ACTA ARITHMETICA
120.3 (2005)

Tetragonal modular curves

by

Daeyeol Jeon and Euisung Park (Seoul)

0. Introduction. A smooth projective curve X defined over an algebraically closed field k is called *d-gonal* if it admits a map $\phi : X \to \mathbb{P}^1$ over k of degree d. If the genus $g \geq 2$ and $d = 2$ then X is called *hyperelliptic.* We will say that X is *trigonal*, *tetragonal* and *pentagonal* for $d = 3, d = 4$ and $d = 5$ respectively.

Let N be a positive integer, and let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}.$$

Let $X_0(N)$ denote the modular curve corresponding to $\Gamma_0(N)$. Then Zograf [Z] gave a linear bound on the level N of d-gonal modular curves $X_0(N)$. Also Nguyen and Saito [N-Sa] proved an analogue of the strong Uniform Boundedness Conjecture for elliptic curves defined over function fields of dimension one by using the connection with giving a bound on the level N of d-gonal modular curves $X_0(N)$.

Recently, Hasegawa and Shimura [H-S] gave a highly sharpened upper bound for $3 \leq d \leq 5$ by trying to determine d-gonal modular curves $X_0(N)$ for such d. For $d = 2$ it was done by Ogg [O]. Actually Hasegawa and Shimura succeeded in determining all trigonal modular curves $X_0(N)$ but failed for tetragonal and pentagonal $X_0(N)$.

The following lists N for which they did not know whether $X_0(N)$ was tetragonal or not:

76, 82, 84, 88, 90, 93, 97, 99, 106, 108, 109, 113, 115, 128, 133, 137, 157, 169.

In this work we prove that $X_0(N)$ is tetragonal only for $N = 88, 99, 109$ among the above 18 numbers. Combining with the result of [H-S] we get the following:

2000 *Mathematics Subject Classification*: 11G18, 11G30.
Key words and phrases: modular curve, tetragonal.

THEOREM 0.1. *The modular curve* $X_0(N)$ *has* $\mathrm{Gon}(X_0(N)) = 4$ *if and only if*

$$\begin{aligned}
g = 5: &\quad N = 42, 51, 52, 55, 56, 57, 63, 65, 67, 72, 73, 75,\\
g = 6: &\quad N = 58, 79, 121,\\
g = 7: &\quad N = 60, 62, 68, 69, 77, 80, 83, 85, 89, 91, 98, 100,\\
g = 8: &\quad N = 74, 101, 103, 109, 125,\\
g = 9: &\quad N = 66, 70, 87, 88, 95, 96, 99, 107,\\
g = 10: &\quad N = 92,\\
g = 11: &\quad N = 78, 94, 104, 111, 119, 131,\\
g = 13: &\quad N = 143,\\
g = 14: &\quad N = 167,\\
g = 16: &\quad N = 191,\\
g = 17: &\quad N = 142.
\end{aligned}$$

REMARK 0.2. Note that rational, elliptic and hyperelliptic curves always admit tetragonal maps. One can find a list of those curves $X_0(N)$ in [O]. For $d \geq \frac{1}{2}g + 1$, any curve of genus g has a d-gonal map [K-L]. Since all the modular curves $X_0(N)$ with $\mathrm{Gon}(X_0(N)) = 3$ are of genus $g = 3, 4$ (see [H-S]), they are tetragonal. Combining with Theorem 0.1 one can get a complete list of $X_0(N)$ which are tetragonal.

1. Definitions and general results. Let X be a smooth projective curve of genus $g \geq 4$.

1.1. *Gonality.* For a line bundle $L \in \operatorname{Pic} X$, a subspace $V \subset H^0(X, L)$ is said to be a g_d^r if $\deg(L) = d$ and $\dim(V) = r + 1$. The *gonality of* X is

$$\mathrm{Gon}(X) := \min\{d \mid X \text{ has a } g_d^1\}.$$

In this paper, when we say that X is d-gonal it does not mean $\mathrm{Gon}(X) = d$ but just that X admits a g_d^1. Thus d-gonal curves X may have $\mathrm{Gon}(X) < d$. For example all hyperelliptic curves are automatically tetragonal.

1.2. *Clifford index.* For a line bundle $L \in \operatorname{Pic} X$, the *Clifford index* of L is the integer

$$\mathrm{Cliff}(L) := \deg(L) - 2(h^0(X, L) - 1)$$

and the *Clifford index of* X itself is defined as

$$\mathrm{Cliff}(X) := \min\{\mathrm{Cliff}(L) \mid h^0(X, L) \geq 2,\ h^1(X, L) \geq 2\}.$$

It is well known that $\mathrm{Cliff}(X) + 2 \leq \mathrm{Gon}(X) \leq \mathrm{Cliff}(X) + 3$ (see [C-M]).

1.3. *Clifford dimension.* We will say that $L \in \operatorname{Pic} X$ *contributes to the Clifford index of* X if both $h^0(X, L) \geq 2$, and $h^1(X, L) \geq 2$, and that L

computes the Clifford index of X if in addition $\operatorname{Cliff}(X) = \operatorname{Cliff}(L)$. The *Clifford dimension of* X is defined as

$$\min\{h^0(X,L) - 1 \mid L \text{ computes the Clifford index of } X\}.$$

If a line bundle L achieves the minimum and computes the Clifford index, then we will say that L *computes the Clifford dimension.* In most cases, the Clifford dimension ℓ is equal to 1 and the curves with $\ell \geq 2$ are rather rare. It is a classical result that $\ell = 2$ if and only if X is a smooth plane curve of degree ≥ 5. The case $\ell = 3$ was settled by Martens [M]. He proved that $\ell = 3$ if and only if X is a complete intersection of two irreducible cubic surfaces in $\mathbb{P}^3$, and hence its genus g is 10.

1.4. *Property N_p.* If X is a non-hyperelliptic curve, then the canonical line bundle ω_X defines an embedding $X \hookrightarrow \mathbb{P}H^0(X,\omega_X) = \mathbb{P}^{g-1}$. Consider the minimal free resolution

$$0 \to F_{g-2} \to \cdots \to F_2 \to F_1 \to S \to S_X \to 0$$

of the homogeneous coordinate ring $S_X = S/I_X$ as an S-module where $S = \mathbb{C}[X_0, X_1, \ldots, X_{g-1}]$ and $F_i = \bigoplus_{j\in\mathbb{Z}} S(-i-j)^{\beta_{i,j}}$. We call $\beta_{i,j}$ the *graded Betti numbers.* Due to Green and Lazarsfeld [G-L], $X \hookrightarrow \mathbb{P}^{g-1}$ is said to have *property N_p* if the resolution is of the form

$$\cdots \to S^{\beta_{p,1}}(-p-1) \to \cdots \to S^{\beta_{2,1}}(-3) \to S^{\beta_{1,1}}(-2) \to S \to S_X \to 0.$$

Therefore property N_1 holds if and only if the homogeneous ideal is generated by quadrics, and property N_p holds for $p \geq 2$ if and only if it has property N_1 and the kth syzygies among the quadrics are generated by linear syzygies for all $1 \leq k \leq p-1$. Now we recall the following:

Theorem 1.1 (M. Green and R. Lazarsfeld, Appendix in [G]). *Let X be a smooth non-hyperelliptic curve of genus $g \geq 3$. Then the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ fails to have property N_p for $p \geq \operatorname{Cliff}(X)$.*

Thus if the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ has property N_p, then $\operatorname{Cliff}(X) \geq p+1$ and $\operatorname{Gon}(X) \geq p+3$.

1.5. *Betti numbers.* When $\operatorname{Cliff}(X) \leq 2$, Schreyer clarified the relation between the minimal free resolution of the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ and the existence of special linear series of divisors on X. For $g \leq 7$, see Table 1 in [Sch1]. For $g \geq 8$, the graded Betti number $\beta_{2,2}$ has one of the values given in the following table [Sch2]:

$\beta_{2,2}$	$(g-4)(g-2)$	$\binom{g-2}{2}-1$	$g-4$	0
Linear series	$\exists\ g^1_3$	$\exists\ g^2_6$ or g^3_8	$\exists$ a single g^1_4	$\operatorname{Cliff}(C) \geq 3$

2. Proof. We proceed to prove that $X_0(N)$ is tetragonal only for $N = 88, 99, 109$ among the 18 numbers N in §0. For this purpose we compute the graded Betti numbers of the canonical embedding of $X_0(N)$. We use the computer programs "Maple" and "Singular". First we calculate the homogeneous ideal of the canonical embedding of $X_0(N)$ by using Maple.

Note that, for such N, $X_0(N)$ is not hyperelliptic [O]. Thus $X_0(N)$ can be identified with the canonical curve which is the image of the canonical embedding

$$X_0(N) \ni P \mapsto (f_1(P) : \cdots : f_g(P)) \in \mathbb{P}^{g-1}$$

where $\{f_1, \ldots, f_g\}$ is a basis of the space of cusp forms of weight 2 on $X_0(N)$. One can get such a basis and the corresponding Fourier coefficients from [St]. Then to obtain the minimal generating system of the homogeneous ideal $I(X_0(N))$, we only have to compute the relations of the $f_i f_j$ $(1 \le i, j \le g)$ by Petri's theorem. Since there are $(g-2)(g-3)/2$ linear relations among the $f_i f_j$, we get quadric generators $Q_k(x_1, \ldots, x_g)$ with $1 \le k \le (g-2)(g-3)/2$ by assigning x_i to f_i (for details see [H-S]).

Now we compute the Betti numbers by using Singular. In fact when the genus of $X_0(N)$ is big then Singular does not work efficiently. Note that since the canonical embedding is always projectively Cohen–Macaulay, the Betti numbers of the canonical curve are equal to those of the hyperplane section, which allows us to get Betti numbers easier.

We exhibit the so-called Betti table of the canonical embedding for our cases in Table 1.

Case 1. The canonical embeddings of $X_0(N)$ for

$$N = 76, 82, 84, 90, 93, 97, 106, 108, 113, 115, 128, 133, 137, 157, 169$$

have property N_p for $p \ge 2$. Thus $\mathrm{Gon}(X_0(N)) \ge 5$ by §1.4.

Case 2. The curve $X_0(88)$ is of genus 9 and $\beta_{2,2} = 9 - 4 = 5$. Thus it is tetragonal by §1.5.

Case 3. For $N = 99$ and 109, $\mathrm{Cliff}(X_0(N)) = 2$. Indeed, by §1.4, $\mathrm{Cliff}(X_0(N)) \ge 2$ because the canonical embeddings of these curves have property N_1. Also $\mathrm{Cliff}(X_0(N)) \le 2$ since they have a g_6^2 or a g_8^3 by §1.4. Therefore $\mathrm{Cliff}(X_0(N))$ is computed by a $g_{2+2\ell}^{\ell}$ where ℓ denotes the Clifford dimension. It remains to show that $\ell = 1$. We already know that $\ell \le 3$ from the existence of a g_6^2 or a g_8^3. Since a smooth plane curve of degree d is of genus $(d-1)(d-2)/2$, the curves $X_0(99)$ and $X_0(109)$ cannot be plane curves. Thus $\ell \ne 2$ by §1.3. Also $\ell \ne 3$ because their genera are less than 10. Therefore $\mathrm{Cliff}(X_0(N))$ is computed by a g_4^1 and hence $X_0(99)$ and $X_0(109)$ are tetragonal. ∎

Table 1. The graded Betti numbers for the canonical embedding

Genus	$X_0(N)$	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	Genus	$X_0(N)$	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$
		$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$			$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$
7	$X_0(97)$	0	0	16	9	$X_0(128)$	0	0	0
		10	16	0			21	64	70
8	$X_0(76)$	0	0	21	10	$X_0(108)$	0	0	20
		15	35	21			28	105	162
	$X_0(109)$	0	14	35	11	$X_0(84)$	0	0	0
		15	35	35			36	160	315
	$X_0(169)$	0	0	21		$X_0(90)$	0	0	0
		15	35	21			36	160	315
9	$X_0(82)$	0	0	8		$X_0(115)$	0	0	0
		21	64	70			36	160	315
	$X_0(88)$	0	5	24		$X_0(133)$	0	0	0
		21	64	75			36	160	315
	$X_0(93)$	0	0	0		$X_0(137)$	0	0	0
		21	64	70			36	160	315
	$X_0(99)$	0	20	64	12	$X_0(106)$	0	0	0
		21	64	90			45	231	550
	$X_0(113)$	0	0	8		$X_0(157)$	0	0	0
		21	64	70			45	231	550

References

[C-M] M. Coppens and G. Martens, *Secant spaces and Clifford's theorem*, Compos. Math. 78 (1991), 193–212.

[G] M. Green, *Koszul cohomology and the geometry of projective varieties I*, J. Differential Geom. 19 (1984), 125–171.

[G-L] M. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Compos. Math. 67 (1988), 301–314.

[H-S] Y. Hasegawa and M. Shimura, *Trigonal modular curves*, Acta Arith. 88 (1999), 129–140.

[K-L] S. L. Kleiman and D. Laksov, *Another proof of the existence of special divisors*, Acta Math. 132 (1974), 163–176.

[M] G. Martens, *Über den Clifford-Index algebraischer Kurven*, J. Reine Angew. Math. 336 (1982), 83–90.

[N-Sa] K. V. Nguyen and M.-H. Saito, *d-gonality of modular curves and bounding torsions*, arXiv:alg-geom/9603024 v1.

[O] A. P. Ogg, *Hyperelliptic modular curves*, Bull. Soc. Math. France 102 (1974), 449–462.

[Sch1] F.-O. Schreyer, *Syzygies of canonical curves and special linear series*, Math. Ann. 275 (1986), 105–137.

[Sch2] F.-O. Schreyer, *A standard basis approach to the syzygies of canonical curves*, J. Reine Angew. Math. 421 (1991), 83–123.

[St] W. A. Stein, http://modular.fas.harvard.edu.

[Z] P. G. Zograf, *Small eigenvalues of automorphic Laplacians in spaces of cusp forms*, in: Automorphic Functions and Number Theory, II, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 134 (1984), 157–168 (in Russian).

Korea Institute for Advanced Study (KIAS)
207-43 Cheongnyangni 2-dong
Dongdaemun-gu
Seoul, 130-722 Korea
E-mail: dyjeon@kias.re.kr
puserdos@kias.re.kr

Received on 6.4.2005
and in revised form on 25.7.2005 (4974)