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The greatest prime divisor of a product of consecutive integers

by

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1. Introduction. Let $k \geq 2$ and $n \geq 1$ be integers. We define

$$\Delta(n,k) = n(n+1)\cdots(n+k-1).$$

For an integer $\nu > 1$, we denote by $\omega(\nu)$ and $P(\nu)$ the number of distinct prime divisors of ν and the greatest prime factor of ν, respectively, and we put $\omega(1) = 0$, $P(1) = 1$.

A well known theorem of Sylvester [7] states that

$$P(\Delta(n,k)) > k \quad \text{if } n > k. \tag{1}$$

We observe that $P(\Delta(1,k)) \leq k$ and therefore the assumption $n > k$ in (1) cannot be removed. For $n > k$, Moser [5] sharpened (1) to $P(\Delta(n,k)) > \frac{11}{10}k$ and Hanson [3] to $P(\Delta(n,k)) > 1.5k$ unless $(n,k) = (3,2),(8,2),(6,5)$. Further Faulkner [2] proved that $P(\Delta(n,k)) > 2k$ if n is greater than or equal to the least prime exceeding $2k$ and $(n,k) \neq (8,2),(8,3)$.

In this paper, we sharpen the results of Hanson and Faulkner. We shall not use these results in the proofs of our improvements. We prove

THEOREM 1. *We have*

(a)

$$P(\Delta(n,k)) > 2k \qquad \textit{for } n > \max\left(k+13, \frac{279}{262}k\right). \tag{2}$$

(b)

$$P(\Delta(n,k)) > 1.97k \quad \textit{for } n > k+13. \tag{3}$$

We observe that 1.97 in (3) cannot be replaced by 2 since there are arbitrarily long chains of consecutive composite positive integers. The same reason implies that Theorem 1(a) is not valid under the assumption $n >$

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$k+13$. Further the assumption $n > \frac{279}{262}k$ in Theorem 1(a) is necessary since $P(\Delta(279,262)) \le 2 \cdot 262$.

Now we give a lower bound for $P(\Delta(n,k))$ which is valid for $n > k > 2$ except for an explicitly given finite set. For this, we need some notation. For a pair (n,k) and a positive integer h, we write $[n,k,h]$ for the set of all pairs $(n,k),\dots,(n+h-1,k)$ and we set $[n,k] = [n,k,1] = \{(n,k)\}$. Let

$$A_{10} = \{58\}, \quad A_8 = A_{10} \cup \{59\}, \quad A_6 = A_8 \cup \{60\},$$

$$A_4 = A_6 \cup \{12, 16, 46, 61, 72, 93, 103, 109, 151, 163\},$$

$$A_2 = A_4 \cup \{4, 7, 10, 13, 17, 19, 25, 28, 32, 38, 43, 47, 62, 73, 94, 104, 110, 124, 152, 164, 269\}$$

and $A_{2i+1} = A_{2i}$ for $1 \le i \le 5$. Further let

$$A_1 = A_2 \cup \{3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 23, 26, 29, 33, 35, 39, 41, 44, 48, 50, 53, 56, 63, 68, 74, 78, 81, 86, 89, 95, 105, 111, 125, 146, 153, 165, 173, 270\}.$$

Finally, we set

$$B = [8,3] \cup [5,4,3] \cup [14,13,3] \cup \{(k+1,k) \mid k = 3, 5, 8, 11, 14, 18, 63\}.$$

Then

THEOREM 2. *We have*

$$P(\Delta(n,k)) > 1.95k \quad \text{for } n > k > 2 \tag{4}$$

except when $(n,k) \in [k+1,k,h]$ *for* $k \in A_h$ *with* $1 \le h \le 11$ *or* $(n,k) = (8,3)$.

If $k = 2$, we observe (see Lemma 7) that $P(\Delta(n,k)) > 2k$ unless $n = 3, 8$ and that $P(\Delta(3,2)) = P(\Delta(8,2)) = 3$. Thus the estimate (4) is valid for $k = 2$ whenever $n \ne 3, 8$. We observe that $P(\Delta(k+1,k)) \le 2k$ and therefore 1.95 in (4) cannot be replaced by 2.

There are few exceptions if 1.95 is replaced by 1.8 in Theorem 2. We derive from Theorem 2 the following result.

COROLLARY 1. *We have*

$$P(\Delta(n,k)) > 1.8k \quad \text{for } n > k > 2 \tag{5}$$

except when $(n,k) \in B$.

2. Lemmas. We begin with a well known result due to Levi ben Gerson on a particular case of the Catalan equation.

LEMMA 1. *The solutions of*

$$2^a - 3^b = \pm 1 \quad \text{in integers } a > 0,\ b > 0$$

are given by $(a,b) = (1,1), (2,1), (3,2)$.

Next we state a result of Saradha and Shorey [6] on a lower bound for $\omega(\Delta(n,k))$.

LEMMA 2. *For $n > k > 2$, we have*

$$\omega(\Delta(n,k)) \geq \pi(k) + \left[\frac{1}{3}\pi(k)\right] + 2$$

except when (n,k) belongs to the union of the sets

$$\begin{cases} [4,3],[6,3,3],[16,3],[6,4],[6,5,4],[12,5],[14,5,3],[23,5,2],\\ [7,6,2],[15,6],[8,7,3],[12,7],[14,7,2],[24,7],[9,8],[14,8],\\ [14,13,3],[18,13],[20,13,2],[24,13],[15,14],[20,14],[20,17]. \end{cases}$$

We shall use Lemma 2 only when $k = 3$ or $5 \leq k \leq 8$. Let p_i denote the ith prime number. Then

LEMMA 3. *We have*

$$p_{i+1} - p_i < \begin{cases} 35 & \text{for } p_i \leq 5591,\\ 15 & \text{for } p_i \leq 1123,\ p_i \neq 523, 887, 1069,\\ 21 & \text{for } p_i = 523, 887, 1069,\\ 9 & \text{for } p_i \leq 361,\\ & p_i \neq 113, 139, 181, 199, 211, 241, 283, 293, 317, 337. \end{cases} \tag{6}$$

LEMMA 4. *Let $\mathfrak{N}$ be a positive real number and k_0 a positive integer. Let $I(\mathfrak{N}, k_0) = \{i \mid p_{i+1} - p_i \geq k_0,\ p_i \leq \mathfrak{N}\}$. Then*

$$P(n(n+1)\cdots(n+k-1)) > 2k$$

for $2k \leq n < \mathfrak{N}$ and $k \geq k_0$ except possibly when $p_i < n < n+k-1 < p_{i+1}$ for $i \in I(\mathfrak{N}, k_0)$.

Proof. Let $2k \leq n < \mathfrak{N}$ and $k > k_0$. We may suppose that none of $n, n+1, \ldots, n+k-1$ is a prime, otherwise the result follows. Let $p_i < n < n+k-1 < p_{i+1}$. Then $i = \pi(n)$ and $p_{\pi(n)} < n < \mathfrak{N}$. For $\pi(n) \notin I(\mathfrak{N}, k_0)$, we have

$$k - 1 = n + k - 1 - n < p_{\pi(n)+1} - p_{\pi(n)} < k_0,$$

which implies $k - 1 < k_0 - 1$, a contradiction. Hence the assertion. ∎

The following result on the estimates for primes is due to Dusart [1, p. 14].

LEMMA 5. *For $\nu > 1$, we have*

(i) $\pi(\nu) \leq \dfrac{\nu}{\log \nu}\left(1 + \dfrac{1.2762}{\log \nu}\right)$,

(ii) $\pi(\nu) \geq \dfrac{\nu}{\log \nu - 1}$ *for $\nu \geq 5393$.*

LEMMA 6. *Let $X > 0$ and $0 < \theta < e-1$ be real numbers. For $l \geq 0$, let*

$$X_0 = \max\left(\frac{5393}{1+\theta}, \exp\left(\frac{\log(1+\theta)+0.2762}{\theta}\right)\right),$$

$$X_{l+1} = \max\left(\frac{5393}{1+\theta}, \exp\left(\frac{\log(1+\theta)+0.2762}{\theta + \frac{1.2762(1-\log(1+\theta))}{\log^2 X_l}}\right)\right).$$

Then

$$\pi((1+\theta)X) - \pi(X) > 0 \quad \textit{for } X > X_l.$$

Proof. Let $l \geq 0$ and $X > X_l$. Then $(1+\theta)X \geq 5393$. By Lemma 5, we have

$$\begin{aligned} \delta := \pi((1+\theta)X) - \pi(X) &\geq \frac{(1+\theta)X}{\log(1+\theta)X - 1} - \frac{X}{\log X}\left(1 + \frac{1.2762}{\log X}\right) \\ &\geq \frac{X}{\log(1+\theta)X-1}\left\{1+\theta - \frac{\log(1+\theta)X-1}{\log X}\left(1+\frac{1.2762}{\log X}\right)\right\} \\ &\geq \frac{X}{\log(1+\theta)X-1}\left\{1+\theta - \left(1 - \frac{1-\log(1+\theta)}{\log X}\right)\left(1+\frac{1.2762}{\log X}\right)\right\} \\ &\geq \frac{X}{\log(1+\theta)X-1}\{F(X)+G(X)\} \end{aligned}$$

where

$$F(X) = \theta - \frac{\log(1+\theta)+0.2762}{\log X}, \qquad G(X) = \frac{1.2762(1-\log(1+\theta))}{\log^2 X}.$$

We see that $G(X) > 0$ and is decreasing since $0 < \theta < e-1$. Further we observe that $\{X_i\}$ is a non-increasing sequence. We notice that $\delta > 0$ if $F(X) + G(X) > 0$. But $F(X)+G(X) > F(X) > 0$ for $X > X_0$ by the definition of X_0. Thus $\delta > 0$ for $X > X_0$.

Let now $X \leq X_0$. Then $F(X)+G(X) \geq F(X)+G(X_0)$ and $F(X)+G(X_0) > 0$ if $X > X_1$ by the definition of X_1. Hence $\delta > 0$ for $X > X_1$. Now we proceed inductively as above to see that $\delta > 0$ for $X > X_l$ with $l \geq 2$. ∎

LEMMA 7. *Let $n > k$ and $k \leq 16$. Then*

$$P(\Delta(n,k)) \leq 2k \tag{7}$$

implies that $(n,k) \in \{(8,2),(8,3)\}$ or $(n,k) \in [k+1,k]$ for $k \in \{2,3,5,6,8,9,11,14,15\}$ or $(n,k) \in [k+1,k,3]$ for $k \in \{4,7,10,13\}$ or $(n,k) \in [k+1,k,5]$ for $k \in \{12,16\}$.

Proof. We apply Lemma 1 to derive that (7) is possible only if $n = 3,8$ when $k = 2$ and $n = 5,6,7$ when $k = 4$. For the latter assertion, we apply Lemma 1 after securing $P((n+i)(n+j)) \leq 3$ with $0 \leq i < j \leq 3$ by deleting the terms divisible by 5 and 7 in $n, n+1, n+2$ and $n+3$. For $k = 3$ and $5 \leq k \leq 8$, the assertion follows from Lemma 2.

Thus we may assume that $k \geq 9$. Assume that (7) holds. Then in the product $\Delta(n,k)$, there are at most $1 + [(k-1)/p]$ terms divisible by the prime p. After removing all the terms divisible by $p \geq 7$, we are left with at least four terms only divisible by $2, 3$ and 5. Further out of these terms, for each prime $2, 3$ and 5, we remove a term in which the prime divides to a maximal power. Then we are left with a term $n+i$ such that $n \leq n+i \leq 8 \cdot 9 \cdot 5 = 360$.

Let $n \geq 2k$. We now apply Lemma 4 with $\mathfrak{N} = 361, k_0 = 9$ and (6) to get $P(\Delta(n,k)) > 2k$ for $k \geq 9$ except possibly when $p_i < n < n+k-1 < p_{i+1}$, $p_i = 113, 139, 181, 199, 211, 241, 283, 293, 317, 337$. For these values of n, we check that $P(\Delta(n,k)) > 2k$ is valid for $9 \leq k \leq 16$. Thus it suffices to consider $k < n < 2k$. We calculate $P(\Delta(n,k))$ for (n,k) with $9 \leq k \leq 16$ and $k < n < 2k$. We find that (7) holds only if (n,k) is as given in the statement of Lemma 7. ∎

3. Proof of Theorem 1(a). Let $n > \max\left(k+13, \frac{279}{262}k\right)$. In view of Lemma 7, we may take $k \geq 17$ since $n \leq k+5$ for the exceptions (n,k) given in Lemma 7. It suffices to prove (2) for k such that $2k-1$ is prime. Let $k_1 < k_2$ be such that $2k_1 - 1$ and $2k_2 - 1$ are consecutive primes. Suppose (2) holds at k_1. Then for $k_1 < k < k_2$, we have

$$P(n(n+1)\cdots(n+k-1)) \geq P(n\cdots(n+k_1-1)) > 2k_1,$$

implying $P(\Delta(n,k)) \geq 2k_2 - 1 > 2k$. Therefore we may suppose that $k \geq 19$ since $2k-1$ with $k = 17, 18$ are composites. We assume from now onward in the proof of Theorem 1(a) that $2k-1$ is prime. We put $x = n+k-1$. Then $\Delta(n,k) = x(x-1)\cdots(x-k+1)$. Let $f_1 < \cdots < f_\mu$ be all the integers in $[0,k)$ such that

$$P((x-f_1)\cdots(x-f_\mu)) \leq k. \tag{8}$$

We argue as in the proof of [4, Lemma 4] to get

$$k! > x^{\mu-\pi(k)}\left(1-\frac{k}{x}\right)^{\mu}. \tag{9}$$

We may suppose $\omega(\Delta(n,k)) \leq \pi(2k)$, otherwise (2) follows. Then

$$\mu \geq k - \pi(2k) + \pi(k) \tag{10}$$

which we use as in [4, Lemma 4] to derive from (9) that

$$x < k^{3/2} \text{ for } k \geq 87; \quad x < k^{7/4} \text{ for } k \geq 40; \quad x < k^2 \text{ for } k \geq 19. \tag{11}$$

If $x \geq 7k$ and $k > 57$, then as in [4, Lemma 7] we derive from (10) that $x \geq k^{3/2}$. Thus (11) implies that $x < 7k$ for $k \geq 87$. Putting back $n = x-k+1$, we may assume that $n < 6k+1$ for $k \geq 87$, $n < k^{7/4} - k + 1$ for $40 \leq k < 87$ and $n < k^2 - k + 1$ for $19 \leq k < 40$.

Let $k < 87$. Suppose $n \geq 2k$. Then $2k \leq n < k^{7/4} - k + 1$ for $40 \leq k < 87$ and $2k \leq n < k^2 - k + 1$ for $19 \leq k < 40$. Thus Lemma 4 with $\mathfrak{N} = 87^{7/4} - 87 + 1, k_0 = 35$ and (6) implies that $P(\Delta(n,k)) > 2k$ for $k \geq 35$. We note here that $2k \leq n < \mathfrak{N}$ for $35 \leq k < 40$. Let $k < 35$. Taking $\mathfrak{N} = 34^2 - 34 + 1, k_0 = 21$ for $21 \leq k \leq 34$ and $\mathfrak{N} = 19^2 - 19 + 1, k_0 = 19$ for $k = 19$, we see from Lemma 4 and (6) that $P(\Delta(n,k)) > 2k$ for $k \geq 19$. Here the case $k = 20$ is excluded since $2k - 1$ is composite. Therefore we may assume that $n < 2k$. Further we observe that $\pi(n + k - 1) - \pi(2k) \geq \pi(2k + 13) - \pi(2k)$ since $n > k + 13$. Next we check that $\pi(2k + 13) - \pi(2k) > 0$. This implies that $[2k, n + k - 1]$ contains a prime.

Thus we may assume that $k \geq 87$. Then we write

$$n = \alpha k + 1 \quad \text{with} \quad \begin{cases} 279/262 - 1/k < \alpha \leq 6 & \text{if } k \geq 201, \\ 1 + 12/k < \alpha \leq 6 & \text{if } k < 201. \end{cases}$$

Further we consider $\pi(n + k - 1) - \pi(\max(n - 1, 2k))$, which is

$$\begin{aligned} &= \pi((\alpha+1)k) - \pi(\alpha k) && \text{for } \alpha \geq 2, \\ &\geq \pi\left(\left[\frac{541}{262}k\right]\right) - \pi(2k) && \text{for } \alpha < 2 \text{ and } k \geq 201, \\ &\geq \pi(2k+13) - \pi(2k) && \text{for } \alpha < 2 \text{ and } k < 201. \end{aligned}$$

By using exact values of the π function we check that

$$\begin{aligned} \pi(2k+13) - \pi(2k) > 0 &\quad \text{for } k < 201, \\ \pi\left(\left[\frac{541}{262}k\right]\right) - \pi(2k) > 0 &\quad \text{for } 201 \leq k \leq 2616. \end{aligned}$$

Thus we may suppose that $k > 2616$ if $\alpha < 2$. Also

$$\left[\frac{541}{262}k\right] \geq \frac{540}{262}k \quad \text{for } k > 2616.$$

Now we apply Lemma 6 with $X = \alpha k, \theta = 1/\alpha, l = 0$ if $\alpha \geq 2$ and $X = 2k, \theta = 4/131, l = 1$ if $\alpha < 2$ to get

$$\pi(n + k - 1) - \pi(\max(n - 1, 2k)) > 0$$

for $X > X_0 = 5393/(1 + 1/\alpha)$ if $\alpha \geq 2$ and $X > X_1 = 5393/(1 + 4/131)$ if $\alpha < 2$. Further when $\alpha < 2$, we observe that $X = 2k > X_1$ since $k > 2616$. Thus the assertion follows for $n < 2k$.

It remains to consider the case $\alpha \geq 2$ and $X \leq 5393(1 + 1/\alpha)^{-1}$. Then $2k \leq n < n + k - 1 = X(1 + 1/\alpha) \leq 5393$. Now we apply Lemma 4 with $\mathfrak{N} = 5393, k_0 = 35$ and (6) to conclude that $P(\Delta(n,k)) > 2k$. ∎

4. Proof of Theorem 1(b). In view of Lemma 7 and Theorem 1(a), we may assume that $k \geq 17$ and $k < n \leq \frac{279}{262}k$. Let $X = \frac{279}{262}k,\ \theta = \frac{245}{279},\ l = 0$.

Then for $k < n \le X$, we see from Lemma 6 that

$$\pi(2k) - \pi(n-1) \ge \pi((1+\theta)X) - \pi(X) > 0$$

for $X > X_0 = 5393(1+\theta)^{-1}$ which is satisfied for $k > 2696$ since $(1+\theta)X = 2k$. Thus we may suppose that $k \le 2696$. Now we check with exact values of the π function that $\pi(2k) - \pi\big(\frac{279}{262}k\big) > 0$. Therefore

$$P(\Delta(n,k)) \ge P(n(n+1)\cdots 2k) \ge p_{\pi(2k)}.$$

Further we apply Lemma 6 with $X = 1.97k$, $\theta = 3/197$ and $l = 25$. We calculate that $X_l \le 284000$. We conclude by Lemma 6 that

$$\pi(2k) - \pi(1.97k) = \pi((1+\theta)X) - \pi(X) > 0$$

for $k > 145000$. Let $k \le 145000$. Then we check that $\pi(2k) - \pi(1.97k) > 0$ is valid for $k \ge 680$ by using exact values of the π function. Thus

$$p_{\pi(2k)} > 1.97k \quad \text{for } k \ge 680. \tag{12}$$

Therefore we may suppose that $k < 680$. Now we observe that for $n > k+13$,

$$\pi(n+k-1) - \pi(1.97k) \ge \pi(2k+13) - \pi(1.97k) > 0;$$

the latter inequality can be checked by using exact values of the π function. Hence the assertion follows since $n < 1.97k$. ∎

5. Proof of Theorem 2. By Theorem 1(b), we may assume that $n \le k+13$. Also we may suppose that $k < 680$ by (12). For $k \le 16$, we calculate $P(\Delta(n,k))$ for all the pairs (n,k) given in the statement of Lemma 7. We find that either $P(\Delta(n,k)) > 1.95k$ or (n,k) is an exception stated in Theorem 1(a). Thus we may suppose that $k \ge 17$. Now we check that $\pi(n+k-1) - \pi(1.95k) > 0$ except when $(n,k) \in [k+1,k,h]$ for $k \in A_h$ with $1 \le h \le 11$, and the assertion follows. ∎

6. Proof of Corollary 1. We calculate $P(\Delta(n,k))$ for all (n,k) with $k \le 270$ and $k+1 \le n \le k+11$. This contains the set of exceptions given in Theorem 2. We find that $P(\Delta(n,k)) > 1.8k$ unless $(n,k) \in B$. Hence the assertion (5) follows from Theorem 2. ∎

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