ACTA ARITHMETICA
168.1 (2015)

Asymptotic formulae for partition ranks

by

JEHANNE DOUSSE (Paris) and MICHAEL H. MERTENS (Köln)

1. Introduction and statement of results. A *partition* of n is a non-increasing sequence of natural numbers whose sum is n. For example, there are five partitions of 4: 4, 3+1, 2+2, 2+1+1 and 1+1+1+1. Let $p(n)$ denote the number of partitions of n. One of the most beautiful theorems in partition theory is Ramanujan's congruences for $p(n)$. He proved [9] that for all $n \geq 0$,

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Dyson [6] introduced the rank, defined as the largest part of a partition minus the number of its parts, in order to explain the congruences modulo 5 and 7 combinatorially. He conjectured that for all n, the partitions of $5n+4$ (resp. $7n+5$) can be divided into 5 (resp. 7) different classes of the same size according to their rank modulo 5 (resp. 7). This was later proved by Atkin and Swinnerton-Dyer [2].

However the rank fails to explain the congruences modulo 11. Therefore Dyson conjectured the existence of another statistic which he called the "crank" which would give a combinatorial explanation for all the Ramanujan congruences. The crank was later found by Andrews and Garvan [1, 7]. If for a partition λ, $o(\lambda)$ denotes the number of ones in λ, and $\mu(\lambda)$ is the number of parts strictly larger than $o(\lambda)$, then the *crank* of λ is defined as

$$\mathrm{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases}$$

Denote by $M(m,n)$ the number of partitions of n with crank m, and by $N(m,n)$ the number of partitions of n with rank m.

2010 *Mathematics Subject Classification*: 05A17, 11F03, 11F30, 11F50, 11P55, 11P82.
Key words and phrases: integer partitions, rank, circle method, Appell–Lerch sums.

DOI: 10.4064/aa168-1-5

The first author and Bringmann [3] recently proved a longstanding conjecture of Dyson by using the modularity of the crank generating function and an extension to two variables of Wright's version of the circle method [10].

Theorem 1.1 (Bringmann–Dousse). *If* $|m| \leq \frac{1}{\pi\sqrt{6}}\sqrt{n}\log n$, *we have, as* $n \to \infty$,

$$M(m,n) = \frac{\beta}{4}\operatorname{sech}^2\left(\frac{\beta m}{2}\right)p(n)(1 + O(\beta^{1/2}|m|^{1/3})), \tag{1.1}$$

where $\beta := \pi/\sqrt{6n}$.

For the rank the situation is more complicated since the generating function is not modular but mock modular, which means roughly that there exists some non-holomorphic function such that its sum with the generating function has nice modular properties. Nonetheless it is possible to apply a method similar to [3] in this case. This way we prove that the same formula also holds for the rank.

Theorem 1.2. *If* $|m| \leq \frac{\sqrt{n}\log n}{\pi\sqrt{6}}$, *we have, as* $n \to \infty$,

$$N(m,n) = \frac{\beta}{4}\operatorname{sech}^2\left(\frac{\beta m}{2}\right)p(n)(1 + O(\beta^{1/2}|m|^{1/3})).$$

Remark 1.3. As in [3], we could in fact replace the error term by $O(\beta^{1/2}m\alpha^2(m))$ for any $\alpha(m)$ such that $\frac{\log n}{n^{1/4}} = o(\alpha(m))$ for all $|m| \leq \frac{1}{\pi\sqrt{6}}\sqrt{n}\log n$ and $\beta m\alpha(m) \to 0$ as $n \to \infty$. Here we have chosen $\alpha(m) = |m|^{-1/3}$ to avoid complicated expressions in the proof.

Remark 1.4. After [3], and simultaneously with and independently of the present paper, Parry and Rhoades [8] proved that the same formula holds for all of Garvan's k-ranks. The crank corresponds to the case $k=1$ and the rank to $k = 2$. Their proof uses a completely different method: they use a sieving technique and do not rely on the modularity of the generating function.

The rest of this paper is organized as follows: in Section 2 we recall some important facts about Appell–Lerch sums, Mordell integrals, and also Euler polynomials, which are used in Section 3 to prove some preliminary estimates for the rank generating function. In Section 4, we use these results to prove the estimates close to and far from the dominant pole, which we need in Section 5 to establish our main result, Theorem 1.2.

2. Preliminaries

2.1. (Mock) modular forms. A key ingredient in the proof of our main theorem is the (mock) modularity of the rank generating function, defined as follows (throughout, if not specified otherwise, we always assume $\tau \in \mathbb{H}$, $z \in \mathbb{R}$, $q := e^{2\pi i\tau}$, and $\zeta := e^{2\pi iz}$):

$$R(z;\tau) := \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} N(m,n)\zeta^m q^n = \frac{1-\zeta}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{3n^2+n}{2}}}{1-\zeta q^n}. \tag{2.1}$$

Let us further define

$$\eta(\tau) := q^{1/24}(q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \tag{2.2}$$

$$\theta(z;\tau) := iq^{1/8}\zeta^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1-\zeta q^n)(1-\zeta^{-1}q^{n-1}). \tag{2.3}$$

In this section we are going to collect some transformation properties of η and θ and recall the definition and most important properties of Appell–Lerch sums as studied by Zwegers [12].

Lemma 2.1. *For η and θ as in* (2.2) *and* (2.3) *we have the following transformation laws:*

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau), \tag{2.4}$$

$$\theta\left(\frac{z}{\tau};-\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\,e^{\frac{\pi iz^2}{\tau}}\theta(z;\tau), \tag{2.5}$$

where $\sqrt{\cdot}$ denotes the principal branch of the holomorphic square-root.

Following Chapter 1 of [12] we define the following.

Definition 2.2.

(i) For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, we define the *Mordell integral* as

$$h(z) = h(z;\tau) = \int_{-\infty}^{\infty}\frac{e^{\pi i\tau w^2 - 2\pi zw}}{\cosh(\pi w)}\,dw.$$

(ii) For $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C}\setminus(\mathbb{Z}\oplus\mathbb{Z}\tau)$, we call the expression

$$A_1(u,v;\tau) = e^{\pi iu}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{n^2+n}{2}}e^{2\pi inv}}{1-e^{2\pi iu}q^n} \tag{2.6}$$

an *Appell–Lerch sum.* We also call $\mu(u,v;\tau) := A_1(u,v;\tau)/\theta(v;\tau)$ a *normalized* Appell–Lerch sum.

We need some transformation properties of these functions:

Lemma 2.3 (cf. [12, Proposition 1.2]). *The Mordell integral has the following properties:*

(i) $h(z) + e^{-2\pi iz-\pi i\tau}h(z+\tau) = 2\zeta^{-1/2}q^{-1/8}$,
(ii) $h(z/\tau;-1/\tau) = \sqrt{-i\tau}\, e^{-\pi iz^2/\tau}h(z;\tau)$.

LEMMA 2.4 (cf. [12, Propositions 1.4 and 1.5]).

(i) *One has*

$$\mu(-u,-v) = \mu(u,v).$$

(ii) *Under modular inversion, the Appell–Lerch sum has the following transformation law:*

$$\frac{1}{\sqrt{-i\tau}}e^{\frac{\pi i(u-v)^2}{\tau}}\mu\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right) + \mu(u,v;\tau) = \frac{1}{2i}h(u-v;\tau),$$

or equivalently

$$-\frac{1}{\tau}e^{\frac{\pi i(u^2-2uv)}{\tau}}A_1\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right) + A_1(u,v;\tau) = \frac{1}{2i}h(u-v;\tau)\theta(v;\tau).$$

2.2. Euler polynomials and Euler numbers. We now recall some facts about Euler polynomials. We define the *Euler polynomials* by the generating function

$$\frac{2e^{xz}}{e^z+1} =: \sum_{k=0}^{\infty} E_k(x)\frac{z^k}{k!}. \tag{2.7}$$

Let us recall two lemmas from [3] which will be useful in our proof.

LEMMA 2.5. *We have*

$$-\frac{1}{2}\operatorname{sech}^2\left(\frac{t}{2}\right) = \sum_{r=0}^{\infty} E_{2r+1}(0)\frac{t^{2r}}{(2r)!}.$$

LEMMA 2.6. *For $j \in \mathbb{N}_0$ set*

$$\mathcal{E}_j := \int_0^{\infty} \frac{z^{2j+1}}{\sinh(\pi z)}\,dz. \tag{2.8}$$

Then

$$\mathcal{E}_j = (-1)^{j+1}E_{2j+1}(0)/2.$$

3. Transformation formulae. In this section, we split $R(z;\tau)$ into several summands to determine its transformation behaviour under $\tau \mapsto -1/\tau$.

LEMMA 3.1. *For all $\tau \in \mathbb{H}$ and $z \in \mathbb{R}$, we have*

$$R(z;\tau) = \frac{q^{1/24}}{\eta(\tau)}\Bigg[\frac{i(\zeta^{1/2}-\zeta^{-1/2})\eta^3(3\tau)}{\theta(3z;3\tau)} - \zeta^{-1}(\zeta^{1/2}-\zeta^{-1/2})A_1(3z,-\tau;3\tau) - \zeta(\zeta^{1/2}-\zeta^{-1/2})A_1(3z,\tau;3\tau)\Bigg] \tag{3.1}$$

with A_1 as in (2.6).

This was first mentioned in [11, Theorem 7.1], but contained a slight typo there. To be precise, the factor i in front of the first summand was missing and the sign in front of the second and third was wrong.

Now we want to determine some asymptotic expressions for the three summands in (3.1). To do so, write $\tau = is/(2\pi)$ and $s = \beta(1 + ixm^{-1/3})$ with $x \in \mathbb{R}$ satisfying $|x| \leq \pi m^{1/3}/\beta$.

LEMMA 3.2. *Assume that $|z| < 1/3$. Then for $|x| \leq 1$ we have, as $n \to \infty$,*

$$i\frac{\eta^3(3\tau)}{\theta(3z;3\tau)} = \frac{-i\pi e^{\frac{6\pi^2 z^2}{s}}}{3s\sinh\left(\frac{2\pi^2 z}{s}\right)}\left[1+O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1-3z)}\right)+O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1+3z)}\right)\right],$$

Proof. By the transformation formulae from Lemma 2.1,

$$\begin{aligned}
i\frac{\eta^3(3\tau)}{\theta(3z;3\tau)} &= i\frac{\left(\frac{1}{\sqrt{-3i\tau}}\right)^3\eta^3\left(-\frac{1}{3\tau}\right)}{\frac{i}{\sqrt{-3i\tau}}e^{-\pi i\frac{(3z)^2}{3\tau}}\theta\left(\frac{z}{\tau};-\frac{1}{3\tau}\right)} \\
&= i\frac{\eta^3\left(-\frac{1}{3\tau}\right)}{3\tau e^{-3\pi i\frac{z^2}{\tau}}\theta\left(\frac{z}{\tau};-\frac{1}{3\tau}\right)} = \frac{2\pi\eta^3\left(\frac{2\pi i}{3s}\right)e^{\frac{6\pi^2 z^2}{s}}}{3s\theta\left(\frac{2\pi z}{is};\frac{2\pi i}{3s}\right)} \\
&= \frac{2\pi e^{\frac{6\pi^2 z^2}{s}}e^{-\frac{\pi^2}{6s}}}{3ise^{\frac{2\pi^2 z}{s}}e^{-\frac{\pi^2}{6s}}}\prod_{k=1}^{\infty}\frac{\left(1-e^{-\frac{4\pi^2 k}{3s}}\right)^2}{\left(1-e^{\frac{4\pi^2 z}{s}-\frac{4\pi^2 k}{3s}}\right)\left(1-e^{-\frac{4\pi^2 z}{s}-\frac{4\pi^2(k-1)}{3s}}\right)} \\
&= \frac{2\pi e^{\frac{6\pi^2 z^2}{s}}}{3ise^{\frac{2\pi^2 z}{s}}\left(1-e^{-\frac{4\pi^2 z}{s}}\right)}\prod_{k=1}^{\infty}\frac{\left(1-e^{-\frac{4\pi^2 k}{3s}}\right)^2}{\left(1-e^{\frac{4\pi^2 z}{s}-\frac{4\pi^2 k}{3s}}\right)\left(1-e^{-\frac{4\pi^2 z}{s}-\frac{4\pi^2 k}{3s}}\right)} \\
&= \frac{2\pi e^{\frac{6\pi^2 z^2}{s}}}{3is\left(e^{\frac{2\pi^2 z}{s}}-e^{-\frac{2\pi^2 z}{s}}\right)}\prod_{k=1}^{\infty}\frac{\left(1-e^{-\frac{4\pi^2 k}{3s}}\right)^2}{\left(1-e^{\frac{4\pi^2 z}{s}-\frac{4\pi^2 k}{3s}}\right)\left(1-e^{-\frac{4\pi^2 z}{s}-\frac{4\pi^2 k}{3s}}\right)} \\
&= \frac{-i\pi e^{\frac{6\pi^2 z^2}{s}}}{3s\sinh\left(\frac{2\pi^2 z}{s}\right)}\left[1+O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1-3z)}\right)+O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1+3z)}\right)\right]. \blacksquare
\end{aligned}$$

Before estimating the two last summands of (3.1), we need two more lemmas about A_1 and h.

LEMMA 3.3. *Let $z \in \mathbb{R}$ with $|z| < 1/3$. Then for $|x| \leq 1$ we have, as $n \to \infty$,*

$$A_1\left(\frac{2\pi z}{is},\mp\frac{1}{3};\frac{2\pi i}{3s}\right) = \frac{-1}{2\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(e^{\frac{-2\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(2-3z)}\right) + O\left(e^{\frac{-2\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(2+3z)}\right).$$

Proof. In the proof, we assume that ζ and q are such that $|\zeta q^n| < 1$ if $n > 0$ and $|\zeta q^n| > 1$ if $n < 0$. By applying the geometric series

$$\frac{1}{1-x} = \begin{cases} \sum_{k=0}^{\infty} x^k & \text{if } |x| < 1, \\ -\sum_{k=1}^{\infty} x^{-k} & \text{if } |x| > 1, \end{cases}$$

we find (writing $\rho = e^{2\pi i/3}$)

$$\begin{aligned} \zeta^{-1/2} A_1(z, \mp 1/3; \tau) = \frac{1}{1-\zeta} &+ \sum_{n=1}^{\infty} (-1)^n \rho^{\mp n} q^{\frac{n^2+n}{2}} \sum_{k=0}^{\infty} \zeta^k q^{nk} \\ &- \sum_{n=1}^{\infty} (-1)^n \rho^{\pm n} q^{\frac{n^2-n}{2}} \sum_{k=1}^{\infty} \zeta^{-k} q^{(-n)\cdot(-k)}. \end{aligned}$$

If we see the above as a power series in q, we find that when $n \to \infty$,

$$\zeta^{-1/2} A_1(z, \mp 1/3; \tau) = \frac{1}{1-\zeta} + O(q) + O(\zeta^{-1} q).$$

Thus

$$A_1(z, \mp 1/3; \tau) = \frac{-1}{\zeta^{1/2} - \zeta^{-1/2}} + O(\zeta^{1/2} q) + O(\zeta^{-1/2} q).$$

Plugging in $\zeta = e^{4\pi^2 z/s}$ and $q = e^{-4\pi^2/(3s)}$ (which satisfy our condition that $|\zeta q^n| < 1$ if $n > 0$ and $|\zeta q^n| > 1$ if $n < 0$), we find

$$\begin{aligned} &A_1\left(\frac{2\pi z}{is}, \mp\frac{1}{3}; \frac{2\pi i}{3s}\right) \\ &= \frac{-1}{e^{\frac{2\pi^2 z}{s}} - e^{\frac{-2\pi^2 z}{s}}} + O\big(e^{2\pi^2 z \operatorname{Re}(\frac{1}{s})} e^{-\frac{4\pi^2}{3}\operatorname{Re}(\frac{1}{s})}\big) + O\big(e^{-2\pi^2 z \operatorname{Re}(\frac{1}{s})} e^{-\frac{4\pi^2}{3}\operatorname{Re}(\frac{1}{s})}\big) \\ &= \frac{-1}{2\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\big(e^{\frac{-2\pi^2}{3}\operatorname{Re}(\frac{1}{s})(2-3z)}\big) + O\big(e^{\frac{-2\pi^2}{3}\operatorname{Re}(\frac{1}{s})(2+3z)}\big). \ \blacksquare \end{aligned}$$

We now turn to the Mordell integral.

LEMMA 3.4. *For $|x| \le 1$ we have, as $n \to \infty$,*

$$\left| h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\right| \ll e^{-\beta/6}.$$

Proof. We apply Lemma 3.4 of [5] with $\ell = 0$, $k = 2$, $h = \mp 1$, $u = 0$, $z = \pi/(3s)$ and $\alpha = 3z$. This gives

$$\left| h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\right| \ll e^{\frac{-\pi}{18}\operatorname{Re}(\frac{3s}{\pi})}.$$

The result follows. ∎

With this, we can now prove the following estimate for the Appell–Lerch sums.

Lemma 3.5. *For $|z| \leq 1/6$ and $|x| \leq 1$ we have, as $n \to \infty$,*

$$A_1(3z, \mp\tau; 3\tau) = \frac{i\pi}{3s} \frac{\zeta^{\pm 1} e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6}\operatorname{Re}(\frac{1}{s})}\right).$$

Proof. We use the transformation properties of A_1 to obtain

$$\begin{aligned}
&A_1(3z, \mp\tau; 3\tau)\\
&= \frac{1}{2i} h(3z \pm \tau; 3\tau)\theta(\mp\tau; 3\tau) + \frac{1}{3\tau} e^{\frac{\pi i(3z^2 \pm 2z\tau)}{\tau}} A_1\left(\frac{z}{\tau}, \mp\frac{1}{3}; -\frac{1}{3\tau}\right)\\
&= \frac{1}{2i} h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\theta\left(\mp\frac{is}{2\pi}; \frac{3is}{2\pi}\right) + \frac{2\pi}{3is} e^{\frac{2\pi^2(3z^2 \pm \frac{2izs}{2\pi})}{s}} A_1\left(\frac{2\pi z}{is}, \mp\frac{1}{3}; \frac{2\pi i}{3s}\right)\\
&= \pm\frac{1}{2} e^{s/6} h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\eta\left(\frac{is}{2\pi}\right) - \frac{2\pi i}{3s} e^{\frac{6\pi^2 z^2}{s}} \zeta^{\pm 1} A_1\left(\frac{2\pi z}{is}, \mp\frac{1}{3}; \frac{2\pi i}{3s}\right),
\end{aligned}$$

by Lemmas 2.3 and 2.4. In the last equality we have additionally used

$$\theta(\mp\tau; 3\tau) = \pm i q^{-1/6}\eta(\tau),$$

which is easily deduced from the definition of θ in (2.3). By Lemmas 3.4 and 2.1, we have

$$\left|\frac{1}{2} e^{s/6} h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\eta\left(\frac{is}{2\pi}\right)\right| \ll e^{\beta/6 - \beta/6}\left|\eta\left(\frac{is}{2\pi}\right)\right| \ll \frac{1}{|s|^{1/2}} e^{\frac{-\pi^2}{6}\operatorname{Re}(\frac{1}{s})}.$$

By Lemma 3.3,

$$\begin{aligned}
&-\frac{2\pi i}{3s} e^{\frac{6\pi^2 z^2}{s}} \zeta^{\pm 1} A_1\left(\frac{2\pi z}{is}, \mp\frac{1}{3}; \frac{2\pi i}{3s}\right)\\
&\qquad = \frac{\pi i}{3s} \frac{e^{\frac{6\pi^2 z^2}{s}} \zeta^{\pm 1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(e^{-\pi^2 \operatorname{Re}(\frac{1}{s})(\frac{4}{3} - 2z - 6z^2)}\right) + O\left(e^{-\pi^2 \operatorname{Re}(\frac{1}{s})(\frac{4}{3} + 2z - 6z^2)}\right).
\end{aligned}$$

For $|z| \leq 1/6$, we have $4/3 - 2z - 6z^2 > 1/6$ and $4/3 + 2z - 6z^2 > 1/6$. Therefore

$$e^{-\pi^2 \operatorname{Re}(\frac{1}{s})(\frac{4}{3} + 2z - 6z^2)} \ll \frac{1}{|s|^{1/2}} e^{\frac{-\pi^2}{6}\operatorname{Re}(\frac{1}{s})},$$

$$e^{-\pi^2 \operatorname{Re}(\frac{1}{s})(\frac{4}{3} - 2z - 6z^2)} \ll \frac{1}{|s|^{1/2}} e^{\frac{-\pi^2}{6}\operatorname{Re}(\frac{1}{s})}.$$

Thus the dominant error term comes from $\pm\frac{1}{2} e^{s/6} h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\eta\left(\frac{is}{2\pi}\right)$. The lemma follows. ∎

4. Asymptotic behavior. Since $N(m, n) = N(-m, n)$ for all m and n, we assume from now on that $m \geq 0$. In this section we want to study the asymptotic behaviour of the generating function of $N(m,n)$. Define

$$R_m(\tau) := \int_{-1/2}^{1/2} R(z;\tau)e^{-2\pi imz}\,dz.$$

Let us recall that $\tau = is/(2\pi)$ and $s = \beta(1+ixm^{-1/3})$ with $x \in \mathbb{R}$ satisfying $|x| \le \pi m^{1/3}/\beta$. To simplify the forthcoming calculations, we need the following lemma.

LEMMA 4.1. *We have*

$$R_m(\tau) = 3\frac{q^{1/24}}{\eta(\tau)}\int_{-1/6}^{1/6} g_m(z;\tau)e^{-2\pi imz}\,dz,$$

where

$$g_m(z;\tau) := \begin{cases} -A_1(3z,\tau;3\tau)e^{3\pi iz} + A_1(3z,-\tau;3\tau)e^{-3\pi iz} & \text{for } m \equiv 0 \pmod 3, \\ -A_1(3z,-\tau;3\tau)e^{-\pi iz} - i\dfrac{\eta^3(3\tau)}{\theta(3z;3\tau)}e^{-\pi iz} & \text{for } m \equiv 1 \pmod 3, \\ A_1(3z,\tau;3\tau)e^{\pi iz} + i\dfrac{\eta^3(3\tau)}{\theta(3z;3\tau)}e^{\pi iz} & \text{for } m \equiv 2 \pmod 3. \end{cases}$$

Proof. By (3.1), write

$$R_m(\tau) = \frac{q^{1/24}}{\eta(\tau)}(I_1 - I_2 - I_3),$$

where

$$I_1 := \int_{-1/2}^{1/2} \frac{i(\zeta^{1/2}-\zeta^{-1/2})\eta^3(3\tau)}{\theta(3z;3\tau)}e^{-2\pi imz}\,dz,$$

$$I_2 := \int_{-1/2}^{1/2} \zeta^{-1}(\zeta^{1/2}-\zeta^{-1/2})A_1(3z,-\tau;3\tau)e^{-2\pi imz}\,dz,$$

$$I_3 := \int_{-1/2}^{1/2} \zeta(\zeta^{1/2}-\zeta^{-1/2})A_1(3z,\tau;3\tau)e^{-2\pi imz}\,dz.$$

First, using (2.3) and (2.6), notice that

$$\theta(3z+1;3\tau) = -\theta(3z;3\tau), \tag{4.1}$$

$$A_1(3z+1,\tau;3\tau) = -A_1(3z,\tau;3\tau), \tag{4.2}$$

$$A_1(3z+1,-\tau;3\tau) = -A_1(3z,-\tau;3\tau). \tag{4.3}$$

Thus by (4.1),

$$\begin{aligned}
I_1 &= \Big(\int_{-1/2}^{-1/6} + \int_{-1/6}^{1/6} + \int_{1/6}^{1/2}\Big) \frac{i(\zeta^{1/2}-\zeta^{-1/2})\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi imz}\,dz \\
&= -i\int_{-1/6}^{1/6} \big(e^{\pi i(z-1/3)} - e^{-\pi i(z-1/3)}\big)\frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi im(z-1/3)}\,dz \\
&\quad + i\int_{-1/6}^{1/6} \big(e^{\pi iz} - e^{-\pi iz}\big)\frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi imz}\,dz \\
&\quad - i\int_{-1/6}^{1/6} \big(e^{\pi i(z+1/3)} - e^{-\pi i(z+1/3)}\big)\frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi im(z+1/3)}\,dz \\
&= \int_{-1/6}^{1/6} \Big[e^{\pi iz}\big(-e^{\frac{\pi i}{3}(2m-1)} + 1 - e^{\frac{\pi i}{3}(-2m+1)}\big) \\
&\quad - e^{-\pi iz}\big(-e^{\frac{\pi i}{3}(2m+1)} + 1 - e^{\frac{\pi i}{3}(-2m-1)}\big)\Big] i\frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi imz}\,dz.
\end{aligned}$$

Therefore

$$I_1 = \begin{cases} 0 & \text{for } m\equiv 0 \pmod 3, \\ -3i\displaystyle\int_{-1/6}^{1/6} \frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-\pi iz(2m+1)}\,dz & \text{for } m\equiv 1 \pmod 3, \\ 3i\displaystyle\int_{-1/6}^{1/6} \frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-\pi iz(2m-1)}\,dz & \text{for } m\equiv 2 \pmod 3. \end{cases} \tag{4.4}$$

By the same method and using (4.2) and (4.3), we obtain

$$I_2 = \begin{cases} -3\displaystyle\int_{-1/6}^{1/6} A_1(3z,-\tau;3\tau) e^{-\pi iz(2m+3)}\,dz & \text{for } m\equiv 0 \pmod 3, \\ 3\displaystyle\int_{-1/6}^{1/6} A_1(3z,-\tau;3\tau) e^{-\pi iz(2m+1)}\,dz & \text{for } m\equiv 1 \pmod 3, \\ 0 & \text{for } m\equiv 2 \pmod 3, \end{cases} \tag{4.5}$$

$$(4.6)\quad I_3 = \begin{cases} 3\int_{-1/6}^{1/6} A_1(3z,\tau;3\tau)e^{-\pi iz(2m-3)}\,dz & \text{for } m\equiv 0 \pmod 3,\\ 0 & \text{for } m\equiv 1 \pmod 3,\\ -3\int_{-1/6}^{1/6} A_1(3z,\tau;3\tau)e^{-\pi iz(2m-1)}\,dz & \text{for } m\equiv 2 \pmod 3.\end{cases}$$

The result follows. ∎

4.1. Bounds near the dominant pole. In this section we consider the range $|x|\le 1$. We start by determining the main term of g_m.

LEMMA 4.2. *For all $m\ge 0$, $-1/6\le z\le 1/6$ and $|x|\le 1$ we have, as $n\to\infty$,*

$$g_m\left(z;\frac{is}{2\pi}\right) = \frac{2\pi\sin(\pi z)e^{\frac{6\pi^2z^2}{s}}}{3s\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}}e^{-\frac{\pi^2}{6}\operatorname{Re}(\frac{1}{s})}\right).$$

Proof. If $m\equiv 0 \pmod 3$, by Lemma 3.5 we have

$$\begin{aligned} g_m(z;\tau) &= -A_1(3z,\tau;3\tau)e^{3\pi iz} + A_1(3z,-\tau;3\tau)e^{-3\pi iz}\\ &= -\frac{i\pi}{3s}\frac{e^{\pi iz}e^{\frac{6\pi^2z^2}{s}}}{\sinh\left(\frac{2\pi^2z}{s}\right)} + \frac{i\pi}{3s}\frac{e^{-\pi iz}e^{\frac{6\pi^2z^2}{s}}}{\sinh\left(\frac{2\pi^2z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}}e^{-\frac{\pi^2}{6}\operatorname{Re}(\frac{1}{s})}\right)\\ &= \frac{i\pi}{3s}\frac{e^{\frac{6\pi^2z^2}{s}}}{\sinh\left(\frac{2\pi^2z}{s}\right)}\left(-e^{\pi iz}+e^{-\pi iz}\right) + O\left(\frac{1}{|s|^{1/2}}e^{-\frac{\pi^2}{6}\operatorname{Re}(\frac{1}{s})}\right)\\ &= \frac{2\pi\sin(\pi z)e^{\frac{6\pi^2z^2}{s}}}{3s\sinh\left(\frac{2\pi^2z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}}e^{-\frac{\pi^2}{6}\operatorname{Re}(\frac{1}{s})}\right). \end{aligned}$$

If $m\equiv 1 \pmod 3$, by Lemmas 3.2 and 3.5 we have

$$\begin{aligned} g_m(z;\tau) &= -A_1(3z,-\tau;3\tau)e^{-\pi iz} - i\frac{\eta^3(3\tau)}{\theta(3z;3\tau)}e^{-\pi iz}\\ &= -\frac{i\pi}{3s}\frac{e^{\pi iz}e^{\frac{6\pi^2z^2}{s}}}{\sinh\left(\frac{2\pi^2z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}}e^{-\frac{\pi^2}{6}\operatorname{Re}(\frac{1}{s})}\right)\\ &\quad + \frac{i\pi e^{-\pi iz}e^{\frac{6\pi^2z^2}{s}}}{3s\sinh\left(\frac{2\pi^2z}{s}\right)}\left[1+O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}(\frac{1}{s})(1-3z)}\right)+O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}(\frac{1}{s})(1+3z)}\right)\right] \end{aligned}$$

$$= \frac{i\pi}{3s} \frac{e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)}(-e^{\pi iz} + e^{-\pi iz}) + O\left(\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}\right)$$
$$= \frac{2\pi \sin(\pi z) e^{\frac{6\pi^2 z^2}{s}}}{3s \sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}\right).$$

Finally, if $m \equiv 2 \pmod 3$, by Lemmas 3.2 and 3.5 we have

$$g_m(z;\tau) = A_1(3z, \tau; 3\tau) e^{\pi iz} + i\frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{\pi iz}$$
$$= \frac{i\pi}{3s} \frac{e^{-\pi iz} e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}\right)$$
$$- \frac{i\pi e^{\pi iz} e^{\frac{6\pi^2 z^2}{s}}}{3s \sinh\left(\frac{2\pi^2 z}{s}\right)} \left[1 + O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1-3z)}\right) + O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1+3z)}\right)\right]$$
$$= \frac{i\pi}{3s} \frac{e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)}(e^{-\pi iz} - e^{\pi iz}) + O\left(\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}\right)$$
$$= \frac{2\pi \sin(\pi z) e^{\frac{6\pi^2 z^2}{s}}}{3s \sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}\right). \quad \blacksquare$$

In view of Lemma 4.2 it is natural to define

$$\mathcal{G}_{m,1}(s) := \frac{2\pi}{s} \int_{-1/6}^{1/6} \frac{\sin(\pi z) e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} e^{-2\pi imz}\, dz,$$
$$\mathcal{G}_{m,2}(s) := 3 \int_{-1/6}^{1/6} \left(g_m\left(z; \frac{is}{2\pi}\right) - \frac{2\pi \sin(\pi z) e^{\frac{6\pi^2 z^2}{s}}}{3s \sinh\left(\frac{2\pi^2 z}{s}\right)}\right) e^{-2\pi imz}\, dz.$$

Thus

$$R_m(\tau) = \frac{q^{1/24}}{\eta(\tau)}(\mathcal{G}_{m,1}(s) + \mathcal{G}_{m,2}(s)). \tag{4.7}$$

Note that we can rewrite $\mathcal{G}_{m,1}(s)$ as

$$\mathcal{G}_{m,1}(s) = \frac{4\pi}{s} \int_0^{1/6} \frac{\sin(\pi z) e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} \cos(2\pi mz)\, dz.$$

Lemma 4.3. *Assume that $|x| \le 1$ and $m \le \frac{1}{6\beta}\log n$. Then, as $n \to \infty$,*

$$\mathcal{G}_{m,1}(s) = \frac{s}{4}\operatorname{sech}^2\left(\frac{\beta m}{2}\right) + O\left(\beta^2 m^{2/3} \operatorname{sech}^2\left(\frac{\beta m}{2}\right)\right).$$

Proof. We use the same method as in [3]. Inserting the Taylor expansion of $\sin(\pi z)$, $\exp(6\pi^2 z^2/s)$, and $\cos(2\pi m z)$ in the definition of $\mathcal{G}_{m,1}(s)$, we find that

$$\sin(\pi z)e^{\frac{6\pi^2 z^2}{s}}\cos(2\pi m z) = \sum_{j,\nu,r\geq 0}\frac{(-1)^{j+\nu}}{(2j+1)!(2\nu)!r!}\pi^{2j+1}(2\pi m)^{2\nu}\left(\frac{6\pi^2}{s}\right)^r z^{2j+2\nu+2r+1}.$$

This yields

$$\mathcal{G}_{m,1}(s) = \frac{4\pi}{s}\sum_{j,\nu,r\geq 0}\frac{(-1)^{j+\nu}}{(2j+1)!(2\nu)!r!}\pi^{2j+1}(2\pi m)^{2\nu}\left(\frac{6\pi^2}{s}\right)^r \mathcal{I}_{j+\nu+r},$$

where for $\ell \in \mathbb{N}_0$ we define

$$\mathcal{I}_\ell := \int_0^{1/6}\frac{z^{2\ell+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)}\,dz.$$

We next relate $\mathcal{I}_\ell$ to $\mathcal{E}_\ell$ defined in (2.8). For this, we note that

$$\mathcal{I}_\ell = \int_0^{\infty}\frac{z^{2\ell+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)}\,dz - \mathcal{I}'_\ell \tag{4.8}$$

with

$$\mathcal{I}'_\ell := \int_{1/6}^{\infty}\frac{z^{2\ell+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)}\,dz \ll \int_{1/6}^{\infty} z^{2\ell+1}e^{-2\pi^2 z\operatorname{Re}\left(\frac{1}{s}\right)}\,dz$$
$$\ll \left(\operatorname{Re}\left(\frac{1}{s}\right)\right)^{-2\ell-2}\Gamma\left(2\ell+2;\frac{\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)\right),$$

where $\Gamma(\alpha;x) := \int_x^\infty e^{-w}w^{\alpha-1}\,dw$. Since

$$\Gamma(\ell;x) \sim x^{\ell-1}e^{-x} \quad \text{as } x\to\infty, \tag{4.9}$$

this yields

$$\mathcal{I}'_\ell \ll \left(\operatorname{Re}\left(\frac{1}{s}\right)\right)^{-1}e^{-\frac{\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)} \leq e^{-\frac{\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)}.$$

By a substitution in Lemma 2.6, we know that

$$\int_0^{\infty}\frac{z^{2\ell+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)}\,dz = \left(\frac{s}{2\pi}\right)^{2\ell+2}\frac{(-1)^{\ell+1}E_{2\ell+1}(0)}{2}.$$

Thus

$$\begin{aligned}\mathcal{G}_{m,1}(s) &= \sum_{j,\nu,r\geq 0} \frac{(-1)^{r+1}3^r}{2^{2j+r+1}(2j+1)!(2\nu)!r!} m^{2\nu} s^{2j+2\nu+r+1} \\ &\qquad \times \big(E_{2j+2\nu+2r+1}(0) + O\big(|z|^{-2j-2\nu-2r-2} e^{-\frac{\pi^2}{3}\operatorname{Re}(\frac{1}{s})}\big)\big) \\ &= \sum_{\nu=0}^{\infty} \frac{(ms)^{2\nu}}{(2\nu)!}\bigg(-\frac{s}{2}E_{2\nu+1}(0) + O(|s|^2)\bigg) \\ &= \frac{s}{4}\operatorname{sech}^2\bigg(\frac{ms}{2}\bigg) + O(|s|^2\cosh(ms)),\end{aligned}$$

where for the last equality we have used Lemma 2.5. The end of the proof is now exactly the same as in [3, Lemma 3.2]. ■

We now want to bound $\mathcal{G}_{m,2}(s)$.

Lemma 4.4. *Assume that* $|x| \leq 1$. *Then, as* $n \to \infty$,

$$|\mathcal{G}_{m,2}(s)| \ll \frac{1}{\beta^{1/2}} e^{-\frac{\pi^2}{12\beta}}.$$

Proof. By Lemma 4.2, we have

$$|\mathcal{G}_{m,2}(s)| \ll \int_{-1/6}^{1/6} \bigg|\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6}\operatorname{Re}(\frac{1}{s})} e^{-2\pi i m z}\bigg|\, dz \ll \frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6}\operatorname{Re}(\frac{1}{s})}.$$

By the definition of s, we know that $1/|s|^{1/2} \leq 1/\beta^{1/2}$. Furthermore, as $|x| \leq 1$, we have $\operatorname{Re}(1/s) \geq 1/(2\beta)$. This yields the conclusion. ■

Combining Lemmas 4.3 and 4.4, we obtain the following asymptotic estimate of $R_m(\tau)$ near the dominant pole.

Proposition 4.5. *Assume that* $|x| \leq 1$. *Then, as* $n \to \infty$,

$$R_m(\tau) = \frac{s^{3/2}}{4(2\pi)^{1/2}} \operatorname{sech}^2\bigg(\frac{\beta m}{2}\bigg) e^{\frac{k\pi^2}{6s}} + O\bigg(\beta^{5/2} m^{2/3} \operatorname{sech}^2\bigg(\frac{\beta m}{2}\bigg) e^{\pi\sqrt{n/6}}\bigg).$$

Proof. Recall from (4.7) that

$$R_m(\tau) = \frac{q^{1/24}}{\eta(\tau)}(\mathcal{G}_{m,1}(s) + \mathcal{G}_{m,2}(s)).$$

By Lemma 2.1 we see that

$$\frac{q^{1/24}}{\eta(\tau)} = \bigg(\frac{s}{2\pi}\bigg)^{1/2} e^{\frac{\pi^2}{6s}}(1 + O(\beta)).$$

We approximate $\mathcal{G}_{m,1}$ and $\mathcal{G}_{m,2}$ using Lemmas 4.3 and 4.4. The main error term comes from $\mathcal{G}_{m,1}$. We obtain

$$R_m(\tau) = \frac{s^{3/2}}{4(2\pi)^{1/2}} e^{\frac{\pi^2}{6s}} \operatorname{sech}^2\left(\frac{\beta m}{2}\right) + O\left(s^{1/2}\beta^2 m^{2/3} \operatorname{sech}^2\left(\frac{\beta m}{2}\right) e^{\frac{\pi^2}{6s}}\right).$$

The claim follows now using the estimates

$$|s| \ll \beta, \qquad \operatorname{Re}\left(\frac{1}{s}\right) \leq \frac{1}{\beta} = \frac{\sqrt{6n}}{\pi}. \quad \blacksquare$$

4.2. Estimates far from the dominant pole. In the previous section, we have established bounds for the behaviour of $R_m(\tau)$ close to the pole $\tau = 0$. For Wright's version of the circle method, we also need estimates far away from this pole. In this section, we consider the range $1 \leq |x| \leq \pi m^{1/3}/\beta$. First we need a lemma, which follows from an argument similar to the one in [10] (see also [3, Lemma 3.5]).

LEMMA 4.6. *Let $P(q) = q^{1/24}/\eta(\tau)$ be the generating function for partitions. Assume that $\tau = u + iv \in \mathbb{H}$. For $Mv \leq |u| \leq 1/2$ and $v \to 0$, we have*

$$|P(q)| \ll \sqrt{v} \exp\left[\frac{1}{v}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+M^2}}\right)\right)\right].$$

Proof. Let us write the following Taylor rearrangement:

$$\log(P(q)) = -\sum_{n=1}^{\infty} \log(1-q^n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{q^{nm}}{m} = \sum_{m=1}^{\infty} \frac{q^m}{m(1-q^m)}.$$

Then we have the estimate

$$\begin{aligned} |\log(P(q))| &\leq \sum_{m=1}^{\infty} \frac{|q|^m}{m|1-q^m|} \leq \frac{|q|}{|1-q|} - \frac{|q|}{1-|q|} + \sum_{m=1}^{\infty} \frac{|q|^m}{m(1-|q|^m)} \\ &= \log(P(|q|)) - |q|\left(\frac{1}{1-|q|} - \frac{1}{|1-q|}\right). \end{aligned}$$

For $Mv \leq |u| \leq 1/4$, we have $\cos(2\pi u) \leq \cos(2\pi M v)$. Therefore

$$|1-q|^2 = 1 - 2e^{-2\pi v}\cos(2\pi u) + e^{-4\pi v} \geq 1 - 2e^{-2\pi v}\cos(2\pi M v) + e^{-4\pi v}.$$

By a Taylor expansion around $v = 0$ we find that

$$|1-q| \geq 2\pi v\sqrt{1+M^2} + O(v^2). \tag{4.10}$$

When $1/4 \leq |u| \leq 1/2$, we have $\cos(2\pi u) \leq 0$. Therefore

$$|1-q| \geq 1.$$

When $v \to 0$, this is asymptotically larger than (4.10). Hence, for all $Mv \leq |u| \leq 1/2$,

$$|1-q| \geq 2\pi v\sqrt{1+M^2} + O(v^2). \tag{4.11}$$

Furthermore,

$$1-|q| = 1-e^{-2\pi v} = 2\pi v + O(v^2). \tag{4.12}$$

By Lemma 2.1, we have

$$P(|q|) = \frac{e^{\frac{-2\pi v}{24}}}{\eta(iv)} = \sqrt{v}\, e^{\frac{\pi}{12v}}(1+O(v)).$$

Thus

$$\log(P(|q|)) = \frac{\pi}{12v} + \frac{1}{2}\log(v) + O(v). \tag{4.13}$$

Combining (4.11)–(4.13), we finally obtain

$$\begin{aligned}
|\log(P(q))| &\le \log(P(|q|)) - \frac{1}{2\pi v}\left(1-\frac{1}{\sqrt{1+M^2}}\right)(1+O(v)) \\
&= \frac{\pi}{12v} + \frac{1}{2}\log(v) + O(v) - \frac{1}{2\pi v}\left(1-\frac{1}{\sqrt{1+M^2}}\right) + O(1) \\
&= \frac{1}{v}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1-\frac{1}{\sqrt{1+M^2}}\right)\right) + \frac{1}{2}\log(v) + O(1).
\end{aligned}$$

Exponentiating yields the desired result. ■

We are now able to bound $|R_m(\tau)|$ away from $q=1$.

PROPOSITION 4.7. *Assume that* $1 \le |x| \le \pi m^{1/3}/\beta$. *Then, as* $n \to \infty$,

$$|R_m(\tau)| \ll \sqrt{n}\exp\left(\pi\sqrt{\frac{n}{6}} - \frac{\sqrt{6n}}{8\pi} m^{-2/3}\right).$$

Proof. By (2.1), we have

$$\begin{aligned}
R_m(\tau) &= P(q)\int_{-1/2}^{1/2}\left((1-\zeta)\sum_{k\in\mathbb{Z}}\frac{(-1)^k q^{\frac{3k^2+k}{2}}}{1-\zeta q^k}\right)e^{-2\pi i m z}\,dz \\
&= P(q)\int_{-1/2}^{1/2}\Bigg(1+(1-\zeta)\sum_{k\ge1}\frac{(-1)^k q^{\frac{3k^2+k}{2}}}{1-\zeta q^k} \\
&\qquad\qquad + (1-\zeta^{-1})\sum_{k\ge1}\frac{(-1)^k q^{\frac{3k^2+k}{2}}}{1-\zeta^{-1} q^k}\Bigg)e^{-2\pi i m z}\,dz.
\end{aligned}$$

So we may bound $|R_m(\tau)|$ when $n\to\infty$ in the following way:

$$\begin{aligned}|R_m(\tau)| &\ll |P(q)| \int_{-1/2}^{1/2} \sum_{k\geq 1} \frac{|q|^{\frac{3k^2+k}{2}}}{1-|q|^k} |e^{-2\pi i m z}|\,dz\\ &\ll |P(q)| \frac{1}{1-|q|} \sum_{k\geq 1} e^{-\beta\frac{3k^2}{2}}\\ &\ll |P(q)| \frac{1}{1-|q|} \int_{-\infty}^{\infty} e^{-\beta\frac{3x^2}{2}}\,dx\\ &\ll |P(q)| \frac{1}{\beta}\sqrt{\frac{2\pi}{3\beta}} \ll |P(q)| n^{3/4}.\end{aligned}$$

Now we use Lemma 4.6 with $v=\beta/(2\pi)$, $u=\beta m^{-1/3}x/(2\pi)$ and $M=m^{-1/3}$. We obtain, for $1\leq |x|\leq \pi m^{1/3}/\beta$,

$$|P(q)| \ll n^{-1/4} \exp\left[\frac{2\pi}{\beta}\left(\frac{\pi}{12}-\frac{1}{2\pi}\left(1-\frac{1}{\sqrt{1+m^{-2/3}}}\right)\right)\right].$$

Therefore

$$\begin{aligned}|R_m(\tau)| &\ll n^{1/2} \exp\left[\frac{2\pi}{\beta}\left(\frac{\pi}{12}-\frac{1}{2\pi}\left(1-\frac{1}{\sqrt{1+m^{-2/3}}}\right)\right)\right]\\ &\ll n^{1/2} \exp\left[\pi\sqrt{\frac{n}{6}}-\frac{\sqrt{6n}}{\pi}\left(1-\frac{1}{\sqrt{1+m^{-2/3}}}\right)\right]\\ &\ll n^{1/2} \exp\left(\pi\sqrt{\frac{n}{6}}-\frac{\sqrt{6n}}{8\pi}m^{-2/3}\right). \blacksquare\end{aligned}$$

5. The Circle Method. In this section, as in [3], we use Wright's variant of the Circle Method to complete the proof of Theorem 1.2.

Using Cauchy's theorem, we write $N(m,n)$ as an integral of its generating function $R_m(\tau)$:

$$N(m,n) = \frac{1}{2\pi i} \int_C \frac{R_m(\tau)}{q^{n+1}}\,dq,$$

where the contour is the counterclockwise traversal of the circle $C := \{q\in\mathbb{C} : |q| = e^{-\beta}\}$. Recall that $s=\beta(1+ixm^{-1/3})$. Changing variables we may write

$$N(m,n) = \frac{\beta}{2\pi m^{1/3}} \int_{|x|\leq \pi m^{1/3}/\beta} R_m\left(\frac{is}{2\pi}\right) e^{ns}\,dx.$$

We split this integral into two pieces, $N(m,n) = M + E$, with

$$M := \frac{\beta}{2\pi m^{1/3}} \int\limits_{|x|\le 1} R_m\left(\frac{is}{2\pi}\right) e^{ns}\,dx,$$

$$E := \frac{\beta}{2\pi m^{1/3}} \int\limits_{1\le |x|\le \pi m^{1/3}/\beta} R_m\left(\frac{is}{2\pi}\right) e^{ns}\,dx.$$

In the following we show that M contributes to the asymptotic main term, whereas E is part of the error term.

As the estimate of $R_m(\tau)$ close to the dominant pole is exactly the same as the one of $\mathcal{C}_{m,1}(q)$ in [3], the asymptotic behaviour of M here is the same as in [3]:

Proposition 5.1. *We have*

$$M = \frac{\beta}{4}\operatorname{sech}^2\left(\frac{\beta m}{2}\right) p(n)\left(1 + O\left(\frac{m^{1/3}}{n^{1/4}}\right)\right).$$

Let us now turn to the integral E.

Proposition 5.2. *As $n \to \infty$,*

$$E \ll n^{1/2} \exp\left(\pi\sqrt{\frac{2n}{3}} - \frac{\sqrt{6n}}{8\pi} m^{-2/3}\right).$$

Proof. Using Proposition 4.7, we may bound

$$E \ll \frac{\beta}{m^{1/3}} \int\limits_{1\le x\le \pi m^{1/3}/\beta} n^{1/2} \exp\left(\pi\sqrt{\frac{n}{6}} - \frac{\sqrt{6n}}{8\pi} m^{-2/3}\right) e^{\beta n}\,dx$$

$$\ll n^{1/2} \exp\left(\pi\sqrt{\frac{2n}{3}} - \frac{\sqrt{6n}}{8\pi} m^{-2/3}\right). \quad \blacksquare$$

Thus E is exponentially smaller than M. This completes the proof of Theorem 1.2.

Acknowledgements. The second author's research is supported by the DFG Graduiertenkolleg 1269 "Global Structures in Geometry and Analysis" at the University of Cologne. For many helpful discussions and comments on earlier versions of this paper the authors want to thank Kathrin Bringmann, Wolf Jung, Jeremy Lovejoy, Karl Mahlburg, and José Miguel Zapata Rolón.

References

[1] G. Andrews and F. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc. 18 (1988), 167–171.

[2] A. Atkin and H. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc. 4 (1954), 84–106.

[3] K. Bringmann and J. Dousse, *On Dyson's crank conjecture and the uniform asymptotic behavior of some inverse theta functions*, Trans. Amer. Math. Soc., to appear.

[4] K. Bringmann and K. Mahlburg, *Asymptotic inequalities for positive crank and rank moments*, Trans. Amer. Math. Soc. 366 (2014), 1073–1094.

[5] K. Bringmann, K. Mahlburg, and R. C. Rhoades, *Taylor coefficients of mock Jacobi forms and moments of partition statistics*, Math. Proc. Cambridge Philos. Soc. 157 (2014), 231–251.

[6] F. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) 8 (1944), 10–15.

[7] F. Garvan, *New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7, and 11*, Trans. Amer. Math. Soc. 305 (1988), 47–77.

[8] D. Parry and R. C. Rhoades, *Dyson's crank distribution conjecture*, preprint, 2014.

[9] S. Ramanujan, *Congruence properties of partitions*, Math. Z. 9 (1921), 147–153.

[10] E. Wright, *Asymptotic partition formulae II. Weighted partitions*, Proc. London Math. Soc. 36 (1933), 117–141.

[11] D. B. Zagier, *Ramanujan's mock theta functions and their applications [d'après Zwegers and Bringmann–Ono]*, Astérisque 326 (2009), 143–164.

[12] S. Zwegers, *Mock theta functions*, Ph.D. thesis, Universiteit Utrecht, 2002.

Jehanne Dousse
LIAFA, Université Paris Diderot
75025 Paris Cedex 13, France
E-mail: jdousse@liafa.univ-paris-diderot.fr

Michael H. Mertens
Mathematisches Institut
der Universität zu Köln
Weyertal 86-90
D-50931 Köln, Germany
E-mail: mmertens@math.uni-koeln.de

Received on 25.8.2014 (7907)