ACTA ARITHMETICA
134.3 (2008)

# On the error term in Weyl's law for Heisenberg manifolds

by

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**Contents**



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2000 *Mathematics Subject Classification*: 11N37, 35P20, 58J50.

*Key words and phrases*: Heisenberg manifold, Weyl's law, error term, Voronoi's formula, power moment.

This work is supported by National Natural Science Foundation of China (Grant No. 10771127).

 

**1. Introduction.** Let $(M,g)$ be a closed $n$-dimensional Riemannian manifold with metric $g$ and Laplace–Beltrami operator $\Delta$. Let $N(t)$ denote its spectral counting function, which is defined as the number of eigenvalues of $\Delta$ not exceeding $t$. Hörmander [13] proved Weyl's law

$$N(t)=\frac{\operatorname{vol}(B_n)\operatorname{vol}(M)}{(2\pi)^n}\,t^{n/2}+O(t^{(n-1)/2}),\tag{1.1}$$

where $\operatorname{vol}(B_n)$ is the volume of the $n$-dimensional unit ball.

Let

$$R(t)=N(t)-\frac{\operatorname{vol}(B_n)\operatorname{vol}(M)}{(2\pi)^n}\,t^{n/2}.$$

Hörmander's estimate (1.1) is in general sharp, as the well-known example of the sphere $S^n$ with its canonical metric shows [13]. However, it is a difficult problem to determine the optimal bound of $R(t)$ in any given manifold, which depends on the properties of the associated geodesic flow. Many improvements have been obtained for certain types of manifolds (see [1–4, 7, 9, 14, 16, 18, 24, 29]).

**1.1.** *Weyl's law for* $\mathbb{T}^2$: *the Gauss circle problem.* The simplest compact manifold with integrable geodesic flow is the 2-torus $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$. The exponential functions $e(mx+ny)$ $(m,n\in\mathbb{Z})$ form a basis of eigenfunctions of the Laplace operator $\Delta=\partial_x^2+\partial_y^2$, which acts on functions on $\mathbb{T}^2$. The corresponding eigenvalues are $4\pi^2(m^2+n^2)$, $m,n\in\mathbb{Z}$. The spectral counting function

$$N_I(t)=\{\lambda_j\in\operatorname{Spec}(\Delta):\lambda_j\le t\}$$

is equal to the number of lattice points of $\mathbb{Z}^2$ inside a circle of radius $\sqrt{t}/2\pi$. The well-known Gauss circle problem is to study the properties of the error term of the function $N_I(t)$.

In this case, the formula (1.1) becomes

$$N_I(t)=\frac{t}{4\pi}+O(t^{1/2}),\tag{1.2}$$

which is the classical result of Gauss. Let $R_I(t)$ denote the error term in (1.2). Many authors improved the upper bound estimate of $R_I(t)$. The latest result, due to Huxley [14], reads

$$R_I(t)\ll t^{131/416}\log^{26957/8320}t.\tag{1.3}$$

Hardy [10] conjectured that

$$R_I(t)\ll t^{1/4+\varepsilon}.\tag{1.4}$$

Cramér [5] proved that

$$\lim_{T\to\infty}T^{-3/2}\int_1^T|R_I(t)|^2\,dt=C,\qquad C=\frac{1}{6\pi^3}\sum_{n=1}^\infty\frac{r^2(n)}{n^{3/2}},$$

which is consistent with Hardy's conjecture, where $r(n)$ denotes the number of ways $n$ can be written as a sum of two squares.

Ivić [15] first used the large value technique to study the higher power moments of $R_I(t)$. He proved the estimate

$$\int_1^T |R_I(t)|^A\,dt \ll T^{1+A/4+\varepsilon} \tag{1.5}$$

for each fixed $0 \le A \le 35/4$. The value of $A$ for which (1.5) holds is closely related to the upper bound of $R_I(t)$. If we insert the estimate (1.3) into Ivić's machinery, we see that (1.5) holds for $0 \le A \le 262/27$.

Tsang [26] studied the third and fourth moments of $R_I(t)$. He proved the following two asymptotic formulas:

$$\int_1^T R_I^3(t)\,dt = c_3T^{7/4} + O(T^{7/4-1/14+\varepsilon}), \tag{1.6}$$

$$\int_1^T R_I^4(t)\,dt = c_4T^2 + O(T^{2-1/23+\varepsilon}), \tag{1.7}$$

where $c_3$ and $c_4$ are explicit constants.

Heath-Brown [11] proved that the function $t^{-1/4}R_I(t)$ has a distribution function $f(\alpha)$ in the sense that for any interval $I$ we have

$$T^{-1}\operatorname{mes}\{t \in [1,T] : t^{-1/4}R_I(t) \in I\} \to \int_I f(\alpha)\,d\alpha$$

as $T \to \infty$. He also proved that for any real number $k \in [0,9]$ the mean value

$$\lim_{T\to\infty} T^{-1-k/4}\int_1^T |R_I(t)|^k\,dt$$

converges to a finite limit as $T$ tends to infinity. Moreover, the same is true for

$$\lim_{T\to\infty} T^{-1-k/4}\int_1^T R_I^k(t)\,dt$$

with $k = 3,5,7,9$.

In [30], the author proved the following result. Let $A > 9$ be a real number such that (1.5) holds. Then for any integer $3 \le k < A$, we have the asymptotic formula

$$\int_1^T R_I^k(t)\,dt = c_kT^{1+k/4} + O(T^{1+k/4-\delta_k+\varepsilon}), \tag{1.8}$$

where $c_k$ and $\delta_k > 0$ are explicit constants. In particular, the asymptotic formula (1.8) holds for $k = 3, \ldots, 9$.

We remark that we can take $\delta_3 = 7/20$, which is due to Tsang and comes from the third moment of the error term of the Dirichlet divisor problem. However, Tsang did not publish this result.

For the fourth moment of $R_I(t)$, the author [31] proved that we can take $\delta_4 = 3/28$.

**1.2.** *Weyl's law for a* $(2l+1)$*-dimensional Heisenberg manifold.* Let $l \geq 1$ be a fixed integer and $(H_l/\Gamma, g)$ be a $(2l+1)$-dimensional Heisenberg manifold with a metric $g$. When $l = 1$, in [24] Petridis and Toth proved that $R(t) = O(t^{5/6} \log t)$ for a special metric. Later in [4] this bound was improved to $O(t^{119/146+\varepsilon})$ for all left-invariant Heisenberg metrics. For $l > 1$ Khosravi and Petridis [18] proved that $R(t) = O(t^{l-7/41})$ for rational Heisenberg manifolds. Both in [4] and [18], first a $\psi$-expression of $R(t)$ was established and then the van der Corput method of exponential sums was used. Substituting Huxley's result of [14] into the arguments of [4] and [18], we can get the estimate

$$R(t) = O(t^{l-77/416}(\log t)^{26957/8320}) \tag{1.9}$$

for all rational $(2l+1)$-dimensional Heisenberg manifolds.

It was conjectured that for rational Heisenberg manifolds, the pointwise estimate

$$R(t) \ll t^{l-1/4+\varepsilon} \tag{1.10}$$

holds (see Petridis and Toth [24] for $l = 1$ and in Khosravi and Petridis [18] for $l > 1$). As an evidence supporting this conjecture, Petridis and Toth proved the following $L^2$ result for $H_1$:

$$\int_{I^3} \left| N(t;u) - \frac{1}{6\pi^2} \operatorname{vol}(M(u)) t^{3/2} \right|^2 du \leq C_\delta t^{3/2+\delta},$$

where $I = [1-\varepsilon, 1+\varepsilon]$. They also proved

$$\frac{1}{T} \int_T^{2T} \left| N(t) - \frac{1}{6\pi^2} \operatorname{vol}(M) t^{3/2} \right| dt \gg T^{3/4}.$$

Now let $M = (H_l/\Gamma, g_l)$ be a $(2l+1)$-dimensional Heisenberg manifold with the metric

$$g_l := \begin{pmatrix} I_{2l\times 2l} & 0 \\ 0 & 2\pi \end{pmatrix},$$

where $I_{2l\times 2l}$ is the identity matrix. M. Khosravi and J. A. Toth [19] proved

$$\int_1^T |R(t)|^2\,dt = C_{2,l}T^{2l+1/2} + O(T^{2l+1/4+\varepsilon}), \tag{1.11}$$

where $C_{2,l}$ is an explicit constant.

M. Khosravi [17] proved the asymptotic formula

$$\int_1^T R^3(t)\,dt = C_{3,l}T^{3l+1/4} + O(T^{3l+3/14+\varepsilon}) \tag{1.12}$$

for some explicit constant $C_{3,l}$.

The aim of this paper is to prove some power moment results for $R(t)$, analogous to the results for $R_I(t)$ stated in Section 1.1. The plan of this paper is as follows. In Section 2 we state our main results. In Section 3 we give some background on Heisenberg manifolds and a new $\psi$-expression of $R(t)$. In Section 4 we quote some preliminary lemmas. We shall prove our theorems in Sections 5–7.

*Notations.* For a real number $t$, let $[t]$ denote the integer part of $t$, $\{t\} = t - [t]$, $\psi(t) = \{t\} - 1/2$, $\|t\| = \min(\{t\}, 1 - \{t\})$, $e(t) = e^{2\pi it}$. By $\varepsilon$ we always denote a sufficiently small positive constant. $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{N}$ denote the set of complex numbers, of real numbers, of integers, and of positive integers, respectively; $n \sim M$ means that $N < n \le 2N$; $d(n)$ denotes the Dirichlet divisor function, $r(n)$ the number of ways $n$ can be written as a sum of two squares, and $\mu(n)$ the Möbius function. Throughout this paper, $\mathcal{L}$ always denotes $\log T$.

**2. Main results.** From now on, $R(t)$ always denotes the error term in Weyl's law for the $(2l+1)$-dimensional Heisenberg manifold $(H_l/\Gamma, g_l)$.

**2.1.** *New results for power moments of $R(t)$.* Our first result is an analogue of (1.5) for $R_I(t)$.

THEOREM 1. *Suppose $A \ge 0$ is a fixed real number. Then*

$$\int_1^T |R(t)|^A\,dt \ll T^{1+A(l-1/4)}\mathcal{L}^{41} \qquad (0 \le A \le 262/27), \tag{2.1}$$

$$\int_1^T |R(t)|^A\,dt \ll T^{A(l-1)+\frac{154+339A}{416}}\mathcal{L}^{4A+1} \qquad (A > 262/27). \tag{2.2}$$

REMARK 2.1. Let $A_0 > 1$ be a positive constant such that

$$\int_1^T |R(t)|^{A_0}\,dt \ll T^{1+A_0(l-1/4)+\varepsilon}. \tag{2.3}$$

Then Theorem 1 states that we can take $A_0 = 262/27$. The value of $A_0$ for which (2.3) holds is closely related to the upper bound of $R(t)$. The exponent 262/27 follows from (1.9). If the conjecture (1.10) would be true, then obviously (2.3) would be true for any $A_0 > 0$. Conversely, if (2.3) were true for any $A_0 > 0$, then we could show (1.10) as follows. The estimate (2.3) implies

$$\int_{T/2}^{T} |R(2\pi x)|^{A_0}\, dx \ll T^{1+A_0(l-1/4)+\varepsilon}.$$

Suppose $|R(2\pi x)|$ reaches its largest value $V_0$ at some $x_0 \in [T/2, T]$. According to the formula (5.11), we have

$$V_0^{A_0+1}(x_0^{l-1/2}\log x_0)^{-1} \ll \int_{T/2}^{T} |R(2\pi x)|^{A_0}\, dx \ll T^{1+A_0(l-1/4)+\varepsilon},$$

which implies that

$$V_0 \ll T^{\frac{l+1/2+A_0(l-1/4)+2\varepsilon}{A_0+1}}.$$

Now the conjecture (1.10) follows by choosing $A_0$ large.

Before stating our next theorem, we introduce some notations.

Suppose $f : \mathbb{N} \to \mathbb{R}$ is any function such that $f(n) \ll n^{\varepsilon}$, and $k \geq 2$ is a fixed integer. Define

$$\text{(2.4)} \quad s_{k;v}(f) := \sum_{\sqrt{n_1}+\cdots+\sqrt{n_v}=\sqrt{n_{v+1}}+\cdots+\sqrt{n_k}} \frac{f(n_1)\cdots f(n_k)}{(n_1\cdots n_k)^{3/4}} \qquad (1 \leq v < k),$$

$$\text{(2.5)} \quad B_k(f) := \sum_{v=1}^{k-1} \binom{k-1}{v} s_{k;v}(f)\cos(\pi(k-2v)/4),$$

$$\text{(2.6)} \quad \tau_l(n) := \sum_{n=h(2r-h)} \frac{e(lh/2)h^{1/2}}{(2r-h)^{1/2}}\left(1-\frac{h}{2r-h}\right)^{l-1}.$$

We shall use $s_{k;v}(f)$ to denote both the series (2.4) and its value. The convergence of $s_{k;v}(f)$ was already proved in [30]. It is obvious that $|\tau_l(n)| \leq d(n)$.

Suppose $A_0 > 2$ is a real number, and define

$$K_0 := \min\{n \in \mathbb{N} : n \geq A_0,\ 2\,|\,n\},$$

$$s(k) := 2^{k-2} + (k-6)/4, \qquad \sigma(k, A_0) := \frac{A_0 - k}{2(A_0 - 2)},$$

$$\delta_1(k, A_0) := \frac{\sigma(k, A_0)}{2s(K_0)}, \qquad \delta_2(k, A_0) := \frac{\sigma(k, A_0)}{2s(k) + 2\sigma(k, A_0)}.$$

Theorem 2. *Let $A_0 > 9$ be such that* (2.3) *holds. Then for any integer $3 \le k < A_0$, we have the asymptotic formula*

$$\int_1^T R^k(t)\,dt = \frac{2^{3+5k/4-2kl} l^k B_k(\tau_l)}{(l!)^k \pi^{3k/4+kl}(4+k(4l-1))}\, T^{1+k(l-1/4)} + O(T^{1+k(l-1/4)-\delta_1(k,A_0)+\varepsilon}). \tag{2.7}$$

Remark 2.2. From Theorem 1 we see that for $k \in \{3,\dots,9\}$, we can get the asymptotic formula (2.7) with $A_0 = 262/27$. Moreover, (2.7) provides the exact form of the main term for $\int_1^T R^k(t)\,dt$ with $k \ge 10$. From Theorem 2 we see that if the conjecture (1.10) were true, then for all $k \ge 3$ we could get an asymptotic formula for $\int_1^T R^k(t)\,dt$.

When $3 \le k \le 9$, we have the following improvement of Theorem 2.

Theorem 3. *For $3 \le k \le 9$, the asymptotic formula* (2.7) *holds with $\delta_1(k, A_0)$ replaced by $\delta_2(k, 262/27)$.*

Remark 2.3. Note that $\delta_2(3, 262/27) = 181/1402$ and $\delta_2(4, 262/27) = 11/230$, which are small compared to the corresponding results for $R_I(t)$. The following Theorem 4 improves these two exponents.

Theorem 4. *We have*

$$\int_1^T R^3(t)\,dt = \frac{2^{27/4-6l} l^3 B_3(\tau_l)}{(l!)^3 \pi^{3l+9/4}(1+12l)}\, T^{3l+1/4} + O(T^{3l+\varepsilon}), \tag{2.8}$$

$$\int_1^T R^4(t)\,dt = \frac{2^{4-8l} l^3 B_4(\tau_l)}{(l!)^4 \pi^{4l+3}}\, T^{4l} + O(T^{4l-3/28+\varepsilon}). \tag{2.9}$$

The following two theorems are analogous to Heath-Brown's results for $R_I(t)$.

Theorem 5. *The function $t^{-(l-1/4)}R(t)$ has a distribution function $f(\alpha)$ in the sense that for any interval $I \subset \mathbb{R}$ we have*

$$T^{-1}\,\mathrm{mes}\{t \in [1,T] : t^{-(l-1/4)}R(t) \in I\} \to \int_I f(\alpha)\,d\alpha$$

*as $T \to \infty$. The function $f(\alpha)$ and its derivatives satisfy*

$$\frac{d^k}{d\alpha^k} f(\alpha) \ll_{A,k} (1+|\alpha|)^{-A}$$

*for $k = 0,1,2,\dots$ and $f(\alpha)$ can be extended to an entire function.*

THEOREM 6. *For any real number $k \in [0, 262/27)$ the mean value*

$$\lim_{T\to\infty} T^{-1-k(l-1/4)} \int_1^T |R(t)|^k\, dt$$

*converges to a finite limit as $T$ tends to infinity.*

REMARK 2.4. Our approach yields a different proof of the formula (1.11), and the error term estimate $O(T^{2l+1/4+\varepsilon})$ therein can be improved slightly to $O(T^{2l+1/4}\mathcal{L}^3)$. However, it is difficult to improve the exponent $2l+1/4$. For the constant $C_{2,l}$ in (1.11), we have a new expression:

$$C_{2,l} = \frac{2^{9/2-4l} l^2 B_2(\tau_l)}{(l!)^2 \pi^{2l+3/2}(4l+1)} = \frac{2^{9/2-4l} l^2}{(l!)^2 \pi^{2l+3/2}(4l+1)} \sum_{n=1}^{\infty} \frac{\tau_l^2(n)}{n^{3/2}},$$

analogous to the expression of $C$ in Cramér's result for the mean square of $R_I(t)$.

**2.2.** *Idea of the proof.* M. Khosravi and J. A. Toth [19] introduced an extra parameter and then used the Poisson summation formula to write the corresponding error term in a form which can be estimated by the method of stationary phase. Finally, they eliminated the parameter and got the asymptotic formula (1.11). This approach worked well for the mean square of $R(t)$. M. Khosravi [17] used the same approach for the third moment of $R(t)$ and proved (1.12). But it is difficult to use this approach for the $k$th moment when $k \geq 4$.

In this paper we shall use a different approach. We establish an analogy between $R_I(t)$ and $R(t)$ and then prove our theorems by using this analogy.

It is well-known that $R_I(t)$ has the following truncated Voronoi formula:

$$(2.10)\qquad R_I(t) = -\frac{1}{2^{1/2}\pi^{3/2}} \sum_{n\leq N} r(n) n^{-3/4} t^{1/4} \cos(2\sqrt{nt} + \pi/4) + O(t^{1/2+\varepsilon} N^{-1/2})$$

for $1 \leq N \ll t$, which follows from Lemma 3 of Müller [22]. The formula (2.10) plays an essential role in the proofs of all results stated in Section 1.1.

$R(t)$ is a sum involving the row-teeth-function $\psi(u)$ and does not have such a direct and easily usable Voronoi's formula at hand. However, by using the finite expression of $\psi(u)$ (see Lemma 4.2) and van der Corput's B-process (see Lemma 4.3) we can prove a formula (see Proposition 6.1) analogous to (2.10). This formula, although weak in comparison to (2.10), is enough to prove all our results.

Nowak [23] and Kühleitner and Nowak [20] first studied mean squares of error terms involving $\psi(u)$, where they used another finite Fourier expansion of $\psi(u)$, much more precise than Lemma 4.1. However, in our paper, we only use Lemma 4.1 to study the upper bound of $R(t)$. For the asymptotic results for $R(t)$, we shall apply Lemma 4.2, which seems more convenient. The method of this paper can be used to study the power moments of other error terms involving the function $\psi(u)$.

**3. Background on Heisenberg manifolds and the $\psi$-expression of $R(t)$.** In this section, we first review some background on Heisenberg manifolds (see [6], [8], [25] for more details). Finally, we give a $\psi$-expression of $R(t)$.

**3.1.** *Heisenberg manifolds.* Suppose $x \in \mathbb{R}^l$ is a row vector and $y \in \mathbb{R}^l$ is a column vector. Define

$$\gamma(x,y,t) = \begin{pmatrix} 1 & x & t \\ 0 & I_l & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad X(x,y,t) = \begin{pmatrix} 0 & x & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

The $(2l+1)$-dimensional Heisenberg group $H_l$ is defined by

$$H_l = \{\gamma(x,y,t) : x, y \in \mathbb{R}^l,\ t \in \mathbb{R}\},$$

its Lie algebra is

$$\mathfrak{H}_l = \{X(x,y,t) : x, y \in \mathbb{R}^l,\ t \in \mathbb{R}\}.$$

We say that $\Gamma$ is a *uniform discrete subgroup* of $H_l$ if $H_l/\Gamma$ is compact. A $(2l+1)$-*dimensional Heisenberg manifold* is a pair $(H_l/\Gamma, g)$, where $\Gamma$ is a uniform discrete subgroup of $H_l$ and $g$ is a left $H_l$-invariant metric.

For every $l$-tuple $r = (r_1, \dots, r_l) \in \mathbb{N}^l$ such that $r_j \mid r_{j+1}$ $(j = 1, \dots, l-1)$, let $r\mathbb{Z}^l$ denote the $l$-tuples $x = (x_1, \dots, x_l)$ with $x_j \in r_j\mathbb{Z}$. Define

$$\Gamma_r = \{\gamma(x,y,t) : x, y \in r\mathbb{Z}^l,\ t \in \mathbb{Z}\}.$$

It is clear that $\Gamma_r$ is a uniform discrete subgroup of $H_l$. According to Theorem 2.4 of [8], the subgroup $\Gamma_r$ classifies all the uniform discrete subgroups of $H_l$ up to automorphism. Thus (see [8, Corollary 2.5]) given any Riemannian Heisenberg manifold $M = (H_l/\Gamma, g)$, there exists a unique $l$-tuple $r$ as before and a left-invariant metric $\widetilde{g}$ on $H_l$ such that $M$ is isometric to $(H_l/\Gamma, \widetilde{g})$. So (see [8, 2.6(b)]) we can replace the metric $g$ by $\phi^* g$, where $\phi$ is an inner automorphism such that the direct sum split of the Lie algebra $\mathfrak{H}_l = \mathbb{R}^{2l} \oplus \mathfrak{Z}$ is orthogonal. Here $\mathfrak{Z}$ is the center of the Lie algebra and

$$\mathbb{R}^{2l} = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y \in \mathbb{R}^l \right\}.$$

With respect to this orthogonal split of $\mathfrak{H}_l$ the metric $g$ has the form

$$\begin{pmatrix} h & 0 \\ 0 & g_{2l+1} \end{pmatrix},$$

where $h$ is a positive-definite $2l \times 2l$ matrix and $g_{2l+1} > 0$ is a real number.

The volume of the Heisenberg manifold is given by

$$\mathrm{vol}(H_l/\Gamma, g) = |\Gamma_r|\sqrt{\det(g)}$$

with $|\Gamma_r| = r_1 \cdots r_l$ for $r = (r_1, \ldots, r_l)$.

**3.2.** *The spectrum of Heisenberg manifolds.* Let $\Sigma$ be the spectrum of the Laplacian on $M = (H_l/\Gamma, g_l)$, where the eigenvalues are counted with multiplicities. According to [8, p. 258], $\Sigma$ can be divided into two parts $\Sigma_1$ and $\Sigma_2$, where $\Sigma_1$ is the spectrum of the $2l$-dimensional torus and $\Sigma_2$ contains all eigenvalues of the form

$$2\pi m^2 + 2\pi m(2n_1 + \cdots + 2n_l + l), \qquad m \in \mathbb{N},\ n_j \in \mathbb{N} \cup \{0\},$$

each counted with multiplicity $2m^l$.

**3.3.** *The $\psi$-expression of $R(t)$.* In this section we shall prove the following $\psi$-expression of $R(t)$.

Lemma 3.1. *We have*

$$R(2\pi x) = -\frac{4}{2^l(l-1)!}\sum_{1\le m\le\sqrt{x}} m(x-m^2)^{l-1}\psi\left(\frac{x}{2m} - \frac{m}{2} - \frac{l}{2}\right) + O(x^{l-1/2}). \tag{3.1}$$

Let

$$N(t) = N_I(t) + N_{II}(t), \tag{3.2}$$

where

$$N_I(t) := \#\{\lambda \in \Sigma_1 : \lambda \le t\}, \qquad N_{II}(t) := \#\{\lambda \in \Sigma_2 : \lambda \le t\}.$$

For $N_I(t)$ by Hörmander's theorem we have

$$N_I(t) = \frac{1}{l!2^l}\left(\frac{t}{2\pi}\right)^l + O(t^{l-1/2}). \tag{3.3}$$

**3.3.1.** *Proof of Lemma 3.1 for $l = 1$.* Suppose $l = 1$. It is easily seen that

$$N_{II}(t) = \sum_{\substack{m^2+m(2n+1)\le t/2\pi \\ m\ge1,\, n\ge0}} 2m. \tag{3.4}$$

Thus we get

$$\begin{aligned}
N_{II}(2\pi x) &= \sum_{\substack{m^2+m(2n+1)\le x\\ m\ge 1,\, n\ge 0}} 2m = \sum_{\substack{m^2+m(2n+1)\le x\\ 1\le m\le\sqrt{x},\, n\ge 0}} 2m + O(\sqrt{x})\\
&= \sum_{1\le m\le\sqrt{x}} 2m \sum_{0\le n\le x/2m-m/2-1/2} 1 + O(\sqrt{x})\\
&= \sum_{1\le m\le\sqrt{x}} 2m\left[\frac{x}{2m}-\frac{m}{2}+\frac{1}{2}\right] + O(\sqrt{x})\\
&= \sum_{1\le m\le\sqrt{x}} 2m\left(\frac{x}{2m}-\frac{m}{2}\right)\\
&\quad - \sum_{1\le m\le\sqrt{x}} 2m\psi\left(\frac{x}{2m}-\frac{m}{2}+\frac{1}{2}\right) + O(\sqrt{x})\\
&= \frac{2}{3}x^{3/2}-\frac{x}{2}-\sum_{1\le m\le\sqrt{x}} 2m\psi\left(\frac{x}{2m}-\frac{m}{2}+\frac{1}{2}\right) + O(\sqrt{x}),
\end{aligned}$$

which combined with (3.3) for $l=1$ proves Lemma 3.1 for the case $l=1$.

**3.3.2.** *Proof of Lemma 3.1 for $l>1$.* Suppose now $l>1$. In this case for $N_{II}(t)$ we have

$$\begin{aligned}
N_{II}(t) &= \sum_{\substack{m^2+m(2n_1+\cdots+2n_l+l)\le t/2\pi\\ m>0,\, n_j\ge 0}} 2m^l\\
&= \sum_{\substack{m^2+m(2n+l)\le t/2\pi\\ m>0,\, n\ge 0}} 2m^l \sum_{\substack{n=n_1+\cdots+n_l\\ n_j\ge 0}} 1\\
&= \sum_{\substack{m^2+m(2n+l)\le t/2\pi\\ m>0,\, n\ge 0}} 2m^l \binom{n+l-1}{l-1}\\
&= \frac{2}{(l-1)!}\sum_{\substack{m^2+m(2n+l)\le t/2\pi\\ m>0,\, n\ge 0}} m^l n^{l-1}\\
&\quad + \frac{l}{(l-2)!}\sum_{\substack{m^2+m(2n+l)\le t/2\pi\\ m>0,\, n\ge 0}} m^l n^{l-2} + O(t^{l-1/2})\\
&= \Sigma_{1,l}(t)+\Sigma_{2,l}(t)+O(t^{l-1/2}),
\end{aligned} \tag{3.5}$$

say. In order to evaluate $\Sigma_{1,l}(t)$ and $\Sigma_{2,l}(t)$, we need the following

LEMMA 3.2 (Euler–Maclaurin summation formula). *If* $f \in C^1[a,y]$, $a \in \mathbb{Z}$, *then*

$$\sum_{a\le n\le y} f(n) = \int_a^y (f(t)+\psi(t)f'(t))\,dt - \psi(y)f(y) + f(a)/2.$$

Suppose $d \ge 0$ is a fixed integer. By Lemma 3.2 we get

$$\sum_{0\le n\le y} n^d = \begin{cases} y-\psi(y)+1/2 & \text{if } d=0,\\ \dfrac{y^{d+1}}{d+1}-\psi(y)y^d+O(y^{d-1}) & \text{if } d\ge 1.\end{cases} \tag{3.6}$$

By (3.6) we get

$$\begin{aligned} \Sigma_{1,l}(2\pi x) = {} & \frac{2}{l!}\sum_{1\le m\le\sqrt{x}} m^l\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)^l \\ & - \frac{2}{(l-1)!}\sum_{1\le m\le\sqrt{x}} m^l\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)^{l-1}\psi\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right) \\ & + O(x^{l-1/2}) \end{aligned} \tag{3.7}$$

and

$$\Sigma_{2,l}(2\pi x) = \frac{l}{(l-1)!}\sum_{1\le m\le\sqrt{x}} m^l\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)^{l-1} + O(x^{l-1/2}). \tag{3.8}$$

Write

$$\sum_{1\le m\le\sqrt{x}} m^l\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)^l = 2^{-l}\sum_{0\le m\le\sqrt{x}}(x-m^2-ml)^l - 2^{-l}x^l. \tag{3.9}$$

Let $u(t)=(x-t^2-tl)^l$. It is easy to check that

$$\int_0^{\sqrt{x}} u(t)\,dt = \frac{(l!)^2 2^{2l}}{(2l+1)!}\,x^{l+1/2} - \frac{l}{2}\,x^l + O(x^{l-1/2}), \qquad \int_0^{\sqrt{x}} \psi(t)u'(t)\,dt \ll x^{l-1/2}.$$

By Lemma 3.2 we have

$$\begin{aligned} \sum_{0\le m\le\sqrt{x}} & (x-m^2-ml)^l \\ & = \int_0^{\sqrt{x}} (u(t)+\psi(t)u'(t))\,dt - \psi(\sqrt{x})u(\sqrt{x}) + u(0)/2 \\ & = \frac{(l!)^2 2^{2l}}{(2l+1)!}\,x^{l+1/2} - \frac{l-1}{2}\,x^l + O(x^{l-1/2}). \end{aligned} \tag{3.10}$$

From (3.9) and (3.10) we get

$$\sum_{1\le m\le\sqrt{x}} m^l\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)^l = \frac{(l!)^2 2^l}{(2l+1)!}\,x^{l+1/2}-\frac{l+1}{2^{l+1}}\,x^l+O(x^{l-1/2}). \tag{3.11}$$

Write

$$\sum_{1\le m\le\sqrt{x}} m^l\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)^{l-1} = 2^{-l+1}\sum_{1\le m\le\sqrt{x}} m(x-m^2-ml)^{l-1}. \tag{3.12}$$

Let $v(t)=t(x-t^2-tl)^{l-1}$. It is easy to check that

$$\int_1^{\sqrt{x}} v(t)\,dt=\frac{1}{2l}\,x^l+O(x^{l-1/2}),\qquad \int_1^{\sqrt{x}}\psi(t)v'(t)\,dt\ll x^{l-1}.$$

By Lemma 3.2 we have

$$\sum_{1\le m\le\sqrt{x}} m(x-m^2-ml)^{l-1} = \int_1^{\sqrt{x}}(v(t)+\psi(t)v'(t))\,dt-\psi(\sqrt{x})v(\sqrt{x})+v(1)/2=\frac{1}{2l}\,x^l+O(x^{l-1/2}). \tag{3.13}$$

From (3.12) and (3.13) we get

$$\sum_{1\le m\le\sqrt{x}} m^l\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)^{l-1}=\frac{1}{2^l l}\,x^l+O(x^{l-1/2}). \tag{3.14}$$

Since

$$\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)^{l-1}-\left(\frac{x}{2m}-\frac{m}{2}\right)^{l-1}\ll\left(\frac{x}{m}\right)^{l-2}$$

for $m\ll\sqrt{x}$, we have

$$\begin{aligned}\sum_{m^2+ml\le x} m^l\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)^{l-1}\psi\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right) &= \sum_{m^2+ml\le x} m^l\left(\frac{x}{2m}-\frac{m}{2}\right)^{l-1}\psi\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)+O(x^{l-1/2})\\ &= \sum_{1\le m\le\sqrt{x}} m^l\left(\frac{x}{2m}-\frac{m}{2}\right)^{l-1}\psi\left(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\right)+O(x^{l-1/2}).\end{aligned} \tag{3.15}$$

Now Lemma 3.1 follows from (3.3), (3.5), (3.7), (3.8), (3.11), (3.14) and (3.15).

**3.4.** *A weighted lattice point problem.* For any positive integer $d$, define

$$f_R(d) := \sum_{\substack{d=m(m+2n+l)\\ m>0,\, n\geq 0}} m^l n^{l-1}.$$

From the proof of Lemma 3.1 it is easy to see that

$$R(2\pi x) = \frac{2}{(l-1)!}\sum_{d\leq x} f_R(d) - \frac{2^{l+1}l!}{(2l+1)!}\, x^{l+1/2} - \frac{l+1}{2^l l!}\, x^l + O(x^{l-1/2}). \tag{3.16}$$

So evaluation of the counting function $N(2\pi x)$ is equivalent to studying the asymptotic behavior of the mean value $\sum_{d\leq x} f_R(d)$.

**4. Some preliminary lemmas.** We need the following lemmas. Lemma 4.1 is due to Vaaler [27]. Lemma 4.2 is well-known; see for example, Heath-Brown [12]. Lemma 4.3 is Theorem 2.2 of Min [21] (see also Lemma 6 of Chapter 1 in [28]). A weaker version of Lemma 4.3 can be found in [20], which also suffices for our proof. Lemma 4.4 is Lemma 3.1 of [30]. Lemma 4.5 is the first derivative test. Lemma 4.6 is the famous Halász–Montgomery inequality (see for example, Ivić [15]).

Lemma 4.1. *Let $H\geq 2$ be any real number. Then*

$$\psi(u) = \sum_{1\leq |h|\leq H} a(h)e(hu) + O\Big(\sum_{0\leq |h|\leq H} b(h)e(hu)\Big),$$

*where $a(h)$ and $b(h)$ are functions such that $a(h)\ll 1/|h|$ and $b(h)\ll 1/H$.*

Lemma 4.2. *Let $H\geq 2$ be any real number. Then*

$$\psi(u) = -\sum_{1\leq |h|\leq H} \frac{e(hu)}{2\pi i h} + O\bigg(\min\Big(1, \frac{1}{H\|u\|}\Big)\bigg).$$

Lemma 4.3. *Suppose $A_1,\ldots,A_5$ are absolute positive constants, $f(x)$ and $g(x)$ are algebraic functions in $[a,b]$ and*

$$\frac{A_1}{R}\leq |f''(x)|\leq \frac{A_2}{R},\qquad |f'''(x)|\leq \frac{A_3}{RU},\qquad U\geq 1,$$
$$|g(x)|\leq A_4 G,\qquad |g'(x)|\leq A_5 G U_1^{-1},\qquad U_1\geq 1,$$

*and $[\alpha,\beta]$ is the image of $[a,b]$ under the mapping $y=f'(x)$. Then*

$$\begin{aligned}\sum_{a<n\leq b} g(n)e(f(n)) = e^{\pi i/4}&\sum_{\alpha<u\leq\beta} b_u \frac{g(n_u)}{\sqrt{f''(n_u)}}\, e(f(n_u)-un_u)\\ &+ O(G\log(\beta-\alpha+2) + G(b-a+R)(U^{-1}+U_1^{-1}))\\ &+ O\bigg(G\min\Big[\sqrt{R}, \max\Big(\frac{1}{\langle\alpha\rangle}, \frac{1}{\langle\beta\rangle}\Big)\Big]\bigg),\end{aligned}$$

*where* $n_u$ *is the solution of* $f'(n) = u$,

$$\langle t \rangle = \begin{cases} \|t\| & \text{if } t \text{ is not an integer}, \\ \beta - \alpha & \text{if } t \text{ is an integer}, \end{cases}$$

$$b_u = \begin{cases} 1 & \text{if } \alpha < u < \beta, \text{ or } \alpha, \beta \text{ are not integers}, \\ 1/2 & \text{if } \alpha \text{ or } \beta \text{ are integers}, \end{cases}$$

$$\sqrt{f''} = \begin{cases} \sqrt{f''} & \text{if } f'' > 0, \\ i\sqrt{|f''|} & \text{if } f'' < 0. \end{cases}$$

LEMMA 4.4. *Suppose* $f(n)$ *is an arithmetic function such that* $f(n) \ll n^{\varepsilon}$, $1 \le v < k$ *are fixed integers*, $y > 1$ *is a large parameter*, *and define*

$$s_{k;v}(f;y) := \sum_{\substack{\sqrt{n_1}+\cdots+\sqrt{n_v}=\sqrt{n_{v+1}}+\cdots+\sqrt{n_k} \\ n_1,\ldots,n_k \le y}} \frac{f(n_1)\cdots f(n_k)}{(n_1\cdots n_k)^{3/4}}, \qquad 1 \le v < k.$$

*Then*

$$|s_{k;v}(f) - s_{k;v}(f;y)| \ll y^{-1/2+\varepsilon}, \qquad 1 \le v < k.$$

LEMMA 4.5. *Suppose* $A, B \in \mathbb{R}, A \ne 0$. *Then*

$$\int_T^{2T} \cos(A\sqrt{t} + B)\, dt \ll T^{1/2}|A|^{-1}.$$

LEMMA 4.6. *Let* $\mathcal{S}$ *be a vector space over* $\mathbb{C}$ *with inner product* $(a, b)$, *and* $\|a\|_0 = \sqrt{(a, a)}$. *For any* $\xi, \varphi_1, \ldots, \varphi_R \in \mathcal{S}$,

$$\sum_{l \le R} |(\xi, \varphi_l)|^2 \le \|\xi\|_0^2 \max_{l_1 \le R} \sum_{l_2 \le R} |(\varphi_{l_1}, \varphi_{l_2})|.$$

**5. Proof of Theorem 1.** In order to prove Theorem 1, we shall prove a large value estimate of $R(x)$. For simplicity and convenience, we consider the function $R_*(x) := R(2\pi x)$.

**5.1.** *Large value estimate of* $R_*(x)$. In this subsection, we shall prove the following

THEOREM 5.1. *Suppose* $T \le x_1 < \cdots < x_M \le 2T$ *satisfy* $|R_*(x_s)| \gg VT^{l-1/2}$ $(s = 1, \ldots, M)$ *and* $|x_j - x_i| \ge V \gg T^{7/32}\mathcal{L}^4$ $(i \ne j)$. *Then*

$$M \ll TV^{-3}\mathcal{L}^9 + T^{15/4}V^{-12}\mathcal{L}^{41}. \tag{5.1}$$

*Proof.* Let $V < T_0$ be a parameter to be determined later. Let $I$ be any subinterval of $[T, 2T]$ of length not exceeding $T_0$ and set $G = I \cap \{x_1, \ldots, x_M\}$. Without loss of generality, suppose $G = \{x_1, \ldots, x_{M_0}\}$.

Now let $J=\left[\frac{\mathcal{L}/2+\log\mathcal{L}-\log V}{2\log 2}\right]$. Then by Lemma 3.1 we have

$$\begin{aligned}R_*(x) = &-\frac{4}{2^l(l-1)!}\sum_{j=0}^{J}\sum_{m\sim x^{1/2}/2^{j+1}} m(x-m^2)^{l-1}\psi\Big(\frac{x}{2m}-\frac{m}{2}+\frac{1}{2}\Big)\\ &+O\Big(\frac{VT^{l-1/2}}{\mathcal{L}}\Big)\\ = &-\frac{4}{2^l(l-1)!}\sum_{j_1=0}^{l-1}(-1)^{j_1}\binom{l-1}{j_1}\sum_{j=0}^{J}\sum_{m\sim x^{1/2}/2^{j+1}} x^{l-1-j_1}m^{2j_1+1}\\ &\times\psi\Big(\frac{x}{2m}-\frac{m}{2}+\frac{1}{2}\Big)+O\Big(\frac{VT^{l-1/2}}{\mathcal{L}}\Big).\end{aligned}$$

By Cauchy's inequality we get

$$R_*^2(x)\ll\mathcal{L}\sum_{j_1=0}^{l-1}\sum_{j=0}^{J}\Big|\sum_{m\sim x^{1/2}/2^{j+1}} x^{l-1-j_1}m^{2j_1+1}\psi\Big(\frac{x}{2m}-\frac{m}{2}+\frac{1}{2}\Big)\Big|^2+\frac{V^2T^{2l-1}}{\mathcal{L}^2}.$$

Summing over the set $G$ we get

$$\begin{aligned}(5.2)\qquad M_0V^2T^{2l-1} &\ll \sum_{s\le M_0}|R_*(x_s)|^2\\ &\ll \mathcal{L}\sum_{j_1=0}^{l-1}\sum_{j=0}^{J}\sum_{s\le M_0}\Big|\sum_{m\sim x_s^{1/2}/2^{j+1}} x^{l-1-j_1}m^{2j_1+1}\psi\Big(\frac{x_s}{2m}-\frac{m}{2}+\frac{1}{2}\Big)\Big|^2\\ &\ll \mathcal{L}^2\sum_{s\le M_0}\Big|\sum_{m\sim x_s^{1/2}/2^{j+1}} x^{l-1-j_1}m^{2j_1+1}\psi\Big(\frac{x_s}{2m}-\frac{m}{2}+\frac{1}{2}\Big)\Big|^2\end{aligned}$$

for some fixed pair $(j_1,j)$ with $0\le j_1\le l-1$ and $0\le j\le J$. For this pair $(j_1,j)$, let $N=T^{1/2}2^{-j}$ and $H=N^{2j_1+2}V^{-1}T^{-1/2-j_1}\mathcal{L}^2$. By Lemma 4.1 we get

$$\begin{aligned}(5.3)\qquad &M_0V^2T^{2l-1}\\ &\ll \mathcal{L}^2\sum_{s\le M_0}\Big|\sum_{1\le h\le H}c(h)\sum_{m\sim x_s^{1/2}/2^{j+1}} x_s^{l-1-j_1}m^{2j_1+1}e\Big(-h\Big(\frac{x_s}{2m}-\frac{hm}{2}\Big)\Big)\Big|^2,\end{aligned}$$

where $c(h)$ is some function such that $c(h)\ll 1/h$.

For any integer $h>0$, define

$$S(x;h,j_1,j):=\sum_{m\sim x^{1/2}/2^{j+1}} x^{l-1-j_1}m^{2j_1+1}e\Big(-h\Big(\frac{x}{2m}-\frac{hm}{2}\Big)\Big).$$

Taking

$$f(m) = -h\left(\frac{x}{2m} - \frac{hm}{2}\right), \qquad g(m) = x^{l-1-j_1} m^{2j_1+1}$$

in Lemma 4.3 we get

$$\begin{aligned}(5.4)\quad & S(x; h, j_1, j) \\ &= \frac{e^{\pi i/4}}{i} \sum_{h(2^{2j-1}+1/2)<r\le h(2^{2j+1}+1/2)} \frac{b_r x^{l-1/4} h^{j_1+3/4}}{(2r-h)^{j_1+5/4}}\, e(-\sqrt{xh(2r-h)}) \\ &\quad + O(T^{l-1/2}\mathcal{L}).\end{aligned}$$

Insert (5.4) into (5.3) to get

$$\begin{aligned}(5.5)\qquad M_0V^2T^{2l-1} &\ll \mathcal{L}^2 \sum_{s\le M_0} |D(x_s; H, j_1, j)|^2 + M_0T^{2l-1}\mathcal{L}^4 \\ &\ll \mathcal{L}^2 \sum_{s\le M_0} |D(x_s; H, j_1, j)|^2,\end{aligned}$$

where

$$\begin{aligned}& D(x; H, j_1, j) \\ &= \sum_{1\le h\le H} c(h) \sum_{h(2^{2j-1}+1/2)<r\le h(2^{2j+1}+1/2)} b_r \frac{x^{l-1/4}h^{j_1+3/4}}{(2r-h)^{j_1+5/4}}\, e(-\sqrt{xh(2r-h)}).\end{aligned}$$

Let

$$\gamma(n; H, j_1, j) = \sum_{\substack{n=h(2r-h),1\le h\le H \\ h(2^{2j-1}+1/2)<r\le h(2^{2j+1}+1/2)}} \frac{b_r c(h) h^{j_1+3/2}}{(2r-h)^{j_1+1/2}}$$

and $N_0 = H^2(2^{2j+1} + 1/2)$. Then it is easy to see that $\gamma(n; H, j_1, j) \ll d(n)$ and $N_0 \ll TV^{-2}\mathcal{L}^4$. Thus we have

$$\begin{aligned}(5.6)\quad M_0V^2 &\ll \mathcal{L}^2T^{1/2} \sum_{s\le M_0} \Bigg| \sum_{n\le N_0} \frac{\gamma(n; H, j_1, j)}{n^{3/4}}\, e(-\sqrt{nx_s}) \Bigg|^2 \\ &\ll \mathcal{L}^2T^{1/2} \sum_{s\le M_0} \Bigg| \sum_{v} \sum_{n\sim N_02^{-v-1}} \frac{\gamma(n; H, j_1, j)}{n^{3/4}}\, e(-\sqrt{nx_s}) \Bigg|^2 \\ &\ll \mathcal{L}^3T^{1/2} \sum_{v} \sum_{s\le M_0} \Bigg| \sum_{n\sim N_02^{-v-1}} \frac{\gamma(n; H, j_1, j)}{n^{3/4}}\, e(-\sqrt{nx_s}) \Bigg|^2 \\ &\ll \mathcal{L}^4T^{1/2} \sum_{s\le M_0} \Bigg| \sum_{n\sim N_02^{-v-1}} \frac{\gamma(n; H, j_1, j)}{n^{3/4}}\, e(-\sqrt{nx_s}) \Bigg|^2\end{aligned}$$

for some $0 \le v \ll \mathcal{L}$, where in the third "$\ll$" we used Cauchy's inequality again. Let $N_1 = N_0 2^{-v-1}$. Then $N_1 \ll TV^{-2}\mathcal{L}^4$.

The procedure below follows the proof of Theorem 13.8 in Ivić [15], so we give only an outline. Take $\xi = \{\xi_n\}_{n=1}^{\infty}$ with $\xi_n = \gamma(n; H, j_1, j)n^{-3/4}$ for $n \sim N_1$ and zero otherwise, and take $\varphi_s = \{\varphi_{s,n}\}_{n=1}^{\infty}$ with $\varphi_{s,n} = e(-\sqrt{nx_s})$ for $n \sim N_1$ and zero otherwise. Then

$$\begin{aligned}
(\xi, \varphi_s) &= \sum_{n\sim N_1} \frac{\gamma(n; H, j_1, j)}{n^{3/4}} e(-\sqrt{nx_s}),\\
(\varphi_{l_1}, \varphi_{l_2}) &= \sum_{n\sim N_1} e(\sqrt{n}(\sqrt{x_{s_1}} - \sqrt{x_{s_2}})),\\
\|\xi\|_0^2 &= \sum_{n\sim N_1} \frac{|\gamma(n; H, j_1, j)|^2}{n^{3/2}} \ll N_1^{-3/2} \sum_{n\sim N_1} d^2(n) \ll N_1^{-1/2}\mathcal{L}^3.
\end{aligned}$$

By Lemma 4.6 we get

$$\begin{aligned}
(5.7)\quad M_0V^2 &\ll \mathcal{L}^7 T^{1/2} N_1^{-1/2} \max_{s_1\le M_0} \sum_{s_2\le M_0} \Big|\sum_{n\sim N_1} e(\sqrt{n}(\sqrt{x_{s_1}} - \sqrt{x_{s_2}}))\Big|\\
&\ll \mathcal{L}^7 T^{1/2} N_1^{1/2} + \mathcal{L}^7 T^{1/2} N_1^{-1/2}\\
&\quad \times \max_{s_1\le M_0} \sum_{s_2\le M_0,\, s_2\ne s_1} \Big|\sum_{n\sim N_1} e(\sqrt{n}(\sqrt{x_{s_1}} - \sqrt{x_{s_2}}))\Big|.
\end{aligned}$$

By the Kuz'min–Landau inequality and the exponent pair $(4/18, 11/18)$ we get

$$\begin{aligned}
\sum_{n\sim N_1} e(\sqrt{n}(\sqrt{x_{s_1}} - \sqrt{x_{s_2}})) &\ll \frac{\sqrt{N_1}}{|\sqrt{x_{s_1}} - \sqrt{x_{s_2}}|} + \left(\frac{|\sqrt{x_{s_1}} - \sqrt{x_{s_2}}|}{\sqrt{N_1}}\right)^{4/18} N_1^{11/18}\\
&\ll \frac{\sqrt{N_1T}}{|x_{s_1} - x_{s_2}|} + \left(\frac{|x_{s_1} - x_{s_2}|}{\sqrt{N_1T}}\right)^{4/18} N_1^{11/18}\\
&\ll \frac{\sqrt{N_1T}}{|x_{s_1} - x_{s_2}|} + T^{-1/9}T_0^{2/9}N_1^{1/2},
\end{aligned}$$

where we used the mean value theorem and the estimate $|x_{s_1} - x_{s_2}| \le T_0$.

Insert this estimate into (5.7) to get

$$\begin{aligned}
(5.8)\quad M_0V^2 &\ll \mathcal{L}^7 T^{1/2} N_1^{1/2} + \mathcal{L}^7 T^{1/2} N_1^{-1/2}\\
&\quad \times \max_{s_1\le M_0} \sum_{s_2\le M_0,\, s_2\ne s_1} \left(\frac{\sqrt{N_1T}}{|x_{s_1} - x_{s_2}|} + T^{-1/9}T_0^{2/9}N_1^{1/2}\right)\\
&\ll \mathcal{L}^7 T^{1/2} N_1^{1/2} + \mathcal{L}^7 TV^{-1} + \mathcal{L}^7 M_0 T^{1/2-1/9} T_0^{2/9}\\
&\ll \mathcal{L}^9 TV^{-1} + \mathcal{L}^7 M_0 T^{7/18} T_0^{2/9},
\end{aligned}$$

where we used the facts that $\{x_s\}$ is $V$-spaced and $N_1 \ll TV^{-2}\mathcal{L}^4$. Take $T_0 = V^9T^{-7/4}\mathcal{L}^{-32}$. It is easy to check that $T_0 \gg V$ if $V \gg T^{7/32}\mathcal{L}^4$. For this $T_0$ we get

$$M_0 \ll \mathcal{L}^9TV^{-3}.$$

Now we divide the interval $[T, 2T]$ into $O(1+T/T_0)$ subintervals of length not exceeding $T_0$. In each of them, the number of $x_s$ is at most $O(\mathcal{L}^9TV^{-3})$. So we have

$$M \ll \mathcal{L}^9TV^{-3}(1 + T/T_0) \ll \mathcal{L}^9TV^{-3} + \mathcal{L}^{41}T^{15/4}V^{-12}. \tag{5.9}$$

This completes the proof of Theorem 5.1.

**5.2.** *Proof of Theorem 1.* When $A = 0$, Theorem 1 is trivial. When $0 < A < 2$, it follows from (1.11) and Hölder's inequality. So later we always suppose $A > 2$. It suffices to prove the estimate

$$\begin{aligned}\int_T^{2T} |R_*(x)|^A\,dx \\ \ll \begin{cases} T^{1+3A/4+A(l-1)}\log^{41} T & \text{if } 2 < A \le 262/27, \\ T^{\frac{154+339A}{416}+A(l-1)}\log^{4A+1} T & \text{if } A > 262/27. \end{cases}\end{aligned} \tag{5.10}$$

Suppose $x^\varepsilon < y \le x/2$. By (3.16) we get

$$\begin{aligned}&|R_*(x+y) - R_*(x)| \\ &\quad\ll \sum_{x<n\le x+y} f_R(n) + |(x+y)^{l+1/2} - x^{l+1/2}| + |(x+y)^l - x^l| \\ &\quad\ll x^{l-1/2}\sum_{x<n\le x+y} d(n) + x^{l-1/2}y \ll x^{l-1/2}y\log x,\end{aligned}$$

where we used the well-known estimate

$$\sum_{x<n\le x+y} d(n) \ll y\log x$$

and the obvious bound $f_R(n) \ll n^{l-1/2}d(n)$. So there exists an absolute constant $c_0$ such that

$$|R_*(x+y) - R_*(x)| \le c_0x^{l-1/2}y\log x,$$

which implies that if $|R_*(x)| \ge 2c_0x^{l-1/2}y\log x$, then

$$|R_*(x+y)| \ge |R_*(x)| - |R_*(x+y) - R_*(x)| \ge c_0x^{l-1/2}y\log x. \tag{5.11}$$

From (5.11) and a similar argument to (13.70) of Ivić [15] we may write

$$\int_T^{2T} |R_*(x)|^A\,dx \ll T^{1+A(l-1/4)}\mathcal{L} + \sum_V V\sum_{r\le N_V} |R_*(x_r)|^A, \tag{5.12}$$

where $T^{1/4} \le V = 2^m \le T^{131/416}\mathcal{L}^4$, $VT^{l-1/2} < |R_*(x_r)| \le 2VT^{l-1/2}$ $(r = 1, \ldots, N_V)$ and $|t_r - t_s| \ge V$ for $r \ne s \le N = N_V$.

If $2 < A \le 11$, then by Theorem 5.1 we get (recall $V \ll T^{131/416}\mathcal{L}^4$)

$$\begin{aligned}(5.13)\qquad V\sum_{r\le N_V}|R_*(x_r)|^A &\ll T^{A(l-1/2)}N_V V^{A+1}\\ &\ll T^{A(l-1/2)}(\mathcal{L}^9TV^{A-2}+\mathcal{L}^{41}T^{15/4}V^{A-11})\\ &\ll T^{A(l-1/2)}(T^{1+\frac{131}{416}(A-2)}\mathcal{L}^{4A+1}+T^{1+A/4}\mathcal{L}^{41})\\ &\ll T^{A(l-1/2)}(T^{\frac{154+131A}{416}}\mathcal{L}^{4A+1}+T^{1+A/4}\mathcal{L}^{41}).\end{aligned}$$

If $A > 11$, then by Theorem 5.1 we get

$$\begin{aligned}(5.14)\qquad V\sum_{r\le N_V}|R_*(x_r)|^A &\ll T^{A(l-1/2)}N_V V^{A+1}\\ &\ll T^{A(l-1/2)}(\mathcal{L}^9TV^{A-2}+\mathcal{L}^{41}T^{15/4}V^{A-11})\\ &\ll T^{A(l-1/2)}(T^{1+\frac{131}{416}(A-2)}\mathcal{L}^{4A+1}+T^{15/4+\frac{131}{416}(A-11)}\mathcal{L}^{4A-3})\\ &\ll T^{A(l-1/2)+\frac{154+131A}{416}}\mathcal{L}^{4A+1}.\end{aligned}$$

Now (5.10) follows from (5.12)–(5.14) by noting that $(154 + 131A)/416 \le 1+A/4$ for $2 < A \le 262/27$ and $(154+131A)/416 > 1+A/4$ for $A > 262/27$.

**6. Proof of Theorem 2.** Suppose that $3 \le k < A_0$, where $A_0 > 9$ is a fixed real number such that (2.3) holds. To prove Theorem 2, it suffices to evaluate the integral $\int_T^{2T} R_*^k(x)\,dx$, where $T$ is a large real number.

Suppose $H$ is a large parameter to be determined later. By Lemmas 3.1 and 4.2 we have

$$(6.1)\qquad R_*(x) = F(x) + O(T^{l-1/2}G(x)),$$

where

$$F(x) = \frac{2^{1-l}}{(l-1)!\pi i}\sum_{1\le|h|\le H}\frac{1}{h}\sum_{m\le\sqrt{x}} m(x-m^2)^{l-1}e\biggl(h\biggl(\frac{x}{2m}-\frac{m}{2}+\frac{l}{2}\biggr)\biggr),$$

$$G(x) = \sum_{m\le\sqrt{2T}}\min\biggl(1, \frac{1}{H\|x/2m - m/2 + l/2\|}\biggr).$$

**6.1.** *Upper bound of* $\int_T^{2T} G(x)\,dx$. In this subsection we prove

LEMMA 6.1. *We have*

$$(6.2)\qquad \int_T^{2T} G(x)\,dx \ll T^{3/2}H^{-1}\mathcal{L}.$$

*Proof.* Obviously we have

$$G(x) \ll \sum_{2m\le\sqrt{2T}} \min\Big(1, \frac{1}{H\|x/4m+1/2\|}\Big) + \sum_{2m-1\le\sqrt{2T}} \min\Big(1, \frac{1}{H\|x/2(2m-1)\|}\Big) \ll G_1(x)+G_2(x), \tag{6.3}$$

where

$$G_1(x) = \sum_{m\le 4\sqrt{T}} \min\Big(1, \frac{1}{H\|x/m+1/2\|}\Big),$$
$$G_2(x) = \sum_{m\le 4\sqrt{T}} \min\Big(1, \frac{1}{H\|x/m\|}\Big).$$

Thus

$$\int_T^{2T} G(x)\,dx \ll \int_T^{2T} G_1(x)\,dx + \int_T^{2T} G_2(x)\,dx. \tag{6.4}$$

We have

$$\begin{aligned}\int_T^{2T} G_1(x)\,dx &\ll \sum_{m\le 4\sqrt{T}} \int_T^{2T} \min\Big(1, \frac{1}{H\|x/m+1/2\|}\Big)\,dx \\ &\ll \sum_{m\le 4\sqrt{T}} m \int_{T/m}^{2T/m} \min\Big(1, \frac{1}{H\|x+1/2\|}\Big)\,dx \\ &\ll T \sum_{m\le 4\sqrt{T}} \int_0^{1/2} \Big(1, \frac{1}{H\|x+1/2\|}\Big)\,dx \\ &\ll T^{3/2} \int_0^{1/2} \Big(1, \frac{1}{H\|x+1/2\|}\Big)\,dx \ll T^{3/2}H^{-1}\mathcal{L}.\end{aligned} \tag{6.5}$$

Similarly,

$$\int_T^{2T} G_2(x)\,dx \ll T^{3/2}H^{-1}\mathcal{L}. \tag{6.6}$$

Now Lemma 6.1 follows from (6.3)–(6.6).

**6.2.** *An analogue of Voronoi's formula for $R_*(x)$.* To prove an analogue of Voronoi's formula for $R_*(x)$, our main tools are Lemmas 4.2 and 4.3.

From the definition of $F(x)$ we have

$$
\begin{aligned}
(6.7)\qquad & F(x)\\
&= \frac{2^{1-l}}{(l-1)!\pi i}\sum_{1\le|h|\le H}\frac{1}{h}\sum_{m\le\sqrt{x}} m(x-m^2)^{l-1}e\Big(h\Big(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\Big)\Big)\\
&= \frac{2^{1-l}}{(l-1)!\pi i}\sum_{j_1=0}^{l-1}(-1)^{j_1}\binom{l-1}{j_1}\sum_{1\le|h|\le H}\frac{1}{h}\sum_{m\le\sqrt{x}} x^{l-1-j_1}m^{2j_1+1}\\
&\qquad\qquad \times e\Big(h\Big(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\Big)\Big)\\
&= \frac{2^{1-l}}{(l-1)!}\sum_{j_1=0}^{l-1}(-1)^{j_1}\binom{l-1}{j_1}F(x;j_1),
\end{aligned}
$$

say, where

$$
F(x;j_1) := \frac{1}{\pi i}\sum_{1\le|h|\le H}\frac{1}{h}\sum_{m\le\sqrt{x}} x^{l-1-j_1}m^{2j_1+1}e\Big(h\Big(\frac{x}{2m}-\frac{m}{2}-\frac{l}{2}\Big)\Big).
$$

Let $J=[(\mathcal{L}-\log\mathcal{L})/2\log 2]$. We get

$$
\begin{aligned}
(6.8)\qquad & F(x;j_1)\\
&= \frac{1}{\pi i}\sum_{-H\le h\le -1}\frac{e(-lh/2)}{h}\sum_{j=0}^{J}\sum_{m\sim\sqrt{x}2^{-j-1}} x^{l-1-j_1}m^{2j_1+1}e\Big(h\Big(\frac{x}{2m}-\frac{m}{2}\Big)\Big)\\
&\quad+\frac{1}{\pi i}\sum_{1\le h\le H}\frac{e(-lh/2)}{h}\sum_{j=0}^{J}\sum_{m\sim\sqrt{x}2^{-j-1}} x^{l-1-j_1}m^{2j_1+1}e\Big(h\Big(\frac{x}{2m}-\frac{m}{2}\Big)\Big)\\
&\quad+O(x^{l-1}\mathcal{L}^2)\\
&= -\frac{1}{\pi i}\sum_{1\le h\le H}\frac{e(lh/2)}{h}\sum_{j=0}^{J}\sum_{m\sim\sqrt{x}2^{-j-1}} x^{l-1-j_1}m^{2j_1+1}e\Big(-h\Big(\frac{x}{2m}-\frac{m}{2}\Big)\Big)\\
&\quad+\frac{1}{\pi i}\sum_{1\le h\le H}\frac{e(-lh/2)}{h}\sum_{j=0}^{J}\sum_{m\sim\sqrt{x}2^{-j-1}} x^{l-1-j_1}m^{2j_1+1}e\Big(h\Big(\frac{x}{2m}-\frac{m}{2}\Big)\Big)\\
&\quad+O(x^{l-1}\mathcal{L}^2)\\
&= -\frac{\Sigma_3}{\pi i}+\frac{\overline{\Sigma}_3}{\pi i}+O(x^{l-1}\mathcal{L}^2),
\end{aligned}
$$

where

$$
\Sigma_3=\sum_{1\le h\le H}\frac{e(lh/2)}{h}\sum_{j=0}^{J}S(x;h,j_1,j)
$$

with $S(x;h,j_1,j)$ defined in Section 5.1.

Inserting the formula (5.4) into $\Sigma_3$ we get

$$\Sigma_3 = e^{-\pi i/4} \sum_{1\le h\le H} \frac{e(lh/2)}{h} \sum_{j=0}^{J} \sum_{h(2^{2j-1}+1/2)<r\le h(2^{2j+1}+1/2)} \frac{b_r x^{l-1/4} h^{j_1+3/4}}{(2r-h)^{j_1+5/4}}\, e(-\sqrt{xh(2r-h)}) + O(T^{l-1/2}\mathcal{L}^3)$$
$$= e^{-\pi i/4} \sum_{1\le h\le H} \frac{e(lh/2)}{h} \sideset{}{'}\sum_{h<r\le h(2^{2J+1}+1/2)} \frac{x^{l-1/4} h^{j_1+3/4}}{(2r-h)^{j_1+5/4}}\, e(-\sqrt{xh(2r-h)}) + O(T^{l-1/2}\mathcal{L}^3), \tag{6.9}$$

where the prime on the last sum means that if $h$ is an even integer, then the term $r = h(2^{2J+1} + 1/2)$ should be halved.

Inserting (6.9) into (6.8) we get

$$F(x;j_1) = \frac{1}{\pi i} \sum_{1\le h\le H} \frac{e(lh/2)}{h} \sideset{}{'}\sum_{h<r\le h(2^{2J+1}+1/2)} \frac{x^{l-1/4} h^{j_1+3/4}}{(2r-h)^{j_1+5/4}} \times (e(\sqrt{xh(2r-h)} + 1/8) - e(\sqrt{xh(2r-h)} - 1/8)) + O(x^{l-1/2}\mathcal{L}^3)$$
$$= \frac{2x^{l-1/4}}{\pi} \sum_{1\le h\le H} \frac{e(lh/2)}{h} \sideset{}{'}\sum_{h<r\le h(2^{2J+1}+1/2)} \frac{h^{j_1+3/4}}{(2r-h)^{j_1+5/4}} \times \sin(2\pi\sqrt{xh(2r-h)} + \pi/4) + O(x^{l-1/2}\mathcal{L}^3)$$
$$= \frac{2x^{l-1/4}}{\pi} \sum_{1\le h\le H} \frac{e(lh/2)}{h} \sideset{}{'}\sum_{h<r\le h(2^{2J+1}+1/2)} \frac{h^{j_1+3/4}}{(2r-h)^{j_1+5/4}} \times \cos(2\pi\sqrt{xh(2r-h)} - \pi/4) + O(x^{l-1/2}\mathcal{L}^3). \tag{6.10}$$

From (6.7) and (6.10) we get

$$F(x) = \frac{2^{2-l} x^{l-1/4}}{(l-1)!\pi} \sum_{1\le h\le H} \sideset{}{'}\sum_{h<r\le h(2^{2J+1}+1/2)} \cos(2\pi\sqrt{xh(2r-h)} - \pi/4) \times \sum_{j_1=0}^{l-1} (-1)^{j_1} \binom{l-1}{j_1} \frac{e(lh/2) h^{j_1-1/4}}{(2r-h)^{j_1+5/4}} + O(x^{l-1/2}\mathcal{L}^3)$$
$$= \frac{2^{2-l} x^{l-1/4}}{(l-1)!\pi} \sum_{1\le h\le H} \sideset{}{'}\sum_{h<r\le h(2^{2J+1}+1/2)} \frac{e(lh/2)}{h^{1/4}(2r-h)^{5/4}} \left(1 - \frac{h}{2r-h}\right)^{l-1} \times \cos(2\pi\sqrt{xh(2r-h)} - \pi/4) + O(x^{l-1/2}\mathcal{L}^3).$$

Define

$$\tau_l(n;H,T) := \sum_{\substack{n=h(2r-h),\,1\le h\le H\\ h<r\le h(2^{2J+1}+1/2)}} \frac{e(lh/2)h^{1/2}}{(2r-h)^{1/2}}\left(1-\frac{h}{2r-h}\right)^{l-1}.$$

We then have

$$F(x) = \frac{2^{2-l}x^{l-1/4}}{(l-1)!\pi} \sum_{1\le n\le H^2(2^{2J+1}+1/2)} \frac{\tau_l(n;H,T)}{n^{3/4}}\cos(2\pi\sqrt{xn}-\pi/4) + O(x^{l-1/2}\mathcal{L}^3), \tag{6.11}$$

where the "primed" terms for which $2\mid h$ and $r=h(2^{2J+1}+1/2)$ are absorbed into the error term.

Obviously, $|\tau_l(n;H,T)|\le d(n)$. By the definition of $J$ we see that if $n\le\min(H,T\mathcal{L}^{-2})$, then $\tau_l(n;H,T)=\tau_l(n)$, where $\tau_l(n)$ was defined in Section 2.1.

From (6.1) and (6.11) we get the following proposition, which is an analogue of Voronoi's formula.

Proposition 6.1. *Suppose* $T\le x\le 2T$, $H\ge T$, $J=[(\mathcal{L}-\log\mathcal{L})/2\log 2]$. *Then*

$$R_*(x) = \frac{2^{2-l}x^{l-1/4}}{(l-1)!\pi} \sum_{1\le n\le H^2(2^{2J+1}+1/2)} \frac{\tau(n;H,T)}{n^{3/4}}\cos(2\pi\sqrt{xn}-\pi/4) + O(T^{l-1/2}G(x)+T^{l-1/2}\mathcal{L}^3), \tag{6.12}$$

where $\tau_l(n;H,T)=\tau_l(n)$ for $n\le T\mathcal{L}^{-2}$.

Now suppose $T^{\varepsilon}<y\le T\mathcal{L}^{-2}$ is a parameter to be determined. Define

$$\begin{aligned} F_1(x) &:= \theta_l x^{l-1/4}\sum_{1\le n\le y}\frac{\tau_l(n)}{n^{3/4}}\cos(2\pi\sqrt{xn}-\pi/4),\\ F_2(x) &:= F(x)-F_1(x), \end{aligned} \tag{6.13}$$

where $\theta_l := 2^{2-l}/(l-1)!\pi$.

**6.3.** *Evaluation of* $\int_T^{2T}F_1^k(x)\,dx$. The $k$th moment of $F_1(x)$ provides the main term in Theorem 2, so in this subsection we shall evaluate $\int_T^{2T}F_1^k(x)\,dx$. For simplicity we set $\mathbb{I}=\{0,1\}$. For each $\mathbf{i}=(i_1,\dots,i_{k-1})\in\mathbb{I}^{k-1}$, put $|\mathbf{i}|=i_1+\dots+i_{k-1}$. By the elementary formula

$$\cos a_1\cdots\cos a_k = \frac{1}{2^{k-1}}\sum_{\mathbf{i}\in\mathbb{I}^{k-1}}\cos(a_1+(-1)^{i_1}a_2+(-1)^{i_2}a_3+\dots+(-1)^{i_{k-1}}a_k) \tag{6.14}$$

we have

$$F_1^k(x) = \theta_l^k x^{k(l-1/4)} \sum_{n_1\le y} \cdots \sum_{n_k\le y} \frac{\tau_l(n_1)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}} \prod_{j=1}^{k} \cos(2\pi\sqrt{n_j x} - \pi/4)$$
$$= \frac{\theta_l^k x^{k(l-1/4)}}{2^{k-1}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{n_1\le y} \cdots \sum_{n_k\le y} \frac{\tau_l(n_1)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}} \cos(2\pi\sqrt{x}\alpha(\mathbf{n};\mathbf{i}) - \pi\beta(\mathbf{i})/4),$$

where $\mathbf{n} = (n_1,\dots,n_k) \in \mathbb{N}^k$ and

$$\alpha(\mathbf{n};\mathbf{i}) := \sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + \cdots + (-1)^{i_{k-1}}\sqrt{n_k},$$
$$\beta(\mathbf{i}) := 1 + (-1)^{i_1} + \cdots + (-1)^{i_{k-1}}.$$

Thus we can write

$$F_1^k(x) = \frac{\theta_l^k}{2^{k-1}} (S_1(x) + S_2(x)), \tag{6.15}$$

where

$$S_1(x) := x^{k(l-1/4)} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \cos(-\pi\beta(\mathbf{i})/4) \sum_{\substack{n_j\le y,\,1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\tau_l(n_1)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}},$$
$$S_2(x) := x^{k(l-1/4)} \times \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{\substack{n_j\le y,\,1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{\tau_l(n_1)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}} \cos(2\pi\alpha(\mathbf{n};\mathbf{i})\sqrt{x} - \pi\beta(\mathbf{i})/4).$$

First consider the contribution of $S_1(x)$. We have

$$\int_T^{2T} S_1(x)\,dx = \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \cos(-\pi\beta(\mathbf{i})/4) \sum_{\substack{n_j\le y,\,1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\tau_l(n_1)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}} \int_T^{2T} x^{k(l-1/4)}\,dx. \tag{6.16}$$

It is easily seen that if $\alpha(\mathbf{n};\mathbf{i}) = 0$, then $1 \le |\mathbf{i}| \le k-1$. So

$$\sum_{\substack{n_j\le y,\,1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\tau_l(n_1)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}} = s_{k;|\mathbf{i}|}(\tau_l; y). \tag{6.17}$$

By Lemma 4.4 we get

$$\int_T^{2T} S_1(x)\,dx = B_k^*(\tau_l) \int_T^{2T} x^{k(l-1/4)}\,dx + O(T^{1+k(l-1/4)+\varepsilon} y^{-1/2}), \tag{6.18}$$

where

$$B_k^*(\tau_l) := \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \cos(-\pi\beta(\mathbf{i})/4) \sum_{\substack{\mathbf{n}\in\mathbb{N}^k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\tau_l(n_1)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}}.$$

For any $\mathbf{i} \in \mathbb{I}^{k-1} \setminus \mathbf{0}$, let

$$S(\tau_l;\mathbf{i}) := \sum_{\substack{\mathbf{n}\in\mathbb{N}^k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\tau_l(n_1)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}}.$$

It is easily seen that if $|\mathbf{i}| = |\mathbf{i}'|$ or $|\mathbf{i}| + |\mathbf{i}'| = k$, then

$$S(\tau_l;\mathbf{i}) = S(\tau_l;\mathbf{i}') = s_{k;|\mathbf{i}|}(\tau_l).$$

From $(-1)^j = 1 - 2j$ $(j = 0, 1)$ we also have $\beta(\mathbf{i}) = k - 2|\mathbf{i}|$. So we get

$$\begin{aligned}(6.19)\qquad B_k^*(\tau_l) &= \sum_{v=1}^{k-1}\sum_{|\mathbf{i}|=v} \cos(-\pi\beta(\mathbf{i})/4) S(\tau_l;\mathbf{i})\\ &= \sum_{v=1}^{k-1} s_{k;v}(\tau_l)\cos(\pi(k-2v)/4)\sum_{|\mathbf{i}|=v} 1\\ &= \sum_{v=1}^{k-1}\binom{k-1}{v} s_{k;v}(\tau_l)\cos(\pi(k-2v)/4) = B_k(\tau_l).\end{aligned}$$

Now we consider the contribution of $S_2(x)$. By Lemma 4.5 we get

$$(6.20)\qquad \int_T^{2T} S_2(x)\,dx \ll T^{1/2+k(l-1/4)} U_k(y),$$

where

$$U_k(y) := \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{\substack{n_j\le y,\,1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}} \frac{d(n_1)\cdots d(n_k)}{(n_1\cdots n_k)^{3/4}|\alpha(\mathbf{n};\mathbf{i})|}.$$

In [30] the author proved

$$(6.21)\qquad U_k(y) \ll y^{s(k)+\varepsilon},$$

where $s(k)$ was defined in Section 2.1. Thus

$$(6.22)\qquad \int_T^{2T} S_2(x)\,dx \ll T^{1/2+k(l-1/4)+\varepsilon} y^{s(k)}.$$

Hence from (6.15)–(6.22) we get

LEMMA 6.2. *For fixed $k \ge 3$, we have*

$$\begin{aligned}(6.23)\qquad \int_T^{2T} F_1^k(x)\,dx &= \frac{2^{1+k-lk} l^k B_k(\tau_l)}{(l!)^k \pi^k} \int_T^{2T} x^{k(l-1/4)}\,dx\\ &\quad + O(T^{1+k(l-1/4)+\varepsilon} y^{-1/2} + T^{1/2+k(l-1/4)+\varepsilon} y^{s(k)}).\end{aligned}$$

**6.4.** *Upper bound of* $\int_T^{2T} F_1^{k-1}(x)F_2(x)\,dx$. Let

$$N_2 := H^2(2^{2J+1}+1/2), \qquad J := [(\mathcal{L}-\log\mathcal{L})/2\log 2].$$

By (6.14) we have

$$F_1^{k-1}(x)F_2(x) = \theta_l^k x^{k(l-1/4)} \sum_{y<n_1\le N_2}\sum_{n_2\le y}\cdots\sum_{n_k\le y} \frac{\tau_l(n_1;H,T)\tau_l(n_2)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}} \times \prod_{j=1}^{k}\cos(2\pi\sqrt{n_j x}-\pi/4)$$

$$= \frac{\theta_l^k x^{k(l-1/4)}}{2^{k-1}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}}\sum_{y<n_1\le N_2}\sum_{n_2\le y}\cdots\sum_{n_k\le y} \frac{\tau_l(n_1;H,T)\tau_l(n_2)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}} \times \cos(2\pi\sqrt{x}\,\alpha(\mathbf{n};\mathbf{i})-\pi\beta(\mathbf{i})/4).$$

Thus

$$F_1^{k-1}(x)F_2(x) = \frac{\theta_l^k}{2^{k-1}}\,(S_3(x)+S_4(x)), \tag{6.24}$$

where

$$S_3(x) = x^{k(l-1/4)} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \cos(-\pi\beta(\mathbf{i})/4) \times \sum_{y<n_1\le N_2} \sum_{\substack{n_j\le y,\,2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{\tau_l(n_1;H,T)\tau_l(n_2)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}},$$

$$S_4(x) = x^{k(l-1/4)} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{y<n_1\le N_2} \sum_{\substack{n_j\le y,\,2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}} \frac{\tau_l(n_1;H,T)\tau_l(n_2)\cdots\tau_l(n_k)}{(n_1\cdots n_k)^{3/4}} \times \cos(2\pi\alpha(\mathbf{n};\mathbf{i})\sqrt{x}-\pi\beta(\mathbf{i})/4).$$

By Lemma 4.4 we have

$$\begin{aligned}
\int_T^{2T} S_3(x)\,dx
&\ll \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{y<n_1\le N_2} \sum_{\substack{n_j\le y,\,2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{d(n_1)\cdots d(n_k)}{(n_1\cdots n_k)^{3/4}} \int_T^{2T} x^{k(l-1/4)}\,dx \\
&\ll T^{1+k(l-1/4)} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{y<n_1\le N_2} \sum_{\substack{n_j\le y,\,2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{d(n_1)\cdots d(n_k)}{(n_1\cdots n_k)^{3/4}} \\
&\ll T^{1+k(l-1/4)} \sum_{v=1}^{k-1} |s_{k;v}(d;y)-s_{k;v}(d)| \ll T^{1+k(l-1/4)+\varepsilon}y^{-1/2}.
\end{aligned} \tag{6.25}$$

Now we consider the contribution of $S_4(x)$. By Lemma 4.5 we get

$$\int_T^{2T} S_4(x)\,dx \ll T^{1/2+k(l-1/4)}(\Sigma_4+\Sigma_5), \tag{6.26}$$

where

$$\Sigma_4 = \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{y<n_1\le k^2y} \sum_{\substack{n_j\le y,\,2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}} \frac{d(n_1)\cdots d(n_k)}{(n_1\cdots n_k)^{3/4}|\alpha(\mathbf{n};\mathbf{i})|},$$

$$\Sigma_5 = \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{k^2y<n_1\le N_2} \sum_{\substack{n_j\le y,\,2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}} \frac{d(n_1)\cdots d(n_k)}{(n_1\cdots n_k)^{3/4}|\alpha(\mathbf{n};\mathbf{i})|}.$$

By (6.21) we have

$$\Sigma_4 \ll U_k(k^2y) \ll y^{s(k)+\varepsilon}. \tag{6.27}$$

When $n_1 > k^2y$, it is easy to show that $|\alpha(\mathbf{n};\mathbf{i})| \gg n_1^{1/2}$, which implies that

$$\Sigma_5 \ll \sum_{k^2y<n_1\le N_2} \sum_{n_j\le y,\,2\le j\le k} \frac{d(n_1)\cdots d(n_k)}{(n_2\cdots n_k)^{3/4} n_1^{5/4}} \ll y^{(k-2)/4}\mathcal{L}^k \tag{6.28}$$

on noting that

$$\sum_{n\le y} d(n)n^{-3/4} \ll y^{1/4}\log y, \qquad \sum_{n>y} d(n)n^{-5/4} \ll y^{-1/4}\log y.$$

From (6.24)–(6.28) we have

$$\int_T^{2T} F_1^{k-1}(x)F_2(x)\,dx \ll T^{1+k(l-1/4)+\varepsilon}y^{-1/2} + T^{1/2+k(l-1/4)+\varepsilon}y^{s(k)}. \tag{6.29}$$

**6.5.** *Higher moments of $F_2(x)$.* From now on, we take $H := T^{A_0}$.

We first study the mean square of $F_2(x)$. Recall that $N_2 = H^2(2^{2J+1}+1/2)$. Since $\tau(n;H,T) = \tau(n)$ for $n \le y \le T\mathcal{L}^{-2}$, we have

$$\begin{aligned} F_2(x) &= \theta_l x^{l-1/4} \sum_{y<n\le N_2} \frac{\tau_l(n;H,T)}{n^{3/4}} \cos(2\pi\sqrt{nx}-\pi/4) + O(T^{l-1/2}\mathcal{L}^3) \\ &\ll T^{l-1/4}\Big|\sum_{y<n\le N_2} \frac{\tau_l(n;H,T)}{n^{3/4}} e(2\sqrt{nx})\Big| + T^{l-1/2}\mathcal{L}^3, \end{aligned}$$

which implies

(6.30)

$$\int_T^{2T} F_2^2(x)\,dx \ll T^{2l-1/2}\int_T^{2T}\Big|\sum_{y<n\le N_2} \frac{\tau(n;H,T)}{n^{3/4}} e(2\sqrt{nx})\Big|^2 dx + T^{2l}\mathcal{L}^3$$

$$\ll T^{2l+1/2}\sum_{y<n\le N_2}\frac{d^2(n)}{n^{3/2}}+T^{2l}\sum_{y<m<n\le N_2}\frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n}-\sqrt{m})}\ll\frac{T^{2l+1/2}\mathcal{L}^3}{y^{1/2}},$$

where we used the estimate $\sum_{n\le u}d^2(n)\ll u\log^3 u$ and the well-known Hilbert inequality.

Now suppose $y^{2b(K_0)}\le T$. Then from Lemma 6.2 we get

$$\int_T^{2T}|F_1(x)|^{K_0}\,dx\ll T^{1+K_0(l-1/4)+\varepsilon},$$

which implies

$$\int_T^{2T}|F_1(x)|^{A_0}\,dx\ll T^{1+A_0(l-1/4)+\varepsilon}\tag{6.31}$$

since $A_0\le K_0$. Trivially we have

$$G(x)=\sum_{m\le\sqrt{2T}}\min\Big(1,\frac{1}{H\|x/2m-m/2+1/2\|}\Big)\ll T^{1/2}.$$

By Lemma 6.1 we have

$$\int_T^{2T}G^{A_0}(x)\,dx\ll T^{(A_0-1)/2}\int_T^{2T}G(x)\,dx\ll T^{A_0/2+1}H^{-1}\mathcal{L}\ll T\mathcal{L}.\tag{6.32}$$

From (2.3), (6.31) and (6.32) we get

$$\begin{aligned}\int_T^{2T}|F_2(x)|^{A_0}\,dx&\ll\int_T^{2T}(|R_*(x)|^{A_0}+|F_1(x)|^{A_0}+T^{A_0(l-1/2)}G^{A_0}(x))\,dx\\&\ll T^{1+A_0(l-1/4)+\varepsilon}.\end{aligned}\tag{6.33}$$

For any $2<A<A_0$, from (6.30), (6.33) and Hölder's inequality we get

$$\begin{aligned}\int_T^{2T}|F_2(x)|^{A}\,dx&=\int_T^{2T}|F_2(x)|^{\frac{2(A_0-A)}{A_0-2}+\frac{A_0(A-2)}{A_0-2}}\,dx\\&\ll\Big(\int_T^{2T}F_2^2(x)\,dx\Big)^{\frac{A_0-A}{A_0-2}}\Big(\int_T^{2T}|F_2(x)|^{A_0}\,dx\Big)^{\frac{A-2}{A_0-2}}\\&\ll T^{1+A(l-1/4)+\varepsilon}y^{-\frac{A_0-A}{2(A_0-2)}}.\end{aligned}\tag{6.34}$$

This yields

LEMMA 6.3. *Suppose $T^{\varepsilon} \le y \le T^{1/2b(K_0)}$ and $2 < A < A_0$. Then*

$$\int_T^{2T} |F_2(x)|^A\,dx \ll T^{1+A(l-1/4)+\varepsilon} y^{-\frac{A_0-A}{2(A_0-2)}}. \tag{6.35}$$

**6.6.** *Evaluation of $\int_T^{2T} F^k(x)\,dx$.* Suppose $3 \le k < A_0$ and $T^{\varepsilon} \le y \le T^{1/2b(K_0)}$. By the elementary formula

$$(a+b)^k = a^k + ka^{k-1}b + O(|a^{k-2}b^2| + |b|^k)$$

we get

$$\int_T^{2T} F^k(x)\,dx = \int_T^{2T} F_1^k(x)\,dx + k\int_T^{2T} F_1^{k-1}(x)F_2(x)\,dx + O\Big(\int_T^{2T} |F_1^{k-2}(x)F_2^2(x)|\,dx + \int_T^{2T} |F_2(x)|^k\,dx\Big). \tag{6.36}$$

By (6.31), Lemma 6.3 and Hölder's inequality we get

$$\begin{aligned}\int_T^{2T} &|F_1^{k-2}(x)F_2^2(x)|\,dx \\ &\ll \Big(\int_T^{2T} |F_1(x)|^{A_0}\,dx\Big)^{\frac{k-2}{A_0}} \Big(\int_T^{2T} |F_2(x)|^{\frac{2A_0}{A_0-k+2}}\,dx\Big)^{\frac{A_0-k+2}{A_0}} \\ &\ll T^{1+k(l-1/4)+\varepsilon} y^{-\frac{A_0-k}{2(A_0-2)}}.\end{aligned} \tag{6.37}$$

Now from (6.29), (6.37), and Lemmas 6.2 and 6.3 ($A = k$) we get

$$\begin{aligned}\int_T^{2T} &F^k(x)\,dx \\ &= \frac{2^{1+k-lk} l^k B_k(\tau_l)}{(l!)^k \pi^k} \int_T^{2T} x^{k(l-1/4)}\,dx + O(T^{1+k(l-1/4)-\frac{A_0-k}{4(A_0-2)s(K_0)}+\varepsilon}) \\ &= \frac{2^{1+k-lk} l^k B_k(\tau_l)}{(l!)^k \pi^k} \int_T^{2T} x^{k(l-1/4)}\,dx + O(T^{1+k(l-1/4)-\delta_1(k,A_0)+\varepsilon})\end{aligned} \tag{6.38}$$

by choosing $y = T^{1/2s(K_0)}$.

**6.7.** *Proof of Theorem 2.* By Proposition 6.1 and the elementary formula $(a+b)^k = a^k + O(|a|^{k-1}|b| + |b|^k)$ we get

$$R_*^k(x) = F^k(x) + O(|F(x)|^{k-1}T^{l-1/2}G(x) + |F(x)|^{k-1}T^{l-1/2}\mathcal{L}^3) + O(T^{k(l-1/2)}G^k(x) + T^{k/2}\mathcal{L}^{3k}). \tag{6.39}$$

Lemma 6.1 and Hölder's inequality yield

$$\begin{aligned}\int_T^{2T} |F(x)|^{k-1}T^{l-1/2}G(x)\,dx &\ll T^{l-1/2}\Big(\int_T^{2T} |F(x)|^{A_0}\,dx\Big)^{\frac{k-1}{A_0}}\Big(\int_T^{2T} G(x)^{\frac{A_0}{A_0-k+1}}\,dx\Big)^{\frac{A_0-k+1}{A_0}} \\ &\ll T^{l-1/2}T^{(1+A_0(l-1/4)+\varepsilon)\frac{k-1}{A_0}}\Big(T^{\frac{k-1}{2(A_0-k+1)}}\int_T^{2T} G(x)\,dx\Big)^{\frac{A_0-k+1}{A_0}} \\ &\ll T^{l-1/2}T^{(1+3A_0/4+\varepsilon)\frac{k-1}{A_0}}\big(T^{\frac{k-1}{A_0-k+1}}T^{-(A_0-2)}\big)^{\frac{A_0-k+1}{A_0}} \\ &\ll T^{1/4+k(l-1/4)}.\end{aligned} \tag{6.40}$$

By Hölder's inequality again we get

$$\begin{aligned}\int_T^{2T} |F(x)|^{k-1}T^{l-1/2}\mathcal{L}^3\,dx &\ll T^{l-1/2}\mathcal{L}^3\Big(\int_T^{2T} |F(x)|^{A_0}\,dx\Big)^{\frac{k-1}{A_0}} T^{\frac{A_0-k+1}{A_0}} \\ &\ll T^{1+k(l-1/4)-1/4+\varepsilon}.\end{aligned} \tag{6.41}$$

From (6.39)–(6.41) we get

$$\int_T^{2T} R_*^k(x)\,dx = \frac{2^{1+k-lk}l^kB_k(\tau_l)}{(l!)^k\pi^k}\int_T^{2T} x^{k(l-1/4)}\,dx + O(T^{1+k(l-1/4)-\delta_1(k,A_0)+\varepsilon}),$$

which implies that

$$\begin{aligned}\int_1^{T} R_*^k(x)\,dx &= \frac{2^{1+k-lk}l^kB_k(\tau_l)}{(l!)^k\pi^k}\int_1^{T} x^{k(l-1/4)}\,dx + O(T^{1+k(l-1/4)-\delta_1(k,A_0)+\varepsilon}) \\ &= \frac{2^{1+k-lk}l^kB_k(\tau_l)}{(l!)^k\pi^k(k(l-1/4)+1)}T^{1+k(l-1/4)} + O(T^{1+k(l-1/4)-\delta_1(k,A_0)+\varepsilon}).\end{aligned} \tag{6.42}$$

Now Theorem 2 follows from (6.42) on noting that

$$\int_1^{T} R^k(t)\,dt = 2\pi\int_1^{T/2\pi} R_*^k(x)\,dx + O(1). \tag{6.43}$$

**7. Proofs of Theorems 3 and 4.** Throughout this section, let $A_0 = 262/27$ and $H = T^{262/27}$.

**7.1.** *Proof of Theorem 3.* The proof is almost the same as that of Theorem 2. So we only give an outline.

Applying the argument of the proof of Theorem 5.1 to $F_1(x)$ directly, we can show that

$$\int_T^{2T} |F_1(x)|^{A_0}\,dx \ll T^{1+A_0(l-1/4)}\mathcal{L}^{50} \tag{7.1}$$

for $y \le T^{77/208}$. Here we remark that if we want to get the result of the type (7.1) we have to assume $yT^{2\theta} \ll T$ when recalling the formula (5.8), where $\theta = 131/416$.

From (7.1), (6.32) and (2.3) we get

$$\int_T^{2T} |F_2(x)|^{A_0}\,dx \ll T^{1+A_0(l-1/4)+\varepsilon}. \tag{7.2}$$

From (7.1) and (7.2) we can show that the conclusion of Lemma 6.2 holds for $y \le T^{77/208}$. Using other estimates in the proof of Theorem 2, by choosing $y = T^{(k-2)/2(A_0-2)s(k)}$ we get

$$\begin{aligned}\int_T^{2T} R_*^k(x)\,dx &= \frac{2^{1+k-lk}l^k B_k(\tau_l)}{(l!)^k\pi^k}\int_T^{2T} x^{k(l-1/4)}\,dx + O(T^{1+k(l-1/4)+\varepsilon}y^{-\frac{A_0-k}{2(A_0-2)}})\\ &\quad + O(T^{1/2+k(l-1/4)+\varepsilon}y^{s(k)})\\ &= \frac{2^{1+k-lk}l^k B_k(\tau_l)}{(l!)^k\pi^k}\int_T^{2T} x^{k(l-1/4)}\,dx + O(T^{1+k(l-1/4)-\delta_2(k,262/27)+\varepsilon}),\end{aligned}$$

which implies Theorem 3.

**7.2.** *Proof of Theorem 4 (the case $k = 3$).* By the argument of Section 6.3 we get

$$\begin{aligned}\int_T^{2T} F_1^3(x)\,dx &= \frac{2^{4-2l}l^3 B_3(\tau_l)}{(l!)^3\pi^3}\int_T^{2T} x^{3(l-1/4)}\,dx\\ &\quad + O(T^{3l+1/4+\varepsilon}y^{-1/2}) + O(T^{3l-1/4}U_3(y)),\end{aligned} \tag{7.3}$$

where $U_3(y)$ was defined in Section 6.3. In Lemma 2.6 of [32], the author proved

$$U_3(y) \ll y^{1/4+\varepsilon}. \tag{7.4}$$

By the argument of Section 6.4 we get

$$\int_T^{2T} F_1^2(x)F_2(x)\,dx \ll T^{3l+1/4+\varepsilon}y^{-1/2}+T^{3l-1/4}y^{1/4}\mathcal{L}^3+T^{3l-1/4}U_3^*(y), \tag{7.5}$$

where

$$U_3^*(y)=\sum_{\mathbf{i}\in\mathbb{I}^2}\sum_{y<n_1\le 9y}\sum_{\substack{n_2\le y,\,n_3\le y\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}}\frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}|\alpha(\mathbf{n};\mathbf{i})|}.$$

By (7.4) we get

$$U_3^*(y)\ll U_3(9y)\ll y^{1/4+\varepsilon}. \tag{7.6}$$

Trivially we have $F_1(x)\ll x^{l-1/4}y^{1/4}\mathcal{L}$. So by (6.30) we obtain

$$\int_T^{2T}|F_1(x)F_2^2(x)|\,dx\ll T^{l-1/4}y^{1/4}\mathcal{L}\int_T^{2T}|F_2^2(x)|\,dx\ll T^{3l+1/4}y^{-1/4}\mathcal{L}^4 \tag{7.7}$$

for $y\le T$. The trivial estimate $F_2(x)\ll x^l$ and (6.30) yield

$$\int_T^{2T}|F_2^3(x)|\,dx\ll T^l\int_T^{2T}|F_2(x)|^2\,dx\ll T^{3l+1/2}y^{-1/2}\mathcal{L}^3. \tag{7.8}$$

From (6.36) with $k=3$ and (7.3)–(7.8) with $y=T\mathcal{L}^{-2}$ we get

$$\int_T^{2T}F^3(x)\,dx=\frac{2^{4-2l}l^3B_3(\tau_l)}{(l!)^3\pi^3}\int_T^{2T}x^{3(l-1/4)}\,dx+O(T^{3l+\varepsilon}),$$

which combined with the arguments of Section 6.7 gives

$$\int_T^{2T}R_*^3(x)\,dx=\frac{2^{4-2l}l^3B_3(\tau_l)}{(l!)^3\pi^3}\int_T^{2T}x^{3(l-1/4)}\,dx+O(T^{3l+\varepsilon}). \tag{7.9}$$

The formula (2.8) follows from (7.9).

**7.3.** *Proof of Theorem 4 (the case $k=4$).* By the argument of Section 6.3 we get

$$\int_T^{2T}F_1^4(x)\,dx=\frac{2^{5-4l}l^4B_4(\tau_l)}{(l!)^4\pi^4}\int_T^{2T}x^{4l-1}\,dx +O(T^{4l+\varepsilon}y^{-1/2})+O(T^{4l-1}V_{1,4}(y)), \tag{7.10}$$

where

$$V_{1,4}(y)=\sum_{\mathbf{i}\in\mathbb{I}^3}\sum_{\substack{n_j\le y,\,1\le j\le 4\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}}\frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1n_2n_3n_4)^{3/4}}\min\Big(T,\frac{T^{1/2}}{|\alpha(\mathbf{n};\mathbf{i})|}\Big).$$

In [31] the author proved that if $y \ll T^{3/4}$, then

$$V_{1,4}(y) \ll T^{1-3/28+\varepsilon}. \tag{7.11}$$

From (7.10) and (7.11) we get

$$\int_T^{2T} F_1^4(x)\,dx = \frac{2^{5-4l}l^4B_4(\tau_l)}{(l!)^4\pi^4}\int_T^{2T} x^{4l-1}\,dx + O(T^{4l-3/28+\varepsilon}). \tag{7.12}$$

By the argument of Section 6.4 we get

$$\int_T^{2T} F_1^3(x)F_2(x)\,dx \ll T^{4l+\varepsilon}y^{-1/2} + T^{4l-1/2}y^{1/2}\mathcal{L}^4 + T^{4l-1}V_{2,4}(y), \tag{7.13}$$

where

$$V_{2,4}(y) = \sum_{\mathbf{i}\in\mathbb{I}^3}\sum_{y<n_1\le 16y}\sum_{\substack{n_j\le y,\,2\le j\le 4\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1n_2n_3n_4)^{3/4}}\min\left(T, \frac{T^{1/2}}{|\alpha(\mathbf{n};\mathbf{i})|}\right).$$

From (7.11) we find that if $y \ll T^{3/4}$, then

$$V_{2,4}(y) \ll V_{1,4}(16y) \ll T^{1-3/28+\varepsilon}. \tag{7.14}$$

From Section 7.1 we know that the estimate of Lemma 6.3 holds for $y \le T^{77/208}$. Taking $A = 4$ in Lemma 6.3 we get

$$\int_T^{2T} F_2^4(x)\,dx \ll T^{4l+\varepsilon}y^{-77/208}. \tag{7.15}$$

Taking $k = 4$ in (6.37) we see that

$$\int_T^{2T} F_1^2(x)F_2^2(x)\,dx \ll T^{4l+\varepsilon}y^{-77/208} \tag{7.16}$$

for $y \le T^{77/208}$.

From (7.10)–(7.16) and taking $y = T^{1/3}$ we get

$$\int_T^{2T} F^4(x)\,dx = \frac{2^{5-4l}l^4B_4(\tau_l)}{(l!)^4\pi^4}\int_T^{2T} x^{4l-1}\,dx + O(T^{4l-3/28+\varepsilon}),$$

which combined with the arguments of Section 6.7 gives

$$\int_T^{2T} R_*^4(x)\,dx = \frac{2^{5-4l}l^4B_4(\tau_l)}{(l!)^4\pi^4}\int_T^{2T} x^{4l-1}\,dx + O(T^{4l-3/28+\varepsilon}). \tag{7.17}$$

The formula (2.9) follows from (7.17).

**8. Proofs of Theorems 5 and 6.** We shall follow Heath-Brown's argument [11]. We first quote some results from [11]. The following Hypothesis (H), Lemma 8.1 and Lemma 8.2 are Hypothesis (H), Theorem 5 and Theorem 6 of [11], respectively.

HYPOTHESIS (H). Let $M(t)$ be a real-valued function, $a_1(t), a_2(t), \ldots$ be continuous real-valued functions with period 1, and suppose there are nonzero constants $\gamma_1, \gamma_2, \ldots$ such that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \min\Big(1, \Big|M(t) - \sum_{n\le N} a_n(\gamma_n t)\Big|\Big)\, dt = 0.$$

LEMMA 8.1. *Suppose $M(t)$ satisfies* (H) *and the constants $\gamma_i$ are linearly independent over $\mathbb{Q}$. Suppose further*

$$\int_0^1 a_n(t)\, dt = 0 \quad (n \in \mathbb{N}), \qquad \sum_{n=1}^{\infty}\int_0^1 a_n^2(t)\, dt < \infty,$$

*and there is a constant $\mu > 1$ such that*

$$\max_{t\in[0,1]} |a_n(t)| \ll n^{1-\mu}, \qquad \lim_{n\to\infty} n^{\mu}\int_0^1 a_n^2(t)\, dt = \infty.$$

*Then $M(t)$ has a distribution function $f(\alpha)$ with the properties described in Theorem* 5.

LEMMA 8.2. *Suppose $M(t)$ satisfies* (H) *and*

$$\int_0^T |M(t)|^K\, dt \ll T$$

*for some $K > 0$. Then for any real number $k \in [0, K)$, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |M(t)|^k\, dt$$

*exists.*

Suppose $T \le x \le 2T$, $H = T^2$, $J = [(\mathcal{L} - \log\mathcal{L})/2\log 2]$. Define

$$\begin{aligned} M(x) &= x^{-(2l-1/2)} R_*(x^2),\\ a_n(x) &= \frac{\mu^2(n)}{n^{3/4}} \sum_{r=1}^{\infty} \frac{\tau_l(nr^2)}{r^{3/2}} \cos(2\pi r x - \pi/4),\\ \gamma_n &= \sqrt{n}. \end{aligned}$$

It is easy to check that $a_n(x)$ satisfies all conditions of Lemma 8.1 for any fixed constant $3/2 < \mu < 7/4$. By Proposition 6.1 we have

$$M(x) = M_1(x) + M_2(x) + M_3(x),$$

where

$$M_1(x) = \frac{2^{2-l}}{(l-1)!\pi} \sum_{1\le n\le T\mathcal{L}^{-2}} \frac{\tau_l(n)}{n^{3/4}} \cos(2\pi x\sqrt{n} - \pi/4),$$

$$M_2(x) = \frac{2^{2-l}}{(l-1)!\pi} \sum_{T\mathcal{L}^{-2}<n\le H^2(2^{2J+1}+1/2)} \frac{\tau_l(n;H,T)}{n^{3/4}} \cos(2\pi x\sqrt{n} - \pi/4),$$

$$M_3(x) = O(T^{-1/2}G(x^2) + T^{-1/2}\mathcal{L}^3).$$

It is easy to see that for any integer $N \le T^{1/3}$ we have (recall that $\tau_l(n;H,T) \ll d(n)$)

$$\begin{aligned}
&\Big|M(x) - \sum_{n\le N} a_n(\gamma_n x)\Big| \\
&\quad\ll \Big|\sideset{}{'}\sum_{n\le T\mathcal{L}^{-2}} \frac{\tau_l(n)}{n^{3/4}} \cos(2\pi x\sqrt{n} - \pi/4)\Big| \\
&\qquad + \sum_{n\le N} \frac{1}{n^{3/4}} \sum_{r>\sqrt{T}/\sqrt{n}} \frac{d(nr^2)}{r^{3/2}} + |M_2(x)| + M_3(x) \\
&\quad\ll \Big|\sideset{}{'}\sum_{n\le T\mathcal{L}^{-2}} \frac{\tau_l(n)}{n^{3/4}} \cos(2\pi x\sqrt{n} - \pi/4)\Big| + N^{1/2}T^{-1/4}\mathcal{L}^5 + |M_2(x)| + M_3(x),
\end{aligned}$$

where $\sum'$ means that $n$ has a square-free kernel greater that $N$.

We have (note that $\tau_l(n) \ll d(n)$)

$$\begin{aligned}
&\int_T^{2T} \Big|\sideset{}{'}\sum_{n\le T\mathcal{L}^{-2}} \frac{\tau_l(n)}{n^{3/4}} \cos(2\pi x\sqrt{n} - \pi/4)\Big|^2 dx \\
&\quad = T \sideset{}{'}\sum_{n\le T\mathcal{L}^{-2}} \frac{d^2(n)}{n^{3/2}} + O(T^\varepsilon) \ll T\sum_{n>N} \frac{d^2(n)}{n^{3/2}} + T^\varepsilon \ll TN^{-1/2}\log^3 N + T^\varepsilon.
\end{aligned}$$

Similarly to (6.30) we have

$$\int_T^{2T} |M_2(x)|^2\,dx \ll TN^{-1/2}\log^3 N.$$

By the trivial estimate $G(u) \ll \sqrt{u}$ $(u \sim T)$ and Lemma 6.1 we get

$$\begin{aligned}
\int_T^{2T} |M_3(x)|^2\,dx &\ll T^{-1}\int_T^{2T} G^2(x^2)\,dx + \mathcal{L}^6 \ll \int_T^{2T} G(x^2)\,dx + \mathcal{L}^6 \\
&\ll \int_{\sqrt{T}}^{\sqrt{2T}} G(y)y^{-1/2}\,dy + \mathcal{L}^6 \ll T^{1/2}H^{-1}\mathcal{L} + \mathcal{L}^6 \ll \mathcal{L}^6.
\end{aligned}$$

From the above estimates and Cauchy's inequality we get

$$\limsup_{T\to\infty}\frac{1}{T}\int_T^{2T}\Big|M(x)-\sum_{n\le N}a_n(\gamma_n x)\Big|^2\,dx\ll N^{-1/2}\log^3 N,$$

whence Hypothesis (H) follows. From Lemma 8.1 with $\mu = 5/3$ we get Theorem 5.

Now we prove Theorem 6. According to Lemma 8.2, it suffices to show that

$$\int_0^T x^{-k(l-1/4)}|R(x)|^k\,dx\ll T \tag{8.1}$$

for any real number $0\le k<A_0:=262/27$.

Fix $2<k<A_0$. Suppose $x\sim T$, $y=T^{1/s(10)}$, $H=T^5$. We have

$$R(x)\ll|F_1(x)|+|F_2(x)|+T^{l-1/2}G(x)+T^{l-1/2}\mathcal{L}^3,$$

where $F_1(x), F_2(x)$ and $G(x)$ were defined in Section 6. By Lemma 6.2

$$\int_T^{2T}|F_1(x)|^{10}\,dx\ll T^{1+10(l-1/4)},$$

which implies that

$$\int_T^{2T}|F_1(x)|^k\,dx\ll T^{1+k(l-1/4)}.$$

By Lemma 6.3,

$$\int_T^{2T}|F_2(x)|^k\,dx\ll T^{1+k(l-1/4)-(A_0-k)/2(A_0-2)s(10)+\varepsilon}\ll T^{1+k(l-1/4)}.$$

By Lemma 6.1,

$$\int_T^{2T}T^{k(l-1/2)}G^k(x)\,dx\ll T^{k(l-1/2)}T^{(k-1)/2}\int_T^{2T}G(x)\,dx\ll T^{k(l-1/4)-2}.$$

From the above estimates we get

$$\int_T^{2T}|R(x)|^k\,dx\ll T^{1+k(l-1/4)}$$

and correspondingly,

$$\int_0^T|R(x)|^k\,dx\ll T^{1+k(l-1/4)},$$

which implies (8.1) by partial summation. This completes the proof of Theorem 6.

**References**


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*Received on 6.8.2007* (5484)