ACTA ARITHMETICA
157.2 (2013)

On the cycle map for products of elliptic curves over a p-adic field

by

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1. Introduction. Let $X = E \times E'$ be the product of elliptic curves E and E' defined over a finite extension K of the p-adic field $\mathbb{Q}_p$. The main objective of this note is to study the Chow group $\mathrm{CH}_0(X)$ of 0-cycles on X modulo rational equivalence. Let $A_0(X)$ be the kernel of the degree map $\mathrm{CH}_0(X) \to \mathbb{Z}$ and $T(X)$ the kernel of the Albanese map $A_0(X) \to X(K)$, called the *Albanese kernel* for X. These maps are surjective, and we have $\mathrm{CH}_0(X)/T(X) \simeq \mathbb{Z} \oplus X(K)$. If we assume the p^n-torsion points $E[p^n]$ and $E'[p^n]$ are K-rational, Mattuck's theorem [9] on $X(K)$ implies $\mathrm{CH}_0(X)/p^n \simeq (\mathbb{Z}/p^n)^{\oplus(2[K:\mathbb{Q}_p]+5)} \oplus T(X)/p^n$. Raskind–Spieß [12] showed the injectivity of the cycle map $\rho : T(X)/p^n \to H^4(X, \mathbb{Z}/p^n(2))$ to the étale cohomology group of X with coefficients $\mathbb{Z}/p^n(2) = \mu_{p^n} \otimes \mu_{p^n}$ when E and E' have ordinary or split multiplicative reduction. Although it is difficult to find the kernel of ρ in general (the injectivity fails for certain surfaces, see [11, Sect. 8]), one can calculate the structure of its image. This is the main theorem of this note:

Main Theorem 1.1 (Thm. 3.4). *Let E and E' be elliptic curves over K with good or split multiplicative reduction, and assume that $E[p^n]$ and $E'[p^n]$ are K-rational. The structure of the image of $T(X)/p^n$ for $X = E \times E'$ by the cycle map ρ is*

(i) $\mathbb{Z}/p^n$ *if both E and E' have ordinary or split multiplicative reduction,*
(ii) $\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n$ *if E and E' have different reduction types.*

Our computation works well in the remaining case where both E and E' have supersingular reduction. The image may vary according to the pth coefficients of multiplication p formula of the formal completion of the curves along the origin (cf. Prop. 3.6). For arbitrary elliptic curves E, E' over K and $X = E \times E'$, the base change $X' := X \otimes_K K'$ to some sufficiently large

2010 *Mathematics Subject Classification*: Primary 11G07; Secondary 19D45.
Key words and phrases: elliptic curves, cycle map, local fields.

DOI: 10.4064/aa157-2-1

extension field K' over K satisfies the assumptions in our main theorem above. Since the kernel of multiplication by p^n on $\mathrm{CH}_0(X)$ is finite (due to Colliot-Thélène [4]), we have a surjection $\mathrm{CH}_0(X')/p^n \to \mathrm{CH}_0(X)/p^n$ with finite kernel if we admit the Raskind and Spieß conjecture ([12, Conj. 3.5.4]); the finiteness of the kernel of the cycle map on X' (see also [12, Cor. 3.5.2]).

The estimation of the difference of the image of $T(X)/p^n$ and $T(X')/p^n$ by the cycle maps is also a problem. Murre and Ramakrishnan ([10, Thm. A]) gave an answer to this problem in the case of $n = 1$ for the self-product $X = E \times E$ of an elliptic curve E over K with ordinary good reduction. In this case, they proved that the structure of the image is at most $\mathbb{Z}/p$ and is exactly $\mathbb{Z}/p$ if and only if the definition field $K(E[p])$ over K is unramified such that the prime-to-p part of $[K(E[p]) : K]$ is ≤ 2 and K has a pth root of unity ζ_p. Note also any elliptic curve with multiplicative reduction has indeed split multiplicative reduction if we assume $p \neq 2$ and the p-torsion points are K-rational ([8, Lem. 4.1.2]). Therefore, we are considering elliptic curves with semi-stable reduction essentially.

The results in our main theorem are known from Takemoto [16] in the case of ordinary reduction or split multiplicative reduction (but the current paper contains an alternative proof in this case, cf. Sect. 3). So our main interest is in supersingular elliptic curves. In Section 2 we study the image of the Kummer homomorphism associated with an isogeny of formal groups. The main ingredient is the structure of the graded quotients of a filtration on formal groups (Prop. 2.8). As a special case, we obtain the structure of the graded quotients associated with a filtration on the multiplicative group modulo p^n. The proof of the main theorem is given in Section 3.

For a discrete valuation field K, we denote by $\mathcal{O}_K$ the valuation ring of K, by $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$, by $k := \mathcal{O}_K/\mathfrak{m}_K$ the residue field of $\mathcal{O}_K$, by v_K the normalized valuation of $\mathcal{O}_K$, by $\mathcal{O}_K^\times$ the group of units in $\mathcal{O}_K$, by $\overline{K}$ a fixed separable closure of K and by $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. For an abelian group A and a non-zero integer m, let $A[m]$ be the kernel and A/m the cokernel of the map $m : A \to A$ defined by multiplication by m.

2. Formal groups. Let K be a complete discrete valuation field of characteristic 0, and k its perfect residue field of characteristic $p > 0$. In this section, we determine the image of the Kummer map associated with an isogeny of formal groups (Thm. 2.11).

First we recall some basic notions concerning formal groups from [6]. Throughout this section, all formal groups are commutative of dimension one. Let F be a formal group over the valuation ring $\mathcal{O}_K$. The elements of the maximal ideal $\mathfrak{m}_{\overline{K}}$ of $\mathcal{O}_{\overline{K}}$ form a G_K-module denoted by $F(\overline{K})$ under

the operation $x + y := F(x, y)$. Similarly, for a finite extension L/K, the maximal ideal $\mathfrak{m}_L$ forms a subgroup of $F(\bar{K})$ denoted by $F(L)$.

For an isogeny $\phi : F \to G$ of formal groups defined over $\mathcal{O}_K$, we regard it as a power series $\phi(T) = a_1T + a_2T^2 + \cdots + a_pT^p + \cdots \in \mathcal{O}_K[\![T]\!]$. The coefficient of T in $\phi(T)$ is denoted by $D(\phi) := a_1$. The *height* of ϕ is defined to be a positive integer n such that $\phi(T) \equiv \psi(T^{p^n}) \bmod \mathfrak{m}_K$ for some $\psi \in \mathcal{O}_K[\![T]\!]$ with $v_K(D(\psi)) = 0$ (cf. [8, 2.1]). It is known that the induced homomorphism $F(\bar{K}) \to G(\bar{K})$ from the isogeny $\phi : F \to G$ is surjective and the kernel of ϕ (= the kernel of the homomorphism $F(\bar{K}) \to G(\bar{K})$ induced by ϕ) is a finite group of order p^n, where n is the height of ϕ. For any integer $m \geq 1$, $F^m(K)$ is the subgroup of $F(K)$ consisting of the set $\mathfrak{m}_K^m$. Fix a uniformizer π of K. For any $m \geq 1$, we have an isomorphism

$$\rho : k \xrightarrow{\sim} \mathrm{gr}^m(F) := F^m(K)/F^{m+1}(K) \tag{2.1}$$

defined by $x \mapsto \widetilde{x}\pi^m$, where $\widetilde{x} \in \mathcal{O}_K^\times$ is a lift of $x \in k \smallsetminus \{0\}$. Recall the behavior of the operation on the graded quotients of raising to an isogeny $\phi : F \to G$ with height 1.

LEMMA 2.1 ([1]; [8, Lem. 2.1.2]). *Let $\phi(T) := a_1T + a_2T^2 + \cdots$ be an isogeny $F \to G$ of formal groups defined over $\mathcal{O}_K$ with height 1.*

(i) *The pth coefficient a_p is a unit in $\mathcal{O}_K$.*
(ii) *For m such that $p \nmid m$, we have $a_1 \mid a_m$.*

The following lemma is proved essentially in the same way as in the case of $F = \widehat{\mathbb{G}}_m$, the multiplicative group (e.g., [5, Chap. I, Sect. 5]).

LEMMA 2.2. *Let $\phi(T) := a_1T + a_2T^2 + \cdots$ be an isogeny $F \to G$ of formal groups defined over $\mathcal{O}_K$ with height 1. Define $t := v_K(a_1)$ and let a be the residue class of $a_1\pi^{-t}$ and $m \geq 1$ an integer. Then $\phi(F^m(K)) \subset G^{mp}(K)$ for $m \leq t/(p-1)$ and $\phi(F^m(K)) \subset G^{m+t}(K)$ for $m > t/(p-1)$. The isogeny ϕ induces the following commutative diagrams:*

(i) *If $m < t/(p-1)$, we have*

$$\begin{array}{ccc} \mathrm{gr}^m(F) & \xrightarrow{\ \phi\ } & \mathrm{gr}^{mp}(G) \\ {\scriptstyle\rho}\uparrow & & \uparrow{\scriptstyle\rho} \\ k & \xrightarrow{\ \overline{a}_pC^{-1}\ } & k \end{array}$$

where $\overline{a}_p \in k$ is the residue class of $a_p \in \mathcal{O}_K^\times$ and $C^{-1} : k \to k$ is "the inverse Cartier operator" defined by $x \mapsto x^p$. The horizontal homomorphisms are bijective.

(ii) *If $m = t/(p-1)$ is in $\mathbb{Z}$, we have*

$$\begin{array}{ccc} \mathrm{gr}^{t/(p-1)}(F) & \xrightarrow{\phi} & \mathrm{gr}^{t+t/(p-1)}(G) \\ \uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \rho} \\ k & \xrightarrow{a+\overline{a}_p C^{-1}} & k \end{array}$$

where the bottom map is defined by $x \mapsto ax + \overline{a}_p x^p$.

(iii) *If $m > t/(p-1)$, we have*

$$\begin{array}{ccc} \mathrm{gr}^{m}(F) & \xrightarrow{\phi} & \mathrm{gr}^{m+t}(G) \\ \uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \rho} \\ k & \xrightarrow{a} & k \end{array}$$

where the bottom map is defined by $x \mapsto ax$. The horizontal homomorphisms are bijective. Furthermore, $G^{m+t}(K) \subset \phi F^m(K)$.

Proof. Take any $u\pi^m \in F^m(K)$ with $u \in \mathcal{O}_K^\times$. From Lemma 2.1, we have $v_K(\phi(u\pi^m)) \geq \min\{t+m, pm\}$ (equality holds if $m \neq t/(p-1)$). Moreover,

$$\phi(u\pi^m) \equiv \begin{cases} \overline{a}_p u^p \pi^{mp} \bmod \pi^{mp+1} & \text{if } m < t/(p-1), \\ (au + \overline{a}_p u^p)\pi^{t+t/(p-1)} \bmod \pi^{t+t/(p-1)+1} & \text{if } m = t/(p-1), \\ au\pi^{m+t} \bmod \pi^{m+t+1} & \text{if } m > t/(p-1). \end{cases}$$

All assertions except the last one follow from this. Using the completeness of K, we obtain $G^{m+t}(K) \subset G^{m+t+1}(K) + \phi F^m(K) \subset G^{m+t+2}(K) + \phi F^m(K) \subset \cdots \subset \phi F^m(K)$ if $m > t/(p-1)$. ∎

Corollary 2.3. *Let $\phi : F \to G$ be an isogeny of formal groups defined over $\mathcal{O}_K$ with height 1. Assume $F[\phi] := \mathrm{Ker}(\phi) \subset F(K)$. For any non-zero element $x \in F[\phi]$, we have $v_K(x) = t/(p-1) \in \mathbb{Z}$. The kernel of $\phi : \mathrm{gr}^{t/(p-1)}(F) \to \mathrm{gr}^{t+t/(p-1)}(G)$ is of order p.*

Proof. For any non-zero $x \in F[\phi]$, we have $\phi(x) = a_1 x + a_2 x^2 + \cdots = 0$. Hence $t + v_K(x) = v_K(a_1 x) = v_K(a_p x^p) = p v_K(x)$ and $v_K(x) = t/(p-1)$. The kernel of $x \mapsto ax + \overline{a}_p x^p$ is $\sqrt[p-1]{-a/\overline{a}_p}\,\mathbb{F}_p$. ∎

The filtration $G^m(\phi)$ on $G(\phi) := G(K)/\phi F(K)$ is defined by the image of the filtration $G^m(K)$. For an isogeny $\phi : F \to G$ with height 1, its graded quotients $\mathrm{gr}^m(\phi) := G^m(\phi)/G^{m+1}(\phi)$ describe the cokernels of ϕ in Lemma 2.2 as follows:

Lemma 2.4. *Let $\phi : F \to G$ be an isogeny over $\mathcal{O}_K$ with height 1 and $t := v_K(D(\phi))$.*

(i) *If $m < t + t/(p-1)$, the sequence*

$$0 \to \mathrm{gr}^{m/p}(F) \xrightarrow{\phi} \mathrm{gr}^m(G) \to \mathrm{gr}^m(\phi) \to 0$$

is exact, where $\mathrm{gr}^x(F) = 0$ if $x \notin \mathbb{Z}$ by convention.

(ii) *If* $m = t + t/(p-1)$ *is in* $\mathbb{Z}$, *then*

$$\mathrm{gr}^{t/(p-1)}(F) \xrightarrow{\phi} \mathrm{gr}^{t+t/(p-1)}(G) \to \mathrm{gr}^{t+t/(p-1)}(\phi) \to 0$$

is exact.

(iii) *If* $m > t + t/(p-1)$, *then*

$$0 \to \mathrm{gr}^{m-t}(F) \xrightarrow{\phi} \mathrm{gr}^{m}(G) \to \mathrm{gr}^{m}(\phi) \to 0$$

is exact.

Proof. Note that $\mathrm{gr}^m(\phi) \simeq G^m(K)/(\phi F(K) \cap G^m(K) + G^{m+1}(K))$. Consider cases (i), (ii). For any $\phi(x) \in G^m(K)$ with $x \in F(K)$, we have $v_K(\phi(x)) \geq \min\{t+r, pr\}$, where $r = v_K(x)$ (equality holds if $r \neq t/(p-1)$) by Lemma 2.1. To show the injectivity of $\mathrm{gr}^m(G) \to \mathrm{gr}^m(\phi)$ if $p \nmid m$, it is enough to show $\phi F(K) \cap G^m(K) \subset G^{m+1}(K)$. For any $\phi(x) \in G^m(K) \cap \phi F(K)$, assume $m = v_K(\phi(x))$. By Lemma 2.1 as above, $m = pr$ if $t/(p-1) > r$. Otherwise, $m = pt/(p-1)$. This contradicts $p \nmid m$. Thus $v_K(\phi(x)) > m$ and we obtain $\phi(x) \in G^{m+1}(K)$. In the case of $p \mid m$, take any $\phi(x) \in G^m(K) \cap \phi F(K)$ with $m = v_K(\phi(x))$. From the above (in)equality, we have $v_K(x) = m/p$. The other assertions follow from this.

Next we consider case (iii). For any $\phi(x) \in G^m(K) \cap \phi F(K)$ with $m = v_K(\phi(x))$. If $t/(p-1) > r$ then $m = pr < pt/(p-1)$ and this contradicts $m > pt/(p-1)$. Otherwise $m \geq t + r$. Hence $r \leq m - t$ and thus $x \in F^{m-t}(K)$. ∎

Recall that a perfect field is said to be *quasi-finite* if its absolute Galois group is isomorphic to $\widehat{\mathbb{Z}}$ (cf. [13, Chap. XIII, Sect. 2]).

Corollary 2.5 (cf. [1, Lem. 1.1.2]; [8, Lem. 2.1.3]). *Let* $\phi(T) := a_1 T + a_2 T^2 + \cdots$ *be an isogeny* $F \to G$ *of formal groups defined over* $\mathcal{O}_K$ *with height* 1. *Assume* $F[\phi] \subset F(K)$. *Define* $t := v_K(a_1)$ *and let* $m \geq 1$ *be an integer.*

(i) *If* $m < t + t/(p-1)$, *then*

$$\mathrm{gr}^m(\phi) \simeq \begin{cases} k & \text{if } p \nmid m, \\ 0 & \text{if } p \mid m. \end{cases}$$

(ii) *If* $m = t + t/(p-1)$, *then* $\mathrm{gr}^{t+t/(p-1)}(\phi) \simeq k/(a + \overline{a}_p C^{-1})k$, *where* a *is the residue class of* $a_1 \pi^{-t}$ *and* $\overline{a}_p$ *is the image of* a_p *in* k. *If we further assume that* k *is separably closed, then* $\mathrm{gr}^{t+t/(p-1)}(\phi) = 0$. *If* k *is quasi-finite, then* $\mathrm{gr}^{t+t/(p-1)}(\phi) \simeq \mathbb{Z}/p\mathbb{Z}$.

(iii) *If* $m > t + t/(p-1)$, *then* $G^m(\phi) = 0$. *In particular,* $\mathrm{gr}^m(\phi) = 0$.

Proof. The proof is taken from [8]. The assertions follow from Lemmas 2.2 and 2.4. If k is quasi-finite, then the homomorphism $\phi : \mathrm{gr}^{t/(p-1)}(F) \to \mathrm{gr}^{t+t/(p-1)}(G)$ extend to $\phi : \overline{k} \to \overline{k}$. Since $H^1(k, \overline{k}) = 1$ and $\mathrm{Ker}(\phi) \simeq \mathbb{Z}/p\mathbb{Z}$ as G_k-modules, we have $k/\phi(k) \simeq H^1(k, \mathrm{Ker}(\phi)) \simeq \mathbb{Z}/p\mathbb{Z}$. ∎

Let $\phi : F \to G$ be an isogeny with finite height $n > 1$ and *assume* $F[\phi]$ is cyclic and $F[\phi] \subset F(K)$. Let $x_0 \in F(K)$ be the generator of the cyclic group $F[\phi]$. The subgroup $pF[\phi] \subset F[\phi]$ generated by $[p]x_0$ has order p^{n-1}, where $[p]$ is multiplication by p on F. From the theorem of Lubin ([6, Chap. IV, Thm. 4]), there exists a formal group $G_1 := F/pF[\phi]$ defined over $\mathcal{O}_K$ and the isogeny ϕ factors as $\phi = \phi_1 \circ \psi$, where $\psi : F \to G_1$ is an isogeny over $\mathcal{O}_K$ such that $F[\psi] = pF[\phi]$ (thus ψ is an isogeny with height $n-1$ and ϕ_1 has height 1). Note that the kernel $G_1[\phi_1]$ is generated by $\psi(x_0)$. From the following lemma, the structure of $\mathrm{gr}^m(\phi)$ is obtained from that in the case of height 1 (Cor. 2.5).

LEMMA 2.6. *Put* $t_1 := v_K(D(\phi_1))$.

(i) *If* $m < t_1 + t_1/(p-1)$, *the sequence*

$$0 \to \mathrm{gr}^{m/p}(\psi) \xrightarrow{\phi_1} \mathrm{gr}^m(\phi) \to \mathrm{gr}^m(\phi_1) \to 0$$

is exact, where $\mathrm{gr}^x(\psi) = 0$ *if* $x \notin \mathbb{Z}$ *by convention.*

(ii) *If* $m = t_1 + t_1/(p-1)$, *the sequence*

$$\mathrm{gr}^{t_1/(p-1)}(\psi) \xrightarrow{\phi_1} \mathrm{gr}^{t_1+t_1/(p-1)}(\phi) \to \mathrm{gr}^{t_1+t_1/(p-1)}\,\mathrm{gr}(\phi_1) \to 0$$

is exact.

(iii) *If* $m > t_1 + t_1/(p-1)$, *the sequence*

$$0 \to \mathrm{gr}^{m-t_1}(\psi) \xrightarrow{\phi_1} \mathrm{gr}^m(\phi) \to \mathrm{gr}^m(\phi_1) \to 0$$

is exact. In particular, $\mathrm{gr}^{m-t_1}(\psi) \simeq \mathrm{gr}^m(\phi)$.

Proof. (i) & (ii); $m \le t_1 + t_1/(p-1)$. In the commutative diagram

$$\begin{array}{ccccccc} \mathrm{gr}^{m/p}(G_1) & \xrightarrow{\phi_1} & \mathrm{gr}^m(G) & \longrightarrow & \mathrm{gr}^m(\phi_1) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \| & & \\ \mathrm{gr}^{m/p}(\psi) & \xrightarrow{\phi_1} & \mathrm{gr}^m(\phi) & \longrightarrow & \mathrm{gr}^m(\phi_1) & \longrightarrow & 0 \end{array} \tag{2.2}$$

the top horizontal row is exact by Lemma 2.4 and the vertical arrows are surjective. We show the injectivity of $\phi_1 : \mathrm{gr}^{m/p}(\psi) \to \mathrm{gr}^m(\phi)$ when $m < t_1 + t_1/(p-1)$ and $m \,|\, p$. In this case, the map $\phi_1 : \mathrm{gr}^{m/p}(G_1) \to \mathrm{gr}^m(G)$ in (2.2) is injective. Thus, it is enough to show the surjectivity of

$$\phi_1 : G_1^{m/p}(K) \cap \psi F(K)/G_1^{m/p+1}(K) \cap \psi F(K) \to G^m(K) \cap \phi F(K)/G^{m+1}(K) \cap \phi F(K).$$

For any $\phi(x) = \phi_1 \circ \psi(x)$ in $G^m(K) \cap \phi F(K)/G^{m+1}(K)$, there exists $y \in G_1^{m/p}(K)$ such that $\phi_1(y) = \phi(x)$ by Lemma 2.4. Hence, we obtain $y = \psi(x) \in G_1^{m/p}(K) \cap \psi F(K)/G_1^{m/p+1}(K) \cap \psi F(K)$.

(iii) follows by a similar argument. ∎

Inductively, one can find isogenies $\phi_i : G_i \to G_{i-1}$ with height 1 such that $\phi = \phi_1 \circ \cdots \circ \phi_n$ and $G_i = F/p^i F[\phi]$, where $p^i F[\phi]$ is the subgroup of $F[\phi]$ generated by $[p^i]x_0$ (we set $F = G_n$ and $G = G_0$ by convention). Define $t_i := v_K(D(\phi_i))$ and put $t_0 := 0$.

LEMMA 2.7. *For $1 \le i < n$, we have $t_i \le t_{i+1}$ and $p^{n-i} \mid t_i$. The equality $t_i = t_{i+1}$ does not hold if the height of F is > 1.*

Proof. By induction on n, it is enough to show the case $n = 2$; $\phi = \phi_1 \circ \phi_2$ has height 2. Recall $[p]x_0 \in F[\phi_2]$ and $\phi_2(x_0) \in G_1[\phi_1]$. From Lemma 2.1, we obtain $t_1/(p-1) = v_K(\phi_2(x_0))$ and

$$t_2/(p-1) = v_K([p]x_0) = v_K(\widehat{\phi_2} \circ \phi_2(x_0)) \ge v_K(\phi_2(x_0)). \tag{2.3}$$

Hence $t_2 \ge t_1$ and $v_K(\phi_2(x_0)) = pv_K(x_0)$. From (2.3), if the height of F is > 1, then $t_i < t_{i+1}$. ∎

PROPOSITION 2.8. *Let $\phi = \phi_1 \circ \cdots \circ \phi_n : F \to G$ and t_i be as above. Put $c_i(\phi) := t_0 + t_1 + \cdots + t_i + t_i/(p-1)$ for $0 \le i \le n$.*

(i) *If $c_i(\phi) < m < c_{i+1}(\phi)$ for some $0 \le i < n$, then*

$$\mathrm{gr}^m(\phi) \simeq \mathrm{gr}^{m-(t_1+t_2+\cdots+t_i)}(\phi_{i+1} \circ \cdots \circ \phi_n) \simeq \begin{cases} k & \text{if } p^{n-i} \nmid m, \\ 0 & \text{if } p^{n-i} \mid m. \end{cases}$$

(ii) *If $m = c_{i+1}(\phi)$ for some $0 \le i < n$, then*

$$\mathrm{gr}^{c_{i+1}(\phi)}(\phi) \simeq \mathrm{gr}^{pt_{i+1}/(p-1)}(\phi_{i+1} \circ \cdots \circ \phi_n) \simeq k/(a + \overline{a}_p C^{-1})k,$$

where a_p is the coefficient of T^p in $\phi_{i+1} \in \mathcal{O}_K[\![T]\!]$ and a is the residue class of $D(\phi_{i+1})\pi^{-t_{i+1}}$. If we further assume that k is separably closed, then $\mathrm{gr}^{c_{i+1}(\phi)}(\phi) = 0$. If k is quasi-finite, then $\mathrm{gr}^{c_{i+1}(\phi)}(\phi) \simeq \mathbb{Z}/p\mathbb{Z}$.

(iii) *If $m > c_n(\phi)$ then $G^m(\phi) = 0$. In particular $\mathrm{gr}^m(\phi) = 0$.*

Proof. From Lemma 2.5, we may assume $n > 1$. Put $\psi = \phi_2 \circ \cdots \circ \phi_n$. First we consider the case $0 < m < c_1(\phi)$ in (i). In the exact sequence (Lem. 2.6(i))

$$0 \to \mathrm{gr}^{m/p}(\psi) \to \mathrm{gr}^m(\phi) \to \mathrm{gr}^m(\phi_1) \to 0,$$

the isogeny ϕ_1 has height 1 and ψ has height $n-1$. Thus we obtain the structure of $\mathrm{gr}^m(\phi)$ for $m < c_1(\phi)$ by induction on n and Lemma 2.5(i). If $m = c_1(\phi)$, then

$$\mathrm{gr}^{t_1/(p-1)}(\psi) \to \mathrm{gr}^{t_1+t_1/(p-1)}(\phi) \to \mathrm{gr}^{t_1+t_1/(p-1)}(\phi_1) \to 0$$

by Lemma 2.6(ii). From Lemma 2.7, we have $p^{n-1} \mid t_1$. By (i) and the case $m < c_1(\phi)$, $\mathrm{gr}^{t_1+t_1/(p-1)}(\psi) = 0$ and thus $\mathrm{gr}^{t_1+t_1/(p-1)}(\phi) \simeq \mathrm{gr}^{t_1+t_1/(p-1)}(\phi_1)$. The assertion follows from Lemma 2.5(ii). Consider the case $m > c_1(\phi)$ in (i) and (ii). By Lemma 2.6(iii), $\mathrm{gr}^{m-t_1}(\psi) \simeq \mathrm{gr}^m(\phi)$. From the induction on n, the assertions are reduced to the case $m \le c_1(\phi)$. ∎

Let L/K be a finite Galois extension with Galois group $H = \mathrm{Gal}(L/K)$. Recall that x is a *jump* for the ramification filtration $(H_j)_{j\geq -1}$ in the lower numbering (resp. $(H^j)_{j\geq -1}$ in the upper numbering) of H if $H_x \neq H_{x+\varepsilon}$ (resp. $H^x \neq H^{x+\varepsilon}$) for all $\varepsilon > 0$ (for definition of the ramification subgroups, see [13, Chap. IV]).

PROPOSITION 2.9. *For $y \in G^m(\phi) \setminus G^{m+1}(\phi)$, take $x \in F(\overline{K})$ with $\phi(x) = y$ in $G(\phi)$. If $1 \leq m < c_1(\phi)$ and $p \nmid m$, then the field of definition $L = K(x)$ of x over K is a totally ramified Galois extension of degree p^n. The jumps of the ramification subgroups of $H := \mathrm{Gal}(L/K)$ in the upper numbering are $c_1(\phi) - m, \ldots, c_n(\phi) - m$. In particular, $H^{c_n(\phi)-m} \neq 1$.*

Proof. For $n = 1$, when $\phi = \phi_1$ has height 1, the assertion follows from [8, Lemma 2.1.5]. For $n > 1$, for the isogeny $\phi = \phi_1 \circ \psi$ ($\psi = \phi_2 \circ \cdots \circ \phi_n$), we have $y' \in G_1(\overline{K})$ such that $\psi(x) = y'$ and $\phi_1(y') = y$. The isogeny ϕ_1 has height 1, and the extension $K' := K(y')/K$ is totally ramified of degree p. The jump is $pt_1/(p-1) - m$. Since $m < c_1(\phi)$ and $p \nmid m$, $v_{K'}(y) = pv_K(y) = v_K(\phi_1(y')) = pv_{K'}(y')$. Hence $v_{K'}(y') = v_K(y) = m$.

By induction on n, the extension L/K' is a totally ramified extension of degree p^{n-1}. The jumps of $\mathrm{Gal}(L/K')$ in the lower numbering are $p^2t_2/(p-1) - m, p^3t_3/(p-1) - m, \ldots, p^nt_n/(p-1) - m$. Since the ramification subgroups in the lower numbering commute with subgroups and in the upper numbering commute with quotients ([13, Chap. IV]), the jumps of the ramification subgroups of H in the lower numbering are $p^it_i/(p-1) - m$ for $1 \leq i \leq n$. The ramification subgroups H^s of H in the upper numbering are defined by using the Herbrand function φ of H as $H_j = H^{\varphi(j)}$. For the positive integer m, we have $\varphi(m) + 1 = \sum_{i=0}^{m} \#(H_i/H_0)$. Thus $\varphi(p^it_i/(p-1) - m) = c_i(\phi) - m$. ∎

The isogeny $[p^n] : \widehat{\mathbb{G}}_m \to \widehat{\mathbb{G}}_m$ defined by multiplication by p^n on the multiplicative group $\widehat{\mathbb{G}}_m$ has the kernel $\widehat{\mathbb{G}}_m[p^n] = \mu_{p^n}$, which is cyclic of order p^n. If K contains a p^nth root of unity ζ_{p^n}, then $\widehat{\mathbb{G}}_m[p^n] \subset \widehat{\mathbb{G}}_m(K)$. Note also that the filtration $\widehat{\mathbb{G}}_m^j([p^n])$ of $\widehat{\mathbb{G}}_m([p^n]) = \widehat{\mathbb{G}}_m(K)/[p^n]\widehat{\mathbb{G}}_m(K) \subset K^\times/p^n$ associated with $[p^n]$ is the image of the higher unit group $U_K^j = 1 + \mathfrak{m}_K^j$ in $K^\times/p^n$ which is also denoted by U_n^j, namely, $\widehat{\mathbb{G}}_m^j([p^n]) = U_n^j := U_K^j/((K^\times)^{p^n} \cap U_K^j)$. Put $U_n^0 := K^\times/p^n$ and let $\mathrm{gr}(p^n)$ be the graded group associated with the filtration $(U_n^m)_{m\geq 0}$:

$$\mathrm{gr}(p^n) := \bigoplus_{m\geq 0} \mathrm{gr}^m(p^n), \quad \mathrm{gr}^m(p^n) := U_n^m/U_n^{m+1}. \tag{2.4}$$

The isogeny $[p^n]$ factors as $[p^n] = [p] \circ \cdots \circ [p]$ (n times). In particular, $c_i := c_i([p^n]) = ie + e_0$, where $e := v_K(p)$ and $e_0 := e/(p-1)$. Let $\phi : F \to G$

be an isogeny with height n as in Proposition 2.8. Fix an isomorphism $F[\phi] \simeq \mu_{p^n} = \widehat{\mathbb{G}}_m[p^n]$. The isogeny induces the Kummer homomorphism $\delta : G(\phi) \to H^1(K, F[\phi]) = K^\times/p^n$. We compare the filtration $G^m(\phi)$ and the filtration $\widehat{\mathbb{G}}^j_m([p^n]) = U^j_n$ on $K^\times/p^n$. In the case of height 1 we have the following theorem:

THEOREM 2.10 ([8, Thm. 2.1.6]). *Let $\phi : F \to G$ be an isogeny defined over $\mathcal{O}_K$ of height 1, and $t := v_K(D(\phi))$. Assume that $F[\phi] \subset F(K)$ and $\zeta_p \in K$. Then the Kummer map δ induces a bijection $\delta : G^m(\phi) \xrightarrow{\simeq} U_1^{pe_0 - pt/(p-1)+m}$ for any $m \geq 1$.*

We extend the above theorem to the case of height > 1 assuming $\zeta_{p^n} \in K$. Let $\phi = \phi_1 \circ \cdots \circ \phi_n : F \to G$ and t_i be as in Proposition 2.8. Put $c_i(\phi) := t_0 + t_1 + \cdots + t_i + t_i/(p-1)$ for $0 \leq i \leq n$. First we show $\delta(G^m(\phi)) \subset U_n^{pe_0 - pt_1/(p-1)+m}$ for $m < c_1(\phi) = t_1 + t_1/(p-1)$ with $m \nmid p$. Let j be the greatest integer such that $\delta(G^m(\phi)) \subset U^j_n$. Since $p \nmid m$, δ induces a non-zero homomorphism $\mathrm{gr}^m(\phi) \to \mathrm{gr}^j(p^n)$. From Lemma 2.6(i), we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{gr}^m(\phi) & \xrightarrow{\simeq} & \mathrm{gr}^m(\phi_1) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ \mathrm{gr}^j(p^n) & \longrightarrow & \mathrm{gr}^j(p) \end{array}$$

where the top horizontal homomorphism is an isomorphism. Since the left δ is non-zero, Theorem 2.10 implies $j = pe_0 - pt_1/(p-1) + m$. In particular we obtain $\delta(G(\phi)) \subset U_n^{e+e_0-pt_1/(p-1)+1}$.

Next, we show that δ induces a bijection $\mathrm{gr}^m(\phi) \xrightarrow{\simeq} \mathrm{gr}^{c_i - c_i(\phi)+m}(p^n)$ on the graded groups by induction on i, where $c_i = ie + e_0$. From Proposition 2.8 and the above observation (although we do not discuss m with $p \,|\, m$), the map δ induces $\mathrm{gr}^m(\phi) \simeq \mathrm{gr}^{pe_0 - pt_1/(p-1)+m}(p^n)$ for any $m < c_1(\phi)$. For $m = c_1(\phi)$, let j be the greatest integer such that $\delta(G^{t_1+t_1/(p-1)}(\phi)) \subset U^j_n$ as above. From Lemma 2.6(b), we have

$$\begin{array}{ccc} \mathrm{gr}^{t_1+t_1/(p-1)}(\phi) & \xrightarrow{\simeq} & \mathrm{gr}^{t_1+t_1/(p-1)}(\phi_1) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ \mathrm{gr}^j(p^n) & \longrightarrow & \mathrm{gr}^j(p) \end{array}$$

Thus $j = e + e_0 = e + e_0 - pt_1/(p-1) + m$. We obtain

$$\mathrm{gr}^m(\phi) \xrightarrow{\simeq} \mathrm{gr}^{e+e_0-pt_1/(p-1)+m}(p^n) \tag{2.5}$$

for $m \leq c_1(\phi)$. For $c_i(\phi) < m \leq c_{i+1}(\phi)$ with $i > 1$, let j be the greatest integer such that $\delta(G^m(\phi)) \subset U^j_n$ again. From the induction hypothesis,

$j > c_i = ie + e_0$. By Proposition 2.8 we have the commutative diagram

$$\begin{array}{ccc} \mathrm{gr}^{m-(t_1+t_2+\cdots+t_i)}(\phi_{i+1}\circ\cdots\circ\phi_n) & \xrightarrow{\simeq} & \mathrm{gr}^m(\phi) \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ \mathrm{gr}^{j-ie}(p^{n-i}) & \xrightarrow{\simeq} & \mathrm{gr}^j(p^n) \end{array}$$

Hence $m-(t_1+t_2+\cdots+t_i) \le pt_{i+1}/(p-1)$. By the argument above (2.5), we obtain $j = c_{i+1} - c_{i+1}(\phi) + m$. From Proposition 2.8, we obtain the following theorem:

THEOREM 2.11. *The image of the Kummer map $\delta : G(\phi) \to K^\times/p^n$ is contained in $U_n^{c_1-c_1(\phi)+1}$ and δ induces a bijection $\mathrm{gr}^m(\phi) \xrightarrow{\simeq} \mathrm{gr}^{c_i-c_i(\phi)+m}(p^n)$ for $c_{i-1}(\phi) < m \le c_i(\phi)$.*

3. Cycle map. Let K be a finite extension field over $\mathbb{Q}_p$. Let $X = E \times E'$ be the product of two elliptic curves E and E' over K with $E[p^n]$ and $E'[p^n]$ K-rational. The goal of this section is to calculate the image of the Albanese kernel $T(X) := \mathrm{Ker}(A_0(X) \to X(K))$ under the cycle map $\rho : A_0(X) \to H^4(X, \mathbb{Z}/p^n(2))$.

From the argument below which is same as in [16] or in the proof of Theorem 4.3 in [17], the study of the image of $T(X)/p^n$ boils down to the calculation of the image of the Kummer map $\delta : E(K) \to H^1(K, E[p^n])$ and the Hilbert symbol: The image of $T(X)$ is contained in $H^2(K, E[p^n] \otimes E'[p^n])$, a direct summand of the étale cohomology group $H^4(X, \mathbb{Z}/p^n(2))$. The Albanese kernel $T(X)$ is isomorphic to the *Somekawa K-group* $K(K; E, E')$ defined by some quotient of $\bigoplus_{K'/K} E(K') \otimes E'(K')$, where K' runs through all finite extensions of K (for definition, see [15], [12]). Thus there is a natural surjection $\bigoplus_{K'/K} E(K') \otimes E'(K') \to T(X)/p^n$. The cycle map also induces the following commutative diagram (cf. proof of Prop. 2.4 in [17]):

$$\begin{array}{ccc} \bigoplus_{K'/K} E(K') \otimes E'(K') & \xrightarrow{\delta\otimes\delta'} & \bigoplus_{K'/K} H^1(K', E[p^n]) \otimes H^1(K', E'[p^n]) \\ & & \downarrow{\scriptstyle\cup} \\ & & \bigoplus_{K'/K} H^2(K', E[p^n] \otimes E'[p^n]) \\ & & \downarrow{\scriptstyle N} \\ T(X)/p^n & \xrightarrow{\rho} & H^2(K, E[p^n] \otimes E'[p^n]), \end{array} \tag{3.1}$$

where δ (resp. δ') is the Kummer map $\delta : E(K') \to H^1(K', E[p^n])$ (resp. $\delta : E'(K') \to H^1(K', E'[p^n])$), $\cup$ is the cup product and N is the norm map. From the calculation below, the image of the cup product does not depend on the extension K'/K (except when both E and E' have super-

singular reduction, cf. the last example in this section). So we consider the case $K' = K$ only.

If we fix isomorphisms $E[p^n] \simeq \mu_{p^n} \oplus \mu_{p^n}$ and $E'[p^n] \simeq \mu_{p^n} \oplus \mu_{p^n}$, the cup product is characterized by the Hilbert symbol $(\ ,\ )_n : K^\times \times K^\times \to \mu_{p^n}$ as follows (cf. [13, Chap. XIV, Prop. 5]):

$$\begin{array}{ccc} H^1(K, E[p^n]) \otimes H^1(K, E'[p^n]) & \xrightarrow{\cup} & H^2(K, E[p^n] \otimes E'[p^n]) \\ \big\downarrow \simeq & & \big\downarrow \simeq \\ (K^\times/p^n)^{\oplus 2} \otimes (K^\times/p^n)^{\oplus 2} & \xrightarrow{(\ ,\ )_n^{\oplus 4}} & (\mu_{p^n})^{\oplus 4} \end{array}$$

The Hilbert symbol is defined by $(a,b)_n := \rho_K(a)(\sqrt[p^n]{b})/(\sqrt[p^n]{b})$ for $a, b \in K^\times$, where $\rho_K : K^\times \to G_K^{\mathrm{ab}}$ is the reciprocity map. Put $c_i := ie + e_0$ for $1 \le i \le n$ as before, and $c_0 := 0$, $c_{n+1} := \infty$. The filtration U_n^j on $K^\times/p^n$ is defined by the image of U_K^j in $K^\times/p^n$ for $j \ge 1$ and $U_n^0 := K^\times/p^n$. The orders of their images in μ_{p^n} under the Hilbert symbol are calculated as follows:

LEMMA 3.1 ([16, Prop. 2.8]). *For any positive integers s, t, we have*

(i) $\#(U_n^s, K^\times/p^n)_n = p^{n-i}$ *for* $c_i < s \le c_{i+1}$.
(ii) *If* $p \nmid s$, *then* $\#(U_n^s, U_n^t)_n = p^{n-i}$ *for* $c_i < s + t \le c_{i+1}$.
(iii) *If* $p \mid s$ *and* $p \mid t$, *then* $\#(U_n^s, U_n^t)_n = p^{n-i}$ *for* $c_i \le s + t < c_{i+1}$.

A proof of Lemma 3.1 is found in [16]. It is conducted by direct computation of the Herbrand function of the Kummer extension $K(\sqrt[p^n]{b})$ over K for some $b \in K^\times$. We present another proof using the results of the preceding section.

Proof of Lemma 3.1. (i) From Proposition 2.8, $s > c_n$ if and only if $U_n^s = 1$. Since the symbol $(\ ,\ )_n$ is non-degenerate, the condition $s > c_n$ is equivalent to $(U_n^s, K^\times/p^n)_n = 1$ for any n. It is known that $(a,b)_n^p = (a,b)_{n-1}$ for $a, b \in K^\times$ (cf. [5, Chap. IV, (5.1)]). Because $s > c_1$, multiplication by p induces $U_{n-1}^{s-e} \simeq U_n^s$ (Lem. 2.6(c)). By induction on n and i, $(U_n^s, K^\times/p^n)_n \subset \mu_{p^{n-i}}$ if and only if $s > c_i$ for any n and $1 \le i \le n$.

(ii) As in the proof of (i), it is enough to show that, for any n, $(U_n^s, U_n^t)_n = 1$ if and only if $s + t > c_n$. For $a, b \in \mathcal{O}_K$, we have

$$(3.2)\quad (1 + a\pi^s, 1 + b\pi^t)_n = \left(1 + \frac{ab\pi^{s+t}}{1 + a\pi^s}, -a\pi^s\right)_n^{-1} (1 + ab\pi^{s+t}, 1 + b\pi^t)_n^{-1}.$$

Thus $(U_n^s, U_n^t)_n \subset (U_n^{s+t}, K^\times/p^n)_n$ (cf. [2, Lem. 4.1]). If we assume $s+t > c_n$, then $(U_n^s, U_n^t)_n \subset (U_n^{s+t}, K^\times/p^n)_n = 1$ by (i).

Conversely, we show $(U_n^s, U_n^t)_n \ne 1$ for $s + t \le c_n$. We may assume $s \ge t$ and $s + t = c_n$ and hence $p \nmid t$. For $n = 1, 2$ and $b \in U_n^t \smallsetminus U_n^{t+1}$, Proposition 2.9 says $\mathrm{Gal}(K(\sqrt[p^n]{b})/K)^s \ne 1$. Since the reciprocity homomorphism

$\rho_K : K^\times \to \mathrm{Gal}(K(\sqrt[p^n]{b})/K)$ maps the higher unit group U_K^s onto the ramification subgroup $\mathrm{Gal}(K(\sqrt[p^n]{b})/K)^s$ ([13, Chap. XV, Cor. 3]), we obtain $(U_n^s, U_n^t)_n \neq 1$.

For $n > 2$, we have $s > c_1$. Therefore, $(U_{n-1}^{s-e})^p = U_n^s$ (Lem. 2.6(c)). By induction on $n > 2$, there exist $a \in U_{n-1}^{s-e}$ and $b \in U_{n-1}^t$ such that $(a,b)_{n-1} \neq 1$. The assertion follows from $(a^p, b)_n = (a,b)_n^p = (a,b)_{n-1} \neq 1$.

(iii) As in the proof of (ii), it is enough to show that $(U_n^s, U_n^t)_n = 1$ if and only if $s + t \geq c_n$. If $s + t < c_n$, then $(U_n^s, U_n^t)_n \subset (U_n^{s+1}, U_n^t)_n \neq 1$ from (ii). Suppose $s + t \geq c_n$. By (ii) we may assume $s + t = c_n$. From (3.2), for $1 + a\pi^s \in U_K^s, 1 + b\pi^t \in U_K^t$, we have

$$(1 + a\pi^s, 1 + b\pi^t)_n^{-1} = (1 + ab\pi^{s+t}/(1 + a\pi^s), -a\pi^s)_n (1 + ab\pi^{s+t}, 1 + b\pi^t)_n.$$

From (ii) and $p \mid s$, we obtain $(1 + a\pi^s, 1 + b\pi^t)_n = 1$. ∎

From the above lemma, the Hilbert symbol induces a homomorphism of graded groups: Let $M^0 := \mu_{p^n}$ and $M^m := \mu_{p^{n-i}}$ if $c_i < m \leq c_{i+1}$. The filtration $(M^m)_{m \geq 0}$ makes μ_{p^n} a filtered group. The associated graded group $\mathrm{gr}(\mu_{p^n})$ is defined by $\mathrm{gr}(\mu_{p^n}) := \bigoplus_{m \geq 0} \mathrm{gr}^m(\mu_{p^n})$, where $\mathrm{gr}^m(\mu_{p^n}) := M^m/M^{m+1}$. On the other hand, let $\mathrm{gr}(p^n)$ be the graded group associated with the filtration $(U_n^m)_{m \geq 0}$ defined in (2.4). If $s, t > 0$, $p \mid s$ and $p \mid t$, the Hilbert symbol gives $(U_n^s, U_n^t)_n = M^{s+t+1}$. Otherwise $(U_n^s, U_n^t)_n = M^{s+t}$ (Lem. 3.1). We modify the structure of the graded tensor product $\mathrm{gr}(p^n) \otimes \mathrm{gr}(p^n)$ of the graded groups as follows: $\mathrm{gr}(p^n \otimes p^n) := \bigoplus_{m \geq 0} \mathrm{gr}^m(p^n \otimes p^n)$, where

$$\mathrm{gr}^m(p^n \otimes p^n) := \bigoplus_{\substack{m = s+t, \\ p \nmid s \text{ or } p \nmid t}} \mathrm{gr}^s(p^n) \otimes \mathrm{gr}^t(p^n) \oplus \bigoplus_{\substack{m = s+t+1, \\ p \mid s \text{ and } p \mid t}} \mathrm{gr}^s(p^n) \otimes \mathrm{gr}^t(p^n).$$

The symbol $(\ ,\ )_n : K^\times/p^n \otimes K^\times/p^n \to \mu_{p^n}$ induces $\mathrm{gr}(\ ,\ )_n : \mathrm{gr}(p^n \otimes p^n) \to \mathrm{gr}(\mu_{p^n})$. For any subgroups U and U' of $K^\times/p^n$, the induced graded subgroups $\mathrm{gr}(U) \subset \mathrm{gr}(p^n)$ and $\mathrm{gr}(U') \subset \mathrm{gr}(p^n)$ give the graded subgroup of $\mathrm{gr}(p^n \otimes p^n)$ denoted by $\mathrm{gr}(U \otimes U')$. The order of the image $(U, U')_n$ coincides with that of the image of $\mathrm{gr}(U \otimes U')$ under $\mathrm{gr}(\ ,\ )_n$. Since the graded quotient $\mathrm{gr}^m(\mu_{p^n})$ is isomorphic to $\mathbb{Z}/p$ if $m = c_i$ for i and otherwise $\mathrm{gr}^m(\mu_{p^n}) = 0$, this order is

$$\#(U, U')_n = p^\alpha, \text{ where } \alpha := \#\{i \mid \mathrm{gr}^{c_i}(U \otimes U') \neq 0 \text{ for } 0 < i \leq n\}. \tag{3.3}$$

Next, we study the image of the map $\delta : E(K) \to H^1(K, E[p^n]) = K^\times/p^n \oplus K^\times/p^n$. When E has *split multiplicative reduction*, the uniformization theorem gives $K^\times/q^{\mathbb{Z}} \simeq E(K)$ for some $q \in K$.

THEOREM 3.2 ([17, Lem. 4.5]). *Let E and F be elliptic curves over K which have split multiplicative reduction. Let $\phi : E \to F$ be an isogeny over K of degree p^n with cyclic kernel $E[\phi]$. Assume that $E[\phi]$ and the kernel*

$F[\widehat{\phi}]$ of the dual isogeny $\widehat{\phi} : F \to E$ are K-rational. Then the image of the Kummer map $\delta_\phi : E(K) \to H^1(K, E[\phi]) = K^\times/p^n$ is

$$\mathrm{Im}(\delta_\phi) = \begin{cases} K^\times/p^n & \text{if } \sqrt[p^n]{q} \notin E[\phi], \\ 1 & \text{if } \sqrt[p^n]{q} \in E[\phi]. \end{cases}$$

We choose an isomorphism $E[p^n] \simeq \mu_{p^n} \oplus \mu_{p^n}$ which maps $E[p^n] \supset \mathbb{G}_m[p^n] = \mu_{p^n}$ onto the second factor of $\mu_{p^n} \oplus \mu_{p^n}$. From the above theorem, we have

$$\mathrm{Im}(\delta) = K^\times/p^n \oplus 1 \tag{3.4}$$

when E has split multiplicative reduction. Here 1 means the trivial subgroup in $K^\times/p^n$.

We assume that E has *ordinary good reduction*. Let $\mathcal{E}$ be the Néron model of E over $\mathcal{O}_K$, $\widetilde{E}$ the neutral component of the special fiber of $\mathcal{E}$, and $\pi : E(K) = \mathcal{E}(\mathcal{O}_K) \to \widetilde{E}(k)$ the specialization map. The group $E(K) = \mathcal{E}(\mathcal{O}_K)$ has a filtration $E^i(K)$ $(i \geq 0)$ defined by $E^0(K) := \mathcal{E}(\mathcal{O}_K)$, $E^1(K) := \mathrm{Ker}(\pi)$ and for $i \geq 1$, $E^i(K) := \{(x, y) \in E(K) \mid v_K(x) \leq -2i\} \cup \{\mathcal{O}\}$, where $\mathcal{O}$ is the origin on E. This filtration coincides with $\widehat{E}^i(K)$ of the formal group $\widehat{E}(K)$ defined in the previous section. So we may identify these filtrations as $E^i(K) = \widehat{E}^i(K)$. Choose an isomorphism $E[p^n] \simeq \mu_{p^n} \oplus \mu_{p^n}$ and $x_0 \in E[p^n]$ corresponding to $(\zeta_{p^n}, 1)$ in $\mu_{p^n} \oplus \mu_{p^n}$. Let Φ be the subgroup of $E[p^n]$ generated by x_0. If $x_0 \in \widehat{E}^1[p^n]$, the isogeny $\phi : E \to F := E/\Phi$ has the cyclic kernel $E[\phi] = \widehat{E}^1[p^n]$. Since $E^1(K)$ is isomorphic to the formal group $\widehat{E}(K)$ and the height of $\widehat{E}$ (= the height of $[p]$) is 1, the isogeny $\phi : E \to F$ induces $[p^n] : \widehat{E} \to \widehat{E} \simeq \widehat{F}$. The first factor of the image of $\delta : E(K) \to H^1(K, E[p^n]) = K^\times/p^n \oplus K^\times/p^n$ coincides with the image of the Kummer map $\delta^1 : \widehat{E}(K) \to H^1(K, \widehat{E}[p^n]) = K^\times/p^n$. By Theorem 2.11, the image is U_n^1. On the other hand, if $x_0 \notin E^1[p^n]$, the isogeny $\phi : E \to F := E/\Phi$ has the kernel $E[\phi] \simeq \widetilde{E}[p^n]$. Hence, the image of $\delta_\phi : E(K) \to H^1(K, E[\phi])$ is contained in

$$H^1_{\mathrm{ur}}(K, E[\phi]) := \mathrm{Ker}(\mathrm{Res} : H^1(K, E[\phi]) \to H^1(K^{\mathrm{ur}}, E[\phi])),$$

where K^{ur} is the completion of the maximal unramified extension of K and Res is the restriction map. The image of δ is contained in $U_n^1 \oplus H^1_{\mathrm{ur}}(K, \mu_{p^n})$. Mattuck's theorem [9] says $\#E(K)/p^n = ([K : \mathbb{Q}_p] + 2)p^n$. The order of $H^1_{\mathrm{ur}}(K, \mu_{p^n}) \simeq H^1(k, \mathbb{Z}/p^n)$ is p^n and $\#U_n^1 = ([K : \mathbb{Q}_p] + 1)p^n$. Thus

$$\mathrm{Im}(\delta) = U_n^1 \oplus H^1_{\mathrm{ur}}(K, \mu_{p^n}). \tag{3.5}$$

For the second factor, the restriction map $\mathrm{Res} : H^1(K, \mu_{p^n}) \to H^1(K^{\mathrm{ur}}, \mu_{p^n})$ induces $\mathrm{Res}^j : U_n^j/U_n^{j+1} \to U_n^{\mathrm{ur},j}/U_n^{\mathrm{ur},j}$, where $U_n^{\mathrm{ur},j}$ is the image of $U^j_{K^{\mathrm{ur}}}$ in $(K^{\mathrm{ur}})^\times/p^n$. Proposition 2.8 implies that Res^j is bijective if $j \neq ie + e_0$ for some $i \leq n$ and $\mathrm{Ker}(\mathrm{Res}^{c_i}) = U_n^{c_i}/U_n^{c_i+1} = \mathrm{gr}^{c_i}(p^n)$.

Finally, we assume that E has a *supersingular good reduction*. Let Φ be a subgroup of $E[p^n]$ generated by x_0 as above and we denote by $\phi : E \to F := E/\Phi$ the induced isogeny. The first factor of the image of $\delta : E(K) \to H^1(K, E[p^n]) = K^\times/p^n \oplus K^\times/p^n$ is the image of the Kummer map $\delta_\phi : F(K) \to H^1(K, E[\phi]) = K^\times/p^n$ and another one is the image of the Kummer map $\delta_{\widehat{\phi}}$ associated with the dual isogeny $\widehat{\phi}$. Since the elliptic curve E has supersingular reduction, $F(K)/\phi E(K)$ is isomorphic to $F^1(K)/(\phi E(K) \cap F^1(K)) \simeq \widehat{F}(\phi)$ ([8, Lem. 3.2.3]).

As in the previous section, ϕ factors as $\phi = \phi_1 \circ \cdots \circ \phi_n$ into height 1 isogenies ϕ_i. The invariants $t_i := D(\phi_i)$ satisfy $t_0 := 0 < t_1 < \cdots < t_n < e$ (Lem. 2.7, see also Thm. 3.5). Theorem 2.11 says that the image of $\delta : E(K) \to K^\times/p^n$ is contained in $U_n^{e+e_0-(t_1+t_1/(p-1))+1}$. More precisely, one can describe the image in terms of the graded groups as follows: From Theorem 2.11, the graded quotient $\mathrm{gr}^m E := E^m(K)/E^{m+1}(K)$ maps onto $\mathrm{gr}^{c_i-c_i(\phi)+m}(p^n)$ for $c_{i-1}(\phi) < m \le c_i(\phi)$, where $c_i(\phi) := t_0 + t_1 + \cdots + t_i + t_i/(p-1)$ and $c_i := ie + e_0$. Hence δ induces a surjection

$$\mathrm{gr}(\delta) : \mathrm{gr}\, E := \bigoplus_{m\ge 0} \mathrm{gr}^m E \to \bigoplus_{i=1}^{n} \bigoplus_{c_{i-1}(\phi)<m\le c_i(\phi)} \mathrm{gr}^{c_i-c_i(\phi)+m}(p^n).$$

Similarly, the dual isogeny $\widehat{\phi}$ is described by the dual isogenies $\widehat{\phi}_i$ of ϕ_i as $\widehat{\phi} = \widehat{\phi}_n \circ \cdots \circ \widehat{\phi}_1$. The invariants $\widehat{t}_i := D(\widehat{\phi}_{n-i+1}) = e - t_{n-i+1}$ satisfy $\widehat{t}_0 := 0 < \widehat{t}_1 < \cdots < \widehat{t}_n < e$. Thus $c_i(\widehat{\phi}) = \widehat{t}_0 + \widehat{t}_1 + \cdots + \widehat{t}_i + \widehat{t}_i/(p-1)$. Summarizing the above observations in terms of the graded groups, we have:

THEOREM 3.3. *The Kummer map $\delta : E(K) \to H^1(K, E[p^n]) = K^\times/p^n \oplus K^\times/p^n$ induces $\mathrm{gr}(\delta) : \mathrm{gr}\, E \to \mathrm{gr}(p^n) \oplus \mathrm{gr}(p^n)$ on graded groups, where $\mathrm{gr}(p^n) := \bigoplus_{j\ge 0} \mathrm{gr}^j(p^n)$.*

(i) *If E has split multiplicative reduction, then $\mathrm{Im}(\mathrm{gr}(\delta)) = \mathrm{gr}(p^n) \oplus 1$.*

(ii) *If E has ordinary reduction, then*

$$\mathrm{Im}(\mathrm{gr}(\delta)) = \bigoplus_{j\ge 1} \mathrm{gr}^j(p^n) \oplus \bigoplus_{i=1}^{n} \mathrm{gr}^{c_i}(p^n).$$

(iii) *If E has supersingular reduction, then*

$$\mathrm{Im}(\mathrm{gr}(\delta)) = \bigoplus_{i=1}^{n} \bigoplus_{c_{i-1}(\phi)<m\le c_i(\phi)} \mathrm{gr}^{c_i-c_i(\phi)+m}(p^n) \oplus \bigoplus_{i=1}^{n} \bigoplus_{c_{i-1}(\widehat{\phi})<m\le c_i(\widehat{\phi})} \mathrm{gr}^{c_i-c_i(\widehat{\phi})+m}(p^n).$$

Now we complete the proof of the main theorem.

THEOREM 3.4. *Let E and E' be elliptic curves over K with good or split multiplicative reduction, and assume $E[p^n]$ and $E'[p^n]$ are K-rational. The structure of the image of $T(X)/p^n$ for $X = E \times E'$ under the cycle map ρ is*

(i) $\mathbb{Z}/p^n$ *if both E and E' have ordinary or split multiplicative reduction,*
(ii) $\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n$ *if E and E' have different reduction types.*

Proof. We denote the subsets of $\mathbb{N} := \mathbb{Z}_{\geq 0}$ which indicate the indices of the graded quotients of $\mathrm{Im}(\mathrm{gr}(\delta))$ by $M := \{m \geq 0\}$, $O := \{m \geq 1\}$, $O_{\mathrm{ur}} := \{m = c_i \mid 0 < i \leq n\}$, and

$$S := \bigcup_{i=1}^{n} \left\{ c_i - \frac{pt_i - t_{i-1}}{p-1} < m \leq c_i \right\},$$

$$\widehat{S} := \bigcup_{i=1}^{n} \left\{ c_{i-1} + \frac{pt_{n-i+1} - t_{n-i+2}}{p-1} < m \leq c_i \right\},$$

where $t_{n+1} := e$ by convention. Define

$$d_j : E(K) \otimes E'(K) \xrightarrow{\delta \otimes \delta'} (K^\times/p^n \otimes K^\times/p^n)^{\oplus 4} \xrightarrow{\mathrm{pr}_j} K^\times/p^n \otimes K^\times/p^n,$$

where pr_j is the jth projection. We calculate the order of the image of the composition $(\ ,\ )_n \circ d_j : E(K) \otimes E'(K) \to \mu_{p^n}$ for each j in the following five cases:

(a) Both E and E' have split multiplicative reduction.
(b) Both E and E' have ordinary reduction.
(c) E has ordinary reduction and E' has split multiplicative reduction.
(d) E has supersingular reduction and E' has split multiplicative reduction.
(e) E has supersingular reduction and E' has ordinary reduction.

(a) From (3.4), the images of d_j are $K^\times/p^n \otimes K^\times/p^n$, $K^\times/p^n \otimes 1$, $1 \otimes K^\times/p^n$ and $1 \otimes 1$. By Lemma 3.1, the image of the cycle map is isomorphic to $\mathbb{Z}/p^n$.

(b) From (3.5), renaming the indices j if necessary, we have $\mathrm{Im}(d_1) = U_n^1 \otimes U_n^1$, $\mathrm{Im}(d_2) = U_n^1 \otimes H^1_{\mathrm{ur}}(K, \mu_{p^n})$, $\mathrm{Im}(d_3) = H^1_{\mathrm{ur}}(K, \mu_{p^n}) \otimes U_n^1$ and $\mathrm{Im}(d_4) = H^1_{\mathrm{ur}}(K, \mu_{p^n}) \otimes H^1_{\mathrm{ur}}(K, \mu_{p^n})$. The image of $\mathrm{Im}(d_1)$ under the Hilbert symbol is μ_{p^n} (Lem. 3.1). We calculate the order of the image of d_2 in the graded groups. A subset R_i of $\mathbb{N} \times \mathbb{N}$ is defined by

$$(3.6)\quad R_i := \{(s, ie+e_0-s) \mid 0 < s < ie+e_0,\ p \nmid s\} \cup \{(0, ie+e_0), (ie+e_0, 0)\}.$$

By (3.3), the order of $\mathrm{Im}(d_2)$ is p^α, where $\alpha = \#\{i \mid (O \times O_{\mathrm{ur}}) \cap R_i \neq \emptyset\}$. However, $(O \times O_{\mathrm{ur}}) \cap R_i = \emptyset$ for all i. Thus $\#\,\mathrm{Im}(d_2) = \#\,\mathrm{Im}(d_3) = 0$. Because $O_{\mathrm{ur}} \subset O$, we also obtain $\#\,\mathrm{Im}(d_4) = 0$.

(c) It is enough to consider the image of $U_n^1 \otimes K^\times/p^n$ and $H^1_{\mathrm{ur}}(K, \mu_{p^n}) \otimes K^\times/p^n$ under the Hilbert symbol. For the latter, the order is p^α, where

$\alpha = \#\{i \mid (O_{\mathrm{ur}} \times M) \cap R_i \neq \emptyset\}$. Since $O_{\mathrm{ur}} \times O \subset O_{\mathrm{ur}} \times M$, $\alpha = n$ from (b). As $O_{\mathrm{ur}} \times M \subset O \times M$, the order of the image of $U_n^1 \otimes K^\times/p^n$ is also p^n.

(d) Since $O_{\mathrm{ur}} \times M \subset S \times M, \widehat{S} \times M$, the image is isomorphic to $\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n$ by (c).

(e) For each i, $(ie + e_0 - 1, 1) \in (S \times O) \cap R_i$ and $(ie + e_0 - 1, 1) \in (\widehat{S} \times O) \cap R_i$. On the other hand $S \times O_{\mathrm{ur}}, \widehat{S} \times O_{\mathrm{ur}} \subset O \times O_{\mathrm{ur}}$. Thus the image is isomorphic to $\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n$. ∎

When both E and E' have supersingular reduction, the computation of the image $\rho(T(X)/p^n)$ uses a similar argument. The results depend on the invariants $t_1 < \cdots < t_n$ associated with the formal group $\widehat{E}$ and $t'_1 < \cdots < t'_n$ associated with $\widehat{E}'$, defined in the previous section. These invariants are calculated from the theory of the *canonical subgroup* due to Katz–Lubin. The canonical subgroup $H(\widehat{E})$ of an elliptic curve E (when it exists) is a distinguished subgroup of order p in $\widehat{E}[p]$ which plays a crucial role in the theory of overconvergent modular forms.

THEOREM 3.5 ([7, Thm. 3.10.7]; [3, Thm. 3.3]). *Let E be an elliptic curve over K with supersingular reduction. Let $v_K(\widehat{E})$ be the valuation of the pth coefficient of multiplication p formula $[p](T)$ of the formal group $\widehat{E}$ by v_K.*

(i) *If $v_K(\widehat{E}) < pe/(p+1)$, then the canonical subgroup $H(\widehat{E}) \subset \widehat{E}[p]$ exists. For any non-zero $x \in \widehat{E}[p]$,*

$$v_K(x) = \begin{cases} \dfrac{e - v_K(\widehat{E})}{p-1} & \text{if } x \in H(\widehat{E}), \\ \dfrac{v_K(\widehat{E})}{p^2 - p} & \text{otherwise.} \end{cases}$$

For a subgroup $H \neq H(\widehat{E})$ of $\widehat{E}[p]$, $v_K(\widehat{E}/H) = v_K(\widehat{E})/p$ and the canonical subgroup $H(\widehat{E}/H)$ of the quotient $\widehat{E}/H$ is the canonical image of $\widehat{E}[p]$ in $\widehat{E}/H$. Moreover:

(a) *If $v_K(\widehat{E}) < e/(p+1)$, then $v_K(\widehat{E}/H(\widehat{E})) = pv_K(\widehat{E})$. The canonical image of $\widehat{E}[p]$ in $\widehat{E}/H(\widehat{E})$ is not the canonical subgroup of $\widehat{E}/H(\widehat{E})$.*

(b) *If $v_K(\widehat{E}) = e/(p+1)$, then $v_K(\widehat{E}/H(\widehat{E})) \geq pe/(p+1)$.*

(c) *If $e/(p+1) < v_K(\widehat{E}) < pe/(p+1)$, then $v_K(\widehat{E}/H(\widehat{E})) = e - v_K(\widehat{E})$ and the canonical subgroup of $\widehat{E}/H(\widehat{E})$ is $H(\widehat{E}/H(\widehat{E})) = \widehat{E}[p]/H$.*

(ii) *If $v_K(\widehat{E}) \geq pe/(p+1)$, then $v_K(x) = e/(p^2-1)$ for any non-zero $x \in \widehat{E}[p]$. For any subgroup H of $\widehat{E}[p]$, $v_K(\widehat{E}/H) = e/(p+1)$ and the canonical subgroup of $\widehat{E}/H$ is the image of $\widehat{E}[p]$ in $\widehat{E}/H$.*

Here we consider the case $n = 1$. Choose an isomorphism $\widehat{E}[p] \simeq \mu_p^{\oplus 2}$ such that $v_K(x_0) = \max\{v_K(x) \mid 0 \neq x \in \widehat{E}[p]\}$. The induced isogeny $\phi : \widehat{E} \to \widehat{E}/\Phi$ has height 1, where Φ is the subgroup of $\widehat{E}[p]$ generated by x_0 as above. If $v_K(\widehat{E}) < pe/(p+1)$, $\Phi = \widehat{E}[\phi]$ is the canonical subgroup. From Corollary 2.3 and Theorem 3.5, we have $t_1/(p-1) = v_K(x_0) = e_0 - v_K(\widehat{E})/(p-1)$. If $v_K(\widehat{E}) \geq pe/(p+1)$, $t_1/(p-1) = v_K(x_0) = e_0/(p+1)$. In the case $n = 1$ also, the image of the cup product does not depend on the extension K'/K in the diagram (3.1). Thus we obtain

PROPOSITION 3.6 (see also [16, Thm. 1.4]). *The image of $T(X)/p$ for $X = E \times E'$ under the cycle map ρ is isomorphic to*

(i) $\mathbb{Z}/p \oplus \mathbb{Z}/p$ *if* $v_K(\widehat{E}) \neq v_K(\widehat{E}')$ *and* $v_K(\widehat{E}) + v_K(\widehat{E}') \neq e$,
(ii) $\mathbb{Z}/p$ *if* $v_K(\widehat{E}) = v_K(\widehat{E}') \neq e/2$, *or* $v_K(\widehat{E}) \neq v_K(\widehat{E}')$ *and* $v_K(\widehat{E}) + v_K(\widehat{E}') = e$,
(iii) 0 *if* $v_K(\widehat{E}) = v_K(\widehat{E}') = e/2$.

For the case $n > 1$, factor the induced isogeny $\phi : E \to E/\Phi =: F$ as $\phi = \phi_1 \circ \cdots \circ \phi_n$ into height 1 isogenies ϕ_i. One can calculate the invariants t_i using Theorem 3.5 inductively. In this case, the image of the cup product may depend on the base field K, in particular, on the ramification index e of K. However, by taking a finite extension K'/K whose extension degree is prime to p, we may assume that the ramification index e is sufficiently large.

We conclude this note with an example: Put $p = 5$ and suppose that E is an elliptic curve defined by $y^2 = x^3 + ax + b$ over K with $v_K(a) \geq 5e/6$ and $v_K(b) = 0$ (cf. [14, Sect. 1.11]). Let $X = E \times E$. Let Φ be the subgroup of $E[p^2]$ as before. The induced isogeny $\phi : E \to F = E/\Phi$ factors as $\phi = \phi_1 \circ \phi_2$, where $\phi_i : F_i \to F_{i-1}$, $F_1 = E/pF[\phi]$, $F_0 = F$, and $F_2 = E$. By Theorem 3.5, we have $t_2/(p-1) = v_K(px_0) = e_0/(p+1)$, $v_K(\widehat{F}_1) = e/(p+1)$ and $t_1/(p-1) = v_K(\phi_2(x_0)) = e_0/p(p+1)$. Thus we have $S = (29e_0/6, 5e_0] \cup (41e_0/5, 9e_0]$, $\widehat{S} = (5e_0/6, 5e_0] \cup (5e_0, 9e_0]$, where $(s,t]$ is the subset of $\mathbb{N}$ that consists of $n \in \mathbb{N}$ with $s < n \leq t$ and $p \nmid n$. It is easy to see that $R_1 \cap (S \times S) = R_1 \cap (S \times \widehat{S}) = \emptyset$ and $R_1 \cap (\widehat{S} \times \widehat{S}) \neq \emptyset$, while $R_2 \cap (S \times S) = \emptyset$ and $R_2 \cap (\widehat{S} \times \widehat{S}) \neq \emptyset$. Here, the set R_i is defined in (3.6). However, $R_2 \cap (S \times \widehat{S})$ is non-empty if and only if $e_0 > 6$. We obtain $\rho(T(X)/p^2) \simeq \mathbb{Z}/p^2 \oplus \mathbb{Z}/p \oplus \mathbb{Z}/p$.

Acknowledgments. The first author thanks Takahiro Tsushima for his helpful suggestion on the theory of canonical subgroups. A part of this note was written during a stay of the first author at the Duisburg-Essen university. He thanks the institute for its hospitality. The first author has been partially supported by JSPS KAKENHI #21740015.

References

[1] V. G. Berkovič, *Division by an isogeny of the points of an elliptic curve*, Mat. Sb. (N.S.) 93 (135) (1974), 467–486, 488.

[2] S. Bloch and K. Kato, *p-adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. 63 (1986), 107–152.

[3] K. Buzzard, *Analytic continuation of overconvergent eigenforms*, J. Amer. Math. Soc. 16 (2003), 29–55.

[4] J.-L. Colliot-Thélène, *Cycles algébriques de torsion et K-théorie algébrique*, in: Arithmetic Algebraic Geometry (Trento, 1991), Lecture Notes in Math. 1553, Springer, Berlin, 1993, 1–49.

[5] I. B. Fesenko and S. V. Vostokov, *Local Fields and Their Extensions*, 2nd ed., Transl. Math. Monogr. 121, Amer. Math. Soc., Providence, RI, 2002.

[6] A. Fröhlich, *Formal Groups*, Lecture Notes in Math. 74, Springer, Berlin, 1968.

[7] N. M. Katz, *p-adic properties of modular schemes and modular forms*, in: Modular Functions of One Variable, III (Antwerp, 1972), Lecture Notes in Math. 350, Springer, Berlin, 1973, 69–190.

[8] M. Kawachi, *Isogenies of degree p of elliptic curves over local fields and Kummer theory*, Tokyo J. Math. 25 (2002), 247–259.

[9] A. Mattuck, *Abelian varieties over p-adic ground fields*, Ann. of Math. (2) 62 (1955), 92–119.

[10] J. Murre and D. Ramakrishnan, *Local Galois symbols on $E \times E$*, in: Motives and Algebraic Cycles, Fields Inst. Comm. 56, Amer. Math. Soc., Providence, RI, 2009, 257–291.

[11] R. Parimala and V. Suresh, *Zero-cycles on quadric fibrations: finiteness theorems and the cycle map*, Invent. Math. 122 (1995), 83–117.

[12] W. Raskind and M. Spiess, *Milnor K-groups and zero-cycles on products of curves over p-adic fields*, Compos. Math. 121 (2000), 1–33.

[13] J.-P. Serre, *Corps locaux*, 2nd ed., Hermann, Paris, 1968.

[14] J.-P. Serre, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. 15 (1972), 259–331.

[15] M. Somekawa, *On Milnor K-groups attached to semi-abelian varieties*, K-Theory 4 (1990), 105–119.

[16] T. Takemoto, *Zero-cycles on products of elliptic curves over p-adic fields*, Acta Arith. 149 (2011), 201–214.

[17] T. Yamazaki, *On Chow and Brauer groups of a product of Mumford curves*, Math. Ann. 333 (2005), 549–567.

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Received on 27.5.2011
and in revised form on 1.3.2012 (6711)