-2}) + \frac{\omega}{(\log x)^2}\sum_{\varrho}\frac{x^{\beta-\sigma}(1+x^{\beta-\sigma})^2}{\{(\sigma-\beta)^2+(t-\gamma)^2\}^{3/2}} \tag{4.8}$$

where $|\omega|\le 1$.

Now, arguing as in [13], we obtain

$$\frac{x^{\beta-\sigma}(1+x^{\beta-\sigma})^2}{\{(\sigma-\beta)^2+(t-\gamma)^2\}^{3/2}} < \frac{2(\log x)x^{k/4-\sigma/2}}{(\sigma_{x,t}-\beta)^2+(t-\gamma)^2}. \tag{4.9}$$

Therefore from (4.7) and (4.9), we get

$$\sum_{\varrho}\frac{x^{\beta-\sigma}(1+x^{\beta-\sigma})^2}{\{(\sigma-\beta)^2+(t-\gamma)^2\}^{3/2}} < \frac{5}{6}(\log x)^2 x^{k/4-\sigma/2}\left|\frac{L_f'}{L_f}(\sigma_{x,t}+it)\right| + O(x^{k/4-\sigma/2}(\log x)^2\log t). \tag{4.10}$$

Hence from (4.10), (4.8) becomes

$$\frac{L_f'}{L_f}(s) = -\sum_{n<x^3}\frac{\Lambda_{x,f}(n)}{n^s} + O(x^{k/4-\sigma/2}\log t) + \frac{5}{6}\,\omega' x^{k/4-\sigma/2}\,\frac{L_f'}{L_f}(\sigma_{x,t}+it) \tag{4.11}$$

where $|\omega'|<1$. Taking first $\sigma=\sigma_{x,t}$, we get

$$\frac{L_f'}{L_f}(\sigma_{x,t}+it) = O\left(\left|\sum_{n<x^3}\frac{\Lambda_{x,f}(n)}{n^{\sigma_{x,t}+it}}\right|\right) + O(\log t). \tag{4.12}$$

Therefore from (4.7) and (4.12), we get

$$\sum_{\varrho}\frac{\sigma_{x,t}-k/2}{(\sigma_{x,t}-\beta)^2+(t-\gamma)^2} = O\left(\left|\sum_{n<x^3}\frac{\Lambda_{x,f}(n)}{n^{\sigma_{x,t}+it}}\right|\right) + O(\log t) \tag{4.13}$$

and

$$\frac{L_f'}{L_f}(s) = -\sum_{n<x^3}\frac{\Lambda_{x,f}(n)}{n^s} + O\left(x^{k/4-\sigma/2}\left|\sum_{n<x^3}\frac{\Lambda_{x,f}(n)}{n^{\sigma_{x,t}+it}}\right|\right) + O(x^{k/4-\sigma/2}\log t). \tag{4.14}$$

Now,

$$\arg L_f(k/2+it) = -\int_{k/2}^{\infty} \Im \frac{L_f'}{L_f}(\sigma+it)\,d\sigma \tag{4.15}$$
$$= \int_{\sigma_{x,t}}^{\infty} \Im \frac{L_f'}{L_f}(\sigma+it)\,d\sigma - (\sigma_{x,t}-k/2)\Im \frac{L_f'}{L_f}(\sigma_{x,t}+it)$$
$$+ \int_{k/2}^{\sigma_{x,t}} \Im \left\{ \frac{L_f'}{L_f}(\sigma_{x,t}+it) - \frac{L_f'}{L_f}(\sigma+it) \right\} d\sigma$$
$$= I_1 + I_2 + I_3 \quad \text{(say)}.$$

Using (4.14), we find that

$$I_1 = \Im \sum_{n<x^3} \frac{\Lambda_{x,f}(n)}{n^{\sigma_{x,t}+it}\log n} + O\left(\frac{1}{\log x}\left|\sum_{n<x^3} \frac{\Lambda_{x,f}(n)}{n^{\sigma_{x,t}+it}}\right|\right) + O\left(\frac{\log t}{\log x}\right). \tag{4.16}$$

From (4.12), we get

$$I_2 = O\left((\sigma_{x,t}-k/2)\left|\sum_{n<x^3} \frac{\Lambda_{x,f}(n)}{n^{\sigma_{x,t}+it}}\right|\right) + O((\sigma_{x,t}-k/2)\log t). \tag{4.17}$$

From Lemma 3.2, taking the imaginary part of both sides and arguing as in [13], we find that

$$|I_3| < 10(\sigma_{x,t}-k/2)\sum_{\varrho} \frac{\sigma_{x,t}-k/2}{(\sigma_{x,t}-\beta)^2+(t-\gamma)^2} + O((\sigma_{x,t}-k/2)\log t) \tag{4.18}$$
$$= O\left((\sigma_{x,t}-k/2)\left|\sum_{n<x^3} \frac{\Lambda_{x,f}(n)}{n^{\sigma_{x,t}+it}}\right|\right) + O((\sigma_{x,t}-k/2)\log t).$$

Now Theorem 1 follows from (4.15)–(4.18).

5. Proof of Theorem 2. It suffices to show that (for any fixed small positive constant B, $T^{\alpha} \le H \le T$, $B' < \alpha \le 1$ where B' is as in the theorem)

$$\int_{\sigma}^{(k+1)/2} (N_f(\sigma', T+H) - N_f(\sigma', T))\,d\sigma' \ll H\left(\frac{H}{T^{B'}}\right)^{-\frac{B}{1-B'}(\sigma-k/2)} \tag{5.1}$$

for $k/2 + A_{11}/\log\xi \le \sigma \le (k+1)/2$.

Let $\Phi(s) = 1-(L_f(s)M_{\xi^2}(s)-1)^2$. The zeros of $L_f(s)$ occur among those of $\Phi(s)$ with at least the same multiplicities. By Littlewood's lemma regarding the number of zeros of an analytic function in a rectangle (see

[15]), we obtain

$$\int_{\sigma}^{(k+1)/2} (N_f(\sigma', T+H) - N_f(\sigma', T))\, d\sigma' \tag{5.2}$$
$$\leq \frac{1}{2\pi} \int_{T}^{T+H} \log|\Phi(\sigma+it)|\, dt$$
$$+ \frac{1}{2\pi} \int_{\sigma}^{\infty} \arg\Phi(\sigma' + i(T+H))\, d\sigma' - \frac{1}{2\pi} \int_{\sigma}^{\infty} \arg\Phi(\sigma' + iT)\, d\sigma'.$$

In the range $((k+1)/2+4, \infty)$, we see that $\arg\Phi(\sigma'+it) = O(2^{-\sigma})$ and hence

$$\int_{(k+1)/2+4}^{\infty} \arg\Phi(\sigma' + iT)\, d\sigma' = O(1). \tag{5.3}$$

In the range $(k/2, (k+1)/2+4)$, from Jensen's theorem (see [14]) and a standard argument, we find that

$$\arg\Phi(\sigma' + iT) = O(\log T). \tag{5.4}$$

Therefore we get

$$\int_{\sigma}^{\infty} \arg\Phi(\sigma' + iT)\, d\sigma' \ll \log T. \tag{5.5}$$

Similarly we have

$$\int_{\sigma}^{\infty} \arg\Phi(\sigma' + i(T+H))\, d\sigma' \ll \log T. \tag{5.6}$$

Since $\log(1+|x|) \leq |x|$, we obtain

$$\int_{T}^{T+H} \log|\Phi(\sigma+it)|\, dt \leq \int_{T}^{T+H} |L_f(\sigma+it)M_{\xi^2}(\sigma+it) - 1|^2\, dt \tag{5.7}$$
$$\ll H\left(\frac{H}{T^{B'}}\right)^{-\frac{B}{1-B'}(\sigma-k/2)}.$$

Now the inequality (5.1) follows from (5.2) to (5.7). Hence it is enough to assume that $\sigma - k/2 \geq (\log T)^{-1}$. Therefore,

$$N_f(\sigma, T, T+H) \leq (\log T) \int_{\sigma-1/\log T}^{\sigma} \{N_f(\sigma', T+H) - N_f(\sigma', T)\}\, d\sigma' \tag{5.8}$$
$$\ll H\left(\frac{H}{T^{B'}}\right)^{-\frac{B}{1-B'}(\sigma-k/2)} \log T$$

from (5.7) and this proves Theorem 2.

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