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New identities for the Rogers–Ramanujan functions

by

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1. Introduction. The Rogers–Ramanujan functions are defined for $|q| < 1$ by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}, \qquad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}, \tag{1.1}$$

where $(a;q)_0 := 1$ and, for $n \geq 1$,

$$(a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k).$$

These functions satisfy the famous Rogers–Ramanujan identities [11], [13], [9, pp. 214–215]

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \qquad H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.2}$$

where

$$(a;q)_\infty := \lim_{n\to\infty} (a;q)_n, \qquad |q| < 1.$$

At the end of his brief communication [8], [9, p. 231] announcing his proofs of the Rogers–Ramanujan identities (1.2), Ramanujan remarks, "I have now found an algebraic relation between $G(q)$ and $H(q)$, viz.,

$$H(q)\{G(q)\}^{11} - q^2 G(q)\{H(q)\}^{11} = 1 + 11q\{G(q)H(q)\}^6. \tag{1.3}$$

Another noteworthy formula is

$$H(q)G(q^{11}) - q^2 G(q) H(q^{11}) = 1. \tag{1.4}$$

Each of these formulae is the simplest of a large class." Ramanujan did not indicate how he had proved the identities. They are two identities from a set of forty identities for $G(q)$ and $H(q)$ that he never published.

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In the years since then, most of the 40 identities were established in a series of papers by L. J. Rogers [12] in 1921, G. N. Watson [14] in 1934, D. Bressoud [6], [7] in 1977, and A. J. F. Biagioli [3] in 1989. It should be remarked that the complete list of identities was not brought to the mathematical public until 1975 when B. J. Birch [4] found them in the Oxford University Library. Although all the identities can be proved using the theory of modular forms, the method employed by Biagioli, it is more instructive to find proofs that Ramanujan might have found. Outside of the theory of modular forms, Rogers's method, which was generalized by Bressoud, is the only general method that has been devised to prove the identities.

The present authors, along with G. Choi, Y.-S. Choi, H. Hahn, B. P. Yeap, A. J. Yee, and J. Yi [2], have found new proofs in the spirit of Ramanujan for many of the identities. In their *Memoir* [2], in addition to offering new proofs, the authors relate some of the proofs of Rogers, Watson, and Bressoud that they think are also in the spirit of Ramanujan's ideas.

In this paper, we establish new representations for G and H as linear combinations of G and H at different arguments, with theta functions appearing in the coefficients. These linear combinations are used in conjunction with some of the previously proved forty identities to prove new identities for the Rogers–Ramanujan functions. The advantage of our method is that the identities to be proved do not need to be known in advance, in contrast to most methods by previous authors, in particular, those methods utilizing the theory of modular forms. Our new identities for G and H yield new, elegant modular equations, or theta function identities.

The principal results of this paper are the following four theorems, which are proved in the last four sections. Set $\chi(q)=(-q;q^2)_\infty$ and $f(-q)=(q;q)_\infty$.

THEOREM 1.1. *Let*

$$B(q) := G(q^{12})H(-q^7) + qG(-q^7)H(q^{12}), \tag{1.5}$$

$$C(q) := G(q)G(q^{84}) + q^{17}H(q)H(q^{84}), \tag{1.6}$$

$$V(q) := H(-q)G(q^{21}) + q^4G(-q)H(q^{21}), \tag{1.7}$$

$$W(q) := G(q^4)G(q^{21}) + q^5H(q^4)H(q^{21}), \tag{1.8}$$

$$Z(q) := H(q^3)G(q^{28}) - q^5G(q^3)H(q^{28}), \tag{1.9}$$

$$Y(q) := G(q^3)G(-q^7) - q^2H(q^3)H(-q^7). \tag{1.10}$$

Then

$$\frac{C(q^2)}{Y(-q^2)} = \frac{V(-q^2)}{B(-q^2)} = \frac{C(q)}{B(q)} = \frac{f(-q^{12})f(-q^{14})}{f(-q^2)f(-q^{84})}, \tag{1.11}$$

$$\frac{Z(-q)}{W(q)} = \frac{Z(q)}{W(-q)} = \frac{Y(q^2)}{W(q^2)} = \frac{Z(q^2)}{V(q^2)} = \frac{f(-q^4)f(-q^{42})}{f(-q^6)f(-q^{28})}. \tag{1.12}$$

THEOREM 1.2. *We have*

$$\begin{aligned}(1.13)\quad &\frac{G(q)G(-q^{14})-q^3H(q)H(-q^{14})}{G(q^7)H(-q^2)+qH(q^7)G(-q^2)}\\ &=\frac{G(q^{56})H(q)-q^{11}H(q^{56})G(q)}{G(q^7)G(q^8)+q^3H(q^7)H(q^8)}\\ &=\frac{\chi(-q^{14})}{\chi(-q^2)}=\frac{G(q)G(q^{14})+q^3H(q)H(q^{14})}{G(-q^7)H(q^2)+qH(-q^7)G(q^2)}.\end{aligned}$$

THEOREM 1.3. *We have*

$$\begin{aligned}(1.14)\quad &\frac{G(-q^2)G(q^{38})-q^8H(-q^2)H(q^{38})}{G(q^{152})H(q^2)-q^{30}G(q^2)H(q^{152})}\\ &=\frac{G(q^{38})H(q^8)-q^6G(q^8)H(q^{38})}{G(q^2)G(-q^{38})-q^8H(q^2)H(-q^{38})}\\ &=\frac{G(q^{19})H(q^4)-q^3G(q^4)H(q^{19})}{G(q^{76})H(-q)+q^{15}G(-q)H(q^{76})}=\frac{\chi(-q^2)}{\chi(-q^{38})}\end{aligned}$$

and

$$\begin{aligned}(1.15)\quad &\{G(q)G(-q^{19})-q^4H(q)H(-q^{19})\}\{G(-q)G(q^{19})-q^4H(-q)H(q^{19})\}\\ &=\{G(q^{19})H(q^4)-q^3H(q^{19})G(q^4)\}\{G(q^{76})H(q)-q^{15}H(q^{76})G(q)\}\\ &=G(q^2)G(q^{38})+q^8H(q^2)H(q^{38}).\end{aligned}$$

THEOREM 1.4. *Let*

$$(1.16)\qquad A(q):=H(q)G(q^{81})-q^{16}G(q)H(q^{81}).$$

Then

$$(1.17)\qquad 2A(q^4)=\chi(q)\chi(-q^3)\chi(q^{27})\chi(-q^{81})+\chi(-q)\chi(q^3)\chi(-q^{27})\chi(q^{81}).$$

2. Definitions and preliminary results. We first recall Ramanujan's definitions for a general theta function and some of its important special cases. Set

$$(2.1)\qquad f(a,b):=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2},\qquad |ab|<1.$$

Basic properties of $f(a,b)$ include [1, p. 34, Entry 18]

$$(2.2)\qquad f(a,b)=f(b,a),$$

$$(2.3)\qquad f(1,a)=2f(a,a^3),$$

$$(2.4)\qquad f(-1,a)=0,$$

and, if n is an integer,

$$(2.5)\qquad f(a,b)=a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n,b(ab)^{-n}).$$

The basic property (2.2) will be used many times without comment. The function $f(a,b)$ satisfies the well known Jacobi triple product identity [1, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.6}$$

The three most important special cases of (2.1) are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{2.7}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.8}$$

and

$$\begin{aligned} f(-q) := f(-q,-q^2) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \\ &= (q;q)_\infty =: q^{-1/24}\eta(\tau), \end{aligned} \tag{2.9}$$

where $q = \exp(2\pi i\tau)$, $\operatorname{Im}\tau > 0$, and η denotes the Dedekind eta-function. The product representations in (2.7)–(2.9) are special cases of (2.6). Also, after Ramanujan, define

$$\chi(q) := (-q;q^2)_\infty. \tag{2.10}$$

Using (2.6) and (2.9), we can rewrite the Rogers–Ramanujan identities (1.2) in the forms

$$G(q) = \frac{f(-q^2,-q^3)}{f(-q)}, \qquad H(q) = \frac{f(-q,-q^4)}{f(-q)}. \tag{2.11}$$

We shall use (2.11) many times in the remainder of the paper. The odd-even dissections of G and H were given by Watson [14], namely,

$$G(q) = \frac{f(-q^8)}{f(-q^2)}\,(G(q^{16}) + qH(-q^4)), \tag{2.12}$$

$$H(q) = \frac{f(-q^8)}{f(-q^2)}\,(q^3H(q^{16}) + G(-q^4)). \tag{2.13}$$

Basic properties of the functions (2.7)–(2.10) include [1, pp. 39–40, Entries 24, 25(iii)]

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \sqrt[3]{\frac{\varphi(q)}{\psi(-q)}} = \frac{\varphi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)}, \tag{2.14}$$

$$f^3(-q^2) = \varphi(-q)\psi^2(q), \qquad \chi(q)\chi(-q) = \chi(-q^2). \tag{2.15}$$

It is easy to conclude from (2.14) or (2.6) that

$$\psi(-q)=\chi(-q)f(-q^4)=\frac{f(-q)}{\chi(-q^2)},\qquad \chi(q)f(-q)=\varphi(-q^2). \tag{2.16}$$

The function $f(a,b)$ also satisfies a useful addition formula. For each nonnegative integer n, let

$$U_n:=a^{n(n+1)/2}b^{n(n-1)/2},\qquad V_n:=a^{n(n-1)/2}b^{n(n+1)/2}.$$

Then [1, p. 48, Entry 31]

$$f(U_1,V_1)=\sum_{r=0}^{n-1}U_r f\left(\frac{U_{n+r}}{U_r},\frac{V_{n-r}}{U_r}\right). \tag{2.17}$$

We also use a formula of R. Blecksmith, J. Brillhart, and I. Gerst [5] providing a representation for a product of two theta functions as a sum of m products of pairs of theta functions, under certain conditions. This formula generalizes formulas of H. Schröter [1, pp. 65–72], which have been enormously useful in establishing many of Ramanujan's modular equations. The formulation that we give can be found in [1, p. 73]. We first define, for $\varepsilon\in\{0,1\}$ and $|ab|<1$,

$$f_\varepsilon(a,b)=\sum_{n=-\infty}^{\infty}(-1)^{\varepsilon n}(ab)^{n^2/2}(a/b)^{n/2}.$$

THEOREM 2.1. *Let a,b,c,d denote positive numbers with $|ab|,|cd|<1$. Suppose that there exist positive integers α,β, and m such that*

$$(ab)^\beta=(cd)^{\alpha(m-\alpha\beta)}. \tag{2.18}$$

Let $\varepsilon_1,\varepsilon_2\in\{0,1\}$, and define $\delta_1,\delta_2\in\{0,1\}$ by

$$\delta_1\equiv\varepsilon_1-\alpha\varepsilon_2 \pmod 2,\qquad \delta_2\equiv\beta\varepsilon_1+p\varepsilon_2 \pmod 2, \tag{2.19}$$

respectively, where $p=m-\alpha\beta$. Then, if R denotes any complete residue system modulo m,

$$\begin{aligned}&f_{\varepsilon_1}(a,b)f_{\varepsilon_2}(c,d)\\&=\sum_{r\in R}(-1)^{\varepsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2}f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^\alpha},\frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^\alpha}\right)\\&\quad\times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^p},\frac{(a/b)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^p}\right).\end{aligned} \tag{2.20}$$

Next, we record some of Ramanujan's identities that we employ in our proofs.

ENTRY 2.2.

$$G(q^2)G(q^3)+qH(q^2)H(q^3)=\frac{\chi(-q^3)}{\chi(-q)}. \tag{2.21}$$

Entry 2.3.

$$G(q^7)H(q^2) - qG(q^2)H(q^7) = \frac{\chi(-q)}{\chi(-q^7)}. \tag{2.22}$$

Entry 2.4.

$$G(q)G(q^{14}) + q^3H(q)H(q^{14}) = \frac{\chi(-q^7)}{\chi(-q)}. \tag{2.23}$$

Entry 2.5.

$$G(q^8)H(q^3) - qG(q^3)H(q^8) = \frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}. \tag{2.24}$$

Entry 2.6.

$$G(q)H(-q) + G(-q)H(q) = \frac{2}{\chi^2(-q^2)} = \frac{2\psi(q^2)}{f(-q^2)}. \tag{2.25}$$

Entry 2.7.

$$G(q^{36})H(q) - q^7G(q)H(q^{36}) = \frac{\chi(-q^6)\chi(-q^9)}{\chi(-q^2)\chi(-q^3)} = \frac{\chi(q^3)\chi(-q^9)}{\chi(-q^2)}. \tag{2.26}$$

Entry 2.8.

$$G(-q^2)G(-q^3) + qH(-q^2)H(-q^3) = \frac{\chi(q)\chi(q^6)}{\chi(q^2)\chi(q^3)}. \tag{2.27}$$

Entry 2.9.

$$G(q^9)H(q^4) - qG(q^4)H(q^9) = \frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})} = \frac{\chi(-q)\chi(q^3)}{\chi(-q^{18})}. \tag{2.28}$$

Entry 2.10.

$$G(q^3)G(q^7) + q^2H(q^3)H(q^7) = G(q^{21})H(q) - q^4G(q)H(q^{21}). \tag{2.29}$$

Entry 2.11.

$$\frac{G(q^{19})H(q^4) - q^3G(q^4)H(q^{19})}{G(q^{76})H(-q) + q^{15}G(-q)H(q^{76})} = \frac{\chi(-q^2)}{\chi(-q^{38})}. \tag{2.30}$$

Entry 2.12.

$$\begin{aligned} &\{G(q)G(-q^{19}) - q^4H(q)H(-q^{19})\} \\ &\qquad \times \{G(-q)G(q^{19}) - q^4H(-q)H(q^{19})\} \\ &\qquad = G(q^2)G(q^{38}) + q^8H(q^2)H(q^{38}). \end{aligned} \tag{2.31}$$

3. Linear relations for $G(q)$ and $H(q)$

LEMMA 3.1. *With* χ *defined by* (2.10),

$$\chi(-q)\chi(q^3)G(q) = \frac{\chi(q^6)}{\chi(-q^4)}\,G(-q^6) - q^5\,\frac{\chi(q^2)}{\chi(-q^{12})}\,H(q^{24}), \tag{3.1}$$

$$\chi(-q)\chi(q^3)H(q) = -q\,\frac{\chi(q^6)}{\chi(-q^4)}\,H(-q^6) + \frac{\chi(q^2)}{\chi(-q^{12})}\,G(q^{24}). \tag{3.2}$$

Proof. First, we need the even-odd dissection of $\chi(-q^3)/\chi(-q)$. By (2.6), (2.8), and (2.16),

$$\begin{aligned} f(-q,-q^5) &= (q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty = \frac{(q;q^2)_\infty}{(q^3;q^6)_\infty}\,(q^6;q^6)_\infty \\ &= \chi(-q)\psi(q^3) = \chi(-q)\chi(q^3)f(-q^{12}). \end{aligned} \tag{3.3}$$

Employing (2.17) with $a = q$ and $b = q^5$, we also have

$$f(q,q^5) = f(q^8,q^{16}) + qf(q^4,q^{20}). \tag{3.4}$$

It is moreover easily verified that (see [1, p. 350, (2.3)])

$$f(q,q^2) = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{3.5}$$

Therefore, by (3.3)–(3.5), we find that

$$\begin{aligned} \chi(q)\chi(-q^3) &= \frac{f(q,q^5)}{f(-q^{12})} = \frac{f(q^8,q^{16})}{f(-q^{12})} + q\,\frac{f(q^4,q^{20})}{f(-q^{12})} \\ &= \frac{\varphi(-q^{24})}{\chi(-q^8)f(-q^{12})} + q\,\frac{\chi(q^4)\chi(-q^{12})f(-q^{48})}{f(-q^{12})}. \end{aligned} \tag{3.6}$$

Next, by several applications of (2.14), we deduce from (3.6) that

$$\chi(q)\chi(-q^3) = \frac{\chi(q^{12})}{\chi(-q^8)} + q\,\frac{\chi(q^4)}{\chi(-q^{24})}. \tag{3.7}$$

We are now ready to prove Lemma 3.1. By two applications of Entry 2.2, the second with q replaced by $-q$, and by Entry 2.6 with q replaced by q^3,

$$\begin{aligned} &\frac{\chi(-q^3)}{\chi(-q)}\,G(-q^3) - \frac{\chi(q^3)}{\chi(q)}\,G(q^3) \\ &\quad = (G(q^2)G(q^3) + qH(q^2)H(q^3))G(-q^3) \\ &\qquad - (G(q^2)G(-q^3) - qH(q^2)H(-q^3))G(q^3) \\ &\quad = qH(q^2)\{H(q^3)G(-q^3) + H(-q^3)G(q^3)\} = 2q\,\frac{H(q^2)}{\chi^2(-q^6)}, \end{aligned} \tag{3.8}$$

which, by (2.15), simplifies to

$$\chi(q)\chi(-q^3)G(-q^3) - \chi(-q)\chi(q^3)G(q^3) = 2q\,\frac{\chi(-q^2)}{\chi^2(-q^6)}\,H(q^2). \tag{3.9}$$

By employing (2.12) with q replaced by $-q^3$ and q^3, respectively, in (3.9), we find that

$$\begin{aligned} L(q) := {}& \chi(q)\chi(-q^3)\{G(q^{48}) - q^3H(-q^{12})\} \\ &- \chi(-q)\chi(q^3)\{G(q^{48}) + q^3H(-q^{12})\} \\ ={}& 2q\,\frac{f(-q^6)\chi(-q^2)}{f(-q^{24})\chi^2(-q^6)}\,H(q^2) = 2q\chi(-q^2)\chi(q^6)H(q^2), \end{aligned} \tag{3.10}$$

by (2.14). Collecting terms on the left side of (3.10) and using (3.7), we find that

$$\begin{aligned} L(q) ={}& \{\chi(q)\chi(-q^3) - \chi(-q)\chi(q^3)\}G(q^{48}) \\ &- q^3\{\chi(q)\chi(-q^3) + \chi(-q)\chi(q^3)\}H(-q^{12}) \\ ={}& 2q\,\frac{\chi(q^4)}{\chi(-q^{24})}\,G(q^{48}) - 2q^3\,\frac{\chi(q^{12})}{\chi(-q^8)}\,H(-q^{12}). \end{aligned} \tag{3.11}$$

Hence, by (3.10) and (3.11),

$$2q\,\frac{\chi(q^4)}{\chi(-q^{24})}\,G(q^{48}) - 2q^3\,\frac{\chi(q^{12})}{\chi(-q^8)}\,H(-q^{12}) = 2q\chi(-q^2)\chi(q^6)H(q^2).$$

Dividing both sides by $2q$ and then replacing q^2 by q, we deduce (3.2). The companion equality (3.1) is proved in a similar way, and so we omit the details. ∎

Lemma 3.2. *We have*

$$\chi(q)\chi(-q^3)G(q^9) - \chi(-q)\chi(q^3)G(-q^9) = 2q\,\frac{G(q^4)}{\chi(-q^{18})}, \tag{3.12}$$

$$\chi(q)\chi(-q^3)H(q^9) + \chi(-q)\chi(q^3)H(-q^9) = 2\,\frac{H(q^4)}{\chi(-q^{18})}. \tag{3.13}$$

Proof. The proofs of (3.12) and (3.13) are very similar to the proof of (3.9), except that Entry 2.9 is used instead of Entry 2.2. We only prove (3.13), since the proof of (3.12) follows along the same lines.

By two applications of Entry 2.9 and one application of Entry 2.6 with q replaced by q^9,

$$\begin{aligned} &\frac{\chi(q)\chi(-q^3)}{\chi(-q^{18})}\,H(q^9) + \frac{\chi(-q)\chi(q^3)}{\chi(-q^{18})}\,H(-q^9) \\ &\qquad = \{G(-q^9)H(q^4) + qG(q^4)H(-q^9)\}H(q^9) \end{aligned}$$

$$+\{G(q^9)H(q^4)-qG(q^4)H(q^9)\}H(-q^9)$$
$$=H(q^4)\{G(-q^9)H(q^9)+G(q^9)H(-q^9)\}=2\,\frac{H(q^4)}{\chi^2(-q^{18})}.$$

Using (2.15), we complete the proof of (3.13). ∎

4. Proof of Theorem 1.1. Using (3.1) and (3.2) in Entry 2.4, we find that

$$\begin{aligned}(4.1)\quad &\left\{\frac{\chi(q^6)}{\chi(-q^4)}\,G(-q^6)-q^5\frac{\chi(q^2)}{\chi(-q^{12})}\,H(q^{24})\right\}G(q^{14})\\ &\quad+q^3\left\{-q\,\frac{\chi(q^6)}{\chi(-q^4)}\,H(-q^6)+\frac{\chi(q^2)}{\chi(-q^{12})}\,G(q^{24})\right\}H(q^{14})\\ &=\frac{\chi(-q^7)}{\chi(-q)}\,\chi(-q)\,\chi(q^3)=\chi(q^3)\chi(-q^7).\end{aligned}$$

Collecting terms and equating odd parts on both sides of (4.1), we conclude that

$$\begin{aligned}(4.2)\quad &2q^3\{G(q^{24})H(q^{14})-q^2G(q^{14})H(q^{24})\}\\ &=\frac{\chi(-q^{12})}{\chi(q^2)}\,\{\chi(q^3)\chi(-q^7)-\chi(-q^3)\chi(q^7)\}.\end{aligned}$$

At the end of this section, we will prove that

$$\begin{aligned}(4.3)\quad &\chi(q^3)\chi(-q^7)-\chi(-q^3)\chi(q^7)\\ &=2q^3\,\frac{f(q^2)f(-q^{168})}{f(-q^{12})f(-q^{28})}\,\{G(-q^2)G(q^{168})-q^{34}H(-q^2)H(q^{168})\}.\end{aligned}$$

Assuming (4.3) for the time being, we conclude from (4.2) and (4.3) that

$$\begin{aligned}(4.4)\quad \frac{G(q^{24})H(q^{14})-q^2G(q^{14})H(q^{24})}{G(-q^2)G(q^{168})-q^{34}H(-q^2)H(q^{168})}&=\frac{\chi(-q^{12})}{\chi(q^2)}\,\frac{f(q^2)f(-q^{168})}{f(-q^{12})f(-q^{28})}\\ &=\frac{f(-q^4)f(-q^{168})}{f(-q^{24})f(-q^{28})},\end{aligned}$$

by (2.14).

Next, we show that (4.4) yields two more identities of the same type. By (2.12) and (2.13),

$$\begin{aligned}(4.5)\quad &\frac{f(-q^2)}{f(-q^8)}\,\{G(q)G(q^{84})+q^{17}H(q)H(q^{84})\}\\ &=\{G(q^{16})+qH(-q^4)\}G(q^{84})+q^{17}\{G(-q^4)+q^3H(q^{16})\}H(q^{84})\\ &=G(q^{16})G(q^{84})+q^{20}H(q^{16})H(q^{84})\\ &\quad+q\{H(-q^4)G(q^{84})+q^{16}G(-q^4)H(q^{84})\}.\end{aligned}$$

Arguing in the same way, we deduce that

$$\begin{aligned}(4.6)\quad \frac{f(-q^{14})}{f(-q^{56})}\{H(-q^7)G(q^{12})+qG(-q^7)H(q^{12})\} \\ = G(q^{12})G(-q^{28})-q^8H(q^{12})H(-q^{28}) \\ +q\{H(q^{12})G(q^{112})-q^{20}G(q^{12})H(q^{112})\}.\end{aligned}$$

By (1.7)–(1.10) and (4.4)–(4.6), we deduce that

$$\begin{aligned}(4.7)\quad \frac{Y(q^4)+qZ(q^4)}{W(q^4)+qV(q^4)} &= \frac{f(-q^2)f(-q^{84})}{f(-q^{12})f(-q^{14})}\,\frac{f(-q^8)f(-q^{14})}{f(-q^2)f(-q^{56})} \\ &= \frac{f(-q^8)f(-q^{84})}{f(-q^{12})f(-q^{56})}.\end{aligned}$$

Hence, we deduce that

$$(4.8)\quad \frac{Y(q)}{W(q)}=\frac{Z(q)}{V(q)}=\frac{f(-q^2)f(-q^{21})}{f(-q^3)f(-q^{14})}.$$

As promised above, we now verify (4.3). For convenience, let us define, by (2.11),

$$(4.9)\quad \begin{aligned} g(q)&=f(-q^2,-q^3)=f(-q)G(q), \\ h(q)&=f(-q,-q^4)=f(-q)H(q).\end{aligned}$$

By (2.16) and (4.9), (4.3) is clearly equivalent to

$$\begin{aligned}(4.10)\quad \psi(q^3)\psi(-q^7)-\psi(-q^3)\psi(q^7) \\ =2q^3\{g(-q^2)g(q^{168})-q^{34}h(-q^2)h(q^{168})\},\end{aligned}$$

which we now prove.

We employ Theorem 2.1 with the set of parameters $a=q^{21}$, $b=q^{63}$, $c=q$, $d=q^3$, $\alpha=3$, $\beta=1$, $m=10$, $\varepsilon_1=1$, and $\varepsilon_2=0$ to deduce that

$$\begin{aligned}&\psi(-q^{21})\psi(q) \\ &\quad = f(-q^{42},-q^{78})f(-q^{112},-q^{168})+qf(-q^{30},-q^{90})f(-q^{140},-q^{140}) \\ &\qquad +q^6f(-q^{18},-q^{102})f(-q^{112},-q^{168})+q^{15}f(-q^6,-q^{114})f(-q^{84},-q^{196}) \\ &\qquad -q^{22}f(-q^6,-q^{114})f(-q^{56},-q^{224})-q^{27}f(-q^{18},-q^{102})f(-q^{28},-q^{252}) \\ &\qquad +q^{21}f(-q^{42},-q^{78})f(-q^{28},-q^{252})+q^{10}f(-q^{54},-q^{66})f(-q^{56},-q^{224}) \\ &\qquad +q^3f(-q^{54},-q^{66})f(-q^{84},-q^{196}).\end{aligned}$$

Replacing q by $-q$ and adding the resulting equality to that above, we find that

(4.11) $$\psi(q)\psi(-q^{21})+\psi(-q)\psi(q^{21}) = 2f(-q^{112},-q^{168})\{f(-q^{42},-q^{78})+q^6f(-q^{18},-q^{102})\} + 2q^{10}f(-q^{56},-q^{224})\{f(-q^{54},-q^{66})-q^{12}f(-q^6,-q^{114})\}.$$

But, by (4.9) and (2.17), with $n=2$ and $a=-q^2$, $b=q^3$ and $a=q$, $b=-q^4$, respectively, we have

$$g(-q)=f(-q^2,q^3)=f(-q^9,-q^{11})-q^2f(-q,-q^{19}),$$
$$h(-q)=f(q,-q^4)=f(-q^7,-q^{13})+qf(-q^3,-q^{17}).$$

Return to (4.11) and substitute each of the equalities above with q replaced by q^6 to deduce that

(4.12) $$\psi(q)\psi(-q^{21})+\psi(-q)\psi(q^{21})=2\{g(q^{56})h(-q^6)+q^{10}h(q^{56})g(-q^6)\}.$$

In what follows, $J(q)$ will denote an arbitrary power series, usually not the same at each appearance. By (2.17) with $n=3$ in each instance,

(4.13) $$g(q)=f(-q^2,-q^3) = f(-q^{21},-q^{24})-q^2f(-q^9,-q^{36})-q^3f(-q^6,-q^{39}) = J(q^3)-q^2h(q^9),$$

(4.14) $$h(q)=f(-q,-q^4) = f(-q^{18},-q^{27})-qf(-q^{12},-q^{33})-q^4f(-q^3,-q^{42}) = g(q^9)-qJ(q^3),$$

(4.15) $$\psi(q)=f(q^3,q^6)+q\psi(q^9) = J(q^3)+q\psi(q^9),$$

where, in the last application of (2.17), we set $a=1$ and $b=q$ and used (2.3) and (2.8). By (4.15),

(4.16) $$\psi(q)\psi(-q^{21})+\psi(-q)\psi(q^{21}) = \{J(q^3)+q\psi(q^9)\}\psi(-q^{21})+\{J(q^3)-q\psi(-q^9)\}\psi(q^{21}) = J(q^3)+q\{\psi(q^9)\psi(-q^{21})-\psi(-q^9)\psi(q^{21})\}.$$

Similarly, by (4.13) and (4.14) with q replaced by q^{56}, we find that

(4.17) $$2\{g(q^{56})h(-q^6)+q^{10}h(q^{56})g(-q^6)\} = J(q^3)+2q^{10}\{g(-q^6)g(q^{504})-q^{102}h(-q^6)h(q^{504})\}.$$

From these last two equalities and (4.12), we conclude that

(4.18) $$\psi(q^9)\psi(-q^{21})-\psi(-q^9)\psi(q^{21}) = 2q^9\{g(-q^6)g(q^{504})-q^{102}h(-q^6)h(q^{504})\},$$

which is (4.10) with q replaced by q^3.

By Theorem 2.1, we can also verify that

$$\psi(q^3)\psi(-q^7)+\psi(-q^3)\psi(q^7)=2\{g(q^8)g(-q^{42})-q^{10}h(q^8)h(-q^{42})\}. \tag{4.19}$$

Considering the 3-dissection of both sides of (4.19), one similarly obtains

$$\begin{aligned}\psi(q)\psi(-q^{21})-\psi(-q)\psi(q^{21})\\ =2q\{h(-q^{14})g(q^{24})+q^2g(-q^{14})h(q^{24})\}.\end{aligned} \tag{4.20}$$

Further applications of Theorem 2.1 give the identities

$$\varphi(q)\varphi(-q^{21})-\varphi(-q)\varphi(q^{21})=4q\{g(q^{12})g(q^{28})+q^8h(q^{12})h(q^{28})\}, \tag{4.21}$$

$$\varphi(q^3)\varphi(-q^7)-\varphi(-q^3)\varphi(q^7)=4q^3\{g(q^{84})h(q^4)-q^{16}h(q^{84})g(q^4)\}. \tag{4.22}$$

These two identities similarly imply each other, and so they are not independent. However, combining (2.29), (4.21) and (4.22), we see that (2.29) is equivalent to the following identity, which we verify in (4.49):

$$\frac{\varphi(q^3)\varphi(-q^7)-\varphi(-q^3)\varphi(q^7)}{\varphi(q)\varphi(-q^{21})-\varphi(-q)\varphi(q^{21})}=q^2\,\frac{f(-q^4)f(-q^{84})}{f(-q^{12})f(-q^{28})}. \tag{4.23}$$

Starting from (2.23), and arguing as in (3.8), we find that

$$\frac{\chi(-q^7)}{\chi(-q)}\,G(-q)-\frac{\chi(q^7)}{\chi(q)}\,G(q)=2q^3\,\frac{H(q^{14})}{\chi^2(-q^2)}. \tag{4.24}$$

By (2.15), we see that (4.24) simplifies to

$$\chi(q)\chi(-q^7)G(-q)-\chi(-q)\chi(q^7)G(q)=2q^3\,\frac{H(q^{14})}{\chi(-q^2)}. \tag{4.25}$$

Similarly, we can find that

$$\chi(q)\chi(-q^7)H(-q)+\chi(-q)\chi(q^7)H(q)=2\,\frac{G(q^{14})}{\chi(-q^2)}. \tag{4.26}$$

In (1.8), we replace q by q^2 and employ (4.25) and (4.26) with q replaced by q^3 to find that

$$\begin{aligned}&2\,\frac{W(q^2)}{\chi(-q^6)}\\ &=G(q^8)\{\chi(q^3)\chi(-q^{21})H(-q^3)+\chi(-q^3)\chi(q^{21})H(q^3)\}\\ &\quad+qH(q^8)\{\chi(q^3)\chi(-q^{21})G(-q^3)-\chi(-q^3)\chi(q^{21})G(q^3)\}\\ &=\chi(q^3)\chi(-q^{21})\{H(-q^3)G(q^8)+qG(-q^3)H(q^8)\}\\ &\quad+\chi(-q^3)\chi(q^{21})\{H(q^3)G(q^8)-qG(q^3)H(q^8)\}\\ &=\chi(q^3)\chi(-q^{21})\,\frac{\chi(q)\chi(-q^4)}{\chi(q^3)\chi(-q^{12})}+\chi(-q^3)\chi(q^{21})\,\frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})},\end{aligned} \tag{4.27}$$

where in the last step we used (2.24) twice, once with q replaced by $-q$. By (2.15), we conclude from (4.27) that

$$2W(q^2) = \frac{\chi(-q^4)}{\chi(q^6)} \{\chi(q)\chi(-q^{21}) + \chi(-q)\chi(q^{21})\}. \tag{4.28}$$

Similarly, in (1.7), we replace q by q^2 and employ (4.25) and (4.26) with q replaced by q^3, and arguing as in (4.27), we find that

$$\begin{aligned} 2q\,\frac{V(q^2)}{\chi(-q^6)} &= \chi(q^3)\chi(-q^{21})\{G(-q^2)G(-q^3) + qH(-q^2)H(-q^3)\} \\ &\quad - \chi(-q^3)\chi(q^{21})\{G(-q^2)G(q^3) - qH(-q^2)H(q^3)\}. \end{aligned} \tag{4.29}$$

Using (2.27) twice, once with q replaced by $-q$, we find from (4.29) that

$$2q\,\frac{V(q^2)}{\chi(-q^6)} = \chi(q^3)\chi(-q^{21})\,\frac{\chi(q)\chi(q^6)}{\chi(q^2)\chi(q^3)} - \chi(-q^3)\chi(q^{21})\,\frac{\chi(-q)\chi(q^6)}{\chi(q^2)\chi(-q^3)}, \tag{4.30}$$

which, by (2.15), implies that

$$2qV(q^2) = \frac{\chi(-q^{12})}{\chi(q^2)} \{\chi(q)\chi(-q^{21}) - \chi(-q)\chi(q^{21})\}. \tag{4.31}$$

Starting from (2.22), and arguing as in (3.8), we find that

$$\frac{\chi(q)}{\chi(q^7)}\,G(q^7) - \frac{\chi(-q)}{\chi(-q^7)}\,G(-q^7) = 2q\,\frac{G(q^2)}{\chi^2(-q^{14})}. \tag{4.32}$$

By (2.15), we see that (4.32) simplifies to

$$\chi(q)\chi(-q^7)G(q^7) - \chi(-q)\chi(q^7)G(-q^7) = 2q\,\frac{G(q^2)}{\chi(-q^{14})}. \tag{4.33}$$

Similarly, we can find that

$$\chi(q)\chi(-q^7)H(q^7) + \chi(-q)\chi(q^7)H(-q^7) = 2\,\frac{H(q^2)}{\chi(-q^{14})}. \tag{4.34}$$

In (1.10), we replace q by q^2 and employ (4.33) and (4.34) with q replaced by q^3 to find that

$$\begin{aligned} 2q^3\,\frac{Y(q^2)}{\chi(-q^{42})} &= G(-q^{14})\{\chi(q^3)\chi(-q^{21})G(q^{21}) - \chi(-q^3)\chi(q^{21})G(-q^{21})\} \\ &\quad - q^7H(-q^{14})\{\chi(q^3)\chi(-q^{21})H(q^{21}) + \chi(-q^3)\chi(q^{21})H(-q^{21})\} \\ &= \chi(q^3)\chi(-q^{21})\{G(-q^{14})G(q^{21}) - q^7H(-q^{14})H(q^{21})\} \\ &\quad - \chi(-q^3)\chi(q^{21})\{G(-q^{14})G(-q^{21}) + q^7H(-q^{14})H(-q^{21})\} \\ &= \chi(q^3)\chi(-q^{21})\,\frac{\chi(-q^7)\chi(q^{42})}{\chi(q^{14})\chi(-q^{21})} - \chi(-q^3)\chi(q^{21})\,\frac{\chi(q^7)\chi(q^{42})}{\chi(q^{14})\chi(q^{21})}, \end{aligned} \tag{4.35}$$

where in the last step we used (2.27) twice, with q replaced by q^7 and $-q^7$. By (2.15), we conclude from (4.35) that

$$2q^3Y(q^2) = \frac{\chi(-q^{84})}{\chi(q^{14})}\{\chi(q^3)\chi(-q^7) - \chi(-q^3)\chi(q^7)\}. \tag{4.36}$$

Similarly, in (1.9), we replace q by q^2 and employ (4.33) and (4.34) with q replaced by q^3, and arguing as in (4.35), we find that

$$\begin{aligned}2\frac{Z(q^2)}{\chi(-q^{42})} &= \chi(q^3)\chi(-q^{21})\{H(q^{21})G(q^{56}) - q^7G(q^{21})H(q^{56})\}\\ &\quad + \chi(-q^3)\chi(q^{21})\{H(-q^{21})G(q^{56}) + q^7G(-q^{21})H(q^{56})\}.\end{aligned} \tag{4.37}$$

Using (2.24) twice, with q replaced by q^7 and $-q^7$, we find from (4.37) that

$$\begin{aligned}2\frac{Z(q^2)}{\chi(-q^{42})} &= \chi(q^3)\chi(-q^{21})\frac{\chi(-q^7)\chi(-q^{28})}{\chi(-q^{21})\chi(-q^{84})}\\ &\quad + \chi(-q^3)\chi(q^{21})\frac{\chi(q^7)\chi(-q^{28})}{\chi(q^{21})\chi(-q^{84})},\end{aligned} \tag{4.38}$$

which, by (2.15), implies that

$$2Z(q^2) = \frac{\chi(-q^{28})}{\chi(q^{42})}\{\chi(q^3)\chi(-q^7) + \chi(-q^3)\chi(q^7)\}. \tag{4.39}$$

Recall that $B(q)$ and $C(q)$ are defined by (1.5) and (1.6), respectively. By (4.4) with q^2 replaced by $-q$, we have

$$\frac{B(q)}{C(q)} = \frac{f(-q^2)f(-q^{84})}{f(-q^{12})f(-q^{14})}. \tag{4.40}$$

Using (2.16) and (4.9), we can easily express (4.3) and (4.20) in their equivalent forms

$$2q^3C(-q^2) = \frac{f(-q^{12})f(-q^{28})}{f(q^2)f(-q^{168})}\{\chi(q^3)\chi(-q^7) - \chi(-q^3)\chi(q^7)\}, \tag{4.41}$$

$$2qB(q^2) = \frac{f(-q^4)f(-q^{84})}{f(q^{14})f(-q^{24})}\{\chi(q)\chi(-q^{21}) - \chi(-q)\chi(q^{21})\}. \tag{4.42}$$

By (4.41), (4.42), (4.31), and (4.36), we conclude that

$$\frac{C(-q^2)}{Y(q^2)} = \frac{V(q^2)}{B(q^2)} = \frac{f(q^{14})f(-q^{12})}{f(q^2)f(-q^{84})}. \tag{4.43}$$

Hence, combining (4.43) with (4.40), we see that (1.11) is proved.

Recall that $W(q)$ and $Z(q)$ are defined by (1.8) and (1.9), respectively. Using (2.16) and (4.9), we have, by (4.19) and (4.12), respectively,

$$2W(-q^2) = \frac{f(-q^{12})f(-q^{28})}{f(-q^8)f(q^{42})}\{\chi(q^3)\chi(-q^7)+\chi(-q^3)\chi(q^7)\}, \tag{4.44}$$

$$2Z(-q^2) = \frac{f(-q^4)f(-q^{84})}{f(q^6)f(-q^{56})}\{\chi(q)\chi(-q^{21})+\chi(-q)\chi(q^{21})\}. \tag{4.45}$$

By (4.28), (4.39), (4.44), and (4.45), we find that

$$\frac{Z(-q^2)}{W(q^2)} = \frac{Z(q^2)}{W(-q^2)} = \frac{f(-q^8)f(-q^{84})}{f(-q^{12})f(-q^{56})}. \tag{4.46}$$

Hence, combining (4.8) and (4.46), we see that (1.12) is proved.

By (4.39), (4.36), (2.15), and the trivial identity,

$$\chi^2(q) = \frac{\varphi(q)}{f(-q^2)},$$

we find that

$$\begin{aligned} 4q^3Z(q^2)Y(q^2) &= \frac{\chi(-q^{28})\chi(-q^{84})}{\chi(q^{42})\chi(q^{14})}\{\chi^2(q^3)\chi^2(-q^7)-\chi^2(-q^3)\chi^2(q^7)\} \\ &= \frac{\chi(-q^{14})\chi(-q^{42})}{f(-q^6)f(-q^{14})}\{\varphi(q^3)\varphi(-q^7)-\varphi(-q^3)\varphi(q^7)\}. \end{aligned} \tag{4.47}$$

Similarly, by (4.31) and (4.28), we deduce that

$$\begin{aligned} 4qV(q^2)W(q^2) &= \frac{\chi(-q^4)\chi(-q^{12})}{\chi(q^2)\chi(q^6)}\{\chi^2(q)\chi^2(-q^{21})-\chi^2(-q)\chi^2(q^{21})\} \\ &= \frac{\chi(-q^2)\chi(-q^6)}{f(-q^2)f(-q^{42})}\{\varphi(q)\varphi(-q^{21})-\varphi(-q)\varphi(q^{21})\}. \end{aligned} \tag{4.48}$$

By (4.8), we conclude from (4.47), (4.48), and (2.14) that

$$\begin{aligned} &\frac{\varphi(q^3)\varphi(-q^7)-\varphi(-q^3)\varphi(q^7)}{\varphi(q)\varphi(-q^{21})-\varphi(-q)\varphi(q^{21})} \\ &\qquad = q^2\frac{f(-q^6)f(-q^{14})}{\chi(-q^{14})\chi(-q^{42})}\frac{\chi(-q^2)\chi(-q^6)}{f(-q^2)f(-q^{42})}\frac{f^2(-q^4)f^2(-q^{42})}{f^2(-q^6)f^2(-q^{28})} \\ &\qquad = q^2\frac{f(-q^4)f(-q^{84})}{f(-q^{12})f(-q^{28})}, \end{aligned} \tag{4.49}$$

which is (4.23).

Alternatively, we can derive two apparently new theta function identities which yield a factorization of (4.23). Namely,

$$\frac{\chi(q^3)\chi(-q^7)-\chi(-q^3)\chi(q^7)}{\chi(q)\chi(-q^{21})+\chi(-q)\chi(q^{21})}=q^3\,\frac{\varphi(-q^4)\psi(-q^{42})}{\varphi(-q^{12})\psi(-q^{14})},\tag{4.50}$$

$$q\,\frac{\chi(q^3)\chi(-q^7)+\chi(-q^3)\chi(q^7)}{\chi(q)\chi(-q^{21})-\chi(-q)\chi(q^{21})}=\frac{\varphi(-q^{84})\psi(-q^2)}{\varphi(-q^{28})\psi(-q^6)}.\tag{4.51}$$

To prove (4.50) and (4.51), we employ (4.8) with q replaced by q^2 and use the representations for $Z(q^2)$, $Y(q^2)$, $V(q^2)$, and $W(q^2)$ obtained in (4.39), (4.36), (4.31), and (4.28), respectively. Thus, by (4.36), (4.28), and (4.8) with q replaced by q^2,

$$\begin{aligned}\frac{\chi(q^3)\chi(-q^7)-\chi(-q^3)\chi(q^7)}{\chi(q)\chi(-q^{21})+\chi(-q)\chi(q^{21})}&\\ =q^3\,\frac{\chi(q^{14})\chi(-q^4)Y(q^2)}{\chi(-q^{84})\chi(q^6)W(q^2)}&=q^3\,\frac{\chi(q^{14})\chi(-q^4)f(-q^4)f(-q^{42})}{\chi(-q^{84})\chi(q^6)f(-q^6)f(-q^{28})}\\ =q^3\,\frac{f(-q^{42})}{\chi(-q^{84})}\,\frac{1}{\chi(q^6)f(-q^6)}\,f(-q^4)\chi(-q^4)\,\frac{\chi(q^{14})}{f(-q^{28})}&\\ =q^3\psi(-q^{42})\,\frac{1}{\varphi(-q^{12})}\,\varphi(-q^4)\,\frac{1}{\psi(-q^{14})},&\end{aligned}\tag{4.52}$$

where in the last step two applications of (2.16) and (2.14) are used. Therefore, (4.50) has been established. The identity (4.51) is proved in a similar way, and so we omit the details.

5. Proof of Theorem 1.2. Let $N(q)$ and $M(q)$ be defined by Entries 2.3 and 2.4, i.e.,

$$N(q):=G(q^7)H(q^2)-qG(q^2)H(q^7)=\frac{\chi(-q)}{\chi(-q^7)},\tag{5.1}$$

$$M(q):=G(q)G(q^{14})+q^3H(q)H(q^{14})=\frac{\chi(-q^7)}{\chi(-q)}.\tag{5.2}$$

Starting from (5.2) and employing (2.12) and (2.13) as in (4.5), we find that

$$\begin{aligned}\frac{\chi(-q^7)}{\chi(-q)}&=G(q)G(q^{14})+q^3H(q)H(q^{14})\\ &=\frac{f(-q^8)}{f(-q^2)}\{G(q^{14})G(q^{16})+q^6H(q^{14})H(q^{16})\\ &\qquad+q(G(q^{14})H(-q^4)+q^2H(q^{14})G(-q^4))\}.\end{aligned}\tag{5.3}$$

Upon equating the even parts in (5.3) and using (2.16) and (2.15), we find that

$$G(q^{14})G(q^{16}) + q^6 H(q^{14})H(q^{16}) \tag{5.4}$$
$$= \frac{1}{2}\frac{f(-q^2)}{f(-q^8)}\left\{\frac{\chi(q^7)}{\chi(q)} + \frac{\chi(-q^7)}{\chi(-q)}\right\} = \frac{1}{2}\chi(-q^4)\{\chi(q)\chi(-q^7) + \chi(-q)\chi(q^7)\}.$$

Similarly, equating odd parts in (5.3), we deduce that

$$G(q^{14})H(-q^4) + q^2 H(q^{14})G(-q^4) \tag{5.5}$$
$$= \frac{1}{2q}\chi(-q^4)\{\chi(q)\chi(-q^7) - \chi(-q)\chi(q^7)\}.$$

Analogously, starting from (5.1), using (2.12) and (2.13) with q replaced by q^7, and equating even and odd parts on both sides of the resulting equality, we can deduce that

$$G(q^{112})H(q^2) - q^{22}H(q^{112})G(q^2) \tag{5.6}$$
$$= \frac{1}{2}\chi(-q^{28})\{\chi(q)\chi(-q^7) + \chi(-q)\chi(q^7)\},$$

$$G(q^2)G(-q^{28}) - q^6 H(q^2)H(-q^{28}) \tag{5.7}$$
$$= \frac{1}{2q}\chi(-q^{28})\{\chi(q)\chi(-q^7) - \chi(-q)\chi(q^7)\}.$$

Hence, by (5.1), (5.2), (5.4)–(5.7) with q^2 replaced by q, and (2.15), we conclude that

$$\frac{G(q)G(-q^{14}) - q^3H(q)H(-q^{14})}{G(q^7)H(-q^2) + qH(q^7)G(-q^2)} = \frac{G(q^{56})H(q) - q^{11}H(q^{56})G(q)}{G(q^7)G(q^8) + q^3H(q^7)H(q^8)}$$
$$= \frac{\chi(-q^{14})}{\chi(-q^2)} = \frac{M(q)}{N(-q)},$$

and so the proof of Theorem 1.2 is complete.

6. Proof of Theorem 1.3. Observe that (1.14) together with Entry 2.12 implies (1.15), and so we prove (1.14). For simplicity, let us define

$$R(q) := G(q^{19})H(q^4) - q^3H(q^{19})G(q^4), \tag{6.1}$$

$$S(q) := G(q^{76})H(-q) + q^{15}H(q^{76})G(-q), \tag{6.2}$$

$$T(q) := G(q)G(-q^{19}) - q^4H(q)H(-q^{19}). \tag{6.3}$$

We can now restate Entry 2.11 in the form

$$\frac{R(q)}{S(q)} = \frac{\chi(-q^2)}{\chi(-q^{38})}. \tag{6.4}$$

In (6.1), we employ (2.12) and (2.13) with q replaced by q^{19} to find that

$$\begin{aligned}(6.5)\qquad \frac{f(-q^{38})}{f(-q^{152})}R(q) &= (G(q^{304})+q^{19}H(-q^{76}))H(q^4)\\ &\quad - q^3(q^{57}H(q^{304})+G(-q^{76}))G(q^4)\\ &= H(q^4)G(q^{304}) - q^{60}G(q^4)H(q^{304})\\ &\quad - q^3(G(q^4)G(-q^{76}) - q^{16}H(q^4)H(-q^{76}))\\ &= S(-q^4) - q^3T(q^4).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}(6.6)\qquad \frac{f(-q^2)}{f(-q^8)}S(q) &= (-q^3H(q^{16})+G(-q^4))G(q^{76})\\ &\quad + q^{15}(G(q^{16}) - qH(-q^4))H(q^{76})\\ &= T(-q^4) - q^3R(q^4).\end{aligned}$$

Combining (6.4), (6.5), and (6.6), we find that

$$\begin{aligned}(6.7)\qquad \frac{S(-q^4)-q^3T(q^4)}{T(-q^4)-q^3R(q^4)} &= \frac{f(-q^{38})f(-q^8)R(q)}{f(-q^{152})f(-q^2)S(q)}\\ &= \frac{f(-q^{38})f(-q^8)\chi(-q^2)}{f(-q^{152})f(-q^2)\chi(-q^{38})}\\ &= \frac{\chi(-q^{76})}{\chi(-q^4)} = \frac{S(q^2)}{R(q^2)},\end{aligned}$$

where (2.14) was used four times. We conclude from (6.7) that

$$(6.8)\qquad \frac{S(-q^4)}{T(-q^4)} = \frac{T(q^4)}{R(q^4)} = \frac{S(q^2)}{R(q^2)} = \frac{\chi(-q^{76})}{\chi(-q^4)},$$

which is (1.14) with q replaced by q^2.

7. Proof of Theorem 1.4. In (1.16), we replace q by q^4 and employ (3.12) and (3.13) to find that

$$\begin{aligned}(7.1)\quad 2\frac{A(q^4)}{\chi(-q^{18})} &= \{\chi(q)\chi(-q^3)H(q^9)+\chi(-q)\chi(q^3)H(-q^9)\}G(q^{324})\\ &\quad - q^{63}\{\chi(q)\chi(-q^3)G(q^9)-\chi(-q)\chi(q^3)G(-q^9)\}H(q^{324})\\ &= \chi(q)\chi(-q^3)\{H(q^9)G(q^{324}) - q^{63}G(q^9)H(q^{324})\}\\ &\quad + \chi(-q)\chi(q^3)\{H(-q^9)G(q^{324}) + q^{63}G(-q^9)H(q^{324})\}.\end{aligned}$$

Using (2.26), with q replaced by q^9 and $-q^9$, we find by (7.1) that

$$2\frac{A(q^4)}{\chi(-q^{18})} = \chi(q)\chi(-q^3)\frac{\chi(q^{27})\chi(-q^{81})}{\chi(-q^{18})} + \chi(-q)\chi(q^3)\frac{\chi(-q^{27})\chi(q^{81})}{\chi(-q^{18})},$$

from which (1.17) follows, and so the proof of Theorem 1.4 is complete.

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